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A STABLE AND CONVERGENT EXTRAPOLATION PROCEDURE FOR THE SCATTERING AMPLITUDE

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ABSTRACT

The extrapolation of the amplitude out of its physical region being mathematically an "improper problem", it is not sufficient to have convergent expansions, one has to be also sure that the results are stable against small perturbations (experimental errors).

In the present paper, this problem is solved by means of semi-convergent expansions in terms of conformal mappings of double connected domains. An optimization problem leads to an optimal mapping function, whose explicit form is then derived. Although a good deal of the paper is devoted to the mathematical side of this problem, the results can be used directly in the analysis of experimental data.

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1. INTRODUCTION

The usual and almost exclusive way one can take advantage of the analyticity of the scattering amplitude is to compute the amplitude by means of a Cauchy integral (dispersion relation) over its absorptive parts. The reverse procedure, i.e., the computation of the discontinuities along the cuts, starting from some region of the complex energy or cosine plane where the amplitude is known, is also possible, at least in principle. Indeed, an analytic function, including the position of its singularities and the discontinuities along them, is completely determined by the values taken by this function along some continua - wherever this continuum would be placed - and the Regge pole formalism as well as the analytic extrapolations, using conformal mappings discussed below, are good examples of such reverse problems. Nevertheless, the words "at least in principle" had to be used to warn that such analytical continuations although possible represent typical examples of what the mathematicians call "improperly posed problems". Indeed, as is well known (see Section 2), the form of the discontinuity obtained by such analytical continuation is very sensitive to the values taken by the function in that very region from which one originally starts the analytic continuation, being unstable against small changes of the initial data.

In spite of these principal difficulties connected with the mathematical instabilities of the "reverse dispersion relation techniques" [1], and without trying to solve them, many authors [2)-14)] have already tried to use them to extrapolate the experimentally found amplitude out of its physical region.

The basic idea of these works consists in using a suitable function w [2)-4)] which maps the energy (or cosine) complex cut plane into the interior of the circle $|w| \leq 1$ (see Figs. 2 and 3), and to expand then the amplitude (or cross-section) in powers of w. Indeed, it can be shown [5)] under reasonable physical assumptions that these series converge even on the cut (i.e., on the circle $|w| = 1$). Originally, Frazer [3)] and Lovelace [4)] used these series respectively to enhance the convergence of the "extrapolation to the poles" or to find directly the form of the spectral function from the physical cosine dependence of the (pion-nucleon) amplitude, but afterwards these methods were used in a great variety of problems.

i) New effective range formulae

The conformal mapping series can, indeed, be used successfully to express the effect of an unknown cut [for instance, the left-hand cut of the partial waves [3]] in terms of some few numerical constants. In a similar way, Islam [6] obtained an effective range formula for Regge trajectories. Further, Bowcock and Stoddard [7] used these series to expand the phase integral for pion-pion scattering, in order to evaluate the influence of the pion-pion - nucleon-antinucleon channel or pion-nucleon scattering.

ii) Extrapolations to the cuts

Besides the spectral function computations of Lovelace [4], a fruitful field of applications was offered by the "discrepancy function" introduced by the Hamilton group [8]-[10], defined to be the difference between the actual pion-nucleon amplitude and the dispersion integral over the right-hand cut, summarizing hence the influence of the crossed channel cuts. Expanding the discrepancy function for backward scattering in powers of the conformal variable, Atkinson [11] was able to obtain the absorptive part of the pion-pion - nucleon-antinucleon channel and the pion-pion phases from the experimental pion-nucleon data. Later, using a similar extrapolation of the backward scattering amplitude up to its left-hand cut, Lovelace, Heinz and Donnachie [12] found a strong evidence for the existence of the sigma meson.

An analogous problem was that of Levinger, Peierls and Wong [13],[14] which expressed the nucleon form factor in its timelike region (on the cut) starting from the known spacelike values extracted from scattering experiments.

iii)

In some cases one could be interested to extrapolate the amplitude not as far as the cut, but to some other region of interest, lying within the holomorphy domain, the "extrapolation to the poles" [3] being an example of such problems. A continuation of the CGLN type [15] of the nearby forward scattering amplitude in the rest of the physical region, for instance in order to project partial waves, falls in the same class of interest. Although there are no singularities in the physical region, the circle of convergence of pure momentum transfer power series is drastically limited by the singularities existing at non-physical cosines [2], and some suitable conformal mapping technique is to be used. Of course the result can be checked experimentally, and it is worth while to note [16] the close relationship between the range of convergence and the regions of reliable prediction.

iii-0) Among these last items, a separate mention will be made for these very problems for which the extension of the continua out of which the extrapolation starts reduces to zero. A typical example is provided by the use of the amplitude together with all its derivatives at $t = 0$ to express it in the whole physical region. [This was the exact programme of the CGLN paper [15].] Although such a question has little semiphenomenological interest, as an experiment could never be accurate enough to provide information about the higher derivatives, case iii-0) is of special interest in deriving dynamical integral equations starting from forward scattering dispersion relations for the amplitude and its momentum derivatives : a suitable convergent conformal series allows then to express the unitarity condition in terms of forward scattering quantities and the set of equations can then be derived in a straightforward way [2,17].

In the present note we shall analyze critically these extrapolation procedures (Section 2), and then try to give a more rigorous mathematical basis to this subject of growing interest. Although no objection could be made to the few parameters effective range formulae of type i), convergence problems of a special type (see Section 3), as well as instabilities towards small experimental errors, arise when one tries to increase the number of adjustable parameters, in order to fit the experimental data along some non-vanishing continua [problems ii) and iii)].

Although very serious, these difficulties are not at all insurmountable. Indeed, contrary to a common superstition, "improperly posed problems" (unstable problems) are not intractable, and classical physics provides a great deal of examples in which such problems are studied and even applied for practical purposes *).

As is usual with improper problems, supplementary *) physical information is required, mainly in the form of a smoothness condition for the amplitude on the cut. To be more specific, one needs (on the cuts) a Hölder condition

$$|f^{(p)}(z) - f^{(p)}(z_0)| \leq \text{const.}\,|z-z_0|^{\alpha} \tag{1.1}$$

for some p^{th} derivative of the amplitude. p can also be zero, but the larger, the better. One can really guarantee such a Hölder condition for the amplitude or for some suitable combination $f(z)$ with some suitable function [see Ref. 5)], as the single singularities met along the cuts are the threshold singularities, or, if one is interested to extrapolate in the spectral function region, the singularities along the Landau curves [5].

However, in clear-cut contrast to the papers [3)-14)], in order to handle the convergence problems related to the non-vanishing length of the physical region and to control the developing instabilities conformal mappings of double connected domains have to be used (see Sections 3 and 4), together with some given prescription (see Section 5) which limits the number of terms in the series in function of the magnitude ε of the errors affecting the data to be extrapolated. The explicit form for the optimal conformal mapping will be then derived in Section 6.

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*) Ore prospecting by seismic wave reflection is a typical example of such an improper problem; however, if some geophysical information on the smoothness, etc., of the layers is supplemented, the seismologist is able to yield with great accuracy the thickness of the glaciers or the position of the ore beds !

2. THE NEVANLINNA PRINCIPLE AND THE PROBLEM OF THE STABILITY OF ANALYTIC CONTINUATIONS

It is often argued that the Regge extrapolation "is possible" *) because although one has to go very far in the non-physical cosine direction (great s), one is happy to use the conjuncture that the s dependence is enclosed only in the argument of the Legendre function $P_{\alpha(t)}(1-2s/t-4)$ (which, of course, is a well-defined mathematical object) and that the incertitudes affect only some functions of t - the residue of the Regge pole and its trajectory. Being interested only in a limited range of values of t, it is supposed that these not well-known t functions will have only little influence on the correctitude of the extrapolation.

To dissipate the confidence in this a priori stability of such procedures, in what follows we will prove a theorem which could be interpreted [1] as a partial failure of the program in using analytical continuations for physical purposes. Indeed, all observation being subjected to small errors, a physical theory has to be stable against these errors, in the sense that the outcome has to tend to a well-defined limit when the errors tend to zero.

We shall first prove a lemma (the Nevanlinna principle).

Let $F(z)$ be holomorphic and bounded in some domain $(\mathcal{D})$

$$|F(z)| \leq M \tag{2.1a}$$

Let the boundary of $(\mathcal{D})$ be formed by two disjoint curves Γ_1 and Γ and let $F(z)$ satisfy

$$|F(z)| \leq m \qquad \text{on } \Gamma_1 . \tag{2.1b}$$

(See Fig. 1 : the double connectivity of $(\mathcal{D})$ is not a necessary condition, but this will actually be the situation for the problems we will be interested in.)

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*) The word "possible" stands here for "stable", but being aware of the mathematical implication of the word "stable", one usually avoids to use it !

Let us now define the harmonic measure $\omega(x,y)$ to be that (real) function of x and y $(z = x+iy)$ which is harmonic in $(\mathcal{D})$

$$\nabla^2 \omega(x,y) = 0 \quad \text{for } z \in (\mathcal{D}) \tag{2.2}$$

and which is equal to one on Γ and vanishes on Γ_1

$$\omega(x,y) = 1 \quad \text{for } z \in \Gamma \tag{2.3a}$$

$$\omega(x,y) = 0 \quad \text{for } z \in \Gamma_1 \tag{2.3b}$$

Being harmonic, in the interior of $(\mathcal{D})$ ω is a non-vanishing positive function, smaller than one, for all $z \in \mathcal{D}$.

Let us suppose for a while — this condition will be relaxed afterwards — that the holomorphic function $F(z)$ has no zeros in $(\mathcal{D})$. Then $\ln F(z)$ will also be holomorphic in $(\mathcal{D})$, [and hence $\mathrm{Re}\, \ln F(z)$, harmonic] and one can build the following harmonic function

$$\ln|F(z)| - (1-\omega(z))\ln m - \omega(z)\ln M \begin{cases} \leq 0 & \text{on } \Gamma \\ \leq 0 & \text{on } \Gamma_1 \end{cases} \tag{2.4}$$

[These inequalities follow from inequalities (2.1) and (2.3).] The left-hand side of (2.4) being a harmonic function, the inequality (2.4) is valid throughout $(\mathcal{D})$ and hence, (the Nevanlinna principle),

$$|F(z)| \leq m^{1-\omega(z)} M^{\omega(z)} \quad \text{for } z \in \mathcal{D} \cup \Gamma_1 \cup \Gamma \tag{2.5}$$

If now $F(z)$ has some zeros in $(\mathcal{D})$, they can be isolated by small circles which can be added to the boundary of the new defined domain of holomorphy for $\ln F(z)$; $|F(z)|$ being very small in the neighbourhoods of its zeros, the condition (2.4) is fulfilled in these circles as well as in their interiors and hence (2.5) remains valid.

Let us now apply the lemma to the difference

$$F_\varepsilon(z) = f(z) - f_\varepsilon(z) \tag{2.6}$$

of the amplitude (or some combination of it with some known given function) f(z) and some of its approximants $f_\varepsilon(z)$ on the physical region Γ_1.

$$|f(z) - f_\varepsilon(z)| < \varepsilon \quad , z \in \Gamma_1 \tag{2.7}$$

ε is some numerical quantity describing the approximation, which is intended to be reduced afterwards to zero. We suppose that the approximation function $f_\varepsilon(z)$ has (at least) the same domain of analyticity with the amplitude and that, as well as f(z) *), is bounded in $(\mathcal{D})$ by some constant M/2

$$|f(z)| < M/2 \quad , z \in \Gamma \tag{2.8a}$$

$$|f_\varepsilon(z)| < M/2 \quad , z \in \Gamma \tag{2.8b}$$

i.e., using Schwartz inequality, also

$$|f(z) - f_\varepsilon(z)|_\Gamma < M \tag{2.9}$$

Applying to $F_\varepsilon(z)$ the Nevanlinna principle (2.5) one gets

$$|f(z) - f_\varepsilon(z)| < \varepsilon^{1-\omega(z)} M^{\omega(z)} \quad , z \in \mathcal{D}. \tag{2.10}$$

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*) If the (polynomial) behaviour of the amplitude at infinity is known (as well as the position of the bound states) we are always in the position to produce some standard bounded function f(z) ; as it will easily be seen further, this can simply be achieved in the conformal mapping by placing the poles on non-physical sheets.

It is clear that the expression (2.10) tends to zero with ε in all interior points of $(\mathcal{D})$, i.e., the boundedness of $f(z)$ is sufficient for the stability of the approximation. However, this is no more the case on the boundary Γ, as $\omega(z)$ is here [see (2.3a)] equal to one, the difference $|f(z) - f_\varepsilon(z)|$ is now insensitive to the smallness of ε.

Of course, the failure of the Nevanlinna principle (2.5) to ensure the stability of the approximations on the cuts (on Γ), by no means means that we have to give up the hope of finding some correct approximation. On the contrary, the aim of the present paper is to prove that using a suitable conformal mapping such a stable approximation procedure can actually be obtained.

The idea to use conformal mapping continuations comes in a natural way if one analyzes the reason of the break-down of the previous stability theorem on the cuts, i.e., the coincidence of $(\mathcal{D})$ with the physical cut plane which entails the vanishing of the exponent of ε in (2.5) on the cuts. If now one uses a conformal mapping (see Sections 3 and 5) which maps the physical cut plane into the ring $1 \leq |\sigma| \leq R$ and if one takes approximants in form of polynomials in the positive and negative powers of this new variable, their holomorphy domain extends far outside $(\mathcal{D})$: the idea of the proof consists now in applying the Nevanlinna principle to the difference between such an approximant and the truncated exact Laurent series of the amplitude (which - see Section 3 - is known to converge to it on the cuts !) into a much larger ring $(\mathcal{D}')$ so that the cut Γ becomes an interior curve of $(\mathcal{D}')$ and, therefore, the new harmonic measure [of $(\mathcal{D}')$] will be different from one on it !

3. THE CONVERGENCE OF POLYNOMIAL APPROXIMATIONS AND THE NECESSITY OF USING CONFORMAL MAPPINGS OF DOUBLE CONNECTED REGIONS

Prior to studying the stability problem, we shall be faced with a convergence problem specific to polynomial approximations of an analytic function, along some given continua Γ_1.

To give an insight into the kind of problem we will be faced with, we shall remember that the radius of convergence - and therefore the domain of validity of a Taylor series - depends in an essential way on the location of the point around which the expansion is performed. In our case, we have not to expand around some given point, but around some given curve Γ_1 of the complex plane, the "physical region". Indeed, we are interested in finding the n^{th} order polynomial for which the uniform norm on Γ_1 of the error function

$$\| f(z) - t_n(z) \|_{U(\Gamma_1)} \stackrel{def.}{=} \max_{z \in \Gamma_1} . | f(z) - t_n(z) | \tag{3.1a}$$

or the L^2 norm

$$\| f(z) - s_n(z) \|_{L^2(\Gamma_1)} \stackrel{def.}{=} \left\{ \int_{\Gamma_1} | f(z) - s_n(z) |^2 . |dz| \right\}^{1/2} \tag{3.2a}$$

is the least. Owing to the fact that, in general, the experimental information is not given with the same realibility in the different points of the physical region Γ_1, in some cases it would be perhaps preferable to use weighted norms

$$\| f(z) - t_n(z) \|_{U_N(\Gamma_1)} \stackrel{def.}{=} \max_{z \in \Gamma_1} . N(z) | f(z) - t_n(z) | \tag{3.1b}$$

and

$$\| f(z) - s_n(z) \|_{L^2_N(\Gamma_1)} \stackrel{def.}{=} \left\{ \int_{\Gamma_1} | f(z) - s_n(z) |^2 N(z) . |dz| \right\}^{1/2} \tag{3.2b}$$

where $N(z)$ is a non-vanishing positive (continuous) weight function defined on Γ_1. The polynomials $t_n(z)$ which minimizes the uniform norm (3.1a) or (3.1b) and the polynomials $s_n(z)$ which minimize the L^2 norm (3.2a) or (3.2b) of the error function are called, respectively, Tchebycheff *) and Szegö approximants. For the latter, owing to a well-known theorem **), a closed form expression can be provided in the form of a linear combination

$$s_n(z) = \sum_0^n a_k \, p_k(z) \tag{3.2c}$$

of the first $n+1$ orthogonal polynomials $p_k(z)$ on Γ_1

$$\int_{\Gamma_1} p_i(z)\, p_k^*(z)\, N(z)\, |dz| = \delta_{ik} \tag{3.2d}$$

where the coefficients a_k are the "partial waves" of the function $f(z)$ with respect to the orthogonal set $p_k(z)$:

$$a_k \overset{\text{def.}}{=} \int_{\Gamma_1} f(z)\, p_k^*(z)\, N(z)\, |dz| \tag{3.2e}$$

The "best fitting" methods of the experimental data lead to Tchebycheff or Szegö sequences ***), whose domains of convergence, as expected, depend strongly on the shape of the physical region Γ_1 and

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*) Not to be confused with the Tchebycheff polynomials.

**) If $s_n'(z) = \sum_0^n a_k' p_k(z)$ (a_k' = arbitrary) is a general n^{th} degree polynomial, then

$$\| f(z) - s_n'(z) \|_{L^2}^2 = \int_{\Gamma_1} \Big(f(z) - \sum_0^n a_k' p_k(z) \Big) \Big(f^*(z) - \sum_0^n a_j'^* p_j^*(z) \Big) N(z)\, |dz| =$$
$$= \| f \|_{L^2}^2 - \sum_0^n |a_k|^2 + \sum_0^n |a_k' - a_k|^2$$

obviously is least when all a_k' are equal to the "partial waves" a_k.

***) The least squares method leads to Szegö approximants while an optimization in terms of uniform norm leads to Tchbycheff ones.

therefore <u>differs considerably</u> from that of a simple Taylor series : in a complete ununderstandable way, this fact was overlooked in all previous papers using $w(z)$ conformal mappings (see Figs. 2 and 3) for amplitude extrapolations. Indeed, both Tchebycheff and Szegö approximants are "maximal convergent polynomials" on Γ_1, and, according to the theorem of maximal convergent polynomials (see Appendix A), their complete domain of convergence can be found in the following way [18)-20)] :

Let $\mathfrak{Z}(z)$ *) be that conformal mapping which maps the physical region curve Γ_1 on the unit circle and the point from infinity at infinity [see Figs. 3 and 4 : here the w plane can be understood as being the z plane, however, if we are interested in the maximal convergence of polynomials in s, the transformation $\mathfrak{Z}(z)$ has to be applied directly to the variable s]. For instance (Fig. 3), if the physical region extends along the real axis between $z = a$ and $z = b$, then

$$\mathfrak{Z}(z) = z' + i\sqrt{1 - z'^2} \qquad \text{where } z' = \frac{2z - (a+b)}{a-b} \tag{3.3}$$

i.e., it is a Jukowsky mapping.

Further, let the first singularity of $f(z)$ be placed at a distance

$$|\mathfrak{Z}(z_s)| = R \tag{3.4}$$

from the origin of the $\mathfrak{Z}(z)$ complex plane. If now [18)-20)] the p^{th} derivative of $f(z)$ satisfies on the circle Γ_R defined by (3.4) a Hölder condition of order α

$$|f^{(p)}(z) - f^{(p)}(z_0)| \leq \text{const.}\, |z - z_0|^{\alpha} \tag{3.5}$$

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*) From now on we shall use Gothic symbols for all the conformal functions which, as $\mathfrak{Z}(z)$, map the physical region Γ_1 on the unit circumference. This will be the case also for the conformal mapping of double connected domains in terms of which the amplitude will be expanded (see following). Nevertheless we shall keep the notations as close as possible with those used in paper [21)] on which repeated reference will be made, especially with regard to the properties of the function $C(z)$ introduced here.

[such a condition can always [5] be provided, if not for the amplitude itself, then always for some combination with a suitably known function), then the maximum convergence theorem tells that the maximum convergent set of polynomials $t_n(z)$ or $s_n(z)$ converge on Γ_1 as : (see Appendix A)

$$|f(z) - t_n(z)| \leq \frac{A}{n^{p+\alpha} R^n} \quad , z \in \Gamma_1 \tag{3.6}$$

where R was defined in (3.4) and A is some constant. The rate of convergence of the polynomial sequence in some given point P of the z plane (or w for Fig. 3) is given by

$$|f(z) - t_n(z)| \leq \frac{A}{n^{p+\alpha}} \left(\frac{\rho}{R}\right)^n \tag{3.7}$$

where

$$\rho = |\mathfrak{z}(z)| \tag{3.8}$$

is the distance in the $\mathfrak{z}$ plane (see Fig. 4) between the point P and the $\mathfrak{z}$ plane origin (if $P \in \Gamma_1$, then $\rho = 1$). The inequality (3.7) remains valid also in the limiting case when P is placed on the image of the circle Γ_R [defined by (3.4)] in the z plane, where the sequences $t_n(z)$ or $s_n(z)$ are still convergent.

Outside Γ_R [if Γ_1 is a straight line as in the example (3.3), Γ_R is, in the z plane, an ellipse] there is <u>no</u> continuum where the sequences $t_n(z)$ or $s_n(z)$ would still converge *).

Of course, this does not mean that there do not exist, in general, polynomial sets which will converge on larger areas - a counterexample is provided by the Jacobi series

$$f(z) = q_0(z) + q_1(z) p(z) + q_2(z) p^2(z) + \ldots \tag{3.9}$$

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*) Convergence could occur outside Γ_R only exceptionally, in some isolated points.

where

$$p(z) = (z-\beta_1)\dots(z-\beta_\lambda) \tag{3.10}$$

is a polynomial of order λ and $q_i(z)$ are $(\lambda-1)$ degree polynomials chosen in such a way that the sum of the first n terms will coincide with $f(z)$ $\underline{m}$ times in the points β_i. The Jacobi series are in some respect a generalization of the Taylor series, and their domain of convergence extends to the largest lemniscate

$$|(z-\beta_1)\dots(z-\beta_\lambda)| \leq \text{const.} \tag{3.11}$$

which does not include some singularity of $f(z)$. Choosing in a suitable way the polynomial (3.10), the lemniscate could be made large enough, but the series (3.9) will not converge maximally on Γ_1. In other words, if the coefficients of the expansion are obtained on the basis of a best fit, in uniform or in any L^p norm, one has to rearrange continously <u>all</u> the "coefficients" $q_i(z)$, not only the last one.

One could, of course, renounce to the best fit philosophy and, at each step, determine the last $q_n(z)$ only; an alternate way of solving this problem would be to find some new variable which would provide maximal convergence on Γ_1 and an acceptable boundary Γ for the convergence area. This can be achieved by mapping the desired domain of <u>convergence cut along the physical region Γ_1</u> into a ring in the new variable complex plane, Γ_1 corresponding to the inner circle of radius equal to one, in order to make the mapping $\mathfrak{Z}$ superfluous.

4. AN OPTIMIZATION PROBLEM

Let $w_a(z)$ be some (arbitrary) conformal mapping of the z cut plane. As it was emphasized in the preceding section, the rate of convergence of the expansion of the amplitude in powers of $w_a(z)$ - the coefficient of the series being determined by a best fit (in uniform or L^p norm) on the physical region Γ_1 - is the same as those of the maximal convergence polynomials in the variable

$$\vartheta_a(z) = \mathfrak{z}(w_a(z)) \qquad (4.1)$$

which maps Γ_1 and Γ_a (see Fig. 5) in the unit circle and the circle of radius R_a of the ϑ_a plane (Fig. 6). Thus, to ensure the convergence in some given point P, it is sufficient to find a function which maps the domain (D_a) which includes both, the point P and Γ_1, into a ring. As many such domains exist, a natural question arises, namely if there exists some optimal mapping which would provide the quickest convergence in P. As we shall now prove below, the answer is really fortunate, in the sense that this mapping is independent of the position of the point P, i.e., there exists and optimal mapping for which the best converging polynomials on Γ_1 have the greatest rate of convergence (comparatively to the other mapping) simultaneously for all the points of the z cut plane..

Let $\vartheta_a(z)$ and $\vartheta_b(z)$ map the domains (D_a) and (D_b) where (see Fig. 5)

$$(D_a) \subseteq (D_b) \qquad (4.2)$$

respectively in the rings of Figs. 6 and 7. To compare the rates of convergence yielded by the two mappings, it is sufficient to compare the ratios ρ_i/R_i *) where

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*) By construction, the first singularities of $f(z)$ are already met on the exterior frontiers Γ_a and Γ_b of (D_a) and (D_b).

$$\rho_{a,b} = |\mathcal{W}_{a,b}(P)| \tag{4.3}$$

and R_a and R_b are the radii of the circles Γ_a and Γ_b respectively in the complex planes $\mathcal{W}_a$ and $\mathcal{W}_b$. It can be shown by means of conformal invariants (the principle of Groetzsch, see Appendix B) that if (4.2) holds

$$R_a \leq R_b \tag{4.4}$$

Let us now define in the z plane the harmonic functions [in (D_b)]

$$\vartheta_a(z) = \ln\left(|\mathcal{W}_a(z)|/R_a\right) \tag{4.5}$$

$$\vartheta_b(z) = \ln\left(|\mathcal{W}_b(z)|/R_b\right) \tag{4.6}$$

By definition, on Γ_1 we have

$$|\mathcal{W}_a(z)|_{\Gamma_1} = |\mathcal{W}_b(z)|_{\Gamma_1} = 1 \tag{4.7}$$

$$\vartheta_a(z)|_{\Gamma_1} = -\ln R_a\,, \quad \vartheta_b(z)|_{\Gamma_1} = -\ln R_b \quad (z \in \Gamma_1) \tag{4.8}$$

and hence, owing to the principle of Groetzsch (4.4)

$$\vartheta_a(z) \geqslant \vartheta_b(z) \quad \text{for } z \in \Gamma_1\,. \tag{4.9}$$

Let us check now the two harmonic functions on Γ_a

$$\begin{aligned} &\vartheta_a(z) = 0 \\ &\vartheta_b(z) \leq 0 \qquad (z \in \Gamma_a) \end{aligned} \tag{4.10}$$

i.e., again

$$\vartheta_a(z) \geqslant \vartheta_b(z) \qquad (z \in \Gamma_a) \tag{4.11}$$

(ϑ_b has to be negative on Γ_a, as it is a harmonic function which is zero on Γ_b and negative on Γ_1).

$\vartheta_a(z)$ and $\vartheta_b(z)$ being both harmonic, the inequalities (4.9) and (4.11) holding on the whole boundary of $(\mathcal{D}_a)$, remain valid also in the interior points (also in the point P) and hence

$$\frac{\rho_\ell}{R_\ell} \leqslant \frac{\rho_a}{R_a} \tag{4.12}$$

(4.12) tells that the larger the holomorphy domain of $f(z)$ mapped into the ring (whose interior circle corresponds to the physical region) is, the better the convergence. The best convergence is thus attained for the function $\mathfrak{C} = C(z)$ *) which maps the whole z cut plane in the ring of Fig. 8, the convergence being secured, according to (3.7), even on the cut. The explicit form of these mappings will be derived in Section 6.

Before proceeding to the study of the stability problem (Section 5), let us derive two simple but important properties of $f(z)$ as a function of the new variable $\mathfrak{C}$.

In most extrapolation problems one extrpolates either the discrepancy function (see Introduction) in the energy complex plane $(z \equiv s)$, or the real part, the imaginary part of the amplitude or even the differential cross-section as function of the cosine $(z \equiv c)$. In all these cases, $f(z)$ is real on Γ_1 as well as along the segments Δ_i of the real axis outside the physical region where there are no cuts. The $\mathfrak{C}$ images of Δ_i being also real axis segments

$$f(\mathfrak{C}) = f^*(\mathfrak{C}^*) \tag{4.13}$$

while the reality condition on the unit circle Γ_1 from Fig. 8, yields

$$f(\mathfrak{C}) = f^*((\mathfrak{C}^*)^{-1}) \tag{4.14}$$

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*) We maintain here the notations of paper [21] where the properties of this function were studied.

or, combining (4.13) with (4.14)

$$f(\zeta) = f(1/\zeta) \tag{4.15}$$

In other words the power expansions of $f(z)$ have necessarily the form

$$f(z) = a_0 + a_1(\zeta + \zeta^{-1}) + a_2(\zeta^2 + \zeta^{-2}) + \ldots \tag{4.16}$$

where a_i are <u>real</u> coefficients to be determined from theory or experiment.

5. THE STABILITY OF THE $\mathcal{C}(z)$ MAXIMAL CONVERGING EXPANSIONS

To find an expansion procedure whose results would be stable against the small experimental errors ε of the histogram $h(z)$ which expresses the results of the experimental measurements on $f(z)$ in the physical region Γ_1,

$$|f(z) - h(z)| \leq \frac{\varepsilon}{2} \quad , z \in \Gamma_1 \tag{5.1}$$

we shall apply the Nevanlinna principle (2.5) not to the difference $h(z)-f(z)$ itself, but to the difference of two best $\mathcal{C}(z)$ and $\mathcal{C}^{-1}(z)$ n^{th} degree interpolations $t_{f,n}(z)$ and $t_{h,n}(z)$ to $f(z)$ and $h(z)$. As it was pointed out at the end of Section 2., taking advantage of the fact that the rational functions $t_{f,n}$ and $t_{h,n}$ as positive and negative power polynomials (4.16) of the function $\mathcal{C}(z)$, extend their holomorphy far outside the image of the first Riemann sheet of the z variable, could prevent the failure of the stability proof at the cuts.

Let us define the yet unknown polynomials $t_{f,n}$

$$t_{f,n} = a_{f,0} + a_{f,1}(\mathcal{C} + \mathcal{C}^{-1}) + \ldots + a_{f,n}(\mathcal{C}^{n} + \mathcal{C}^{-n}). \tag{5.2}$$

as the best n^{th} degree approximation, in the Tchebycheff's (3.1a) sense, to the yet unknown amplitude $f(z)$, on Γ_1. According to (3.6) *)

$$|f(z) - t_{f,n}(z)|_{\Gamma_1} \leq \frac{A/2}{n^{p+\alpha} R^{n}} \equiv \eta_n/2 \tag{5.3}$$

---

*) The fact that $t_{f,n}$ are not really polynomials but truncated Laurent series is immaterial, as, owing to (5.2), they are nevertheless polynomials in the variable $\mathcal{C} + \mathcal{C}^{-1}$.

where R is the radius of the rings of Fig. 8 *) [see also (6.6)] p and α are defined by the Hölder condition (3.5) and A is a constant. From (5.1) and (5.3) one gets

$$|h(z) - t_{f,n}(z)|_{\Gamma_1} \leq (\varepsilon + \eta_n)/2 \tag{5.4}$$

Although the coefficients are unknown, the very existence of $t_{f,n}(z)$ proves that the polynomial $t_{h,n}(z)$ whose coefficients can be <u>found explicitly</u> through a Tchebycheff optimal interpolation condition of the experimental data

$$\max_{z \in \Gamma_1} |h(z) - t_{h,n}(z)| \longrightarrow \text{minimum} \tag{5.5}$$

has to approximate the experimental histogram at least as well as $t_{f,n}(z)$:

$$|h(z) - t_{h,n}(z)|_{\Gamma_1} \leq (\varepsilon + \eta_n)/2 \tag{5.6}$$

Applying once more the Schwartz inequality one finds

$$|t_{f,n}(z) - t_{h,n}(z)|_{\Gamma_1} \leq \varepsilon + \eta_n \equiv \frac{A + \varepsilon n^{p+\alpha} R^n}{n^{p+\alpha} R^n} \tag{5.7}$$

Of course, the inequality (5.6) remains valid also for the difference between $t_{h,n}(z)$ and $t_{f,n+1}(z)$

$$|t_{f,n+1}(z) - t_{h,n}(z)|_{\Gamma_1} \leq \frac{A + \varepsilon n^{p+\alpha} R^n}{n^{p+\alpha} R^n} \tag{5.7'}$$

---

*) R is a conformal invariant.

which, on the other hand can easily be calculated on a circle $\Gamma_{\mathcal{R}}$ (see Fig. 8) of very large radius

$$\mathcal{R} = R^{1/\varkappa} \quad , \quad \varkappa \to 0 \tag{5.8}$$

where, $t_{f,n+1}(z)$ being by definition the exact Tchebycheff's polynomial of $f(z)$, the highest degree term of $t_{f,n+1}$ behaves as $(\mathcal{R}/R)^{n+1}$:

$$\left| t_{f,n+1}(z) - t_{h,n}(z) \right|_{\Gamma_{\mathcal{R}}} \leq A_1 \, R^{\left(\frac{1}{\varkappa}-1\right)(n+1)} \quad , \varkappa \to 0 \tag{5.9}$$

We now apply the Nevanlinna principle to the difference $t_{h,n}(z) - t_{f,n+1}(z)$ between the circles Γ_1 and $\Gamma_{\mathcal{R}}$, the harmonic measure being

$$\Omega(\mathcal{C}) = \frac{\ln |\mathcal{C}|}{\ln \mathcal{R}} = \varkappa \frac{\ln |\mathcal{C}|}{\ln R} \tag{5.10}$$

which yields for $z \in \Gamma_R$, in the very small $\varkappa$ limit (i.e., for very large $\mathcal{R}$)

$$\left| t_{h,n}(z) - t_{f,n+1}(z) \right|_{\Gamma_R} \leq R \, \frac{A + \varepsilon n^{p+\alpha} R^n}{n^{p+\alpha}} \tag{5.11}$$

As (3.7) the exact Tchebycheff approximants <u>converge</u> to $f(z)$ <u>also on the cut</u>

$$\left| f(z) - t_{f,n+1}(z) \right|_{\Gamma_R} \leq \frac{A/2}{n^{p+\alpha}} \tag{5.12}$$

one gets the final result

$$\left| f(z) - t_{h,n}(z) \right|_{\Gamma_R} \leq B \, \frac{1 + \frac{\varepsilon R}{B} n^{p+\alpha} R^n}{n^{p+\alpha}} \tag{5.13}$$

where B is a new constant, $B = A(R + \frac{1}{2})$.

Let us look at formula (5.13) which accounts for the error on the cuts between the exact amplitude and the polynomials $t_{h,n}(z)$ which extrapolate the experimental data. If $\varepsilon R/B$ is small, the higher n the smaller the right-hand side of (5.13) gets. When n reaches the critical value defined by

$$n_c^{p+\alpha+1} R^{n_c} = \frac{B(p+\alpha)}{\varepsilon R \ln R} \tag{5.14}$$

i.e, *)

$$n_c(\varepsilon) \simeq \ln\left(\frac{B(p+\alpha)}{\varepsilon R \ln R}\right) / \ln\left(R \cdot 1{,}44^{p+\alpha}\right)$$

the right-hand side of (5.13) reaches its lowest value

$$\left| f(z) - t_{h,n_c}(z) \right|_{\Gamma_R} \leq B\left(1 + \frac{p+\alpha}{n_c(\varepsilon) \ln R}\right) / n_c(\varepsilon)^{p+\alpha} \tag{5.15}$$

increasing then again if one takes more terms in $t_{h,n}$ than the allowed critical value: we say that instabilities begin to develop, and the recipe to keep them down is never to interpolate the physical data with $t_{h,n}$ polynomials with more than $n_c(\varepsilon)$ terms. Nevertheless, and this proves the stability of the $t_{h,n_c(\varepsilon)}(z)$ extrapolation procedure, as one sees immediately from (5.14), $n_c(\varepsilon)$ increases monotonously when the experimental error ε decreases, and thus the right-hand side of the inequality (5.15) goes down with ε.

---

*) As, for integer n's, $n^{1/n}$ is bounded by $\sqrt[3]{3}$.

6. EXPLICIT FORM OF THE OPTIMAL MAPPING

We shall derive now the explicit form of the function $\mathcal{C}(z)$ which, according to Section 4, maps the first Riemann sheet of the amplitude into a ring (the inner circle, of radius 1, corresponding to the physical region Γ_1).

We start from a standard z cut plane, the physical region extending from -1 to $+1$ while the cuts are placed symmetrically between $-\infty$ and $-1/k$ (k is a positive subunitary constant) and between $+1/k$ and $+\infty$ (Fig. 9). If we are interested in analytical continuation in the s plane, or, if in the cosine plane the cuts are not symmetrical (unequal mass case), one can always, by means of elementary transformation, transform s or $\cos\theta$ in the standard form of Fig. 9.

Let us fix our attention to the upper z half plane and, as a first step, let us perform the mapping (Fig. 10) :

$$u(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{6.1}$$

the cuts of the square root being taken along Γ_1 and Γ_R (Fig. 9), and let the square root be positive above Γ_1, negative imaginary at $+1 < z < +1/k$ and negative above Γ_R.

If z is small and increases along the real axis (containing an infinitesimal positive imaginary part, as the path proceeds along the upper half plane), u(z) increases from zero to the positive constant

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{6.2}$$

when z reaches the point $z = 1$.

Further, if $1 < z < 1/k$, one can write

$$u(z) = \int_0^z \frac{dt}{\sqrt{\ }} = K(k) + \int_1^z \frac{dt}{\sqrt{\ }} \tag{6.3}$$

where, since the square root is negative imaginary for $1 < t < 1/k$, the second integral from the right-hand side of (6.3) is a pure positive imaginary function, increasing with z. For $z = 1/k$, u reaches the value $K+iK'$ where

$$i\,K'(k) = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{6.4}$$

$K(k)$ and $K'(k)$ are called the complete elliptical integrals, and are tabulated functions.

If z proceeds further along the upper side of Γ_R, the real part of u will decrease monotonically, as the square root is negative. For $z = \infty$, u equals iK'. Performing the same study in the negative direction, one finds that the whole upper plane maps into the rectangle of Fig. 10.

The second and final step consists in transforming the u plane rectangle in the upper half ring of the $\mathcal{T}(z)$ plane (Fig. 11) :

$$\mathcal{T}(z) = i \exp\{-i\pi\, u(z)/2K\} \tag{6.5}$$

The point $u = 0$ transforms into $\mathcal{T} = +i$, $u = K$ into $\mathcal{T} = 1$ and $u = K+iK'$ into $\mathcal{T} = \exp(\pi K'/2K)$ which defines the value of R :

$$R = \exp(\pi K'/2K) \tag{6.6}$$

which depends hence only on the position $(\pm 1/k)$ of the branching points of the z cut plane.

A final remark : the physical region Γ_1 is not actually a cut line for the amplitude $f(z)$, however, the function $\mathcal{T}(z)$ has such a cut; nevertheless, this cut disappears in $f(z)$, owing to the fact that the positive and negative frequencies (4.16) have the same coefficients. Indeed, (4.15) is a continuation relation inside the unit circle of Fig. 8, till the circle of radius $1/R$, which is the image (on the unit circle) of Γ_R.

To conclude, as it was pointed in Section 5, the semiconvergent approximation

$$t_{h,n}(z) = a_{h,0} + a_{h,1}(\zeta + \zeta^{-1}) + \ldots + a_{h,n}(\zeta^{n} + \zeta^{-n}) \qquad (6.7)$$

where the coefficients $a_{h,i}$ are to be found by a Tchebycheff best fit (5.5) of the experimental data [the histogram $h(z)$] and where the number of terms has to be limited by $n_c(\varepsilon)$ (5.14), to converge to the amplitude $f(z)$, even on the cuts. Indeed, the dependence on the cuts of the modulus of the difference $f(z)-t_{a,n_c(\varepsilon)}(z)$ with the experimental errors ε in Γ_1, is given by the joint expressions (5.14) and (5.15).

The condition $n \leq n_c(\varepsilon)$ is somewhat academic, as the right-hand side of (5.14) depends on a constant B, which - in principle - could be determined from the smoothness conditions, but in practice would be hard to find. Nevertheless, (5.13) proves at least the existence of such an optimal number (5.14), beyond which instabilities develop. Therefore, for practical purposes, $n_c(\varepsilon)$ could simply be evaluated on a computer as the critical number of terms beyond which the different fits (6.7) begin to diverge between them when calculated on Γ_R.

We feel that the uniform norm procedure (5.5) to determine the coefficients of the expansion (6.7) is not very suited, as the experiment is usually analyzed by least square methods, which, as was stated at the beginning of Section 3, led to Szegö approximants, not to the Tchebycheff ones. Of course, the method of Section 5 based on the Nevanlinna principle does not apply also to the L^p norm, nevertheless we shall try in a subsequent paper to study also the stability of these approximants constructed on Γ_1 by L^2 norm minimization procedure.

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APPENDIX A

MAXIMAL CONVERGENCE POLYNOMIALS

We shall give here the classical [20] proof for the existence of maximal convergence polynomials (for definitions, see further, Theorem A.1) because this proof has its own interest for the physicist as it makes use of a set of polynomials (Jacobi polynomials) which interpolate the function $f(z)$ on a discrete set of points along the "physical region" Γ_1. At the end of the present Appendix two other theorems will be stated (without proof), relating the convergence rate on the smoothness conditions satisfied by the amplitude on the cut.

EQUIPOTENTIAL CURVES OF THE GREEN FUNCTION

Let Γ_1 be the boundary of a closed limited set of points in the z complex plane, whose complement, with respect to the extended complex plane is connected and regular, in the sense that it possesses a Green's function $G(x,y)$ with pole at infinity. The function $\mathfrak{Z}(z) = \exp\{G+iH\}$, where H is conjugated to G, maps the exterior of Γ_1 conformally but not necessarily uniformly to the exterior of the unit circle. [If the complement of Γ_1 is not simple connected, the function $\mathfrak{Z}(z)$ cannot be more single valued, but its modulus $\exp G(x,y)$ still remains single valued.] Further, we denote by Γ_R the equipotentials

$$(x,y)\in\Gamma_R \;:\; G(x,y) = \ln R \qquad (R>1) \tag{A.1}$$

and we shall now show that they can be arbitrarily well approximated by some conveniently chosen lemniscate

$$|\omega_N(z)| \equiv |(z-z_1)(z-z_2)\dots| = \text{const.} \qquad (z_i \in \Gamma_1) \tag{A.2}$$

Indeed, $G(x,y)$ being the Green's function with pole at infinity, let us write it in the form

$$G(x,y) = \ln (x^2+y^2)^{1/2} - g \text{ + terms vanishing at infinity} \tag{A.3}$$

and let Γ' be a very large circle. The Green's theorem for the angular domain between Γ_1 and Γ' yields

$$G(x,y) = \frac{1}{2\pi} \int_{\Gamma_1} \left(\ln r \frac{\partial G}{\partial n} - G \frac{\partial \ln r}{\partial n}\right) ds + \frac{1}{2\pi} \int_{\Gamma'} \left(\ln r \frac{\partial G}{\partial n} - G \frac{\partial \ln r}{\partial n}\right) ds \tag{A.4}$$

Being a Green's function G is zero on Γ_1; further one can replace, in the integrands, G with $G-\ln r = g$+terms vanishing at infinity, obtaining in the limit $\Gamma' \to \infty$

$$G(x,y) + g = \frac{1}{2\pi} \int_{\Gamma_1} \ln r \frac{\partial G}{\partial n} ds \tag{A.5}$$

$\partial G/\partial n$ being positive along Γ_1, one can define a new variable

$$u(\xi) = \frac{1}{2\pi} \int_0^{s(\xi)} \frac{\partial G}{\partial n} ds \tag{A.6}$$

well defined in all points of Γ_1 and increasing monotonously from $u=0$ to $u=1$, as

$$\frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial G}{\partial n} ds = -\frac{1}{2\pi} \int_{\Gamma'} \frac{\partial G}{\partial n} ds$$

which, in the limit of very large Γ' is equal to

$$= \frac{1}{2\pi} \int_{\Gamma'} \frac{\partial \ln r}{\partial n} ds = 1$$

(A.5) can thus be written as

$$G(x,y) + g = \int_0^1 \ln r \, du \tag{A.7}$$

r being, we recall, the distance between the point $z = x+iy$ and the different points z_i of Γ_1.

The integral can now be approximated by a Darboux series

$$\left| G(x,y) + g - \frac{1}{N}(\ln r_1 + \ln r_2 + \ldots + \ln r_N) \right| < \varepsilon \tag{A.8}$$

where

$$r_i = |z - \zeta(u_i)| \quad , \quad u_1 = \frac{1}{N} \quad , \quad u_2 = \frac{2}{N} \quad , \quad \ldots \quad u_N = 1$$

If the point $z = x+iy$ lies now on the lemniscate L_R^N defined by

$$z \in L_R^N \; : \quad r_1 r_2 \ldots r_N = R^N \exp\{gN\} \tag{A.9}$$

then (A.8) becomes

$$\left| G(x,y) - \ln R \right| < \varepsilon \tag{A.8'}$$

Owing to the definition (A.1) of the equipotential Γ_R as well as to the continuity of the Green's function, (A.8') tells that, if N is sufficiently large (small ε), the lemniscates L_R^N (A.9) approximate arbitrarily well the curves Γ_R.

MAXIMAL CONVERGENCE OF INTERPOLATING POLYNOMIALS

If $n = mN+j$ $(j < N)$, we should like to write down the $(n-1)^{th}$ degree polynomial $P_n(z)$ which interpolates $(m+1)$ times the function $f(z)$ in the points $z_1, z_2, \ldots z_j$ and m times in the other $N-j$ centres $z_{j+1} \ldots z_N$ of the lemniscate (A.9) (all z_i lie on Γ_1). Thus, if

$$\omega_j = (z - z_1) \ldots (z - z_j) \tag{A.10}$$

$$\omega_N = (z - z_1) \ldots (z - z_N) \tag{A.2}$$

and

$$\omega(z) = \omega_j(z)\,\omega_N(z)^m \tag{A.11}$$

then, as one sees, the function

$$f(z) - P_n(z) \overset{\text{def.}}{=} \frac{1}{2\pi i}\int_C \frac{\omega(z)}{\omega(t)}\,\frac{f(t)}{t-z}\,dt \tag{A.12}$$

vanishes the desired number of times at the points z_i, while the function

$$P_n(z) = f(z) - \big(f(z) - P_n(z)\big) = \frac{1}{2\pi i}\int_C \frac{\omega(t)-\omega(z)}{\omega(t)\,(t-z)}\,f(t)\,dt \tag{A.13}$$

is really a polynomial, the integrand having no more poles at $t = z$. If the first singularities of $f(z)$ are situated on Γ_R, for each R' $(1 < R' < R)$ the curves $\Gamma_{R'}$ will be embedded in the holomorphy domain of $f(z)$ as well as the lemniscates $L^N_{R'_\varepsilon}$, $R'_\varepsilon = R' + \varepsilon$, which contain the curve $\Gamma_{R'}$, but approximate it as well as one likes. Integrating the integrals of (A.12) and (A.13) along $L^N_{R'_\varepsilon}$, if $z \in L^N_\rho$ $(1 < \rho < R'_\varepsilon)$:

$$\left|\frac{\omega^m_N(z)}{\omega^m_N(t)}\right| \leqslant \left(\frac{\rho}{R'_\varepsilon}\right)^{mN} = \left(\frac{\rho}{R'_\varepsilon}\right)^{n-j} \tag{A.14a}$$

and, as j is always bounded by N one can define M_1 such that

$$\left|\frac{\omega_j(z)}{\omega_j(t)}\right| < \frac{M_1}{R_\varepsilon^{\,N}} < M_1\left(\frac{\rho}{R_\varepsilon}\right)^j \tag{A.14b}$$

(the second inequality being also trivial, as $1 < \rho < R'$) one gets finally

$$|f(z) - P_n(z)| < \frac{1}{2\pi}\int_{L^N_{R'_\varepsilon}} \left|\frac{\omega(z)}{\omega(t)}\right|\frac{dt}{|t-z|} \leqslant M\left(\frac{\rho}{R'_\varepsilon}\right)^n \tag{A.15}$$

As L_ϱ^N approximates arbitrarily well Γ_ϱ (or Γ_1 for $\varrho = 1$) one can state the following theorem.

Theorem A-1

Let Γ_1 be a closed limited point set whose complement is connected and regular. If the function $f(z)$ is single valued and analytic on Γ_1, there exists a greater number R (finite or infinite) such that $f(z)$ is analytic and single valued in every point interior to Γ_R. If $R' < R$ is arbitrary, there exist polynomials $P_n(z)$ of respective degrees 0, 1, 2,.. such that $(1 \leqslant \varrho < R')$

$$\left| f(z) - P_n(z) \right| \leqslant M \left(\frac{\varrho}{R'} \right)^n \tag{A.16}$$

is valid for z on Γ_ϱ (or on Γ_1, for $\varrho = 1$); but there exist no polynomials $P_n(z)$ such that (A.16) is valid for $z \in \Gamma_\varrho$ with $R' > R$.

The last sentence of the theorem follows immediately from the following lemma :

If a polynomial $p_n(z)$ satisfies

$$\left| p_n(z) \right| < \frac{M}{R'^n} \quad \text{for} \quad z \in \Gamma_1 \tag{A.17}$$

$p_n(z)/\mathfrak{Z}^n(z)$ being a holomorphic function in the complement of Γ_1 (the pole at $z = \infty$ has disappeared) and on Γ_1 the modulus of $\mathfrak{Z}^n(z)$ being equal to one, we have on an arbitrary $\Gamma_{\mathcal{R}}$ $(1 < \mathcal{R})$

$$\left| p_n(z) / \mathfrak{Z}^n(z) \right| < \frac{M}{R'^n}$$

and hence

$$\left| p_n(z) \right| < M \left(\frac{\mathcal{R}}{R'} \right)^n \quad \text{for} \quad z \in \Gamma_{\mathcal{R}} \tag{A.18}$$

We apply the lemma to the difference :

$$p_n(z) = P_n(z) - P_{n-1}(z) = \left(P_n(z) - f(z) \right) - \left(P_{n-1}(z) - f(z) \right) \tag{A.19}$$

the inequality (A.17) following then from the Schwartz inequality. If now R' were larger than R, in virtue of (A.18) the norm of $p_n(z)$ will tend to zero also on the $\Gamma_{\mathcal{R}}$ with $R < \mathcal{R} < R'$ and hence the sequence $P_n(z)$ would converge here to $f(z)$, which would be in contradiction with the assumed analytical properties of $f(z)$.

We state without proof the following theorem, [see 20), page 371].

Theorem A-2

Let Γ_1 be a Jordan curve, and for some $R(>1)$ let $f(z)$ be analytic in the interior of Γ_R and continuous in the closed interior of Γ_R. If the p^{th} derivative $f^{(p)}(z)$ exists and satisfies on Γ_R a Hödler condition of order α, $0 < \alpha \leq 1$, then there exist polynomials $p_n(z)$ of respective degrees n such that

$$|f(z) - p_n(z)| \leq M/n^{p+\alpha}R^n \qquad \text{for } z \text{ on } \Gamma_1$$

$$|f(z) - p_n(z)| \leq M/n^{p+\alpha} \qquad \text{for } z \text{ on } \Gamma_R \tag{A.20}$$

If there exist polynomials such that (A.20) is valid, then $f^{(p)}(z)$ exists on Γ_R and satisfies here a Hölder condition of order α. The Szegö as well as the Tchebycheff approximants to $f(z)$ on Γ_1 are such maximal convergence polynomials satisfying Theorem A-2.

APPENDIX B

THE PRINCIPLE OF GROETZSCH

We shall now prove, making use of the theory of conformal invariants, a theorem (the principle of Groetzsch) which we needed in Section 4 in order to find the optimal conformal mapping. Namely, if the double connected domains (D_a) and (D_b) $[(D_a)$ being included in $(D_b)]$ are mapped into rings of unit inner radii and outer radii R_a and R_b, then

$$R_a \leqslant R_b \tag{B.1}$$

Let $\rho(z) \cdot |dz|$ be a conformal invariant metric, i.e.,

1) $\rho(z) \geqslant 0$

2) $\int_c \rho(z)|dz|$ exists (being finite or infinite), if c is a rectifiable Jordan arc, and, for each pair of local uniformizing parameters z and z', we have there, where the respective neighbourhoods around some given point intersect :

3) $\rho'(z') = \rho(z)|dz/dz'|$.

Further, if (L) is some family of rectifiable curves defined on the Riemann surface (D), let (P_L) be the class of all invariant metrics such that if

$$c \in (L) \text{ and } \rho(z) \in (P_L) \tag{B.2}$$

we should have

$$\int_c \rho(z) \cdot |dz| \geqslant 1 \tag{B.3}$$

while the integral

$$A_\rho(D) = \iint_{(D)} \rho^2(z)\,dx\,dy \tag{B.4}$$

would have a meaning.

We shall then say that the number

$$m(L) = \inf_{\rho \in (P_L)} A_\rho(D) \tag{B.5}$$

is the <u>modulus</u> of the family (L), and the metric $\rho_e \in (P_L)$ (if it exists) for which this minimum is realized

$$\iint_{(D)} \rho_e^2(z)\,dx\,dy = m(L) \tag{B.6}$$

the "extremal metric".

If it exists the extremal metric is unique. Indeed, if ρ_1 and ρ_2 were both extremal

$$m(L) = A_{\rho_1}(D) = A_{\rho_2}(D) \tag{B.7}$$

as

$$\int_C \frac{\rho_1+\rho_2}{2}\,|dz| \geq \frac{1}{2} + \frac{1}{2} = 1 \tag{B.8}$$

the metric $(\rho_1+\rho_2)/2$ would be also a permitted metric, and therefore, owing to the definition (B.5)

$$m(L) \leq A_{\frac{\rho_1+\rho_2}{2}}(D) \tag{B.9}$$

but, as $(\rho_1^2+\rho_2^2)/2 \equiv (\rho_1+\rho_2)^2/4+(\rho_1-\rho_2)^2/4$ one gets

$$m(L) = \iint_{(D)} \left(\frac{\rho_1+\rho_2}{2}\right)^2 dx\,dy + \iint_{(D)} \left(\frac{\rho_1-\rho_2}{2}\right)^2 dx\,dy$$

$$\geq m(L) + \iint_{(D)} \left(\frac{\rho_1-\rho_2}{2}\right)^2 dx\,dy \tag{B.10}$$

which is absurd unless the last integral from the right-hand side of this inequality does not vanish, i.e., unless $\rho_1 = \rho_2$.

Let (D) be double connected and (L) the family of curves which separate the two frontiers of (D). To prove now the principle of Groetzsch, it is sufficient to recognize that the metric which, after mapping (D) into a ring (D') takes the form $1/2\pi|z'|$

$$\rho(z)\frac{|dz|}{|dz'|} = \frac{1}{2\pi\,|z'|} \tag{B.11}$$

is extremal; indeed, first of all

$$\int_{c'} \frac{|dz'|}{2\pi|z'|} = \int_{c'} \frac{\left[dr'^2 + r'^2 d\varphi'^2\right]^{1/2}}{2\pi\, r'} \geq \frac{1}{2\pi}\int_{c'} |d\varphi'| \geq 1 \tag{B.12}$$

and hence $1/2\pi|z'|$ is [see (B.3)] a permitted metric. On the other hand, if $\bar{\rho}'(z')$ were a permitted metric too, one has, on the circles of the family (L), be definition :

$$\int_{\text{circle}} \bar{\rho}'(z')|dz'| \equiv r'\int_{\text{circle}} \bar{\rho}'(z')\,d\varphi' \geq 1$$

and hence

$$\int_{\text{circle}} \bar{\rho}'(z')\,d\varphi' \geq \frac{1}{r'} \tag{B.13}$$

Further,

$$0 \leq \iint_{(D')} \left(\bar{\rho}'(z') - \frac{1}{2\pi r'}\right)^2 r'\,dr'\,d\varphi' = \iint_{(D')} \bar{\rho}'(z')^2\, r'\,dr'\,d\varphi' - \frac{1}{\pi}\int_1^R \frac{r'\,dr'}{r'}\int_0^{2\pi} \bar{\rho}'(z')\,d\varphi' +$$

$$+ \frac{\ln R}{2\pi} \leq \iint_{(D')} \bar{\rho}'^2(z')\,dx\,dy - \frac{\ln R}{2\pi} \tag{B.14}$$

which shows that

$$A_{\bar{\rho}'}(D') \geq A_{\frac{1}{2\pi|z'|}}(D') \tag{B.15}$$

which proves that $1/2\pi|z'|$ is extremal and that

$$m(L) = \frac{1}{2\pi} \ln R \tag{B.16}$$

Being a module, (B.16) is first of all a conformal invariant and secondly [owing to the positiveness of the integrand of (B.4)], if

$$(D_a) \subset (D_b) \tag{B.17}$$

it follows that

$$m_a(L) \leq m_b(L) \tag{B.18}$$

which proves the assertion (B.1).

R E F E R E N C E S

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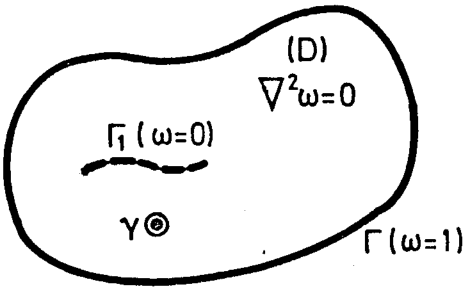
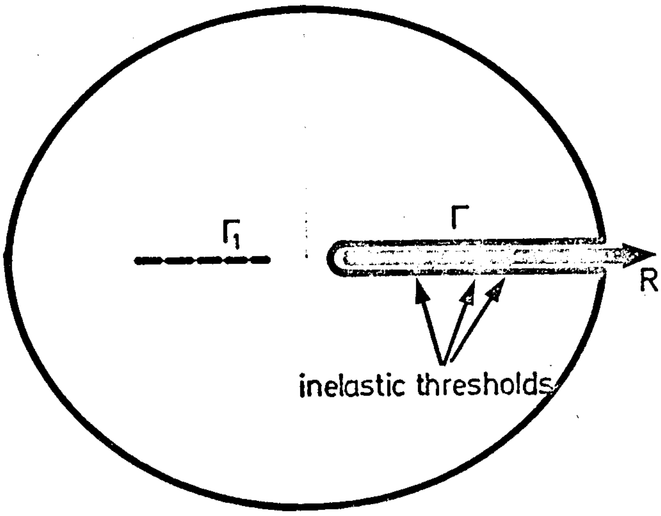


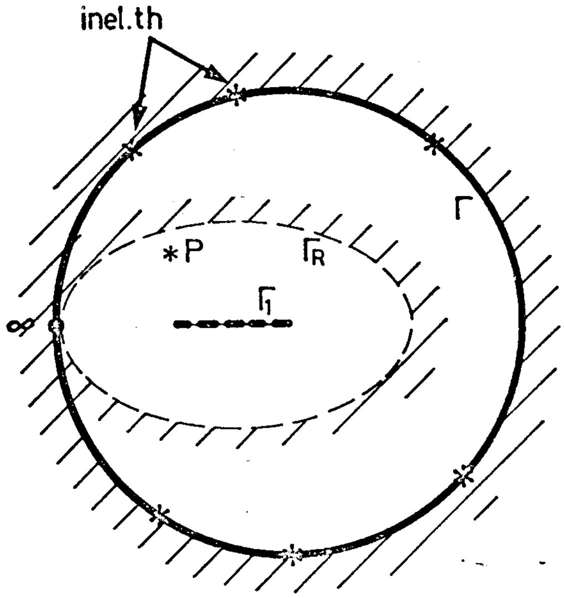
the harmonic measure

FIG.1



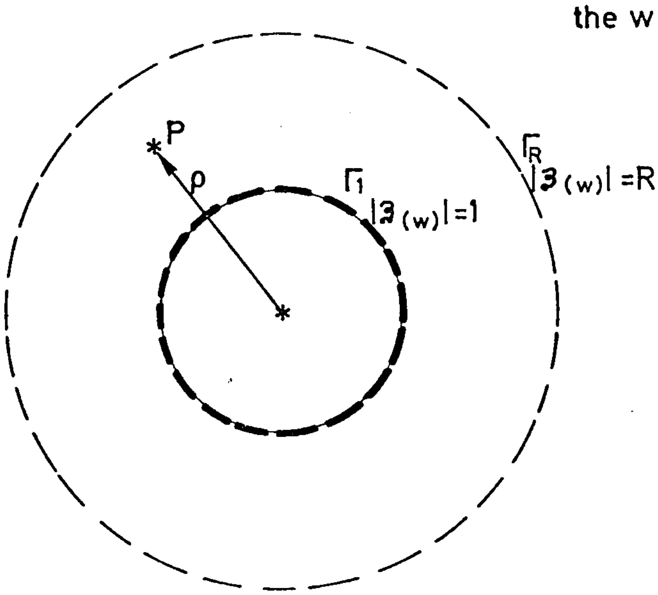
the s complex plane

FIG. 2



the w(s) complex plane

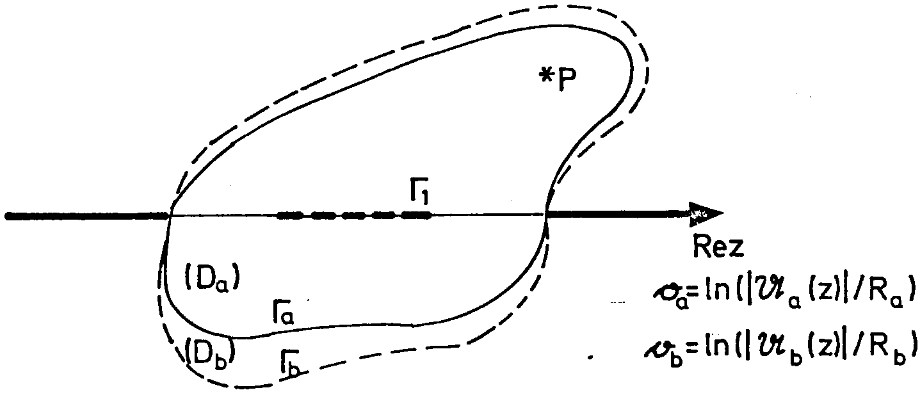
FIG. 3



the $\mathfrak{z}_{(w)}$ complex plane

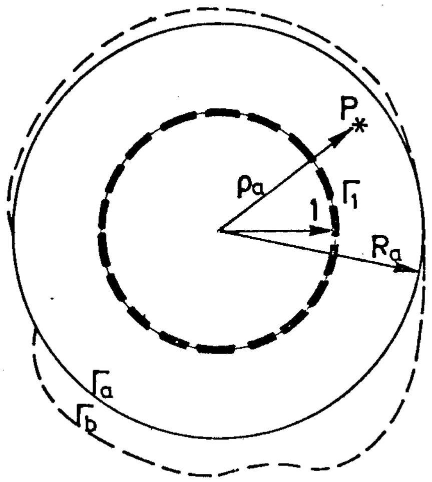
FIG. 4

SIS/K 0 765



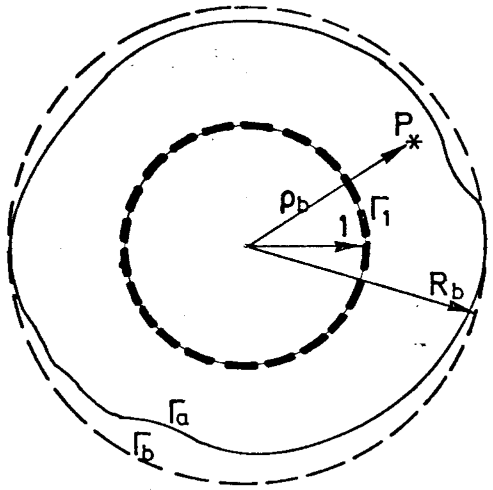
the z complex plane

FIG. 5



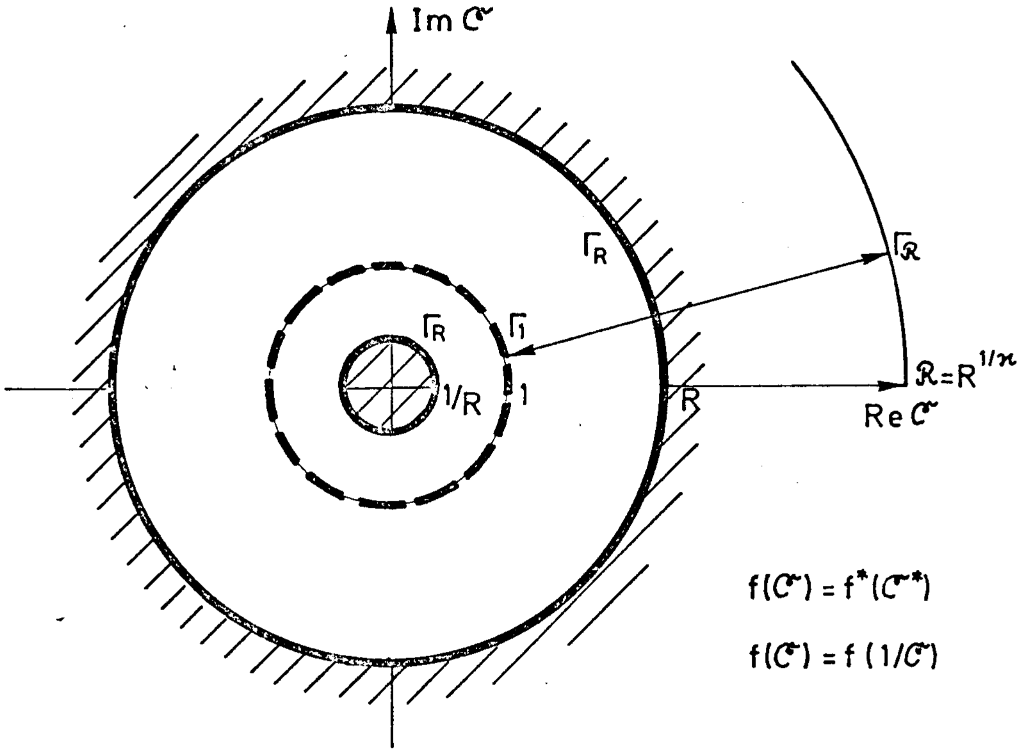
the $\mathcal{W}_a(z)$-plane

FIG. 6



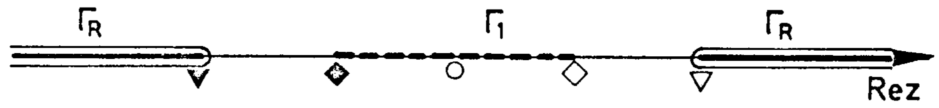
the $\mathcal{W}_b(z)$-plane

FIG. 7



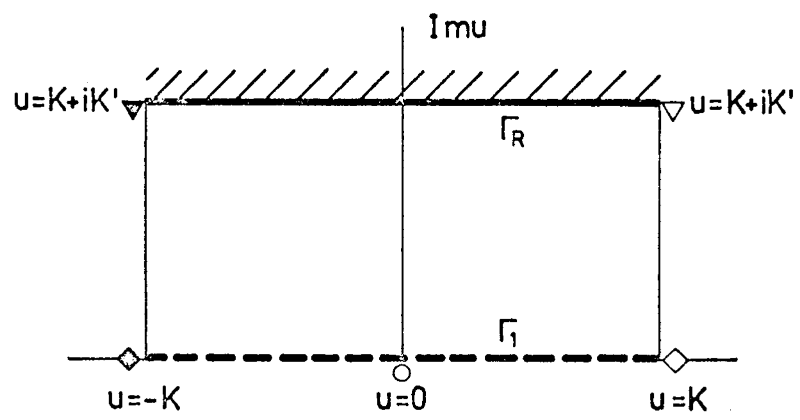
the complex σ-plane

FIG. 8



the z complex plane

FIG.9

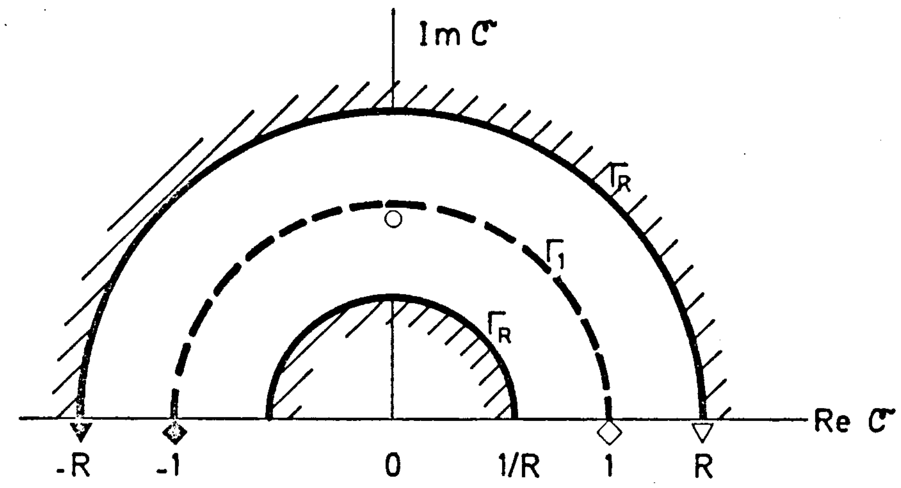


$$u(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad iK' = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

the $u = u(z)$ - plane $\quad [z = \mathrm{sn}\, u]$

FIG.10



$R = \exp.(\pi K'/2K)$

FIG.11

the $\mathcal{C}(z) = i\exp(-i\pi u(z)/2K)$ - plane