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A stabilized finite element method for advection–diffusion equations on surfaces

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A recently developed Eulerian finite element method (FEM) is applied to solve advection–diffusion equations posed on hypersurfaces. When transport processes on a surface dominate over diffusion, FEMs tend to be unstable unless the mesh is sufficiently fine. The paper introduces a stabilized FE formulation based on the streamline upwind Petrov-Galerkin (SUPG) technique. An error analysis of the method is given. Results of numerical experiments are presented, which illustrate the performance of the stabilized method.

Keywords: surface PDE; finite element method; transport equations; advection–diffusion equation; SUPG stabilization.

1. Introduction

Mathematical models involving partial differential equations (PDEs) posed on hypersurfaces occur in many applications. Often surface equations are coupled with other equations that are formulated in a (fixed) domain which contains the surface. This happens, for example, in common models of multiphase fluid dynamics if one takes the so-called surface active agents into account (Gross & Reusken, 2011). The surface transport of such surfactants is typically driven by convection and surface diffusion and the relative strength of these two is measured by the dimensionless surface Peclet number $\mathrm{Pe}_s = UL/D_s$. Here U and L denote typical velocity and length scales, respectively, and D_s is the surface diffusion coefficient. Typical surfactants have surface diffusion coefficients in the range $D_s \sim 10^{-3}$–$10^{-5}\,\mathrm{cm}^2/\mathrm{s}$ (Agrawal & Neumann, 1988), leading to (very) large surface Peclet numbers in many applications. Hence, such applications result in advection–diffusion equations on the surface with dominating advection terms. The surface may evolve in time and be available only implicitly (for example, as a zero level of a level-set function).

It is well known that finite element (FE) discretization methods for advection–diffusion problems need an additional stabilization mechanism, unless the mesh size is sufficiently small to resolve boundary and internal layers in the solution of the differential equation. For the planar case, this topic has been studied extensively in the literature and a variety of stabilization methods has been developed; see, for example, Roos *et al.* (2008). However, we are not aware of any studies of stable finite element methods (FEMs) for advection–diffusion equations posed on surfaces.

In the past decade the study of numerical methods for PDEs on surfaces has been a rapidly growing research area. The development of FEMs for solving elliptic equations on surfaces can be traced back to the paper Dziuk (1988), which considers a piecewise polygonal surface and uses an FE space on a triangulation of this discrete surface. This approach has further been analysed and extended in several directions; see, for example, Dziuk & Elliott (2011) and the references therein. Another approach was introduced in Deckelnick *et al.* (2010) and builds on the ideas of Bertalmio *et al.* (2001). The method in that paper applies to cases in which the surface is given implicitly by some level-set function and the key idea is to solve the PDE on a narrow band around the surface. Unfitted FE spaces on this narrow band are used for discretization. Another surface FEM based on an outer (bulk) mesh was introduced in Olshanskii *et al.* (2009) and further studied in Olshanskii & Reusken (2009) and Demlow & Olshanskii (2012). The main idea of this method is to use FE spaces that are induced by triangulations of an outer domain to discretize the PDE on the surface by considering *traces* of the bulk FE space on the surface, instead of extending the PDE off the surface, as in Bertalmio *et al.* (2001) and Deckelnick *et al.* (2010). The method is particularly suitable for problems in which the surface is given implicitly by a level set or a volume of fluid (VOF) function and in which there is a coupling with a differential equation in a fixed outer domain. If in such problems one uses FE techniques for the discretization of equations in the outer domain, this setting immediately results in an easy-to-implement discretization method for the surface equation. The approach does not require additional surface elements.

In this paper we reconsider the volume mesh FEM from Olshanskii *et al.* (2009) and study a new aspect, which has not been studied in the literature so far, namely the stabilization of advection-dominated problems. We restrict ourselves to the case of a stationary surface. To stabilize the discrete problem for the case of large mesh Peclet numbers, we introduce a surface variant of the SUPG method. For a class of stationary advection–diffusion equations, an error analysis is presented. Although the convergence of the method is studied using a SUPG norm similar to the planar case (Roos *et al.*, 2008), the analysis is not standard and contains new ingredients: some new approximation properties for the traces of FEs are needed and geometric errors require special control. The main theoretical result is given in Theorem 3.11. It yields an error estimate in the SUPG norm which is almost robust in the sense that the dependence on the Peclet number is mild. This dependence is due to some insufficiently controlled geometric errors, as will be explained in Section 3.7.

The remainder of the paper is organized as follows. In Section 2, we recall equations for transport–diffusion processes on surfaces and present the stabilized FEM. Section 3 contains the theoretical results of the paper concerning the approximation properties of the FE space and discretization error bounds for the FEM. Finally, in Section 4 results of numerical experiments are given for both stationary and time-dependent advection-dominated surface transport–diffusion equations, which show that the stabilization performs well and that numerical results are consistent with what is expected from the SUPG method in the planar case.

2. Advection–diffusion equations on surfaces

Let Ω be an open domain in $\mathbb{R}^3$ and Γ be a connected C^2 compact hypersurface contained in Ω. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$, the tangential derivative at Γ is defined by

$$\nabla_\Gamma g = \nabla g - (\nabla g \cdot \mathbf{n}_\Gamma)\mathbf{n}_\Gamma, \tag{2.1}$$

where $\mathbf{n}_\Gamma$ denotes the unit normal to Γ. Denote by Δ_Γ the Laplace–Beltrami operator on Γ. Let $\mathbf{w} : \Omega \to \mathbb{R}^3$ be a given divergence-free ($\operatorname{div} \mathbf{w} = 0$) velocity field in Ω. If the surface Γ evolves with a normal velocity of $\mathbf{w} \cdot \mathbf{n}_\Gamma$, then the conservation of a scalar quantity u with a diffusive flux on $\Gamma(t)$ leads to the surface PDE

$$\dot{u} + (\operatorname{div}_\Gamma \mathbf{w})u - \varepsilon \Delta_\Gamma u = 0 \quad \text{on } \Gamma(t), \tag{2.2}$$

where $\dot{u}$ denotes the advective material derivative and ε is the diffusion coefficient. In Dziuk & Elliott (2007) problem (2.2) was shown to be well posed in a suitable weak sense.

In this paper, we study an FEM for an advection-dominated problem on a *steady* surface. Therefore, we assume that $\mathbf{w} \cdot \mathbf{n}_\Gamma = 0$, that is, the advection velocity is everywhere tangential to the surface. This and $\operatorname{div} \mathbf{w} = 0$ imply $\operatorname{div}_\Gamma \mathbf{w} = 0$, and the surface advection–diffusion equation takes the form

$$u_t + \mathbf{w} \cdot \nabla_\Gamma u - \varepsilon \Delta_\Gamma u = 0 \quad \text{on } \Gamma. \tag{2.3}$$

Although the methodology and numerical examples of the paper are applied to equation (2.3), the error analysis will be presented for the stationary problem

$$-\varepsilon \Delta_\Gamma u + \mathbf{w} \cdot \nabla_\Gamma u + c(\mathbf{x})u = f \quad \text{on } \Gamma, \tag{2.4}$$

with $f \in L^2(\Gamma)$ and $c(\mathbf{x}) \geqslant 0$. To simplify the presentation we assume $c(\cdot)$ to be constant, that is, $c(x) = c \geqslant 0$. The analysis, however, also applies to nonconstant c; cf. Section 3.7. Note that (2.3) and (2.4) can be written in intrinsic surface quantities, since $\mathbf{w} \cdot \nabla_\Gamma u = \mathbf{w}_\Gamma \cdot \nabla_\Gamma u$, with the tangential velocity $\mathbf{w}_\Gamma = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n}_\Gamma)\mathbf{n}_\Gamma$. We assume $\mathbf{w}_\Gamma \in H^{1,\infty}(\Gamma) \cap L^\infty(\Gamma)$ and scale the equation so that $\|\mathbf{w}_\Gamma\|_{L^\infty(\Gamma)} = 1$ holds. Furthermore, since we are interested in the advection-dominated case we take $\varepsilon \in (0, 1]$. Introduce the bilinear form and the functional

$$a(u, v) := \varepsilon \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, \mathrm{d}\mathbf{s} + \int_\Gamma (\mathbf{w} \cdot \nabla_\Gamma u) v \, \mathrm{d}\mathbf{s} + \int_\Gamma c\, uv \, \mathrm{d}\mathbf{s},$$

$$f(v) := \int_\Gamma fv \, \mathrm{d}\mathbf{s}.$$

The weak formulation of (2.4) is as follows: find $u \in V$ such that

$$a(u, v) = f(v) \quad \text{for all } v \in V, \tag{2.5}$$

with

$$V = \begin{cases} \left\{ v \in H^1(\Gamma) \,\middle|\, \int_\Gamma v \, \mathrm{d}\mathbf{s} = 0 \right\} & \text{if } c = 0, \\ H^1(\Gamma) & \text{if } c > 0. \end{cases}$$

Owing to the Lax–Milgram lemma, there exists a unique solution of (2.5). For the case $c = 0$, the following Friedrich's inequality (Sobolev, 1991) holds:

$$\|v\|^2_{L^2(\Gamma)} \leqslant C_{\mathrm{F}} \|\nabla_\Gamma v\|^2_{L^2(\Gamma)} \quad \text{for all } v \in V. \tag{2.6}$$

2.1 *The stabilized volume mesh FEM*

In this section, we recall the volume mesh FEM introduced in Olshanskii *et al.* (2009) and describe its SUPG-type stabilization.

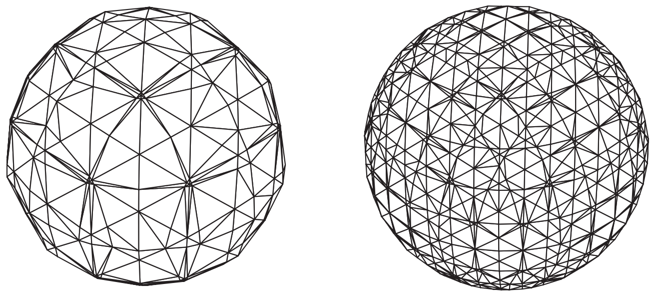

FIG. 1. Approximate interface Γ_h for a sphere, resulting from a coarse tetrahedral triangulation (left) and after one refinement (right).

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of tetrahedral triangulations of the domain Ω. These triangulations are assumed to be regular, consistent and stable. To simplify the presentation we assume that this family of triangulations is quasi-uniform. The latter assumption, however, is not essential for our analysis. We assume that for each $\mathcal{T}_h$ a polygonal approximation of Γ, denoted by Γ_h, is given: Γ_h is a $C^{0,1}$ surface without boundary and Γ_h can be partitioned into planar triangular segments. It is important to note that Γ_h is not a 'triangulation of Γ' in the usual sense (an $\mathcal{O}(h^2)$ approximation of Γ, consisting of regular triangles). Instead, we (only) assume that Γ_h is *consistent with the outer triangulation* $\mathcal{T}_h$ in the following sense. For any tetrahedron $S_T \in \mathcal{T}_h$ such that $\text{meas}_2(S_T \cap \Gamma_h) > 0$, define $T = S_T \cap \Gamma_h$. We assume that every $T \in \Gamma_h$ is a *planar* segment and thus it is either a triangle or a quadrilateral. Each quadrilateral segment can be divided into two triangles, so we may assume that every T is a triangle. An illustration of such a triangulation is given in Fig. 1. The results shown in this figure are obtained by representing a sphere Γ implicitly by its signed distance function, constructing the piecewise linear nodal interpolation of this distance function on a uniform tetrahedral triangulation $\mathcal{T}_h$ of Ω and then considering the zero level of this interpolant.

Let $\mathcal{F}_h$ be the set of all triangular segments T; then Γ_h can be decomposed as

$$\Gamma_h = \bigcup_{T \in \mathcal{F}_h} T. \tag{2.7}$$

Note that the triangulation $\mathcal{F}_h$ is not necessarily regular, that is, elements from T may have very small internal angles and the size of neighbouring triangles can vary strongly; cf. Fig. 1. In applications with level-set functions (that represent Γ implicitly), the approximation Γ_h can be obtained as the zero level of a piecewise linear FE approximation of the level-set function on the tetrahedral triangulation $\mathcal{T}_h$.

The surface FE space is *the space of traces on Γ_h of all piecewise linear continuous functions with respect to the outer triangulation* $\mathcal{T}_h$. This can be formally defined as follows. We define a subdomain that contains Γ_h:

$$\omega_h = \bigcup_{T \in \mathcal{F}_h} S_T, \tag{2.8}$$

and a corresponding volume mesh FE space:

$$V_h := \{v_h \in C(\omega_h) \mid v_h|_{S_T} \in P_1 \ \text{ for all } T \in \mathcal{F}_h\}, \tag{2.9}$$

where P_1 is the space of polynomials of degree one. V_h induces the following space on Γ_h:

$$V_h^\Gamma := \{\psi_h \in H^1(\Gamma_h) \mid \exists v_h \in V_h \text{ such that } \ \psi_h = v_h|_{\Gamma_h}\}. \tag{2.10}$$

When $c = 0$, we require that any function v_h from V_h^Γ satisfies $\int_{\Gamma_h} v_h \,\mathbf{ds} = 0$. Given the surface FE space V_h^Γ, the FE discretization of (2.5) is as follows: find $u_h \in V_h^\Gamma$ such that

$$\varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v \,\mathbf{ds} + \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} u) v \,\mathbf{ds} + \int_{\Gamma_h} c\, uv \,\mathbf{ds} = \int_{\Gamma_h} f^e v \,\mathbf{ds} \tag{2.11}$$

for all $v_h \in V_h^\Gamma$. Here $\mathbf{w}^{\mathrm{e}}$ and f^{e} are the extensions of $\mathbf{w}_\Gamma$ and f, respectively, along normals to Γ (the precise definition is given in the next section). Similarly to the plain Galerkin FE for advection–diffusion equations, method (2.11) is unstable unless the mesh is sufficiently fine such that the mesh Peclet number is less than 1.

We introduce the following stabilized FEM based on the standard SUPG approach (cf. Roos *et al.*, 2008): find $u_h \in V_h^\Gamma$ such that

$$a_h(u_h, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h^\Gamma, \tag{2.12}$$

with

$$\begin{aligned} a_h(u, v) &:= \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v \,\mathbf{ds} + \int_{\Gamma_h} cuv \,\mathbf{ds} \\ &\quad + \frac{1}{2}\left[\int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} u) v \,\mathbf{ds} - \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v) u \,\mathbf{ds}\right] \\ &\quad + \sum_{T \in \mathcal{F}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} u + \mathbf{w}^e \cdot \nabla_{\Gamma_h} u + c\, u) \mathbf{w}^e \cdot \nabla_{\Gamma_h} v \,\mathbf{ds}, \end{aligned} \tag{2.13}$$

$$f_h(v) := \int_{\Gamma_h} f^e v \,\mathbf{ds} + \sum_{T \in \mathcal{F}_h} \delta_T \int_T f^e (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v) \,\mathbf{ds}. \tag{2.14}$$

The stabilization parameter δ_T depends on $T \subset S_T$. The diameter of the tetrahedron S_T is denoted by h_{S_T}. Let $\mathrm{Pe}_T := h_{S_T} \|\mathbf{w}^e\|_{L^\infty(T)} / 2\varepsilon$ be the cell Peclet number. We take

$$\tilde{\delta}_T = \begin{cases} \dfrac{\delta_0 h_{S_T}}{\|\mathbf{w}^e\|_{L^\infty(T)}} & \text{if } \mathsf{Pe}_T > 1, \\ \dfrac{\delta_1 h_{S_T}^2}{\varepsilon} & \text{if } \mathsf{Pe}_T \leqslant 1, \end{cases} \quad \text{and} \quad \delta_T = \min\{\tilde{\delta}_T, c^{-1}\}, \tag{2.15}$$

with some given positive constants $\delta_0, \delta_1 \geqslant 0$.

Since $u_h \in V_h^\Gamma$ is linear on every T we have $\Delta_{\Gamma_h} u_h = 0$ on T, and thus $a_h(u_h, v_h)$ simplifies to

$$\begin{aligned} a_h(u_h, v_h) &= \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \,\mathbf{ds} + \frac{1}{2}\left[\int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} u_h) v_h - (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h) u_h \,\mathbf{ds}\right] \\ &\quad + \int_{\Gamma_h} c\, u_h (v_h + \delta(\mathbf{x}) \mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h) \,\mathbf{ds} + \int_{\Gamma_h} \delta(\mathbf{x}) (\mathbf{w}^e \cdot \nabla_{\Gamma_h} u_h)(\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h) \,\mathbf{ds}, \end{aligned} \tag{2.16}$$

where $\delta(\mathbf{x}) = \delta_T$ for $\mathbf{x} \in T$.

3. Error analysis

The analysis in this section is organized as follows. First we collect some definitions and useful results in Section 3.1. In Section 3.2, we derive a coercivity result. In Section 3.3, we present interpolation error bounds. In Sections 3.4 and 3.5, continuity and consistency results are derived. Combining this analysis we obtain the FE error bound given in Section 3.6. In the error analysis we use the following mesh-dependent norm:

$$\|u\|_* := \left(\varepsilon \int_{\Gamma_h} |\nabla_{\Gamma_h} u|^2 \, \mathrm{d}\mathbf{s} + \int_{\Gamma_h} \delta(\mathbf{x}) |\mathbf{w}^e \cdot \nabla_{\Gamma_h} u|^2 \, \mathrm{d}\mathbf{s} + \int_{\Gamma_h} c\, |u|^2 \, \mathrm{d}\mathbf{s}\right)^{1/2}. \tag{3.1}$$

Here and in the remainder, $|\cdot|$ denotes the Euclidean norm for vectors and the corresponding spectral norm for matrices.

3.1 *Preliminaries*

For the hypersurface Γ, we define its h-neighbourhood,

$$U_h := \{\mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma) < c_0\, h\}, \tag{3.2}$$

and assume that c_0 is sufficiently large such that $\omega_h \subset U_h$, and h sufficiently small such that

$$5 c_0\, h < \left(\max_{i=1,2} \|\kappa_i\|_{L^\infty(\Gamma)}\right)^{-1} \tag{3.3}$$

holds, with κ_i being the principal curvatures of Γ. Here and in what follows, h denotes the maximum diameter for tetrahedra of the outer triangulation: $h = \max_{S \in \omega_h} \mathrm{diam}(S)$.

Let $d : U_h \to \mathbb{R}$ be the signed distance function, $|d(\mathbf{x})| = \mathrm{dist}(\mathbf{x}, \Gamma)$ for all $\mathbf{x} \in U_h$. Thus, Γ is the zero level set of d. We assume $d < 0$ in the interior of Γ and $d > 0$ in the exterior and define $\mathbf{n}(\mathbf{x}) := \nabla d(\mathbf{x})$ for all $\mathbf{x} \in U_h$. Hence, $\mathbf{n} = \mathbf{n}_\Gamma$ on Γ and $|\mathbf{n}(\mathbf{x})| = 1$ for all $\mathbf{x} \in U_h$. The Hessian of d is denoted by

$$\mathbf{H}(\mathbf{x}) := \nabla^2 d(\mathbf{x}) \in \mathbb{R}^{3\times 3}, \quad \mathbf{x} \in U_h. \tag{3.4}$$

The eigenvalues of $\mathbf{H}(\mathbf{x})$ are denoted by $\kappa_1(\mathbf{x})$, $\kappa_2(\mathbf{x})$ and 0. For $\mathbf{x} \in \Gamma$ the eigenvalues $\kappa_i,\ i = 1, 2$ are the principal curvatures.

For each $\mathbf{x} \in U_h$, define the projection $\mathbf{p} : U_h \to \Gamma$ by

$$\mathbf{p}(\mathbf{x}) = \mathbf{x} - d(\mathbf{x})\mathbf{n}(\mathbf{x}). \tag{3.5}$$

Owing to assumption (3.3), the decomposition $\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\mathbf{n}(\mathbf{x})$ is unique. We will need the orthogonal projector

$$\mathbf{P}(\mathbf{x}) := \mathbf{I} - \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x})^{\mathrm{T}} \quad \text{for } \mathbf{x} \in U_h.$$

Note that $\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{p}(\mathbf{x}))$ and $\mathbf{P}(\mathbf{x}) = \mathbf{P}(\mathbf{p}(\mathbf{x}))$ for $\mathbf{x} \in U_h$ holds. The tangential derivative can be written as $\nabla_\Gamma g(\mathbf{x}) = \mathbf{P}\nabla g(\mathbf{x})$ for $\mathbf{x} \in \Gamma$. One can verify that for this projection and for the Hessian $\mathbf{H}$ the relation

$\mathbf{HP} = \mathbf{PH} = \mathbf{H}$ holds. Similarly, define

$$\mathbf{P}_h(\mathbf{x}) := \mathbf{I} - \mathbf{n}_{\Gamma_h}(\mathbf{x})\mathbf{n}_{\Gamma_h}(\mathbf{x})^{\mathrm{T}} \quad \text{for } \mathbf{x} \in \Gamma_h,\ \mathbf{x} \text{ is not on an edge,} \tag{3.6}$$

where $\mathbf{n}_{\Gamma_h}$ is the unit (outward-pointing) normal at $\mathbf{x} \in \Gamma_h$ (not on an edge). The tangential derivative along Γ_h is given by $\nabla_{\Gamma_h} g(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})\nabla g(\mathbf{x})$ (not on an edge).

ASSUMPTION 3.1 In this paper, we assume that for all $T \in \mathcal{F}_h$,

$$\operatorname*{ess\,sup}_{\mathbf{x}\in T} |d(\mathbf{x})| \leqslant c_1 h_{S_T}^2, \tag{3.7}$$

$$\operatorname*{ess\,sup}_{\mathbf{x}\in T} |\mathbf{n}(\mathbf{x}) - \mathbf{n}_{\Gamma_h}(\mathbf{x})| \leqslant c_2 h_{S_T}, \tag{3.8}$$

where h_{S_T} denotes the diameter of the tetrahedron S_T that contains T, that is, $T = S_T \cap \Gamma_h$ and constants c_1 and c_2 are independent of h and T.

Assumptions (3.7) and (3.8) describe how accurate the piecewise planar approximation Γ_h of Γ is. If Γ_h is constructed as the zero level of a piecewise linear interpolation of a level-set function that characterizes Γ (as in Fig. 1), then these assumptions are fulfilled; cf. Gross & Reusken (2011, Section 7.3).

In the remainder, $A \lesssim B$ means $A \leqslant \tilde{c}B$ for some positive constant $\tilde{c}$ independent of h and of the problem parameters ε and c. $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$.

For $\mathbf{x} \in \Gamma_h$, define

$$\mu_h(\mathbf{x}) = (1 - d(\mathbf{x})\kappa_1(\mathbf{x}))(1 - d(\mathbf{x})\kappa_2(\mathbf{x}))\mathbf{n}^{\mathrm{T}}(\mathbf{x})\mathbf{n}_h(\mathbf{x}).$$

The surface measures $\mathrm{d}\mathbf{s}$ and $\mathrm{d}\mathbf{s}_h$ on Γ and Γ_h, respectively, are related by

$$\mu_h(\mathbf{x})\,\mathrm{d}\mathbf{s}_h(\mathbf{x}) = \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in \Gamma_h. \tag{3.9}$$

Assumptions (3.7) and (3.8) imply that

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma_h} (1 - \mu_h) \lesssim h^2; \tag{3.10}$$

cf. Olshanskii *et al.* (2009, (3.37)). The solution of (2.4) is defined on Γ, while its FE approximation $u_h \in V_h^\Gamma$ is defined on Γ_h. We need a suitable extension of a function from Γ to its neighbourhood. For a function v on Γ we define

$$v^e(\mathbf{x}) := v(\mathbf{p}(\mathbf{x})) \quad \text{for all } \mathbf{x} \in U_h. \tag{3.11}$$

The following formulas for this lifting function are known (cf. Demlow & Dziuk, 2007, Section 2.3):

$$\nabla u^e(\mathbf{x}) = (\mathbf{I} - d(\mathbf{x})\mathbf{H})\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on } U_h, \tag{3.12}$$

$$\nabla_{\Gamma_h} u^e(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})(\mathbf{I} - d(\mathbf{x})\mathbf{H})\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on } \Gamma_h, \tag{3.13}$$

with $\mathbf{H} = \mathbf{H}(\mathbf{x})$. By direct computation one derives the relation

$$\begin{aligned}\nabla^2 u^e(\mathbf{x}) = {}& (\mathbf{P} - d(\mathbf{x})\mathbf{H})\nabla_\Gamma^2 u(\mathbf{p}(\mathbf{x}))(\mathbf{P} - d(\mathbf{x})\mathbf{H}) - \mathbf{n}^{\mathrm{T}}\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))\mathbf{H} \\ & - (\mathbf{H}\nabla_\Gamma u(\mathbf{p}(\mathbf{x})))\mathbf{n}^{\mathrm{T}} - \mathbf{n}(\mathbf{H}\nabla_\Gamma u(\mathbf{p}(\mathbf{x})))^{\mathrm{T}} - d\nabla_\Gamma \mathbf{H} : \nabla_\Gamma u(\mathbf{p}(\mathbf{x})).\end{aligned} \tag{3.14}$$

For sufficiently smooth u and $|\mu| \leqslant 2$, using this relation one obtains the estimate

$$|D^\mu u^e(\mathbf{x})| \lesssim \left(\sum_{|\mu|=2} |D^\mu_\Gamma u(\mathbf{p}(\mathbf{x}))| + |\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))| \right) \quad \text{a.e. on } U_h \tag{3.15}$$

(cf. Dziuk, 1988, Lemma 3). This further leads to (cf. Olshanskii *et al.*, 2009, Lemma 3.2)

$$\|D^\mu u^e\|_{L^2(U_h)} \lesssim \sqrt{h}\|u\|_{H^2(\Gamma)}, \quad |\mu| \leqslant 2. \tag{3.16}$$

The next lemma is needed for the analysis in the following section.

LEMMA 3.2 The following holds:

$$\|\mathrm{div}_{\Gamma_h} \mathbf{w}^e\|_{L^\infty(\Gamma_h)} \lesssim h\|\nabla_\Gamma \mathbf{w}\|_{L^\infty(\Gamma)}.$$

Proof. We use the following representation for the tangential divergence:

$$\mathrm{div}_\Gamma \mathbf{w}(\mathbf{x}) = \mathrm{tr}(\nabla_\Gamma \mathbf{w}(\mathbf{x})) = \mathrm{tr}(\mathbf{P}\nabla \mathbf{w}(\mathbf{x})). \tag{3.17}$$

Take $\mathbf{x} \in \Gamma_h$, not lying on an edge. Using (3.12) we obtain

$$\begin{aligned} \mathrm{div}_{\Gamma_h} \mathbf{w}^e(\mathbf{x}) &= \mathrm{tr}(\mathbf{P}_h \nabla \mathbf{w}^e(\mathbf{x})) = \mathrm{tr}(\mathbf{P}_h(\mathbf{I} - d(\mathbf{x})\mathbf{H})\nabla_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x}))) \\ &= \mathrm{tr}(\mathbf{P}\nabla_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x}))) + \mathrm{tr}((\mathbf{P}_h - \mathbf{P})\nabla_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x}))) - d(\mathbf{x})\,\mathrm{tr}(\mathbf{P}_h\mathbf{H}\nabla_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x}))). \end{aligned}$$

The first term vanishes due to $\mathrm{tr}(\mathbf{P}\nabla_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x}))) = \mathrm{div}_\Gamma \mathbf{w}(\mathbf{p}(\mathbf{x})) = 0$. The second and the third terms can be bounded using (3.7) and (3.8):

$$|\mathbf{P}_h - \mathbf{P}| \lesssim h, \quad |d(\mathbf{x})\mathbf{P}_h\mathbf{H}| \lesssim h^2.$$

This proves the lemma. □

3.2 *Coercivity analysis*

In the next lemma we present a coercivity result. We use the norm introduced in (3.1).

LEMMA 3.3 The following holds:

$$a_h(v_h, v_h) \geqslant \tfrac{1}{2}\|v_h\|_*^2 \quad \text{for all } v_h \in V_h^\Gamma. \tag{3.18}$$

Proof. For any $v_h \in V_h^\Gamma$, we have

$$a_h(v_h, v_h) = \|v_h\|_*^2 + \int_{\Gamma_h} c\,\delta(\mathbf{x})v_h(\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\,\mathrm{d}\mathbf{s}. \tag{3.19}$$

The choice of δ_T (cf. (2.15)) implies $c\,\delta(\mathbf{x}) \leqslant 1$. Hence the last term in (3.19) can be estimated as follows:

$$\left| \int_{\Gamma_h} c\delta(\mathbf{x})v_h(\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\,\mathrm{d}\mathbf{s} \right| \leqslant \frac{1}{2}\int_{\Gamma_h} cv_h^2\,\mathrm{d}\mathbf{s} + \frac{1}{2}\int_{\Gamma_h} c\delta(\mathbf{x})^2(\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)^2\,\mathrm{d}\mathbf{s} \leqslant \frac{1}{2}\|v_h\|_*^2.$$

This yields (3.18). □

As a consequence of this result, we obtain the well-posedness of the discrete problem (2.12).

3.3 *Interpolation error bounds*

Let $I_h : C(\bar{\omega}_h) \to V_h$ be the nodal interpolation operator. For any $u \in H^2(\Gamma)$ the surface FE function $(I_h u^e)|_{\Gamma_h} \in V_h^\Gamma$ is an interpolant of u^e in V_h^Γ.

For any $u \in H^2(\Gamma)$ the following estimates hold (Olshanskii *et al.*, 2009):

$$\|u^e - (I_h u^e)|_{\Gamma_h}\|_{L^2(\Gamma_h)} \lesssim h^2 \|u\|_{H^2(\Gamma)}, \tag{3.20}$$

$$\|\nabla_{\Gamma_h} u^e - \nabla_{\Gamma_h}(I_h u^e)|_{\Gamma_h}\|_{L^2(\Gamma_h)} \lesssim h \|u\|_{H^2(\Gamma)}. \tag{3.21}$$

Using these results we easily obtain an interpolation error estimate in the $\|\cdot\|_*$ norm.

LEMMA 3.4 For any $u \in H^2(\Gamma)$ the following holds:

$$\|u^e - (I_h u^e)|_{\Gamma_h}\|_* \lesssim h(\varepsilon^{1/2} + h^{1/2} + c^{1/2} h) \|u\|_{H^2(\Gamma)}. \tag{3.22}$$

Proof. Define $\varphi := u^e - (I_h u^e)|_{\Gamma_h} \in H^1(\Gamma_h)$. Using definition (2.15) of $\delta(\mathbf{x})$, we obtain

$$\begin{aligned} \int_{\Gamma_h} \delta(\mathbf{x}) |\mathbf{w}^e \cdot \nabla_{\Gamma_h} \varphi|^2 \, \mathrm{d}\mathbf{s} &= \sum_{T \in \mathcal{F}_h} \int_T \delta_T |\mathbf{w}^e \cdot \nabla_{\Gamma_h} \varphi|^2 \, \mathrm{d}\mathbf{s} \\ &\lesssim h \|\nabla_{\Gamma_h} \varphi\|^2_{L^2(\Gamma_h)} \lesssim h^3 \|u\|^2_{H^2(\Gamma)}. \end{aligned} \tag{3.23}$$

The remaining two terms in $\|u^e - (I_h u^e)\|_*$ are estimated in a straightforward way using (3.20) and (3.21). This and (3.23) imply the inequality (3.22). □

The next lemma estimates the interpolation error on the edges of the surface triangulation. In the remainder, $\mathcal{E}_h$ denotes the set of all edges in the interface triangulation $\mathcal{F}_h$.

LEMMA 3.5 For all $u \in H^2(\Gamma)$ the following holds:

$$\left(\sum_{E \in \mathcal{E}_h} \int_E (u^e - I_h u^e)|^2_{\Gamma_h} \, \mathrm{d}\mathbf{s} \right)^{1/2} \lesssim h^{3/2} \|u\|_{H^2(\Gamma)}. \tag{3.24}$$

Proof. Define $\phi := u^e - I_h u^e \in H^1(\omega_h)$. Take $E \in \mathcal{E}_h$ and let $T \in \mathcal{F}_h$ be a corresponding planar segment of which E is an edge. Let W be a side of the tetrahedron S_T such that $E \subset W$. From Hansbo & Hansbo (2002, Lemma 3), we have

$$\|\phi\|^2_{L^2(E)} \lesssim h^{-1} \|\phi\|^2_{L^2(W)} + h \|\phi\|^2_{H^1(W)}.$$

From the standard trace inequality

$$\|w\|^2_{L^2(\partial S_T)} \lesssim h^{-1} \|w\|^2_{L^2(S_T)} + h \|w\|^2_{H^1(S_T)} \quad \text{for all } w \in H^1(S_T),$$

applied to ϕ and $\partial_{x_i} \phi$, where $i = 1, 2, 3$, we obtain

$$\begin{aligned} h^{-1} \|\phi\|^2_{L^2(W)} &\lesssim h^{-2} \|\phi\|^2_{L^2(S_T)} + \|\phi\|^2_{H^1(S_T)}, \\ h \|\phi\|^2_{H^1(W)} &\lesssim \|\phi\|^2_{H^1(S_T)} + h^2 \|u^e\|^2_{H^2(S_T)}. \end{aligned}$$

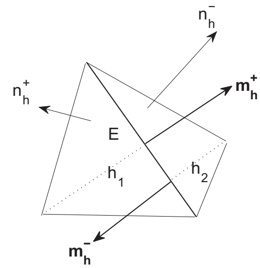

FIG. 2. Two neighboring triangles may not lie in the same plane, so the normal vectors $\mathbf{m}_h^+$ and $\mathbf{m}_h^-$ to a common edge E are not necessarily collinear.

From standard error bounds for the nodal interpolation operator I_h we obtain

$$\|\phi\|_{L^2(E)}^2 \lesssim h^{-2}\|\phi\|_{L^2(S_T)}^2 + \|\phi\|_{H^1(S_T)}^2 + h^2\|u^e\|_{H^2(S_T)}^2 \lesssim h^2\|u^e\|_{H^2(S_T)}^2.$$

Summing over $E \in \mathcal{E}_h$ and using $\|u^e\|_{H^2(\omega_h)} \lesssim h^{1/2}\|u\|_{H^2(\Gamma)}$, (cf. (3.16)) results in

$$\sum_{E\in\mathcal{E}_h} \|\phi\|_{L^2(E)}^2 \lesssim h^2\|u^e\|_{H^2(\omega_h)}^2 \lesssim h^3\|u\|_{H^2(\Gamma)}^2,$$

which completes the proof. □

3.4 *Continuity estimates*

In this section we derive a continuity estimate for the bilinear form $a_h(\cdot,\cdot)$. If one applies partial integration to the integrals that occur in $a_h(\cdot,\cdot)$ then *jumps across the edges* $E \in \mathcal{E}_h$ occur. We start with a lemma that yields bounds for such jump terms. Related to these jump terms we introduce the following notation. For each $T \in \mathcal{F}_h$, denote by $\mathbf{m}_h|_E$ the outer normal to an edge E in the plane which contains element T. Let $[\mathbf{m}_h]|_E = \mathbf{m}_h^+ + \mathbf{m}_h^-$ be the jump of the outer normals to the edge in two neighbouring elements; cf. Fig. 2.

LEMMA 3.6 The following holds:

$$|\mathbf{P}(\mathbf{x})[\mathbf{m}_h](\mathbf{x})| \lesssim h^2 \quad \text{a.e. } \mathbf{x} \in E. \tag{3.25}$$

Proof. Let E be the common side of two elements T_1 and T_2 in $\mathcal{F}_h$, and $\mathbf{n}_h^+$, $\mathbf{n}_h^-$, $\mathbf{m}_h^+$ and $\mathbf{m}_h^-$ are the unit normals as illustrated in Fig. 2. Denote by $\mathbf{s}_h$ the unit (constant) vector along the common side E, which can be represented as $\mathbf{s}_h = \mathbf{n}_h^+ \times \mathbf{m}_h^+ = \mathbf{m}_h^- \times \mathbf{n}_h^-$. The jump across E is given by

$$[\mathbf{m}_h] = \mathbf{s}_h \times (\mathbf{n}_h^+ - \mathbf{n}_h^-).$$

For each $\mathbf{x} \in E$ and $\mathbf{p}(\mathbf{x}) \in \Gamma$, let $\mathbf{n} = \mathbf{n}(\mathbf{p}(\mathbf{x}))$ be the unit normal to Γ at $\mathbf{p}(\mathbf{x})$ and $\mathbf{P} = \mathbf{P}(\mathbf{x}) = \mathbf{I} - \mathbf{n}\mathbf{n}^{\mathrm{T}}$ the corresponding orthogonal projection. Using (3.8), we obtain

$$|\mathbf{n}_h^- - \mathbf{n}_h^+| \leqslant |\mathbf{n}_h^+ - \mathbf{n}| + |\mathbf{n}_h^- - \mathbf{n}| \lesssim h_{S_{T_1}} + h_{S_{T_2}} \lesssim h.$$

Since $|\mathbf{n}_h^-| = |\mathbf{n}_h^+| = |\mathbf{n}| = 1$, the above estimate implies

$$\mathbf{n}_h^+ - \mathbf{n}_h^- = ch^2\mathbf{n} + \mathbf{e}_1, \quad \mathbf{e}_1 \perp \mathbf{n}, \quad |\mathbf{e}_1| \lesssim h.$$

We also have

$$\mathbf{s}_h = \mathbf{n}_h^+ \times \mathbf{m}_h^+ = (\mathbf{n} + (\mathbf{n}_h^+ - \mathbf{n})) \times \mathbf{m}_h^+ = \mathbf{n} \times \mathbf{m}_h^+ + \mathbf{e}_2, \quad |\mathbf{e}_2| \lesssim h.$$

We use the decomposition

$$\mathbf{P}[\mathbf{m}_h] = \mathbf{P}[(\mathbf{n} \times \mathbf{m}_h^+ + \mathbf{e}_2) \times (ch^2\mathbf{n} + \mathbf{e}_1)].$$

Since $\mathbf{e}_1 \perp \mathbf{n}$ we have $(\mathbf{n} \times \mathbf{m}_h^+) \times \mathbf{e}_1 \parallel \mathbf{n}$ and thus $\mathbf{P}((\mathbf{n} \times \mathbf{m}_h^+) \times \mathbf{e}_1) = 0$. Therefore, we obtain

$$|\mathbf{P}[\mathbf{m}_h]| \lesssim h^2 + |\mathbf{e}_1|\,|\mathbf{e}_2| \lesssim h^2, \tag{3.26}$$

that is, the result (3.25) holds. □

In the analysis below, we need an inequality of the form $\|v_h\|_{L^2(\Gamma_h)} \lesssim \|v_h\|_*$ for all $v_h \in V_h^\Gamma$. This result can be obtained as follows. First we consider the case $c = 0$. Then the functions $v_h \in V_h^\Gamma$ satisfy $\int_{\Gamma_h} v_h \,\mathrm{d}\mathbf{s} = 0$. We assume that in V_h^Γ a discrete analogy of Friedrich's inequality (2.6) holds uniformly with respect to h, that is, there exists a constant C_{F} independent of h such that

$$\|v_h\|^2_{L^2(\Gamma_h)} \leqslant C_{\mathrm{F}} \|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)} \quad \text{for all } v_h \in V_h^\Gamma. \tag{3.27}$$

Now we reduce the parameter domain $\varepsilon \in (0, 1]$, $c > 0$ as follows. For a given generic constant c_0 with $0 < c_0 < 1$, in the remainder we restrict our attention to the parameter set

$$\varepsilon \in (0, 1], \quad c \in \{0\} \cup [c_0\varepsilon, \infty). \tag{3.28}$$

For $c > 0$ we then have

$$c_0\|v_h\|^2_{L^2(\Gamma_h)} \leqslant \frac{2c_0}{c_0+1}\frac{1}{c}\|c^{1/2}v_h\|^2_{L^2(\Gamma_h)} \leqslant \frac{2}{c/c_0 + c}\|v_h\|_*^2 \leqslant \frac{2}{\varepsilon + c}\|v_h\|_*^2,$$

and combining this with the result in (3.27) for the case $c = 0$ we obtain

$$\|v_h\|_{L^2(\Gamma_h)} \lesssim \frac{1}{\sqrt{\varepsilon + c}}\|v_h\|_* \quad \text{for all } v_h \in V_h^\Gamma \tag{3.29}$$

and arbitrary $\varepsilon \in (0, 1]$, $c \in \{0\} \cup [c_0\varepsilon, \infty)$.

In the proof of Theorem 3.9 we need a bound for $\|v_h\|_{L^2(\mathcal{E}_h)}$ in terms of $\|v_h\|_*$. Such a result is derived with the help of the following lemma.

Lemma 3.7 Assume that the outer tetrahedra mesh size satisfies $h \leqslant h_0$, with some sufficiently small $h_0 \simeq 1$, depending only on the constant c_2 from (3.8). The following holds:

$$\sum_{E \in \mathcal{E}_h} \int_E v_h^2 \,\mathrm{d}\mathbf{s} \lesssim h^{-1}\|v_h\|^2_{L^2(\Gamma_h)} + h\|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)} \quad \text{for all } v_h \in V_h^\Gamma. \tag{3.30}$$

Proof. Let $E \in \mathcal{E}_h$ be an edge of a triangle $T \in \mathcal{F}_h$ and $S_T \in \mathcal{T}_h$ is the corresponding tetrahedron of the outer triangulation. Consider the patch $\tilde{\omega}(S_T)$ of all $S \in \mathcal{T}_h$ touching S_T. Denote $\omega(S_T) = \tilde{\omega}(S_T) \cap \Gamma_h$. Let v_h be an arbitrary fixed function from V_h^Γ. We shall prove the bound

$$\int_E v_h^2 \, \mathbf{ds} \lesssim h^{-1} \|v_h\|^2_{L^2(\omega(S_T))} + h \|\nabla_{\Gamma_h} v_h\|^2_{L^2(\omega(S_T))}. \tag{3.31}$$

Then summing over all $E \in \mathcal{E}_h$ and using that $\omega(S_T)$ consists of a uniformly bounded number of tetrahedra (due to the regularity of the outer mesh), we obtain (3.30).

Let $\mathbb{P}$ be a plane containing T. We can define, for sufficiently small h, an injective mapping $\phi : \omega(S_T) \to \mathbb{P}$ such that $|\nabla \phi| \lesssim 1$ and $|\nabla(\phi^{-1})| \lesssim 1$. For example, ϕ can be built by the orthogonal projection on $\mathbb{P}$. Then $|\nabla \phi| \lesssim 1$ and $|\nabla(\phi^{-1})(\mathbf{x})| \lesssim (\sin\alpha)^{-1}$, where α is the angle between $\mathbb{P}$ and $\mathbf{n}_h(\phi^{-1}(\mathbf{x}))$. Because of assumption (3.8) we have $1 \lesssim \sin\alpha$ for sufficiently small h. If ϕ is the orthogonal projection on $\mathbb{P}$, then $\phi(E) = E$. Thus, we obtain

$$\begin{cases} \displaystyle\int_E v_h^2 \mathbf{ds} = \int_{\phi(E)} (v_h \circ \phi^{-1})^2 \mathbf{ds}, \\ \|v_h\|_{L^2(\omega(S_T))} \simeq \|v_h \circ \phi^{-1}\|_{L^2(\phi(\omega(S_T)))}, \\ \|\nabla_{\Gamma_h} v_h\|_{L^2(\omega(S_T))} \simeq \|\nabla_{\mathbb{P}}(v_h \circ \phi^{-1})\|_{L^2(\phi(\omega(S_T)))}. \end{cases} \tag{3.32}$$

Owing to the shape regularity of $S \in \tilde{\omega}(S_T)$ we have

$$h \lesssim \operatorname{dist}(E, \partial\, \tilde{\omega}(S_T)) \leqslant \operatorname{dist}(E, \partial\, \omega(S_T)).$$

Hence, from $|\nabla\phi^{-1}| \lesssim 1$ it follows that $h \lesssim \operatorname{dist}(\phi(E), \partial\, \phi(\omega(S_T)))$. Thus, we may consider a rectangle $Q \subset \phi(\omega(S_T))$ such that $E = \phi(E)$ is a side of Q and $|Q| \simeq h|E|$. By the standard trace theorem and a scaling argument we obtain

$$\int_{\phi(E)} (v_h \circ \phi^{-1})^2 \mathbf{ds} \lesssim h^{-1} \|v_h \circ \phi^{-1}\|^2_{L^2(Q)} + h \|\nabla_{\mathbb{P}}(v_h \circ \phi^{-1})\|^2_{L^2(Q)}.$$

This together with (3.32) and $Q \subset \phi(\omega(S_T))$ implies (3.31). □

An immediate consequence of the lemma and (3.29) is the following corollary.

Corollary 3.8 The following estimate holds:

$$h \sum_{E \in \mathcal{E}_h} \int_E v_h^2 \, \mathbf{ds} \lesssim \left(\frac{1}{\varepsilon + c} + \frac{h^2}{\varepsilon} \right) \|v_h\|_*^2 \quad \text{for all } v_h \in V_h^\Gamma. \tag{3.33}$$

We are now in a position to prove a continuity result for the surface FE bilinear form.

Lemma 3.9 For any $u \in H^2(\Gamma)$ and $v_h \in V_h^\Gamma$, we have

$$|a_h(u^e - (I_h u^e)|_{\Gamma_h}, v_h)| \lesssim \left(\varepsilon^{1/2} + h^{1/2} + c^{1/2} h + \frac{h^2}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) h \|u\|_{H^2(\Gamma)} \|v_h\|_*. \tag{3.34}$$

Proof. Define $\phi = u^e - (I_h u^e)|_{\Gamma_h}$, then

$$a_h(\phi, v_h) = \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h}\phi \cdot \nabla_{\Gamma_h} v_h \mathrm{d}\mathbf{s} + \int_{\Gamma_h} \frac{1}{2}((\mathbf{w}^e \cdot \nabla_{\Gamma_h}\phi)v_h - (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\phi) + c\,\phi v_h \,\mathrm{d}\mathbf{s} + \sum_{T\in\mathcal{F}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h}\phi + \mathbf{w}^e \cdot \nabla_{\Gamma_h}\phi + c\phi)\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}\mathbf{s}. \tag{3.35}$$

We estimate $a_h(\phi, v_h)$ term by term. Because of (3.20) and (3.21), for the first term on the right-hand side of (3.35) we obtain

$$\left|\varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h}\phi \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}\mathbf{s}\right| \lesssim \varepsilon h \|u\|_{H^2(\Gamma)} \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} \leqslant \varepsilon^{1/2} h \|u\|_{H^2(\Gamma)} \|v_h\|_*. \tag{3.36}$$

To the second term on the right-hand side of (3.35) we apply integration by parts:

$$\begin{aligned}
&\int_{\Gamma_h} \frac{1}{2}((\mathbf{w}^e \cdot \nabla_{\Gamma_h}\phi)v_h - (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\phi) + c\phi v_h \,\mathrm{d}\mathbf{s}\\
&= \int_{\Gamma_h} c\phi v_h \,\mathrm{d}\mathbf{s} - \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\phi \,\mathrm{d}\mathbf{s}\\
&\quad + \frac{1}{2}\sum_{T\in\mathcal{F}_h} \int_{\partial T} (\mathbf{w}^e \cdot \mathbf{m}_h)\phi v_h \,\mathrm{d}\mathbf{s} - \frac{1}{2}\int_{\Gamma_h} (\mathrm{div}_{\Gamma_h} \mathbf{w}^e)\phi v_h \,\mathrm{d}\mathbf{s}\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.37}$$

The term I_1 can be estimated by

$$|I_1| \lesssim c^{1/2} \|\phi\|_{L^2(\Gamma_h)} \|\sqrt{c} v_h\|_{L^2(\Gamma_h)} \lesssim h^2 c^{1/2} \|u\|_{H^2(\Gamma)} \|v_h\|_*.$$

To estimate I_2, we consider the advection-dominated case and the diffusion-dominated case separately. If $\mathsf{Pe}_T > 1$, we have

$$\begin{aligned}
\int_T (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\phi \,\mathrm{d}\mathbf{s} &\lesssim \delta_T^{-1/2} \|\phi\|_{L^2(T)} \left(\int_T \delta_T (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)^2 \mathrm{d}\mathbf{s}\right)^{1/2}\\
&\lesssim \max(h^{-1/2}, c^{1/2}) \|\phi\|_{L^2(T)} \left(\int_T \delta_T (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)^2 \mathrm{d}\mathbf{s}\right)^{1/2},
\end{aligned}$$

and if $\mathsf{Pe}_T \leqslant 1$,

$$\begin{aligned}
\int_T (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h)\phi \,\mathrm{d}\mathbf{s} &\lesssim \|\mathbf{w}^e\|_{L^\infty(T)} \|\nabla_{\Gamma_h} v_h\|_{L^2(T)} \|\phi\|_{L^2(T)}\\
&\lesssim \varepsilon^{1/2} h^{-1} \|\varepsilon^{1/2} \nabla_{\Gamma_h} v_h\|_{L^2(T)} \|\phi\|_{L^2(T)}.
\end{aligned}$$

Summing over $T \in \mathcal{F}_h$ we obtain

$$|I_2| \lesssim (h^{-1/2} + \varepsilon^{1/2} h^{-1}) \|\phi\|_{L^2(\Gamma_h)} \|v_h\|_* \lesssim h(c^{1/2}h + h^{1/2} + \varepsilon^{1/2}) \|u\|_{H^2(\Gamma)} \|v_h\|_*.$$

The term I_3 is estimated using $\mathbf{P}\mathbf{w}^e = \mathbf{w}^e$, Lemmas 3.5 and 3.6 and Corollary 3.8:

$$\begin{aligned} |I_3| &\lesssim \left| \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{w}^e \cdot [\mathbf{m}_h]) \phi v_h \, \mathrm{d}\mathbf{s} \right| \\ &\lesssim \left(\sum_{E \in \mathcal{E}_h} \int_E |\phi|^2 \mathrm{d}\mathbf{s} \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \int_E |\mathbf{P}[\mathbf{m}_h]|^2 v_h^2 \, \mathrm{d}\mathbf{s} \right)^{1/2} \\ &\lesssim h^3 \|u\|_{H^2(\Gamma)} \left(h \sum_{E \in \mathcal{E}_h} \int_E v_h^2 \, \mathrm{d}\mathbf{s} \right)^{1/2} \lesssim \left(\frac{h^3}{\sqrt{\varepsilon + c}} + \frac{h^4}{\sqrt{\varepsilon}} \right) \|u\|_{H^2(\Gamma)} \|v_h\|_*. \end{aligned} \tag{3.38}$$

The term I_4 in (3.37) can be bounded due to Lemma 3.2, the interpolation bounds and (3.29):

$$\begin{aligned} |I_4| &\leqslant \frac{1}{2} \|\mathrm{div}_{\Gamma_h} \mathbf{w}^e\|_{L^\infty(\Gamma_h)} \|\phi\|_{L^2(\Gamma_h)} \|v_h\|_{L^2(\Gamma_h)} \lesssim h^3 \|u\|_{H^2(\Gamma)} \|v_h\|_{L^2(\Gamma_h)} \\ &\leqslant \frac{h^3}{\sqrt{\varepsilon + c}} \|u\|_{H^2(\Gamma)} \|v_h\|_*. \end{aligned}$$

Finally, we treat the third term on the right-hand side of (3.35). Using $\delta_T \|\mathbf{w}^e\|_{L^\infty(T)} \lesssim h$, $\delta_T \varepsilon \lesssim 1$, $\delta_T c \leqslant 1$ and the interpolation estimates (3.20) and (3.21) we obtain

$$\begin{aligned} &\sum_{T \in \mathcal{F}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} \phi + \mathbf{w}^e \cdot \nabla_{\Gamma_h} \phi + c\phi) \mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h \, \mathrm{d}\mathbf{s} \\ &\lesssim \left(\sum_{T \in \mathcal{F}_h} \delta_T (\varepsilon^2 \|\Delta_{\Gamma_h} u^e\|_{L^2(T)}^2 + \|\mathbf{w}^e \cdot \nabla_{\Gamma_h} \phi\|_{L^2(T)}^2 + c^2 \|\phi\|_{L^2(T)}^2) \right)^{1/2} \|v_h\|_* \\ &\lesssim (\varepsilon^{1/2} + h^{1/2} + c^{1/2} h) h \|u\|_{H^2(\Gamma)} \|v_h\|_*. \end{aligned} \tag{3.39}$$

Combining the inequalities (3.36–3.39) proves the result of the lemma. □

3.5 *Consistency estimate*

The consistency error of the FEM (2.12) is due to geometric errors resulting from the approximation of Γ by Γ_h. To estimate these geometric errors we need a few additional definitions and results, which can be found in, for example, Demlow & Dziuk (2007). For $\mathbf{x} \in \Gamma_h$ define $\tilde{\mathbf{P}}_h(\mathbf{x}) = \mathbf{I} - \mathbf{n}_h(\mathbf{x})\mathbf{n}(\mathbf{x})^{\mathrm{T}} / (\mathbf{n}_h(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}))$. One can represent the surface gradient of $u \in H^1(\Gamma)$ in terms of $\nabla_{\Gamma_h} u^e$ as follows:

$$\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = (\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))^{-1} \tilde{\mathbf{P}}_h(\mathbf{x}) \nabla_{\Gamma_h} u^e(\mathbf{x}) \quad \text{a.e. } \mathbf{x} \in \Gamma_h. \tag{3.40}$$

Owing to (3.9), we obtain

$$\int_{\Gamma} \nabla_{\Gamma} u \nabla_{\Gamma} v \, \mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v^e \, \mathrm{d}\mathbf{s} \quad \text{for all } v \in H^1(\Gamma), \tag{3.41}$$

with $\mathbf{A}_h(\mathbf{x}) = \mu_h(\mathbf{x})\tilde{\mathbf{P}}_h^{\mathrm{T}}(\mathbf{x})(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))^{-2}\tilde{\mathbf{P}}_h(\mathbf{x})$. From $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{w}^e(\mathbf{x}) = \mathbf{w}(\mathbf{p}(\mathbf{x}))$, $\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{p}(\mathbf{x}))$ it follows that $\mathbf{n}(\mathbf{x}) \cdot \mathbf{w}^e(\mathbf{x}) = 0$ and thus $\mathbf{w}(\mathbf{p}(\mathbf{x})) = \tilde{\mathbf{P}}_h(\mathbf{x})\mathbf{w}^e(\mathbf{x})$ holds. Using this, we obtain the relation

$$\int_{\Gamma} (\mathbf{w} \cdot \nabla_{\Gamma} u) v \, \mathrm{d}\mathbf{s} = \int_{\Gamma_h} (\mathbf{B}_h \mathbf{w}^e \cdot \nabla_{\Gamma_h} u^e) v^e \, \mathrm{d}\mathbf{s}, \tag{3.42}$$

with $\mathbf{B}_h = \mu_h(\mathbf{x})\tilde{\mathbf{P}}_h^{\mathrm{T}}(I - d\mathbf{H})^{-1}\tilde{\mathbf{P}}_h$. In the proof we use the lifting procedure $\Gamma_h \to \Gamma$ given by

$$v_h^l(\mathbf{p}(\mathbf{x})) := v_h(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma_h. \tag{3.43}$$

It is easy to see that $v_h^l \in H^1(\Gamma)$.

The following lemma estimates the consistency error of the FEM (2.12).

Lemma 3.10 Let $u \in H^2(\Gamma)$ be the solution of (2.5); then we have

$$\sup_{v_h \in V_h^{\Gamma}} \frac{|f_h(v_h) - a_h(u^e, v_h)|}{\|v_h\|_*} \lesssim \left(h^{1/2} + c^{1/2}h + \frac{h}{\sqrt{c+\varepsilon}} \right) h(\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}). \tag{3.44}$$

Proof. The residual is decomposed as

$$f_h(v_h) - a_h(u^e, v_h) = f_h(v_h) - f(v_h^l) + a(u, v_h^l) - a_h(u^e, v_h). \tag{3.45}$$

The following holds:

$$\begin{aligned}
f(v_h^l) &= \int_{\Gamma} f v_h^l \, \mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mu_h f^e v_h \, \mathrm{d}\mathbf{s}, \\
a(u, v_h^l) &= \varepsilon \int_{\Gamma} \nabla_{\Gamma} u \nabla_{\Gamma} v_h^l \, \mathrm{d}\mathbf{s} + \int_{\Gamma} (\mathbf{w} \cdot \nabla_{\Gamma} u) v_h^l \mathrm{d}\mathbf{s} + \int_{\Gamma} c u v_h^l \, \mathrm{d}\mathbf{s} \\
&= \varepsilon \int_{\Gamma} \nabla_{\Gamma} u \nabla_{\Gamma} v_h^l \, \mathrm{d}\mathbf{s} + \frac{1}{2} \int_{\Gamma} (\mathbf{w} \cdot \nabla_{\Gamma} u) v_h^l - (\mathbf{w} \cdot \nabla_{\Gamma} v_h^l) u \, \mathrm{d}\mathbf{s} + \int_{\Gamma} c u v_h^l \, \mathrm{d}\mathbf{s} \\
&= \varepsilon \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v_h \, \mathrm{d}\mathbf{s} + \frac{1}{2} \int_{\Gamma_h} (\mathbf{B}_h \mathbf{w}^e \cdot \nabla_{\Gamma_h} u^e) v_h \, \mathrm{d}\mathbf{s} \\
&\quad - \frac{1}{2} \int_{\Gamma_h} (\mathbf{B}_h \mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h) u^e \, \mathrm{d}\mathbf{s} + \int_{\Gamma_h} \mu_h c u^e v_h \, \mathrm{d}\mathbf{s}.
\end{aligned}$$

Substituting these relations into (3.45) and using (2.13), (2.14) results in

$$
\begin{aligned}
& f_h(v_h) - a_h(u^e, v_h) \\
&\quad = \int_{\Gamma_h} (1-\mu_h) f^e v_h \, \mathrm{d}\mathbf{s} + \varepsilon \int_{\Gamma_h} (\mathbf{A}_h - \mathbf{P}_h) \nabla_{\Gamma_h} u^e \cdot \nabla_{\Gamma_h} v_h \, \mathrm{d}\mathbf{s} \\
&\qquad + \frac{1}{2} \int_{\Gamma_h} ((\mathbf{B}_h - \mathbf{P}_h)\mathbf{w}^e \cdot \nabla_{\Gamma_h} u^e) v_h \, \mathrm{d}\mathbf{s} - \frac{1}{2} \int_{\Gamma_h} ((\mathbf{B}_h - \mathbf{P}_h)\mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h) u^e \, \mathrm{d}\mathbf{s} \\
&\qquad + \int_{\Gamma_h} (\mu_h - 1) c u^e v_h \, \mathrm{d}\mathbf{s} + \sum_{T \in \mathcal{F}_h} \delta_T \int_T (f^e + \varepsilon \Delta_{\Gamma_h} u^e - \mathbf{w}^e \cdot \nabla_{\Gamma_h} u^e - c u^e) \mathbf{w}^e \cdot \nabla_{\Gamma_h} v_h \, \mathrm{d}\mathbf{s} \\
&\quad =: I_1 + I_2 + I_3 + I_4 + I_5 + \Pi_1.
\end{aligned} \tag{3.46}
$$

We estimate the I_i terms separately. Applying (3.10) and (3.29) we obtain

$$
I_1 \lesssim h^2 \|f^e\|_{L^2(\Gamma_h)} \|v_h\|_{L^2(\Gamma_h)} \lesssim \frac{h^2}{\sqrt{c+\varepsilon}} \|f\|_{L^2(\Gamma)} \|v_h\|_*, \tag{3.47}
$$

$$
I_5 \lesssim h^2 c^{1/2} \|u^e\|_{L^2(\Gamma_h)} \|\sqrt{c} v_h\|_{L^2(\Gamma_h)} \lesssim h^2 c^{1/2} \|u\|_{L^2(\Gamma)} \|v_h\|_*. \tag{3.48}
$$

One can show (cf. Olshanskii *et al.*, 2009, (3.43)) the bound

$$
|\mathbf{P}_h - \mathbf{A}_h| = |\mathbf{P}_h(\mathbf{I} - \mathbf{A}_h)| \lesssim h^2.
$$

Using this we obtain

$$
I_2 \lesssim \varepsilon h^2 \|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)} \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} \lesssim \varepsilon^{1/2} h^2 \|u^e\|_{H^2(\Gamma)} \|v_h\|_*. \tag{3.49}
$$

Since $(\mathbf{I} - d\mathbf{H})^{-1} = \mathbf{I} + \mathcal{O}(h^2)$, we also estimate

$$
|\mathbf{B}_h - \mathbf{P}_h| \lesssim h^2 + |\mathbf{A}_h - \mathbf{P}_h| \lesssim h^2.
$$

This yields

$$
I_3 \lesssim h^2 \|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)} \|v_h\|_{L^2(\Gamma_h)} \lesssim \frac{h^2}{\sqrt{c+\varepsilon}} \|u\|_{H^2(\Gamma)} \|v_h\|_*. \tag{3.50}
$$

To estimate I_4 we use definition (2.15) of δ_T. If $\mathsf{Pe}_T \leqslant 1$, then $\varepsilon^{-1/2}\|\mathbf{w}^e\|_{L^\infty(T)} \leqslant \sqrt{2}\|\mathbf{w}^e\|_{L^\infty(T)}^{1/2} h_{S_T}^{-1/2}$ holds. If $\mathsf{Pe}_T > 1$, then $\delta_T^{-1/2} \leqslant \max(c^{1/2}, \delta_0^{-1/2}\|\mathbf{w}^e\|_{L^\infty(T)}^{1/2} h_{S_T}^{-1/2})$ holds. Using the assumption that the outer triangulation is quasi-uniform we obtain $h_{S_T}^{-1} \lesssim h^{-1}$. Thus, we obtain

$$
\min\{\varepsilon^{-1/2}\|\mathbf{w}^e\|_{L^\infty(T)}, \delta_T^{-1/2}\} \lesssim \max(c^{1/2}, h^{-1/2}) \lesssim c^{1/2} + h^{-1/2}
$$

and

$$
\begin{aligned}
I_4 &\lesssim \max_{x\in\Gamma_h} |\mathbf{B}_h - \mathbf{P}_h| \sum_{T\in\mathcal{F}_h} (\varepsilon^{1/2}\|\nabla_{\Gamma_h} v_h\|_{L^2(T)} + \delta_T^{1/2}\|\mathbf{w}^e_{\Gamma_h}\cdot\nabla_{\Gamma_h} v_h\|_{L^2(T)}) \\
&\quad \times \min\{\varepsilon^{-1/2}\|\mathbf{w}^e\|_{L^\infty(T)}, \delta_T^{-1/2}\}\|u^e\|_{L^2(T)} \\
&\lesssim h(c^{1/2}h + h^{1/2})\|v_h\|_*\|u\|_{L^2(\Gamma)}.
\end{aligned} \tag{3.51}
$$

Now we estimate Π_1. Using the equation $-\varepsilon\Delta_\Gamma u + \mathbf{w}\cdot\nabla_\Gamma u + cu = f$ on Γ we obtain

$$
\begin{aligned}
\Pi_1 &= \sum_{T\in\mathcal{F}_h} \delta_T \int_T (f\circ\mathbf{p} + \varepsilon\Delta_{\Gamma_h}u^e - \mathbf{w}^e\cdot\nabla_{\Gamma_h}u^e - cu^e)\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\,\mathrm{d}\mathbf{s} \\
&= \sum_{T\in\mathcal{F}_h} \delta_T \int_T (-\varepsilon(\Delta_\Gamma u)\circ\mathbf{p} + \varepsilon\Delta_{\Gamma_h}u^e)\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\,\mathrm{d}\mathbf{s} \\
&\quad + \sum_{T\in\mathcal{F}_h} \delta_T \int_T ((\mathbf{w}^e\cdot\nabla_\Gamma u)\circ\mathbf{p} - \mathbf{w}^e\cdot\nabla_{\Gamma_h}u^e)\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\,\mathrm{d}\mathbf{s} \\
&\quad + \sum_{T\in\mathcal{F}_h} \delta_T \int_T (cu\circ\mathbf{p} - cu^e)\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\,\mathrm{d}\mathbf{s} \\
&=: \Pi_1^1 + \Pi_1^2 + \Pi_1^3.
\end{aligned} \tag{3.52}
$$

From $u^e = u\circ\mathbf{p}$ it follows that $\Pi_1^3 = 0$ holds. For Π_1^2 we obtain, using (3.40) and $|\mu_h^{-1}\mathbf{B}_h - \mathbf{P}_h| \leqslant |\mu_h - 1||\mu_h|^{-1}|\mathbf{B}_h| + |\mathbf{B}_h - \mathbf{P}_h| \lesssim h^2$,

$$
\begin{aligned}
\Pi_1^2 &= \sum_{T\in\mathcal{F}_h} \delta_T \int_T ((\mu_h^{-1}\mathbf{B}_h - \mathbf{P}_h)\mathbf{w}^e\cdot\nabla_{\Gamma_h}u^e)\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\,\mathrm{d}\mathbf{s} \\
&\lesssim h^2 \left(\sum_{T\in\mathcal{F}_h} \delta_T\|\mathbf{w}^e\|^2_{L^\infty}\|\nabla_{\Gamma_h}u^e\|^2_{L^2(T)}\right)^{1/2} \left(\sum_{T\in\mathcal{F}_h} \delta_T\|\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\|^2_{L^2(T)}\right)^{1/2} \\
&\lesssim h^{5/2}\|u\|_{H^2(\Gamma)}\|v_h\|_*.
\end{aligned} \tag{3.53}
$$

Since $\delta_T\varepsilon \lesssim h^2$, we obtain

$$
\begin{aligned}
\Pi_1^1 &\lesssim \left(\sum_{T\in\mathcal{F}_h} \varepsilon^2\delta_T\|(\Delta_\Gamma u)\circ\mathbf{p} - \Delta_{\Gamma_h}u^e\|^2_{L^2(T)}\right)^{1/2} \left(\sum_{T\in\mathcal{F}_h} \delta_T\|\mathbf{w}^e\cdot\nabla_{\Gamma_h}v_h\|^2_{L^2(T)}\right)^{1/2} \\
&\lesssim h\varepsilon^{1/2} \left(\sum_{T\in\mathcal{F}_h} \|(\Delta_\Gamma u)\circ\mathbf{p} - \Delta_{\Gamma_h}u^e\|^2_{L^2(T)}\right)^{1/2} \|v_h\|_*.
\end{aligned} \tag{3.54}
$$

We finally consider the term between brackets in (3.54). Using the identity $\operatorname{div}_\Gamma\mathbf{f} = \operatorname{tr}(\nabla_\Gamma\mathbf{f})$ and $\mathbf{n}\cdot\nabla u^e(\mathbf{p}(\mathbf{x})) = 0$ we obtain for $\mathbf{x}\in T$, with $\nabla^2 := \nabla\nabla^{\mathrm{T}}$,

$$
\Delta_\Gamma u(\mathbf{p}(\mathbf{x})) = \operatorname{div}_\Gamma\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = \operatorname{tr}(\mathbf{P}\nabla\mathbf{P}\nabla u^e(\mathbf{p}(\mathbf{x}))) = \operatorname{tr}(\mathbf{P}\nabla^2 u^e(\mathbf{p}(\mathbf{x}))\mathbf{P}). \tag{3.55}
$$

From the same arguments it follows that

$$\Delta_{\Gamma_h} u^e(\mathbf{x}) = \mathrm{tr}(\mathbf{P}_h \nabla \mathbf{P}_h \nabla u^e(\mathbf{x})) = \mathrm{tr}(\mathbf{P}_h \nabla^2 u^e(\mathbf{x}) \mathbf{P}_h) \tag{3.56}$$

holds. Using (3.14) and $|d(\mathbf{x})| \lesssim h^2$, $|\mathbf{P} - \mathbf{P}_h| \lesssim h$, $|\mathbf{H}| \lesssim 1$, $|\nabla \mathbf{H}| \lesssim 1$ we obtain

$$\begin{aligned} \mathbf{P}_h \nabla^2 u^e(\mathbf{x}) \mathbf{P}_h &= \mathbf{P} \nabla^2 u^e(\mathbf{p}(\mathbf{x})) \mathbf{P} + \mathbf{R}, \\ & |\mathbf{R}| \lesssim h(|\nabla^2 u^e(\mathbf{p}(\mathbf{x}))| + |\nabla u^e(\mathbf{p}(\mathbf{x}))|). \end{aligned}$$

Thus, using (3.55) and (3.56), we obtain

$$\begin{aligned} |\Delta_\Gamma u(\mathbf{p}(\mathbf{x})) - \Delta_{\Gamma_h} u^e(\mathbf{x})| &\leqslant |\mathrm{tr}(\mathbf{P} \nabla^2 u^e(\mathbf{p}(\mathbf{x})) \mathbf{P} - \mathbf{P}_h \nabla^2 u^e(\mathbf{x}) \mathbf{P}_h)| \\ &\lesssim h(|\nabla^2 u^e(\mathbf{p}(\mathbf{x}))| + |\nabla u^e(\mathbf{p}(\mathbf{x}))|), \end{aligned}$$

and combining this with (3.54) yields

$$\Pi_1^1 \lesssim h \varepsilon^{1/2} \left(\sum_{T \in \mathcal{F}_h} \|(\Delta_\Gamma u) \circ \mathbf{p} - \Delta_{\Gamma_h} u^e\|_{L^2(T)}^2 \right)^{1/2} \|v_h\|_* \lesssim \varepsilon^{1/2} h^2 \|u\|_{H^2(\Gamma)} \|v_h\|_*.$$

Combining this with the results (3.46–3.51) and (3.53) proves the lemma. □

3.6 *Main theorem*

Now we put together the results derived in the previous sections to prove the main result of the paper.

THEOREM 3.11 Let Assumption 3.1 be satisfied. We consider problem parameters ε and c as in (3.28). Assume that the solution u of (2.5) has regularity $u \in H^2(\Gamma)$. Let u_h be the discrete solution of the SUPG FEM (2.12). Then the following holds:

$$\|u^e - u_h\|_* \lesssim h \left(h^{1/2} + \varepsilon^{1/2} + c^{1/2} h + \frac{h}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) (\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}). \tag{3.57}$$

Proof. The triangle inequality yields

$$\|u^e - u_h\|_* \leqslant \|u^e - (I_h u^e)|_{\Gamma_h}\|_* + \|(I_h u^e)|_{\Gamma_h} - u_h\|_*. \tag{3.58}$$

The second term in the upper bound can be estimated using Lemmas 3.3, 3.9 and 3.10:

$$\begin{aligned} \frac{1}{2} \|(I_h u^e)|_{\Gamma_h} - u_h\|_*^2 &\leqslant a_h((I_h u^e)|_{\Gamma_h} - u_h, (I_h u^e)|_{\Gamma_h} - u_h) \\ &= a_h((I_h u^e)|_{\Gamma_h} - u^e, (I_h u^e)|_{\Gamma_h} - u_h) + a_h(u^e - u_h, (I_h u^e)|_{\Gamma_h} - u_h) \end{aligned}$$

$$
\begin{aligned}
&\lesssim h\left(h^{1/2}+\varepsilon^{1/2}+c^{1/2}h+\frac{h^2}{\sqrt{\varepsilon+c}}+\frac{h^3}{\sqrt{\varepsilon}}\right)\|u\|_{H^2(\Gamma)}\|(I_h u^e)|_{\Gamma_h}-u_h\|_*\\
&\quad+|a_h(u^e,(I_h u^e)|_{\Gamma_h}-u_h)-f_h((I_h u^e)|_{\Gamma_h}-u_h)|\\
&\lesssim h\left(h^{1/2}+\varepsilon^{1/2}+c^{1/2}h+\frac{h}{\sqrt{\varepsilon+c}}+\frac{h^3}{\sqrt{\varepsilon}}\right)\\
&\quad\times(\|u\|_{H^2(\Gamma)}+\|f\|_{L^2(\Gamma)})\|(I_h u^e)|_{\Gamma_h}-u_h\|_*.
\end{aligned}
$$

This results in

$$
\|(I_h u^e)|_{\Gamma_h}-u_h\|_*\lesssim h\left(h^{1/2}+\varepsilon^{1/2}+c^{1/2}h+\frac{h}{\sqrt{\varepsilon+c}}+\frac{h^3}{\sqrt{\varepsilon}}\right)(\|u\|_{H^2(\Gamma)}+\|f\|_{L^2(\Gamma)}). \tag{3.59}
$$

The error estimate (3.57) follows from (3.22) and (3.59).

□

3.7 *Further discussion*

We comment on some of the aspects related to the main theorem. With regard to the analysis, we note that the norm $\|\cdot\|_*$, which measures the error on the left-hand side of (3.57), is the standard SUPG norm as found in standard analyses of planar streamline-diffusion FEMs. The analysis in this paper contains new ingredients compared to the planar case. To control the geometric errors (approximation of Γ by Γ_h), we derived a consistency error bound in Lemma 3.10. To derive a continuity result (Lemma 3.9), as in the planar case, we apply partial integration to the term $\int_{\Gamma_h}(\mathbf{w}^e\cdot\nabla_{\Gamma_h}\phi)v_h\,\mathrm{d}\mathbf{s}$; cf. (3.37). However, differently from the planar case, this results in jumps across the edges $E\in\mathcal{E}_h$ which have to be controlled; cf. (3.38). For this the new results in Lemmas 3.6 and 3.7 are derived. These jump terms across the edges cause the term $h^4/\sqrt{\varepsilon}$ in the error bound in (3.57).

Consider the error reduction factor $h^{3/2}+\varepsilon^{1/2}h+c^{1/2}h^2+h^2/\sqrt{\varepsilon+c}+h^4/\sqrt{\varepsilon}$ on the right-hand side of (3.57). The first three terms are typical for the error analysis of planar SUPG FEMs for $P1$ elements. In the standard literature for the planar case (cf. Roos *et al.*, 2008), one typically considers only the case $c>0$. Our analysis also applies to the case $c=0$; cf. (3.28). Furthermore, the estimates are uniform w.r.t. the size of the parameter c. For a fixed $c>0$ and $\varepsilon\lesssim h$, the first four terms can be estimated by $\lesssim h^{3/2}$, a bound similar to the standard one for the planar case. The only 'suboptimal' term is the last one, which is caused by (our analysis of) the jump terms. Note, however, that $h^4/\sqrt{\varepsilon}\lesssim h^{3/2}$ if $h^5\lesssim\varepsilon$ holds, which is a very mild condition.

The norm $\|\cdot\|_*$ provides a robust control of streamline derivatives of the solution. Crosswind oscillations, however, are not completely suppressed. It is well known that nonlinear stabilization methods can be used to get control over crosswind derivatives as well. Extending such methods to surface PDEs is not within the scope of the present paper.

The error estimates in this paper are in terms of the maximum mesh size over tetrahedra in ω_h, denoted by h. In practice, the stabilization parameter δ_T is based on the *local* Peclet number and the stabilization is switched off or reduced in the regions, where the mesh is 'sufficiently fine'. To prove error estimates accounting for local smoothness of the solution u and the local mesh size (as available for the planar SUPG FE method), one needs local interpolation properties of FE spaces, instead of (3.20) and (3.21). Since our FE space is based on traces of piecewise linear functions, such local estimates are not immediately available. The extension of our analysis to this *non*quasi-uniform case is left for future research.

Finally, we remark on the case of a varying reaction term coefficient c. If the coefficient c in the third term of (2.4) varies, the above analysis is valid with minor modifications. We briefly explain these modifications. The stabilization parameter δ_T should be based on elementwise values $c_T = \max_{\mathbf{x}\in T} c(\mathbf{x})$. For the well-posedness of (2.4), it is sufficient to assume c to be strictly positive on a subset of Γ with positive measure:

$$\mathcal{A} := \mathrm{meas}\{\mathbf{x}\in\Gamma \,:\, c(\mathbf{x}) \geqslant c_0\} > 0,$$

with some $c_0 > 0$. If this is satisfied, the Friedrich's-type inequality (Sobolev, 1991; see, also Olshanskii *et al.*, 2009, Lemma 3.1),

$$\|v\|^2_{L^2(\Gamma)} \leqslant C_{\mathrm{F}}(\|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \|\sqrt{c}v\|^2_{L^2(\Gamma)}) \quad \text{for all } v\in V,$$

holds, with a constant C_{F} depending on c_0 and $\mathcal{A}$. Using this, all arguments in the analysis can be generalized to the case of a varying $c(\mathbf{x})$ with obvious modifications. With $c_{\min} := \operatorname{ess\,inf}_{\mathbf{x}\in\Gamma} c(\mathbf{x})$ and $c_{\max} := \operatorname{ess\,sup}_{\mathbf{x}\in\Gamma} c(\mathbf{x})$, the final error estimate takes the form

$$\|u^e - u_h\|_* \lesssim h\left(h^{1/2} + \varepsilon^{1/2} + c_{\max}^{1/2}h + \frac{h}{\sqrt{\varepsilon + c_{\min}}} + \frac{h^3}{\sqrt{\varepsilon}}\right)(\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}).$$

4. Numerical experiments

In this section we show the results of a few numerical experiments which illustrate the performance of the method.

EXAMPLE 4.1 The stationary problem (2.4) is solved on the unit sphere Γ, with the velocity field

$$\mathbf{w}(\mathbf{x}) = \left(-x_2\sqrt{1-x_3^2}, x_1\sqrt{1-x_3^2}, 0\right)^{\mathrm{T}},$$

which is tangential to the sphere. We set $\varepsilon = 10^{-6}$, $c \equiv 1$ and consider the solution

$$u(\mathbf{x}) = \frac{x_1x_2}{\pi}\arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right).$$

Note that u has a sharp internal layer along the equator of the sphere. The corresponding right-hand side function f is given by

$$f(\mathbf{x}) = \frac{8\varepsilon^{3/2}(2+\varepsilon+2x_3^2)x_1x_2x_3}{\pi(\varepsilon+4x_3^2)^2} + \frac{6\varepsilon x_1x_2 + \sqrt{x_1^2+x_2^2}(x_1^2-x_2^2)}{\pi}\arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right) + u.$$

We consider the standard (unstabilized) FEM in (2.11) and the stabilized method (2.12). A sequence of meshes was obtained by the gradual refinement of the outer triangulation. The induced *surface* FE spaces have dimensions $N = 448, 1864, 7552, 30\,412$. The resulting algebraic systems are solved by a direct sparse solver. FE errors are computed outside the layer: the variation of the quantities $\mathrm{err}_{L^2} = \|u -$

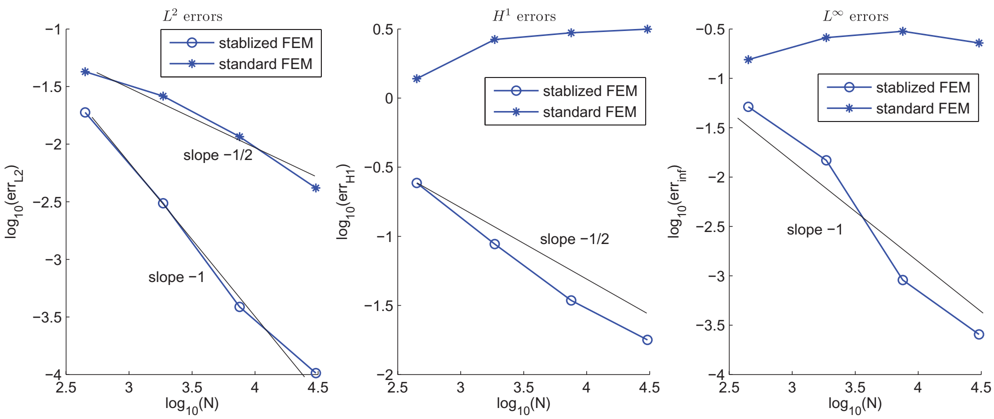

FIG. 3. Discretization errors for Example 4.1.

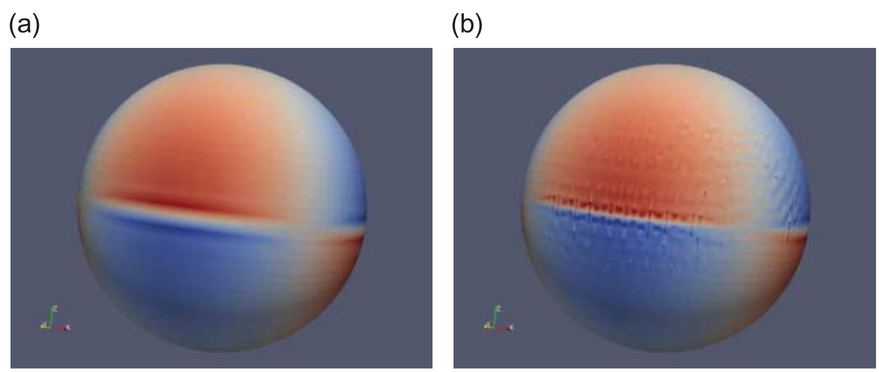

FIG. 4. Example 4.1: solutions using the stabilized method and the standard method (2.11).

$u_h\|_{L^2(D)}$, $\mathrm{err}_{H^1} = \|u - u_h\|_{H^1(D)}$ and $\mathrm{err}_{\mathrm{inf}} = \|u - u_h\|_{L^\infty(D)}$, with $D = \{\mathbf{x} \in \Gamma \ : \ |x_3| > 0.3\}$, are shown in Fig. 3. The results clearly show that the stabilized method performs much better than the standard one. The results for the stabilized method indicate an $\mathcal{O}(h^2)$ convergence in the $L^2(D)$ norm and $L^\infty(D)$ norm. In the $H^1(D)$ norm, first-order convergence is observed. Note that the analysis predicts (only) $\mathcal{O}(h^{3/2})$ order of convergence in the (global) L^2 norm.

Figure 4 shows the computed solutions with the two methods. Since the layer is unresolved, the FEM (2.11) produces a globally oscillating solution. The stabilized method gives a much better approximation, although the layer is slightly smeared, as is typical for the SUPG method.

EXAMPLE 4.2 Now we consider the stationary problem (2.4) with $c \equiv 0$. The problem is posed on the unit sphere Γ, with the same velocity field $\mathbf{w}$ as in Example 4.1. We set $\varepsilon = 10^{-6}$, and consider the solution

$$u(\mathbf{x}) = \frac{x_1 x_2}{\pi} \arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right).$$

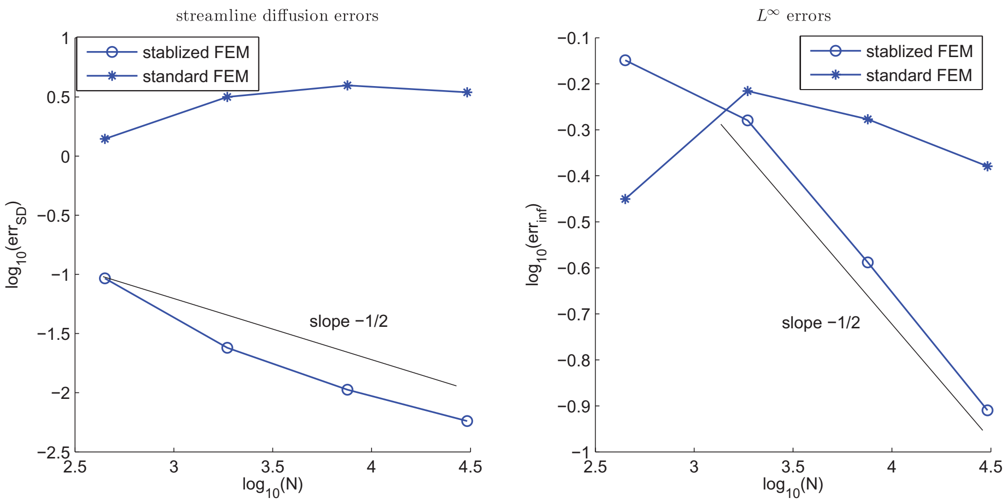

FIG. 5. Discretization errors for Example 4.2.

The corresponding right-hand side function f is now given by

$$f(\mathbf{x}) = \frac{8\varepsilon^{3/2}(2+\varepsilon+2x_3^2)x_1x_2x_3}{\pi(\varepsilon+4x_3^2)^2} + \frac{6\varepsilon x_1x_2 + \sqrt{x_1^2+x_2^2}(x_1^2-x_2^2)}{\pi}\arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right).$$

Note that for $c=0$ one loses explicit control of the L_2 norm in $\|\cdot\|_*$. Thus, we consider the streamline diffusion error,

$$\mathrm{err}_{\mathrm{SD}} = \|\mathbf{w}_\Gamma \cdot \nabla_\Gamma(u-u_h)\|_{L^2(D)}.$$

Results for this error quantity and for $\mathrm{err}_{\mathrm{inf}} = \|u-u_h\|_{L^\infty(D)}$ are shown in Fig. 5. We observe an $\mathcal{O}(h)$ behaviour for the streamline diffusion error, which is consistent with our theoretical analysis. The L^∞ norm of the error also shows first-order convergence.

EXAMPLE 4.3 We show how this stabilization can be applied to a time-dependent problem and illustrate its stabilizing effect. We consider a nonstationary problem (2.3) posed on the torus

$$\Gamma = \left\{(x_1,x_2,x_3) \mid \left(\sqrt{x_1^2+x_2^2}-1\right)^2 + x_3^2 = \frac{1}{16}\right\}. \tag{4.1}$$

We set $\varepsilon = 10^{-6}$ and define the advection field,

$$\mathbf{w}(\mathbf{x}) = \frac{1}{\sqrt{x_1^2+x_2^2}}(-x_2, x_1, 0)^{\mathrm{T}},$$

which is divergence free and satisfies $\mathbf{w}\cdot\mathbf{n}_\Gamma = 0$. The initial condition is

$$u_0(\mathbf{x}) = \frac{x_1x_2}{\pi}\arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right).$$

The function u_0 possesses an internal layer, as shown in Fig. 6(a).

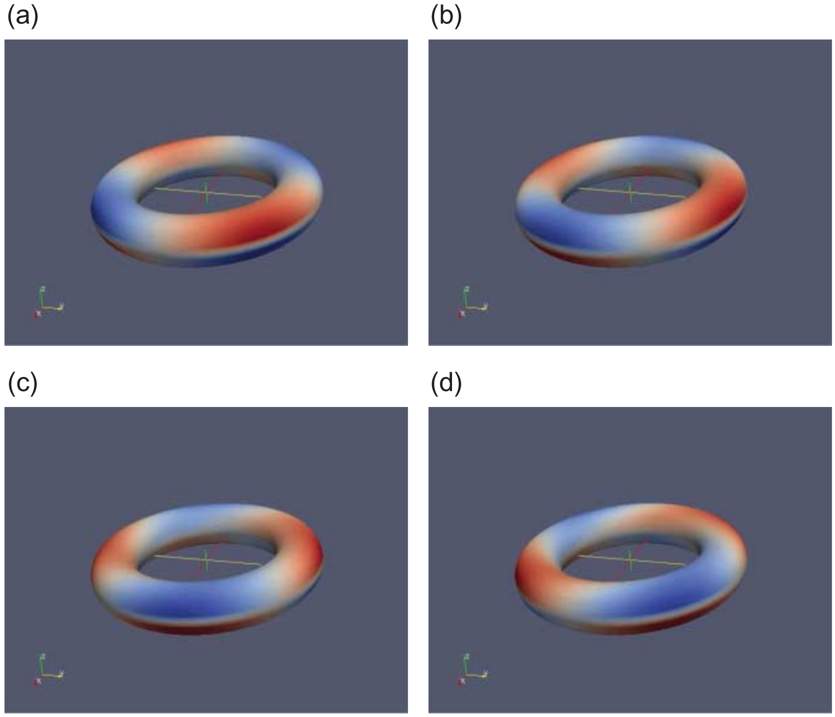

FIG. 6. Example 4.3: solutions for $t = 0, 0.6, 1.2, 1.8$ using the SUPG stabilized FEM.

The stabilized spatial semidiscretization of (2.3) reads as follows: determine $u_h = u_h(t) \in V_h^\Gamma$ such that

$$m(\partial_t u_h, v_h) + \hat{a}_h(u_h, v_h) = 0 \quad \text{for all } v_h \in V_h^\Gamma \tag{4.2}$$

with

$$m(\partial_t u, v) := \int_{\Gamma_h} \partial_t u v \, \mathrm{d}\mathbf{s} + \sum_{T \in \mathcal{F}_h} \delta_T \int_T \partial_t u (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v) \, \mathrm{d}\mathbf{s},$$

$$\hat{a}_h(u, v) := \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v \, \mathrm{d}\mathbf{s} + \frac{1}{2} \left[\int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} u) v \, \mathrm{d}\mathbf{s} - \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_{\Gamma_h} v) u \, \mathrm{d}\mathbf{s} \right]$$
$$+ \sum_{T \in \mathcal{F}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} u + \mathbf{w}^e \cdot \nabla_{\Gamma_h} u) \mathbf{w}^e \cdot \nabla_{\Gamma_h} v \, \mathrm{d}\mathbf{s}.$$

Note that $\hat{a}_h(\cdot, \cdot)$ is the same as $a_h(\cdot, \cdot)$ in (2.13) with $c \equiv 0$. The resulting system of ordinary differential equations is discretized in time by the Crank–Nicolson scheme.

For $\varepsilon = 0$ the exact solution is the transport of $u_0(\mathbf{x})$ by a rotation around the x_3-axis. Thus, the inner layer remains the same for all $t > 0$. For $\varepsilon = 10^{-6}$, the exact solution is similar, unless t is large enough for dissipation to play a noticeable role. The space V_h^Γ is constructed in the same way as in the previous examples. The spatial discretization has 5638 degrees of freedom. The fully discrete problem is obtained

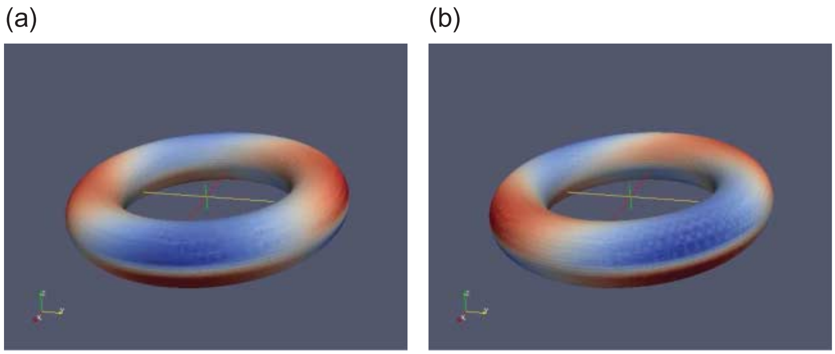

FIG. 7. Example 4.3: solutions for $t = 1.2, 1.8$ using the standard FEM.

by combining the SUPG method in (4.2) and the Crank–Nicolson method with time step $\delta t = 0.1$. The evolution of the solution is illustrated in Fig. 6 demonstrating a smoothly 'rotated' pattern.

We repeated this experiment with $\delta_T = 0$ in the bilinear forms $m(\cdot,\cdot)$ and $\hat{a}_h(\cdot,\cdot)$ in (4.2), that is, the method without stabilization. As expected, we obtain (on the same grid) much less smooth discrete solutions (Fig. 7).

REMARK 4.4 With respect to mass conservation of the scheme we note the following. For $v_h \equiv 1$ in (4.2) we obtain, with $M_h(t) := \int_{\Gamma_h} u_h(x,t)\,\mathrm{d}\mathbf{s}$,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}M_h(t)\right| &= \left|\int_{\Gamma_h} \frac{1}{2}\mathbf{w}^e \cdot \nabla_{\Gamma_h} u_h \,\mathrm{d}\mathbf{s}\right| \\ &= \left|-\frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E \mathbf{w}^e\cdot[\mathbf{m}]u_h\,\mathrm{d}\mathbf{s} + \frac{1}{2}\int_{\Gamma_h}\mathrm{div}_{\Gamma_h}\,\mathbf{w}^e u_h\,\mathrm{d}\mathbf{s}\right| \\ &\lesssim h^2\sum_{E\in\mathcal{E}_h}\int_E |u_h|\,\mathrm{d}\mathbf{s} + Ch\int_{\Gamma_h}|u_h|\,\mathrm{d}\mathbf{s}.\end{aligned}$$

Here we used estimates from Lemmas 3.2 and 3.6. Using Lemma 3.7 we obtain

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\int_E |u_h|\,\mathrm{d}\mathbf{s} &\lesssim h^{-1/2}\left(\sum_{E\in\mathcal{E}_h}\int_E u_h^2\mathrm{d}\mathbf{s}\right)^{1/2} \\ &\lesssim h^{-1}(\|u_h\|_{L^2(\Gamma_h)} + h\|\nabla_{\Gamma_h}u_h\|_{L^2(\Gamma_h)}).\end{aligned}$$

Assume that for the discrete solution we have a bound $\|u_h\|_{L^2(\Gamma_h)} + h\|\nabla_{\Gamma_h}u_h\|_{L^2(\Gamma_h)} \leqslant c$ with c independent of h. Then we obtain $|\frac{\mathrm{d}}{\mathrm{d}t}M(t)| \lesssim h$ and thus $|M_h(t) - M_h(0)| \leqslant cth$, with a constant c independent of h and t. Hence, with respect to mass conservation we have an error that is (only) first order in h. With regard to mass conservation, it would be better to use a discretization in which in the discrete bilinear

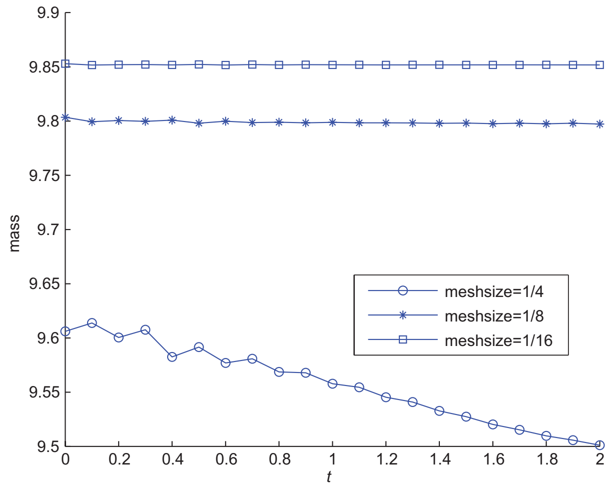

FIG. 8. Total mass variation for Example 4.3.

form in (2.13) one replaces

$$\frac{1}{2}\left[\int_{\Gamma_h}(\mathbf{w}^e\cdot\nabla_{\Gamma_h}u)v\,\mathrm{d}\mathbf{s}-\int_{\Gamma_h}(\mathbf{w}^e\cdot\nabla_{\Gamma_h}v)u\,\mathrm{d}\mathbf{s}\right]\quad\text{by}\quad-\int_{\Gamma_h}(\mathbf{w}^e\cdot\nabla_{\Gamma_h}v)u\,\mathrm{d}\mathbf{s}. \tag{4.3}$$

This method results in optimal mass conservation: $(\mathrm{d}/\mathrm{d}t)M_h(t)=0$. It turns out, however, that (with our approach) it is more difficult to analyse. In particular, it is not clear how to derive a satisfactory coercivity bound.

In numerical experiments, we observed that the behaviour of the two methods (that is, with the two variants given in (4.3)) is very similar. In particular, the mass conservation error bound $|M(t)-M(0)|\leqslant cth$ for the first method seems to be too pessimistic (in many cases). To illustrate this, we show results for the problem described above, but with initial condition

$$u_0(\mathbf{x})=1+\frac{1}{\pi}\arctan\left(\frac{x_3}{\sqrt{\varepsilon}}\right),\qquad\int_{\Gamma}u_0\,\mathrm{d}\mathbf{s}=\pi^2\approx 9.8696.$$

Figure 8 shows the quantity $M_h(t)$ for several mesh sizes h. For $t=0$ we have, due to interpolation of the initial condition u_0, a difference between $M_h(0)$ and $\int_\Gamma u_0\,\mathrm{d}\mathbf{s}$ that is of order h^2. For $t>0$ we see, except for the very coarse mesh with $h=\frac{1}{4}$, very good mass conservation.

EXAMPLE 4.5 As a final illustration we show results for the nonstationary problem (2.3), but now on a surface with a 'less regular' shape. We take the surface given in Dziuk (1988):

$$\Gamma=\{(x_1,x_2,x_3)\mid(x_1-x_3^2)^2+x_2^2+x_3^2=1\}. \tag{4.4}$$

We set $\varepsilon=10^{-6}$ and define the advection field as the Γ-tangential part of $\tilde{\mathbf{w}}=(-1,0,0)^{\mathrm{T}}$. This velocity field does not satisfy the divergence-free condition. The initial condition is taken as $u_0(\mathbf{x})=1$. We apply the same method as in Example 4.3. The mesh size is $\frac{1}{8}$ and the time step is $\delta t=0.1$. Figure 9 shows

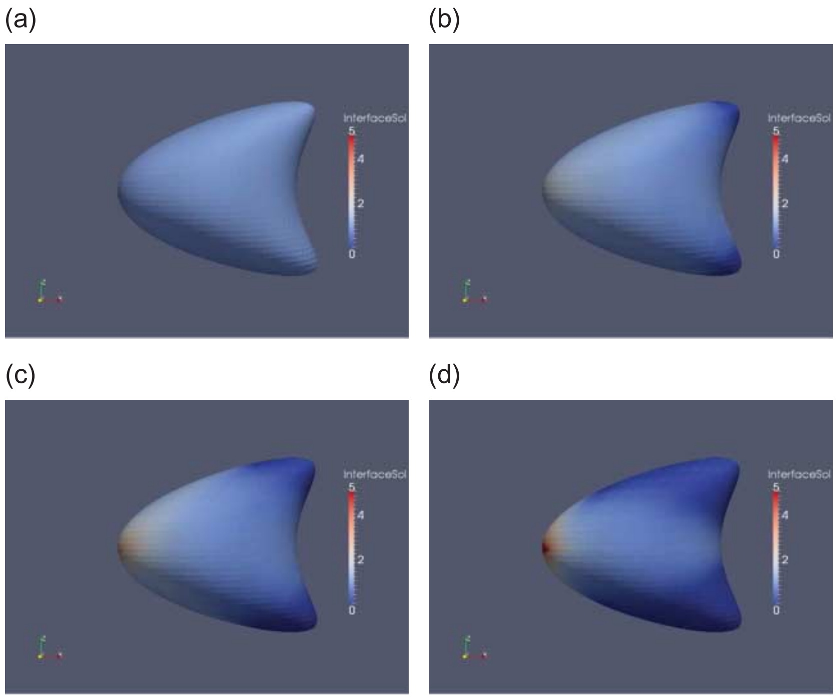

FIG. 9. Example 4.5: solutions for $t = 0, 0.5, 1.0, 2.0$ using the SUPG stabilized FEM.

the solution for several t values. We observe that as time evolves, mass is transported from the two poles on the right to the pole on the left just as expected. Our discretization yields, for this strongly convection-dominated transport problem, a qualitatively good discrete result, even with a (very) low grid resolution.

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