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A Stabilizing Output Based Nonlinear Model Predictive Control Scheme

B.J.P. Roset, M. Lazar, W.P.M.H. Heemels and H. Nijmeijer

Abstract— **In this paper we present an asymptotically stabilizing output feedback control scheme for a class of nonlinear discrete-time systems. The presented scheme consists of an extended observer interconnected with an NMPC controller which represents a possible discontinuous state feedback control law. Local asymptotic stability of the resulting closed-loop system is proven.**

I. Introduction

One of the problems in Nonlinear Model Predictive Control (NMPC) that receives increased attention and has reached a relatively mature stage, consists in guaranteeing closed-loop stability. The approach usually used to ensure nominal closed-loop stability in NMPC is to consider the value function of the NMPC cost as a candidate Lyapunov function, see the survey [1], for an overview. The stability results heavily rely on state space models of the system, and the assumption that the full state of the real system is available for feedback. However, in practice it is rarely the case that the full state of the system is available for feedback. A possible solution to this problem is the use of an observer. An observer can generate an estimate of the full state using knowledge of the output and input of the system. However, nominal stability results for NMPC usually do not guarantee closed-loop stability of an interconnected NMPC-observer combination. One of the potential approaches to guarantee closed-loop stability in the presence of estimation errors in the state, is to employ (inherent) robustness of the model predictive controller. In [2] asymptotic stability of state feedback NMPC is examined in face of asymptotically decaying disturbances. As stated by the authors of [2], their results are also useful for the solution of the output feedback problem, although a formal proof is missing. A stability result on Observer Based Nonlinear Model Predictive Control (OBNMPC) is reported in [3], under the standing assumption that the NMPC value function and the resulting NMPC control law are Lipschitz continuous. However, no nonlinear observer which satisfies the assumptions made on a potential nonlinear observer is given in [3]. The stability problem of OBNMPC is revisited in [4], where only continuity of the NMPC value function is assumed. In [4] robust global asymptotic stability is shown under the assumption that there are no state constraints present in the NMPC problem. Other related results on OBNMPC can be found in [5]. However, in [5] a continuous-time perspective is taken, while we focus on discrete-time nonlinear systems.

B.J.P. Roset, W.P.M.H. Heemels and H. Nijmeijer are with the Department of Mechanical Engineering, and M. Lazar is with the Department of Electrical Engineering, both departments are from the Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB, The Netherlands, E-mails: [b.j.p.roset,m.lazar,m.heemels,h.nijmeijer]@tue.nl. The authors kindly acknowledge The European Community for their financial support through the Network of Excellence HYCON (contract FP6-IST-511368).

In this paper we investigate asymptotic stability of an OBNMPC scheme in the presents of input and state constraints. The novelty of the proposed approach consists in providing a generically applicable observer design method. Furthermore, we employ the Input-to-State Stability (ISS) framework, e.g. see [6], [7] and the references therein, to study the stability of the resulting closed-loop system. The extended observer design methodology from [8] is considered. The extended observer design has the advantage that it works (locally) under a very mild condition which is strong local observability of the system dynamics. However, the drawback is that future information of the controls applied to the system are needed, which are normally not available and result in a causality problem. Since in the NMPC framework *predicted* future controls are available, this framework might be suitable to be employed in combination with the proposed observer theory. This idea has been pointed out in [9]. Still, conditions that guarantee a priori closed-loop stability are lacking. Resolving this issue is the main contribution of the current paper.

The paper is organized as follows. First, some basic definitions and notations are given in Section II, together with basic NMPC notions. The observer theory of [8] is summarized in Section III. In Section IV we briefly explain the proposed NMPC scheme from which the problem set-up follows. In Section V we spell out how to separately design an ISS NMPC controller and nonlinear observer. The stability of the observer-NMPC interconnection is investigated in Section VI. Conclusions are summarized in Section VII.

II. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of real numbers, non-negative reals, integers and non-negative integers, respectively. $\mathbb{Z}_{\geq i}$ denotes the set $\{k \in \mathbb{Z} | k \geq i\}$ for some $i \in \mathbb{Z}$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{K}$*-function* if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{KL}$*-function* if, for each fixed $k \in \mathbb{R}_+$, the function $\beta(\cdot, k)$ is a $\mathscr{K}$*-function*, and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is non-increasing and $\beta(s, k) \to 0$ as $k \to \infty$. A function $q : \mathbb{X} \times \mathbb{S} \to \mathbb{R}^n$ with $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{S} \subseteq \mathbb{R}^{n_s}$ is *Lipschitz* continuous w.r.t. x on the domain $\mathbb{X} \times \mathbb{S}$, if there exists a constant $0 \leq L_q < \infty$ such that $\forall x_1,\ x_2 \in \mathbb{X}$ and $\forall s \in \mathbb{S}$, $|q(x_1, s) - q(x_2, s)| \leq L_q |x_1 - x_2|$. Constant L_q is called a *Lipschitz constant* of q w.r.t. x. A function $f(x)$ of which at least its first derivative exists on in its domain is denoted by $f \in C^1$. A function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$, i.e. $\phi(k)$, is for shorthand notational purposes also denoted as ϕ_k. Furthermore, $\overline{\lim}_{k\to\infty} \phi_k$ is a shorthand notation for $\limsup_{k\to\infty} \phi_k$. For any $x \in \mathbb{R}^n$, x_i with $i \in \{1, 2, .., n\}$ stands for the i^{th} component of x and $|x|$ stands for its Euclidean norm. For a matrix $A \in \mathbb{R}^{n\times m}$ we define $|A| \triangleq \sup_{x\neq 0} |Ax|/|x|$ as the induced matrix norm. A pair of matrices $(C \in \mathbb{R}^{p\times n}, A \in \mathbb{R}^{n\times n})$ is called an *observable pair* if and only if $rank\{C, CA, .., CA^{n-1}\} = n$. For any function $\phi : \mathbb{Z}_+ \to \mathbb{R}^n$, we denote $\|\phi\| = \sup\{|\phi_k| : k \in \mathbb{Z}_+\}$. An $n\times n$ square matrix is called Schur if all its eigenvalues are within the unit disk. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by int$(\mathscr{S})$ its interior. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{S} \sim \mathscr{P} \triangleq \{x \in \mathbb{R}^n | x + \mathscr{P} \subseteq \mathscr{S}\}$ and $\mathscr{S} \oplus \mathscr{P} \triangleq \{x + y | x \in \mathscr{S}, y \in \mathscr{P}\}$ denote their *Pontryagin* difference and *Minkowski* sum, respectively.

A. Systems theory notions

Consider the following discrete-time nonlinear system

$$\begin{aligned}\xi_{k+1} &= \Phi(\xi_k, v_k) \\ \zeta_k &= G(\xi_k, v_k)\end{aligned}, \quad \xi_{k=0} = \xi_0, \quad k \in \mathbb{Z}_+, \tag{1}$$

where $\xi_k \in \mathbb{R}^n$ is the state, $\zeta_k \in \mathbb{R}^l$ the output and $v_k \in \mathbb{V} \subseteq \mathbb{R}^m$ the input at discrete time $k \in \mathbb{Z}_+$. The input v_k can be an unknown disturbance at time $k \in \mathbb{Z}_+$. $\mathbb{V}$ is assumed to be a known compact set with $0 \in$ int$(\mathbb{V})$. For all disturbances (inputs) $v : \mathbb{Z}_+ \to \mathbb{V}$, we use the notation $\mathscr{M}_\mathbb{V}$ to denote a certain subset of disturbance functions v. Furthermore, $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ are nonlinear, possibly discontinuous, functions. We assume $\xi_e = 0$ is an equilibrium of the 0-input system, i.e. $\Phi(0,0) = 0$ and that $G(0,0) = 0$. A solution to (1) for an input function v and initial condition ξ_0 is denoted as $\xi(\cdot, \xi_0, v)$.

Definition II.1 A set $\mathscr{P} \subseteq \mathbb{R}^n$ is called a *Robust Positively Invariant* (RPI) set for system (1) if for all $\xi_k \in \mathscr{P}$ it holds that $\Phi(\xi_k, v_k) \in \mathscr{P}$ for all $v_k \in \mathbb{V}$.

Definition II.2 Let $\mathscr{X}$ be a subset of $\mathbb{R}^n$, with $0 \in$ int$(\mathscr{X})$. Then, system (1) is called locally

i) *input-to-state stable* (ISS) for initial conditions in $\mathscr{X}$ if there exist a $\mathscr{K}\mathscr{L}$*-function* β_ξ and a $\mathscr{K}$*-function* γ_ξ^v such that, for each input function v taking value in $\mathbb{V}$ and each initial condition $\xi_0 \in \mathscr{X}$, it holds that for each $k \in \mathbb{Z}_+$

$$|\xi(k, \xi_0, v)| \leq \beta_\xi(|\xi_0|, k) + \gamma_\xi^v(\|v\|), \tag{2}$$

ii) *input-to-output stable* (IOS) for initial conditions in $\mathscr{X}$ if there exist a $\mathscr{K}\mathscr{L}$*-function* β_ζ and a $\mathscr{K}$*-function* γ_ζ^v such that, for each input function v taking value in $\mathbb{V}$ and each initial condition $\xi_0 \in \mathscr{X}$, it holds that for each $k \in \mathbb{Z}_+$

$$|\zeta(k, \xi_0, v)| \leq \beta_\zeta(|\xi_0|, k) + \gamma_\zeta^v(\|v\|), \tag{3}$$

iii) *stable* for initial conditions in $\mathscr{X}$ if there exists a $\mathscr{K}$*-function* φ such that, for each initial condition $\xi_0 \in \mathscr{X}$, it holds that for each $k \in \mathbb{Z}_+$

$$|\xi(k, \xi_0, v)| \leq \varphi(|\xi_0|), \quad \forall v \in \mathscr{M}_\mathbb{V}, \tag{4}$$

iv) *asymptotically stable* for initial conditions in $\mathscr{X}$ if iii) holds and

$$\overline{\lim_{k\to\infty}} |\xi(k, \xi_0, v)| = 0, \quad \forall v \in \mathscr{M}_\mathbb{V} \tag{5}$$

B. NMPC notions

Consider the following nominal and perturbed discrete-time nonlinear systems

$$x_{k+1} = f(x_k, u_k), \quad k \in \mathbb{Z}_+, \tag{6a}$$

$$\tilde{x}_{k+1} = f(\tilde{x}_k, u_k) + w_k, \quad k \in \mathbb{Z}_+, \tag{6b}$$

where x_k, $\tilde{x}_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the input at discrete-time $k \in \mathbb{Z}_+$, respectively. Furthermore, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f(0,0) = 0$. The vector $w_k \in \mathbb{W} \subseteq \mathbb{R}^n$ denotes an unknown additive disturbance and $\mathbb{W}$ is assumed to be a known compact set with $0 \in$ int$(\mathbb{W})$. The nominal discrete-time nonlinear system (6a) will be used in an NMPC scheme to make an N time steps ahead prediction of the system behavior. The system given by (6b) represents a perturbed discrete-time system to which the NMPC controller based on the nominal model (6a) will be applied. Throughout the paper we assume that the state and the controls are constrained for both systems (6a) and (6b) to some *compact* sets $\mathbb{X} \subseteq \mathbb{R}^n$ with $0 \in$ int$(\mathbb{X})$ and $\mathbb{U} \subseteq \mathbb{R}^m$ with $0 \in$ int$(\mathbb{U})$.

For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\overline{\mathbf{x}}_k^{[1,N]}(\tilde{x}_k, \overline{\mathbf{u}}_k^{[0,N-1]}) \triangleq [x_{k+1|k}^\top, \ldots, x_{k+N|k}^\top]^\top$ denote the state sequence generated by the nominal system (6a) from initial state $x_{k|k} \triangleq \tilde{x}_k$ at time k and by applying the input sequence $\overline{\mathbf{u}}_k^{[0,N-1]} \triangleq [u_{k|k}^\top, \ldots, u_{k+N-1|k}^\top]^\top \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathscr{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin. The class of *admissible input sequences* defined with respect to $\mathscr{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathscr{U}_N(\tilde{x}_k) \triangleq \{\overline{\mathbf{u}}_k^{[0,N-1]} \in \mathbb{U}^N \mid \overline{\mathbf{x}}_k^{[1,N]}(\tilde{x}_k, \overline{\mathbf{u}}_k^{[0,N-1]}) \in \mathbb{X}^N, x_{k+N|k} \in \mathscr{X}_T\}$.

Problem II.3 Let the target set $\mathscr{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ with $L(0,0) = 0$ be continuous bounded mappings. At time $k \in \mathbb{Z}_+$, let $\tilde{x}_k \in \mathbb{X}$ be given and minimize the cost $J(\tilde{x}_k, \overline{\mathbf{u}}_k^{[0,N-1]}) \triangleq F(x_{k+N|k}) + \sum_{i=0}^{N-1} L(x_{k+i|k}, u_{k+i|k})$, with prediction model (6a), over all $\overline{\mathbf{u}}_k^{[0,N-1]} \in \mathscr{U}_N(\tilde{x}_k)$.

We call a state $\tilde{x}_k \in \mathbb{X}$ *feasible* if $\mathscr{U}_N(\tilde{x}_k) \neq \emptyset$. Similarly, Problem II.3 is said to be *feasible* for $\tilde{x}_k \in \mathbb{X}$ if $\mathscr{U}_N(\tilde{x}_k) \neq \emptyset$. Let $\mathscr{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible initial states* with respect to Problem II.3 and let $V_{\text{MPC}} : \mathscr{X}_f(N) \to \mathbb{R}_+$,

$$V_{\text{MPC}}(\tilde{x}_k) \triangleq \inf_{\mathbf{u}_k^{[0,N-1]} \in \mathscr{U}_N(\tilde{x}_k)} J(\tilde{x}_k, \overline{\mathbf{u}}_k^{[0,N-1]}) \tag{7}$$

denote the value function corresponding to Problem II.3. If there exists an optimal sequence of controls $\overline{\mathbf{u}}_k^{[0,N-1]*} \triangleq [u_{k|k}^{*\top}, u_{k+1|k}^{*\top}, \ldots, u_{k+N-1|k}^{*\top}]^\top$ that minimizes (7), see [10], the infimum in (7) is a minimum and $V_{\text{MPC}}(\tilde{x}_k) = J(\tilde{x}_k, \overline{\mathbf{u}}_k^{[0,N-1]*})$. Then, an *optimal* NMPC control law is defined as $u_k = \kappa^{\text{MPC}}(\tilde{x}_k) \triangleq u_{k|k}^*$, $k \in \mathbb{Z}_+$. The NMPC control law κ^{MPC}, can be substituted in (6b) and yields

$$\tilde{x}_{k+1} = f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + w_k, \quad w_k \in \mathbb{W} \subseteq \mathbb{R}^n, \quad k \in \mathbb{Z}_+. \tag{8}$$

III. Extended observer theory: A summary

In this paper we use the extended observer theory proposed in [8]. For notational brevity we consider the theory for the single input single output case, although the theory applies in the multiple input output case as well. Consider the system

$$x_{k+1} = f(x_k, u_k), \;\; y_k = g(x_k), \;\; x_{k=0} = x_0, \;\; k \in \mathbb{Z}_+, \tag{9}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$ and $y_k \in \mathbb{R}$ is the state, the control and the output at discrete-time $k \in \mathbb{Z}_+$, respectively. Furthermore, $f, g \in C^1$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ have the property that $f(0,0) = 0$ and $g(0) = 0$. The observer problem for (9) deals with the question how to reconstruct the state trajectory $x(\cdot, x_0, u)$ on the basis of the knowledge of the input u and the output y of the system. The observer design problem in its full generality is a problem that is not yet fully solved for nonlinear systems of the form (9). A proposed observer candidate applicable for a broad class of discrete-time nonlinear systems is studied in this paper. To be more precise, observer design for a class of systems that can be expressed in the so called *Extended Nonlinear Observer Canonical Form* (ENOCF) is considered. Systems of the form (9) can be transformed, at least in a local sense, into the ENOCF provided system (9) is locally strongly observable [8], [11]. In Section III-B we give more details on this issue. Observers that are based on the ENOCF are denoted by *extended* observers. One of the major characteristics that distinguishes *extended* observers form "conventional" observers, is that not only the output y_k and input u_k at the current time k are employed to obtain an estimate of the state x_k, but, also future inputs are needed.

A. Observers in the ENOCF

A system representation in ENOCF, or the z-dynamics for brevity, reads as

$$\begin{aligned} z_{k+1} &= A_z z_k + f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \mathbf{u}_k^{[1,n]}) \\ y_k &= h_z(C_z z_k, \mathbf{u}_k^{[1-n,0]}) \end{aligned}, \quad z_{k=0} = z_0, \qquad (10)$$

where $\mathbf{y}_k^{[1-n,0]} \triangleq [y_{k-n+1}, \dots, y_k]^\top$, $\mathbf{u}_k^{[1-n,0]} \triangleq [u_{k-n+1}, \dots, u_k]^\top$, $\mathbf{u}_k^{[1,n]} \triangleq [u_{k+1}, \dots, u_{k+n}]^\top$, $z_k \in \mathbb{R}^n$ represent the past output, input, future input and state in z-coordinates at discrete time $k \in \mathbb{Z}_+$, respectively. Furthermore, $f_z : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h_z : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are nonlinear functions, where h_z is, for a fixed input sequence, a one-to-one *invertible* output function for the system in ENOCF and $C_z \triangleq [0, \dots, 0, 1]$. Moreover, the pair (C_z, A_z) is an observable pair. For the exact structure of h_z, f_z and A_z we refer the reader to [8]. Except for the future input sequence, all other sequences are known at time k if input and output variables (measurements) are buffered.

Observer candidates based on the system descriptions in ENOCF were proposed in [8]. One of the observer candidates simply consists of a "copy" of the z-dynamics (10) added with an output injection term (also known as an "innovation" term), i.e.

$$\begin{aligned} \hat{z}_{k+1} = A_z \hat{z}_k &+ f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \mathbf{u}_k^{[1,n]}) + \\ &[\ell_1, \dots, \ell_n]^\top (h^{-1}_{z,\mathbf{u}\text{ fixed}}(y_k, \mathbf{u}_k^{[1-n,0]}) - \hat{z}_{n,k}), \end{aligned} \qquad (11)$$

with $\hat{z}_{n,k} = C_z \hat{z}_k$, $\hat{z}_{k=0} = \hat{z}_0$, and $h^{-1}_{z,\mathbf{u}\text{ fixed}}$ representing for a fixed input sequence $\mathbf{u}_k^{[1-n,0]}$ the inverse function of h_z in (10). Furthermore, $\ell_1, \dots, \ell_n$ represent the observer gains. The observer gains can be used to assign a certain dynamic behavior of the observer z-error dynamics. The z-error dynamics is the dynamics which describes the evolution of the z-error defined at each time $k \in \mathbb{Z}_+$ as $e_{z,k} \triangleq z_k - \hat{z}_k$. Due to the fact that the state z_k of a system representation in ENOCF appears linearly in the system equations and all nonlinearity enters the state equations via the nonlinear function f_z, depending only on input and output sequences of the system, a linear autonomous z-error dynamics is obtained. The z-error dynamics for (10) and (11) reads as

$$e_{z,k+1} = A_e e_{z,k}, \quad \text{with} \quad A_e \triangleq (A_z - [\ell_1, \dots, \ell_n]^\top C_z). \qquad (12)$$

Note that (C_z, A_z) is an *observable pair*, this is sufficient for the existence of observer gains $\ell_1, \dots, \ell_n$ to render A_e Schur.

B. Existence of the ENOCF

We explained that if the dynamics of a system is given in the ENOCF (10), then it is always possible to design an observer for this system. However, the following question remains open: Which systems in the general from (9) can be transformed into the ENOCF (10)?

In order to answer this question, we recall the notion of *strong local observability* [11]. For convenience we first introduce the *observability map* for non-autonomous discrete-time nonlinear systems [12].

Definition III.1 The observability map ψ of the system given by (9) is defined as:

$$\begin{aligned} \psi(x_k, \mathbf{u}_k^{[0,n-2]}) \triangleq \big[g(x_k) \;\; g(f^1(x_k, u_k)) \;\; \dots \\ g(f^{n-1}(x_k, [u_k, \dots, u_{k+n-2}]^\top)) \big]^\top, \quad \text{where} \end{aligned}$$

$f^i(x_k, [u_k, .., u_{k+i-1}]^\top) = f(f(..f(f(x_k, u_k), u_{k+1}), ..), u_{k+i-1})$, with $i \geq 1$.

Definition III.2 i) System (9) is *strongly locally observable at* x_0, if there exists an open neighborhood $\mathscr{N} \subset \mathbb{X}$ around x_0 such that for all states $\breve{x}_0 \in \mathscr{N}$ and all admissible input sequences $\mathbf{u}_0^{[0,n-2]}$ resulting in the same output sequence as obtained by x_0, i.e. $\psi(x_0, \mathbf{u}_0^{[0,n-2]}) = \psi(\breve{x}_0, \mathbf{u}_0^{[0,n-2]})$, implies that $x_0 = \breve{x}_0$.

ii) System (9) is *strongly locally observable on a domain* $\mathbb{X}$, if i) holds for all $x_0 \in \mathbb{X}$.

A sufficient condition for system (9) to be *strongly locally observable at* x_0 is the following rank condition:

$$\text{rank}\{\partial \psi(x_k, \mathbf{u}_k^{[0,n-2]}) / \partial x_k |_{x_k = x_0}\} = n, \;\; \forall \mathbf{u}_k^{[0,n-2]} \in \mathbb{U}^{n-1}.$$

In [8] it is proven that if (9) is strongly locally observable on a domain $\mathbb{X}$, then there exists a function $\Xi : \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^n$, i.e.

$$z_k = \Xi(x_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \mathbf{u}_k^{[1,n-1]}), \qquad (13)$$

which acts for fixed input and output sequences as a one-to-one invertible map around all $x_0 \in \mathbb{X}$ relating state x_k satisfying (9) and the state z_k satisfying a system representation in ENOCF (10). The interested reader can find a detailed proof and the explicit structure of (13) in [8].

IV. The OBNMPC Scheme

Consider the system dynamics given by (9). The full state x_k is assumed not to be available for feedback. For feedback, an estimate of the state, $\hat{x}_k$, is fed to an NMPC controller instead, i.e. $u_k = \kappa^{\text{MPC}}(\hat{x}_k)$. The state estimate $\hat{x}_k$ is obtained by buffering input and output and using this information in observer (11) in combination with the map $\Xi^{-1}_{\mathbf{uy}\text{ fixed}}$ which represents, for fixed input and output sequences, the inverse of Ξ in (13). The observer candidate (11), appears to be a (local) observer for the class of *strongly locally observable* systems under the assumption that the future input sequence $\mathbf{u}_k^{[1,n]}$ is known a priori. Still, the future input sequence is not known a priori. However, under the assumption that the prediction horizon of the NMPC controller is sufficiently long ($N \geq n$), one can employ a part of the *predicted* future input sequence obtained by the NMPC controller at every time step k, denoted by $\overline{\mathbf{u}}_k^{[1,n]*}$, as a guess for the unknown sequence $\mathbf{u}_k^{[1,n]}$ and inject this sequence to the observer.

To show that this OBNMPC strategy can work, we will investigate whether the resulting closed-loop system can be rendered (locally) asymptotically stable to the origin, i.e. $e_z \triangleq z - \hat{z} = 0$ ($e_x = 0$) and $x = 0$. An outline of the approach is given next.

V. Controller & Observer results

A. Controller

Since there are estimation errors present in the estimated state which is injected to model predictive control law, we will synthesize a model predictive controller which is robust to these estimation errors. The input-to-state-stability (ISS) framework is used for this purpose. Once the controller in closed-loop with system (9), i.e.

$$x_{k+1} = f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})), \quad e_{x,k} \in \mathbb{E}_x \subseteq \mathbb{R}^n, \tag{14}$$

$k \in \mathbb{Z}_+$, is locally ISS with respect to the estimation error e_x, it is known that if the estimation error vanishes e.g. $e_{x,k} \to 0$, then also $x_k \to 0$ for $k \to \infty$. This follows directly from the ISS property given in Definition II.2. Our approach to synthesize an NMPC controller that renders (14) ISS, is to employ existing NMPC schemes for which one can a priori guarantee ISS properties with respect to *additive* model uncertainty, like the one in [13], and then use the following result to conclude about ISS of (14).

Theorem V.1 *Let L_f be the Lipschitz constant of the function f on the domain $\mathbb{X} \times \mathbb{U}$ with respect to its first argument. Furthermore, let $\mathbb{W} \triangleq \{w \in \mathbb{R}^n \mid \|w\| \leq \mu\}$ for some $\mu > 0$, $\mathbb{E}_x \triangleq \{e_x \in \mathbb{R}^n \mid \|e_x\| \leq \mu/(L_f+1)\}$ and let $\widetilde{\mathscr{X}_f}(N) \subseteq \mathscr{X}_f(N)$ be an RPI set for closed-loop system (8) with $0 \in \text{int}(\widetilde{\mathscr{X}_f}(N))$ and $\widetilde{\mathscr{X}_f^e}(N) \triangleq \widetilde{\mathscr{X}_f}(N) \sim \mathbb{E}_x$ with $0 \in \text{int}(\widetilde{\mathscr{X}_f^e}(N))$. Suppose $u_k = \kappa^{\text{MPC}}(\tilde{x}_k)$ is an NMPC control law which renders system (8) ISS, i.e. for additive disturbances w in $\mathbb{W}$ and initial conditions $\tilde{x}_0$ in $\widetilde{\mathscr{X}_f}(N)$ there exist a $\mathscr{K}\mathscr{L}$-function $\beta_{\tilde{x}}$ and a $\mathscr{K}$-function $\gamma_{\tilde{x}}^w$ such that for all $k \in \mathbb{Z}_+$*

$$|\tilde{x}(k, \tilde{x}_0, w)| \leq \beta_{\tilde{x}}(|\tilde{x}_0|, k) + \gamma_{\tilde{x}}^w(\|w\|). \tag{15}$$

Then, the NMPC control law $u_k = \kappa^{\text{MPC}}(x_k + e_{x,k})$, renders (14) ISS, i.e. for initial conditions x_0 in $\widetilde{\mathscr{X}_f^e}(N)$ and estimation errors e_x in $\mathbb{E}_x$ we have for each $k \in \mathbb{Z}_+$

$$|x(k, x_0, e_x)| \leq \beta_x(|x_0|, k) + \gamma_x^{e_x}(\|e_x\|), \tag{16}$$

with $\beta_x(|x_0|, k) \triangleq \beta_{\tilde{x}}(2|x_0|, k)$, $\gamma_x^{e_x}(\|e_x\|) \triangleq \beta_{\tilde{x}}(2|x_0|, 0) + \gamma_{\tilde{x}}^w((L_f+1)\|e_x\|) + \|e_x\|$. Moreover, $\widetilde{\mathscr{X}_f^e}(N)$ is rendered RPI for the closed-loop system given in (14) with $e_{x,k} \in \mathbb{E}_x$.

Proof: To prove that $\widetilde{\mathscr{X}_f^e}(N)$ is RPI for (14) we have to show that $x_k \in \widetilde{\mathscr{X}_f^e}(N)$ and $e_{x,k} \in \mathbb{E}_x$ implies $f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})) \in \widetilde{\mathscr{X}_f^e}(N)$. Let $x_k \in \widetilde{\mathscr{X}_f^e}(N)$, we will now show that for any $\widetilde{e}_k \in \mathbb{E}_x$ $f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})) + \widetilde{e}_k \in \widetilde{\mathscr{X}_f}(N)$, which yields $f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})) \in \widetilde{\mathscr{X}_f}(N) \sim \mathbb{E}_x \triangleq \widetilde{\mathscr{X}_f^e}(N)$. Indeed

$$f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})) + \widetilde{e}_k = f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + w_k \tag{17}$$

with $\tilde{x}_k = x_k + e_{x,k}$ and $w_k = f(x_k, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + \widetilde{e}_k$. Utilizing the Lipschitz property of f yields $|f(x_k, u_k) - f(x_k + e_{x,k}, u_k)| \leq L_f |e_{x,k}|$. Thus it holds that for and all $k \in \mathbb{Z}_+$ and $e_{x,k} \in \mathbb{E}_x$

$$\begin{aligned} |w_k| &= |f(x_k, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(x_k + e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) + \widetilde{e}_k| \\ &\leq L_f \|e_x\| + \|\widetilde{e}\| \leq \mu. \end{aligned} \tag{18}$$

Due to (18) and the hypothesis in the Theorem V.1 we have that $f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + w_k \in \widetilde{\mathscr{X}_f}(N)$ and thus (17) implies that for $e_{x,k} \in \mathbb{E}_x$ $f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})) + \widetilde{e}_k \in \widetilde{\mathscr{X}_f}(N)$ for any $\widetilde{e}_k \in \mathbb{E}_x$. We continue proving the ISS property (16) for (14). We perform the following coordinate change on (14), i.e.

$$x_k = \tilde{x}_k - e_{x,k}, \quad \forall k \in \mathbb{Z}_+, \tag{19}$$

which gives

$$\tilde{x}_{k+1} = f(\tilde{x}_k - e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) + e_{x,k+1}. \tag{20}$$

Rewriting (20) as $\tilde{x}_{k+1} = f(\tilde{x}_k - e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) + f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + e_{x,k+1}$ yields

$$\tilde{x}_{k+1} = f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + w_k \tag{21}$$

where $w_k \triangleq f(\tilde{x}_k - e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + e_{x,k+1}$. Thus for all $k \in \mathbb{Z}_+$ and $e_{x,k}, e_{x,k+1} \in \mathbb{E}_x$ it holds that

$$\begin{aligned} |w_k| =& |f(\tilde{x}_k - e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k)) + e_{x,k+1}| \\ \leq& |f(\tilde{x} - e_{x,k}, \kappa^{\text{MPC}}(\tilde{x}_k)) - f(\tilde{x}_k, \kappa^{\text{MPC}}(\tilde{x}_k))| + \|e_x\| \\ \leq& (L_f + 1)\|e_x\| \leq \mu. \end{aligned} \tag{22}$$

Then, from the hypothesis in Theorem V.1 it follows that, for $\tilde{x}_0 \in \widetilde{\mathscr{X}_f}(N)$, (21) is ISS w.r.t. additive disturbance w_k. Substituting (22) into (15) yields

$$|\tilde{x}(k, \tilde{x}_0, e_x)| \leq \beta_{\tilde{x}}(|\tilde{x}_0|, k) + \gamma_{\tilde{x}}^{e_x}(\|e_x\|), \tag{23}$$

where $\gamma_{\tilde{x}}^{e_x}(\|e_x\|) = \gamma_{\tilde{x}}^w((L_f+1)\|e_x\|)$. Since $\widetilde{\mathscr{X}_f}(N)$ is an RPI set for (21), $\|e_x\| \leq \frac{\mu}{L_f+1}$ and relation (22) holds, it follows that property (23) holds for all $k \in \mathbb{Z}_+$, $e_{x,k}, e_{x,k+1} \in \mathbb{E}_x$ and $\tilde{x}_0 \in \widetilde{\mathscr{X}_f}(N)$. Utilizing (19), property (23), the fact that $x_0 \in \widetilde{\mathscr{X}_f^e}(N)$ implies that $\tilde{x}_0 \in \widetilde{\mathscr{X}_f}(N)$ (due to $\widetilde{\mathscr{X}_f^e}(N) \triangleq \widetilde{\mathscr{X}_f}(N) \sim \mathbb{E}_x$ and (19)) and that $\widetilde{\mathscr{X}_f^e}(N)$ is an RPI set for (14) we have that for all $x_0 \in \widetilde{\mathscr{X}_f^e}(N)$, $e_{x,k} \in \mathbb{E}_x$ and $k \in \mathbb{Z}_+$

$$x(k, x_0, e_x) = (\tilde{x}(k, \tilde{x}_0, e_x) - e_{x,k}) \in \widetilde{\mathscr{X}_f^e}(N) \subseteq \mathscr{X}_f(N) \subseteq \mathbb{X}$$

and

$$\begin{aligned} |x(k, x_0, e_x)| &= |\tilde{x}(k, \tilde{x}_0, e_x) - e_{x,k}| \leq |\tilde{x}(k, \tilde{x}_0, e_x)| + |e_{x,k}| \\ &\leq \beta_{\tilde{x}}(|x_0 + e_{x,0}|, k) + \gamma_{\tilde{x}}^{e_x}(\|e_x\|) + |e_{x,k}| \\ &\leq \beta_x(|x_0|, k) + \gamma_x^{e_x}(\|e_x\|). \end{aligned}$$

∎

As mentioned before, once (14) is ISS with respect to the estimation error e_x, the requirement which will lead to asymptotic stability of the OBNMPC scheme is that the estimation error vanishes i.e. $e_{x,k} \to 0$ for $k \to \infty$. In case the *predicted* future input sequence $\overline{\mathbf{u}}_k^{[1,n]*}$ coincides with the actual future input sequence $\mathbf{u}_k^{[1,n]}$, the error dynamics appears to be linear by employing the coordinate change (13). The requirement for asymptotic stability of e_x (e_z) is, in this special case, that the system matrix A_e defining the linear error dynamics is Schur. However, since the *predicted* future input sequence $\overline{\mathbf{u}}_k^{[1,n]*}$ does not coincide with the actual future input sequence $\mathbf{u}_k^{[1,n]}$ in general, the asymptotic stability result of the estimation error dynamics pointed out in Section III does not apply. A possible approach for dealing with this problem is presented in the sequel.

B. Observer

Recall that all nonlinearity of the system in ENOCF appears in the state equations as a nonlinear term, which only depends on the input and output sequences of the system. In case of mismatch between sequences in the applied observer dynamics and the actual nominal dynamics in ENOCF, elimination of the nonlinear term in the derivation of the z-error dynamics is *not* realized. In this situation the error dynamics (12) changes into

$$e_{z,k+1} = A_e e_{z,k} + \Delta f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]*}, \mathbf{e}_{\mathbf{u},k}^{[1,n]}), \quad (24a)$$

$$e_{x,k} = \Delta\Xi(e_{z,k}, \hat{z}_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n-1]*}, \mathbf{e}_{\mathbf{u},k}^{[1,n-1]}), \quad (24b)$$

where $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \triangleq \overline{\mathbf{u}}_k^{[1,n]*} - \mathbf{u}_k^{[1,n]}$,

$$\Delta f_z \triangleq f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]*} - \mathbf{e}_{\mathbf{u},k}^{[1,n]}) - f_z(\cdot,\cdot,\overline{\mathbf{u}}_k^{[1,n]*}),$$

$$\Delta\Xi \triangleq \Xi^{-1}_{\mathbf{uy}\text{ fixed}}(\hat{z}_k - e_{z,k}, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n-1]*} - \mathbf{e}_{\mathbf{u},k}^{[1,n-1]}) - \Xi^{-1}_{\mathbf{uy}\text{ fixed}}(\hat{z}_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]} \overline{\mathbf{u}}_k^{[1,n-1]*})$$

with $\Delta f_z(\cdot,\cdot,\cdot,0) = 0$, $\Delta\Xi(0,\cdot,\cdot,\cdot,\cdot,0) = 0$. Recall that A_e is a matrix that can always be rendered Schur by an appropriate choice of the observer gains. In the sequel we assume A_e in (24a) is Schur. Equations (24a) and (24b) define the error dynamics of the observer candidate in case of feeding an imperfect predicted future input sequence to the observer. The error dynamics (24a) has now become a non-autonomous system. We will continue formulating an ISS and IOS result of the error dynamics (24a), (24b) with respect to $\mathbf{e}_{\mathbf{u},k}^{[1,n-1]}$ as input. Since $\mathbb{U}$ is compact we have that $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ also lives in a compact set, i.e. $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \in \mathbb{E}_{\mathbf{e_u}}$ where $\mathbb{E}_{\mathbf{e_u}} \triangleq \{\mathbf{e}_{\mathbf{u}}^{[1,n]} \in \mathbb{R}^n \mid \|\mathbf{e}_{\mathbf{u}}^{[1,n]}\| \le \varepsilon_{\mathbf{e_u}}\}$ with $\varepsilon_{\mathbf{e_u}} > 0$. Before we formulate the main result in this section we introduce L_{f_z} as being the Lipschitz constant of the function f_z in (10) with respect to the argument $\mathbf{u}_k^{[1,n]}$ on the domain $\mathbb{Y}^n \times \mathbb{U}^n \times \mathbb{U}^n$, where $\mathbb{Y}^n \triangleq \mathbb{Y} \times \ldots \times \mathbb{Y}$ with $\mathbb{Y} \triangleq \{y \in \mathbb{R} \mid y = g(x), x \in \mathbb{X}\}$. Furthermore, L_Ξ denotes a constant such that for all $k \in \mathbb{Z}_+$

$$|\Delta\Xi(e_{z,k},\cdot,\cdot,\cdot,\cdot,\mathbf{e}_{\mathbf{u},k}^{[1,n-1]})| \le L_\Xi(|e_{z,k}| + |\mathbf{e}_{\mathbf{u},k}^{[1,n-1]}|). \quad (25)$$

The constant L_Ξ is directly related to the Lipschitz constant of the function $\Xi^{-1}_{\mathbf{uy}\text{ fixed}}$ on the domain $\mathbb{S}_z \times \mathbb{Y}^{n-1} \times \mathbb{U}^n \times \mathbb{U}^{n-1}$ with respect to the arguments z_k and $\mathbf{u}_k^{[1,n-1]}$, where $\mathbb{S}_z \triangleq \{z \in \mathbb{R}^n \mid z = \Xi(x, \mathbf{y}^{[1-n,-1]}, \mathbf{u}^{[1-n,0]}, \mathbf{u}^{[1,n-1]}),\ x \in \mathbb{X},\ \mathbf{y}^{[1-n,-1]} \in \mathbb{Y}^{n-1},\ \mathbf{u}^{[1-n,0]} \in \mathbb{U}^n,\ \mathbf{u}^{[1,n-1]} \in \mathbb{U}^{n-1}\}$.

Assumption V.2 For the state x_k of (9) (or (14)) there holds $x_k \in \overline{\mathbb{X}}\ \forall k \in \mathbb{Z}_+$ where $\overline{\mathbb{X}}$ is a compact set with $0 \in \text{int}(\overline{\mathbb{X}})$ such that $\overline{\mathbb{X}} \oplus \mathbb{E}_x \subseteq \mathbb{X}$ with $\mathbb{E}_x \triangleq \{e_x \in \mathbb{R}^n \mid \|e_x\| \le L_\Xi \varepsilon_{\mathbf{e_u}}(\frac{\hbar}{1-\eta} L_{f_z}(\varepsilon_z + 1) + 1)\}$ for some $\varepsilon_z > 0$ and with $\hbar \ge 1$, $\eta \in [0,1)$ such that $|A_e^k| \le \hbar\eta^k$ holds for all $k \in \mathbb{Z}_+$.

Theorem V.3 *Let $\mathbf{y}_k^{[1-n,0]} \in \mathbb{Y}^n$, $\mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]*} \in \mathbb{U}^n$ for all $k \in \mathbb{Z}_+$. Suppose Assumption V.2 holds and (9) is strongly locally observable on domain $\mathbb{X}$. Then, the z-error dynamics (24a) is ISS with respect to $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ in $\mathbb{E}_{\mathbf{e_u}}$ as input and initial conditions $e_{z,0}$ in $\mathscr{E}_z \triangleq \{e_{z,0} \in \mathbb{R}^n \mid |e_{z,0}| \le \frac{\varepsilon_z}{1-\eta} L_{f_z} \varepsilon_{\mathbf{e_u}}\}$, i.e. for all $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \in \mathbb{E}_{\mathbf{e_u}}$, $e_{z,0} \in \mathscr{E}_z$ and $k \in \mathbb{Z}_+$ it holds that*

$$|e_z(k, e_{z,0}, \mathbf{e}_{\mathbf{u}}^{[1,n]})| \le \beta_{e_z}(|e_{z,0}|, k) + \gamma_{e_z}^{\mathbf{e_u}}(\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|), \quad (26)$$

where $\beta_{e_z}(|e_{z,0}|, k) \triangleq \hbar\eta^k |e_{z,0}|$, $\gamma_{e_z}^{\mathbf{e_u}}(\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|) \triangleq \frac{\hbar}{1-\eta} L_{f_z} \|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|$. Moreover, the x-error dynamics defined by (24a), (24b) is locally IOS with respect to $\mathbf{e}_{\mathbf{u}}^{[1,n]}$ as input, i.e. for all $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \in \mathbb{E}_{\mathbf{e_u}}$ and $e_{z,0} \in \mathscr{E}_z$ we have that for all $k \in \mathbb{Z}_+$

$$|e_x(k, e_{z,0}, \mathbf{e}_{\mathbf{u}}^{[1,n]})| \le \beta_{e_x}(|e_{z,0}|, k) + \gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|), \quad (27)$$

where $\beta_{e_x}(|e_{z,0}|, k) \triangleq L_\Xi \beta_{e_z}(|e_{z,0}|, k)$, $\gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|) \triangleq L_\Xi(\frac{\hbar}{1-\eta} L_{f_z} + 1)\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|$. Furthermore, $\hat{x}_k \in \overline{\mathbb{X}} \oplus \mathbb{E}_x$ for all $k \in \mathbb{Z}_+$ and for all $e_{z,0} \in \mathscr{E}_z$, $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \in \mathbb{E}_{\mathbf{e_u}}$ and $k \in \mathbb{Z}_+$

$$|e_x(k, e_{z,0}, \mathbf{e}_{\mathbf{u}}^{[1,n]})| \le L_\Xi \varepsilon_{\mathbf{e_u}}((\hbar/(1-\eta)) L_{f_z}(\varepsilon_z + 1) + 1). \quad (28)$$

Proof: Due to lack of space we just give a sketch of the proof. For the full proof, see [14]. Due to the fact that (24a) is linear in z_k, A_e is Schur and f_z is Lipschitz continuous w.r.t. $\mathbf{u}_k^{[1,n]}$ the ISS property of (24a) follows. IOS and boundedness of $e_x(k, e_{z,0}, \mathbf{e}_{\mathbf{u}}^{[1,n]})$ follows since $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ lives in a compact set and Ξ is well defined for all $\hat{x}_k \in \overline{\mathbb{X}} \oplus \mathbb{E}_x \subseteq \mathbb{X}$ and $x_k \in \mathbb{X}$ such that (25) holds. ■

VI. MAIN RESULTS ON THE INTERCONNECTION

So far, we have *separately* designed an NMPC controller which is robust (ISS) to estimation errors (e_x) and obtained an observer for which the error dynamics is robust (IOS) with respect to prediction errors $\mathbf{e}_{\mathbf{u}}^{[1,n]}$ present in the predicted future input sequence $\overline{\mathbf{u}}^{[1,n]*}$ injected to the observer. In this section we investigate the properties of the IOS observer error dynamics interconnected with the ISS NMPC system (14), i.e.

$$\begin{cases} e_{z,k+1} = A_e e_{z,k} + \Delta f_z(\cdot,\cdot,\cdot,\mathbf{e}_{\mathbf{u},k}^{[1,n]}) \\ e_{x,k} = \Delta\Xi(e_{z,k},\cdot,\cdot,\cdot,\cdot,\mathbf{e}_{\mathbf{u},k}^{[1,n]}) \end{cases} \quad (29a)$$

$$x_{k+1} = f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})), \quad (29b)$$

where $[e_{z,k}, x_k]^\top \in \mathbb{R}^{2n}$ and $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \in \mathbb{R}^n$ is the state and the input of the interconnection, respectively. For the interconnection in (29), with "external" input $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ having certain properties, we want to prove (local) asymptotic stability.

Assumption VI.1 i) The NMPC controller admits a *regularity* property, i.e. $\exists\ \theta_1, \theta_2 > 0$ s.t. $|u_{k|k}| \le \theta_1|\hat{x}_k|$ and $|u_{k+i|k}| \le \theta_2|\hat{x}_k|$ for $i = 1, 2, ..., n$;

ii) Lipschitz continuity of f_z with respect to $\mathbf{u}_k^{[1,n]}$ on the domain $\mathbb{Y}^n \times \mathbb{U}^n \times \mathbb{U}^n$; iii) $f, g \in C^1$.

Regularity can be imposed by simply including $|u_{k|k}| \le \theta_1|\hat{x}_k|$ and $|u_{k+i|k}| \le \theta_2|\hat{x}_k|$ for $i = 1, 2, ..., n$ as an additional constrained to the NMPC problem (Problem II.3), for a priori fixed θ_1, θ_2. For ease of exposition we assume the ISS gain of the NMPC controller, i.e. $\gamma_x^{e_x}$, is linear in its argument.

Theorem VI.2 *Let $\mathbb{E}_x \triangleq \{e_x \in \mathbb{R}^n \mid \|e_x\| \le L_\Xi \varepsilon_{\mathbf{e_u}}(\frac{\hbar}{1-\eta} L_{f_z}(\varepsilon_z + 1) + 1)\}$ with $\varepsilon_z > 0$. Suppose Assumption VI.1 holds and (9) is strongly locally observable on domain $\mathbb{X}$. Furthermore, let κ^{MPC} be an NMPC control law, obtained using the result in Theorem V.1 with $N \ge n$, which renders (14) ISS w.r.t. input $e_{x,k}$ in $\mathbb{E}_x$ for initial conditions x_0 in $\widetilde{\mathscr{X}_f^e}(N)$. Then, if*

$$\gamma_{\mathbf{e_u}}^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\gamma_x^{e_x} + 1) < 1 \quad (30)$$

with $\gamma_{\mathbf{e_u}}^{x} \gamma_{e_x}^{\mathbf{e_u}}(\gamma_x^{e_x} + 1) = (\theta_1 + \theta_2) L_\Xi(\frac{\hbar}{1-\eta} L_{f_z} + 1)(\gamma_x^{e_x} + 1)$, system (29) is ***asymptotically stable****, for initial conditions $[e_{z,0}, x_0]^\top$ in $\mathscr{E}_z \times \widetilde{\mathscr{X}_f^e}(N)$ with $\mathscr{E}_z \triangleq \{e_{z,0} \in \mathbb{R}^n \mid |e_{z,0}| \le \frac{\varepsilon_z}{1-\eta} L_{f_z} \varepsilon_{\mathbf{e_u}}\}$, for $\varepsilon_z > 0$.*

To prove the theorem, we first formulate a technical lemma.

Lemma VI.3 *Suppose $N \ge n$ and Assumption VI.1 holds. Then, the signal $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ satisfies*

$$|\mathbf{e}_{\mathbf{u}}^{[1,n]}(k, x, e_x)| \le \gamma_{\mathbf{e_u}}^{x}\|x\| + \gamma_{\mathbf{e_u}}^{e_x}\|e_x\|, \quad \forall k \in \mathbb{Z}_+, \quad (31)$$

where the gains $\gamma_{\mathbf{e_u}}^{x}, \gamma_{\mathbf{e_u}}^{e_x}$ are defined as $\gamma_{\mathbf{e_u}}^{x} = \gamma_{\mathbf{e_u}}^{e_x} \triangleq (\theta_1 + \theta_2)$.

Proof: Using *regularity* (Assumption VI.1 (iii)) and the triangle inequality, the induced norm of the difference between the predicted future inputs and the real inputs can be upper bounded for all $k \in \mathbb{Z}_+$ and $i = 1, \ldots, n$, i.e.

$$|u_{k+i} - u_{k+i|k}| \le |u_{k+i}| + |u_{k+i|k}| \le \theta_1|\hat{x}_{k+i}| + \theta_2|\hat{x}_k|. \quad (32)$$

Since, (32) holds $\forall k \in \mathbb{Z}_+$, $i = 1,..,n$ the result follows. ■

Regularity thus leads to property (31). Employing this property we can prove the statement in Theorem VI.2.

Proof: The first part of the proof consists of proving that the ISS (IOS) properties of subsystems (29a), (29b), as proven in the previous sections, are preserved when they are interconnected into the cascade as described by (29). Secondly stability, i.e. property (4), is proven for (29). The last part of the proof consists in proving attraction of (29), i.e. property (5). *Part 1)* Due to the hypothesis of Theorem VI.2 $\widetilde{\mathscr{X}}_f^e(N)$ is RPI for (29b). This implies that Assumption V.2 holds with $\overline{\mathbb{X}} = \widetilde{\mathscr{X}}_f^e(N)$ such that the results of Theorem V.3 hold. This implies that the trajectory $e_x(k, e_{z,0}, \mathbf{e}_\mathbf{u}^{[1,n]})$, satisfying the dynamics of subsystem (29a), remains for each $k \in \mathbb{Z}_+$ and initial conditions $e_{z,0}$ in $\mathscr{E}_z$ in the set $\mathbb{E}_x$, which implies that the ISS property of subsystem (29b) as stated in the hypothesis of Theorem VI.2 in preserved for all $k \in \mathbb{Z}_+$ and initial conditions x_0 in $\widetilde{\mathscr{X}}_f^e(N)$.

Part 2) We can now conclude that property (16) and (26), (27) of Theorem V.1 and V.3, respectively, hold. Property (16), (26) and (27) imply that for all $k \in \mathbb{Z}_+$ $|x(k, x_0, \mathbf{e}_\mathbf{u}^{[1,n]})| \le$

$$\le \beta_x(|x_0|, 0) + \gamma_x^{e_x}(\beta_{e_x}(|e_{z,0}|, 0) + \gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_\mathbf{u}^{[1,n]}\|)), \quad (33a)$$

$$|e_z(k, e_{z,0}, \mathbf{e}_\mathbf{u}^{[1,n]})| \le \beta_{e_z}(|e_{z,0}|, 0) + \gamma_{e_z}^{\mathbf{e_u}}(\|\mathbf{e}_\mathbf{u}^{[1,n]}\|), \quad (33b)$$

$$|e_x(k, e_{z,0}, \mathbf{e}_\mathbf{u}^{[1,n]})| \le \beta_{e_x}(|e_{z,0}|, 0) + \gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_\mathbf{u}^{[1,n]}\|). \quad (33c)$$

Employing property (31), (33a) and (33c) yields

$$\begin{aligned}\|\mathbf{e}_\mathbf{u}^{[1,n]}\| \le{}& \gamma_{\mathbf{e_u}}^x \beta_x(|x_0|, 0) + \gamma_{\mathbf{e_u}}^x \gamma_x^{e_x} \beta_{e_x}(|e_{z,0}|, 0) + \gamma_{\mathbf{e_u}}^x \gamma_x^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_\mathbf{u}^{[1,n]}\|) \\ &+ \gamma_{\mathbf{e_u}}^{e_x} \beta_{e_x}(|e_{z,0}|, 0) + \gamma_{\mathbf{e_u}}^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\|\mathbf{e}_\mathbf{u}^{[1,n]}\|).\end{aligned} \quad (34)$$

Since $\gamma_{\mathbf{e_u}}^x = \gamma_{\mathbf{e_u}}^{e_x}$ we have

$$\begin{aligned}\|\mathbf{e}_\mathbf{u}^{[1,n]}\| \le{}& \gamma_{\mathbf{e_u}}^{e_x} \beta_x(|x_0|, 0) + \gamma_{\mathbf{e_u}}^{e_x} \gamma_x^{e_x} \beta_{e_x}(|e_{z,0}|, 0) + \gamma_{\mathbf{e_u}}^{e_x} \beta_{e_x}(|e_{z,0}|, 0) \\ &+ \gamma_{\mathbf{e_u}}^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\gamma_x^{e_x} + 1)\|\mathbf{e}_\mathbf{u}^{[1,n]}\|, \quad \text{or} \\ \|\mathbf{e}_\mathbf{u}^{[1,n]}\| \le{}& (1 - \gamma_{\mathbf{e_u}}^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\gamma_x^{e_x} + 1))^{-1}(\gamma_{\mathbf{e_u}}^{e_x} \beta_x(|x_0|, 0) \\ &+ \gamma_{\mathbf{e_u}}^{e_x} \beta_{e_x}(|e_{z,0}|, 0) + \gamma_{\mathbf{e_u}}^{e_x} \beta_{e_x}(|e_{z,0}|, 0)) \triangleq \chi(|e_{z,0}|, |x_0|).\end{aligned} \quad (35)$$

Then, from the expressions obtained by substituting (35) into (33a) and (33b) one can conclude that there exists a $\mathscr{K}$*-function* $\varphi(|[e_{z,0}, x_0]^\top|)$ such that property (4) in Definition II.2 holds for (29) $\forall \mathbf{e}_\mathbf{u}^{[1,n]} \in \mathscr{M}_{\mathbb{E}_{\mathbf{e_u}}}$, where $\mathscr{M}_{\mathbb{E}_{\mathbf{e_u}}} = \{\mathbf{e}_\mathbf{u}^{[1,n]} : \mathbb{Z}_+ \to \mathbb{E}_{\mathbf{e_u}} \mid \|\mathbf{e}_\mathbf{u}^{[1,n]}\| \le \chi(|e_{z,0}|, |x_0|)\}$.

Part 3) Property (16), (27) and (31) of Theorem V.1, V.3 and Lemma VI.3, respectively, imply

$$\overline{\lim_{k\to\infty}} |x(k, x_0, e_x)| \le \gamma_x^{e_x}\left(\overline{\lim_{k\to\infty}} |e_{x,k}|\right), \quad (36a)$$

$$\overline{\lim_{k\to\infty}} |e_x(k, e_{z,0}, \mathbf{e}_\mathbf{u}^{[1,n]})| \le \gamma_{e_x}^{\mathbf{e_u}}\left(\overline{\lim_{k\to\infty}} |\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right), \quad (36b)$$

$$\overline{\lim_{k\to\infty}} |\mathbf{e}_\mathbf{u}^{[1,n]}(k, x, e_x)| \le \gamma_{\mathbf{e_u}}^x\left(\overline{\lim_{k\to\infty}} |x_k|\right) + \gamma_{\mathbf{e_u}}^{e_x}\left(\overline{\lim_{k\to\infty}} |e_{x,k}|\right). \quad (36c)$$

Substitution of (36a) and (36b) in (36c) and subsequently substituting (36a) in the obtained expression and using the fact that $\gamma_{\mathbf{e_u}}^x = \gamma_{\mathbf{e_u}}^{e_x}$, yields

$$\overline{\lim_{k\to\infty}} |\mathbf{e}_\mathbf{u}^{[1,n]}(k, x, e_x)| \le \gamma_{\mathbf{e_u}}^{e_x} \gamma_{e_x}^{\mathbf{e_u}}(\gamma_x^{e_x} + 1)\left(\overline{\lim_{k\to\infty}} |\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right). \quad (37)$$

Due to the small gain property (30) in the hypothesis of Theorem VI.2 and the fact that $\overline{\lim}_{k\to\infty} |\mathbf{e}_{\mathbf{u},k}^{[1,n]}|$ is well defined (due to compactness of $\mathbb{U}$ we know that $\overline{\lim}_{k\to\infty} |\mathbf{e}_{\mathbf{u},k}^{[1,n]}|$ is finite) we have that (37) is true only if

$$\overline{\lim_{k\to\infty}} |\mathbf{e}_\mathbf{u}^{[1,n]}(k, x, e_x)| = 0. \quad (38)$$

Property (38) together with (36a), and property (26) imply

$$\overline{\lim_{k\to\infty}} |[e_z(k, e_{z,0}, \mathbf{e}_\mathbf{u}^{[1,n]}), x(k, x_0, \mathbf{e}_\mathbf{u}^{[1,n]})]^\top| = 0. \quad (39)$$

■

VII. Conclusions

We proposed an observer based nonlinear predictive control scheme for the class of strongly observable nonlinear discrete-time systems. It is proven that a separately designed NMPC state feedback controller, ISS w.r.t. observer errors, and an extended observer in closed-loop with the system results in a (locally) asymptotically stable closed-loop system under the satisfaction of a small gain argument. Furthermore, for the design of an NMPC state feedback controller that is ISS w.r.t. observer errors a result is obtained which enables one to employ state feedback NMPC controllers, designed for rendering the closed-loop system ISS with respect to additive disturbances, in a scenario where the closed-loop system has to be rendered ISS w.r.t. observer errors.

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