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A Stackelberg Game for Pricing Uplink Power in Wide-Band Cognitive Radio Networks

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***Abstract*— We study the problem of pricing uplink power in wide-band cognitive radio networks under the objective of revenue maximization for the service provider and while ensuring incentive compatibility for the users. User utility is modeled as a concave function of the signal-to-noise ratio (SNR) at the base station, and the problem is formulated as a Stackelberg game. Namely, the service provider imposes differentiated prices per unit of transmitting power and the users consequently update their power levels to maximize their net utilities. We devise a pricing policy and give conditions for its optimality when all the users are to be accommodated in the network. We show that there exist infinitely many Nash equilibrium points that reward the service provider with the same revenue. The pricing policy charges more from users that have better channel conditions and more willingness to pay for the provided service. We then study properties of the optimal revenue with respect to different parameters in the network. We show that for regimes with symmetric users who share the same level of willingness to pay, the optimal revenue is concave and increasing in the number of users in the network. We analytically obtain achievable SNRs for this special case, and finally present a numerical study in support of our results.**

I. INTRODUCTION

Given the recent reports on scarcity of available frequency bands and inefficiency in spectrum utilization [1], the importance of cognitive radio paradigm has emerged for allocating valuable wireless resources. Unlike traditional wireless devices, cognitive radio nodes are aware of their capabilities, environment, and intended use, and can also learn new waveforms, models, or operational scenarios [2]. See for example [3] for a comprehensive text on cognitive radio technology. Such smart characteristics of cognitive nodes can also help implementing various operational features of telecommunication networks that have always suffered from lack of flexibility on the operator and user sides. In this context, pricing is one important topic that can be revived by the virtue of cognitive radio technology.

Pricing for telecommunication networks has been embraced in the literature as an effective tool for creating policies to share network resources. Efficient pricing techniques not only increase the performance, but also improve network utilization in light of the rapid growth and variety of network demand. Objectives for pricing communication networks have varied in the literature between social welfare maximization [4], [5], fairness guarantees [6], [7], and revenue maximization [8], [9]. In this context, non-cooperative game theory has proved useful, especially for modeling selfish behavior of network users and introducing utility based pricing techniques. See [10] for a good survey on applying game theory principles on resource allocation problems in communication networks.

We consider a revenue maximization version of the pricing problem with focus on systems that apply spread spectrum technology in the physical layer. Similar systems appear in wireless networks that employ CDMA as the spectrum access mechanism. The problem involves a service provider that accommodates cognitive radio users transmitting on the uplink channels. User utility is modeled as a concave function of the SNR. The user is charged per unit of transmitting power and therefore adapts its power level to maximize the difference between its utility and the cost. The net utility depends on the price imposed by the service provider and the power levels exercised by the other users. We study the problem from the perspective of the service provider under the objective of maximizing revenue from the network.

We formulate the problem as a Stackelberg game. Namely, we use a leader-follower game formulation where the service provider (leader) imposes differentiated prices per unit of transmitting power on the different users in the network. Consequently, the users (followers) update their power levels to maximize their net utilities. The Stackelberg game provides incentive compatibility for the users while maximizing revenue for the service provider. We devise a pricing policy for accommodating all the users in the network and give optimality conditions. The pricing policy suggests charging the users proportional to their uplink channel conditions and willingness to pay for the provided service. We show that participation of all the users in the game is subject to having a large enough spreading factor for the wide-band network. We show properties of the optimal revenue for the service provider with respect to different parameters in the network. We then study the case where the users have identical appreciation for their utilities and analytically obtain achievable SNRs for the suggested pricing policy.

A. *Related Work*

Designing revenue-maximizing network policies based on user-follower game modeling has got considerable attention in the literature. For example, an Internet packet-pricing scheme is devised in [9] for monopolistic service providers with large number of users. Notions like differentiated pric-

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ing and incomplete game information are introduced to the problem and studied in [11] and [12]. The authors consider utility functions for the users to be concave in the flow rates exercised. In addition to the amount charged by the service provider, the user bears an additional cost of delay due to congestion on the links. In our power pricing problem there is no congestion cost to be considered as in classical Internet flow control problems. Moreover, we consider user utility to be a function of the SNR which, unlike flow rate in the cited works, it admits an explicit formula in terms of the actions (power levels) of the other users in the network. This form of the utility functions proved to be insightful for an optimal pricing policy as will be shown later.

Pricing for revenue maximization in wireless networks has been studied jointly with power control. For example, in [13] the authors adopt a utility function in terms of the ratio of throughput to transmitting power. The authors assume that users are charged the same price for unit throughput, and give an approximate form of the revenue-maximizing price. User utility function is taken to be quasi-concave when the bit error rate decays exponentially in the SNR.

This work can be considered as an extension to [14]. While we adopt somehow similar power control game for the users, the game in [14] is solved for arbitrary prices set by the service provider. In this work, we give the problem another dimension by considering revenue-maximizing prices in a Stackelberg formulation.

B. Paper Organization

The rest of the paper is organized as follows: In Section II we define the revenue for the service provider and the utility functions of the users, give the problem setup, and formulate the Stackelberg game. In Sections III and IV we analyze the game and devise an optimal pricing strategy for the problem along with the rest of the results. Numerical results that support our analysis are given in Section V. Finally, conclusions and remarks for future work are provided in Section VI.

II. Problem Setup

Consider a wide-band wireless network that employs CDMA as the spectrum access mechanism and has N cognitive radio nodes numbered $1,\cdots,N$. The service provider charges the i^{th} user the amount λ_i per unit of transmitting power on the uplink channel. Namely, if the transmitting power of user i is p_i, then the amount charged is $\lambda_i p_i$. The total revenue for the service provider is then given by

$$R(\boldsymbol{\lambda},\boldsymbol{p}) = \sum_{i=1}^{N} \lambda_i p_i, \tag{1}$$

where $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\cdots,\lambda_N)$ and $\boldsymbol{p} = (p_1,p_2,\cdots,p_N)$.

The SNR at the base station for the i^{th} user can be represented by the formula

$$\gamma_i(\boldsymbol{p}) = \frac{L p_i h_i}{\sum_{k=1,k\neq i}^{N} p_k h_k + \sigma^2}, \tag{2}$$

where

L: the spreading gain of the CDMA network, $L > 1$.

h_k: the uplink channel gain of the k^{th} user, $0 < h_k < 1$.

σ^2: the ambient noise in the network.

We consider user utility to be logarithmic, hence concave, in its SNR. In particular, by accounting for the amount charged by the service provider, the net utility for the i^{th} user is given by

$$U_i(\boldsymbol{p},\lambda_i) = \alpha_i log\left(1+\gamma_i(\boldsymbol{p})\right) - \lambda_i p_i, \tag{3}$$

where the constant $\alpha_i > 0$ is a factor that converts utility units to currency. Therefore, the higher α_i is the more the user is willing to pay for a unit utility.

The problem involves a non-cooperative setup where each user in the network is interested in maximizing its net utility. In specific, for a given price λ_i, and a vector of power levels of all users except the i^{th} user, denoted by $\boldsymbol{p}_{-i}$, user i objective is to find p_i^* that solves the following problem:

$$\max_{p_i \geq 0} \quad U_i(p_i,\boldsymbol{p}_{-i},\lambda_i). \tag{4}$$

Now consider the problem of the service provider for maximizing revenue characterized by (1). The service provider aims to impose optimal prices on the users given their selfish behavior as represented by (4). The problem in this context can be considered from a game theoretical perspective and formulated as a Stackelberg game. We study a leader-follower game where the service provider, the leader in this case, sets the prices, and consequently the users, or followers, update their power levels in accordance with their own preferences or utilities. The objective of the service provider is to find $\boldsymbol{\lambda}^* = (\lambda_1^*,\lambda_2^*,\cdots,\lambda_N^*)$ that solves the problem

$$\max_{\boldsymbol{\lambda}>\boldsymbol{0}} \quad R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda})), \tag{5}$$

where $\boldsymbol{p}^*(\boldsymbol{\lambda}) = (p_1^*(\boldsymbol{\lambda}),p_2^*(\boldsymbol{\lambda}),\cdots,p_N^*(\boldsymbol{\lambda}))$ such that $p_i^*(\boldsymbol{\lambda})$ is a solution for the i^{th} user's problem (4) for a given vector $\boldsymbol{\lambda}$.

In pursuing a solution for the Stackelberg game, our objective is to find Nash Equilibrium (NE) point(s) where neither the service provider nor any of the users have incentive to deviate unilaterally from that point. We formally state the following NE definition:

Definition 1: (Nash Equilibrium) Let $\boldsymbol{\lambda}^*$ be a solution for the service provider's problem (5) and p_i^* be a solution for the i^{th} user's problem (4). Then the point $(\boldsymbol{\lambda}^*,\boldsymbol{p}^*)$ is a NE for the Stackelberg game if for any $(\boldsymbol{\lambda},\boldsymbol{p})$:

$$\begin{aligned} U_i(p_i^*,\boldsymbol{p}_{-i}^*,\boldsymbol{\lambda}^*) &\geq U_i(p_i,\boldsymbol{p}_{-i}^*,\boldsymbol{\lambda}^*) \quad \forall i, \quad \text{and} \\ R(\boldsymbol{\lambda}^*,\boldsymbol{p}^*) &\geq R(\boldsymbol{\lambda},\boldsymbol{p}^*). \end{aligned}$$

III. Analytical Framework

We base our analysis on networks that have high SNR so that ambient noise is sufficiently small and can be neglected; i.e. $\sigma^2 \simeq 0$. A pricing policy in this case has an appealing form and gives additional insight. Also to avoid any trivialities from dividing by 0 in (2) we assume $N \geq 2$.

In order to solve for the Stackelberg game we use a backward induction technique. We start with the game of the users and derive the best response for each user as a function of the price set by the service provider and the power levels exercised by the other users in the network. Namely, for a given λ_i and $\boldsymbol{p}_{-i}$, the first and second order optimality conditions of the i^{th} user's problem (4) suggest the following best response:

$$\Phi_i(\lambda_i, \boldsymbol{p}_{-i}) = \begin{cases} \frac{1}{h_i}\left[\frac{\alpha_i h_i}{\lambda_i} - \frac{y_{-i}}{L}\right] & \text{if } 0 < \frac{1}{h_i}\left[\frac{\alpha_i h_i}{\lambda_i} - \frac{y_{-i}}{L}\right] \\ 0 & \text{if } \frac{1}{h_i}\left[\frac{\alpha_i h_i}{\lambda_i} - \frac{y_{-i}}{L}\right] \leq 0, \end{cases} \quad (6)$$

where $y_{-i} = \sum_{k=1,k\neq i}^{N} p_k h_k$ for all i. Notice that the second choice is due to the non-negativity constraint on the transmitting power in (4).

We state the following proposition by adopting the results in [14]:

Proposition 1 (Unique NE for the users' game): The power game of the users admits a unique NE for any vector of prices $\boldsymbol{\lambda} > 0$ assigned by the service provider. In particular, index the users such that if $\frac{\alpha_i h_i}{\lambda_i} < \frac{\alpha_j h_j}{\lambda_j}$, then $i > j$ with the ordering to be picked arbitrarily if $\frac{\alpha_i h_i}{\lambda_i} = \frac{\alpha_j h_j}{\lambda_j}$. Let $\hat{M}(\boldsymbol{\lambda})$ be the largest integer $M \leq N$ for which the following condition is satisfied

$$\frac{\alpha_M h_M}{\lambda_M} > \frac{1}{L+M-1}\sum_{j=1}^{M}\frac{\alpha_j h_j}{\lambda_j}. \quad (7)$$

The game admits a unique NE which has the property that users $\hat{M}(\boldsymbol{\lambda})+1, \cdots, N$ have zero power levels, i.e. $p_j^*(\boldsymbol{\lambda}) = 0$ for $j \geq \hat{M}(\boldsymbol{\lambda})+1$. The equilibrium power levels of the first $\hat{M}(\boldsymbol{\lambda})$ users are positive and obtained uniquely by

$$p_i^*(\boldsymbol{\lambda}) = \frac{L}{L-1}\left(\frac{\alpha_i}{\lambda_i} - \frac{1}{h_i(L+\hat{M}(\boldsymbol{\lambda})-1)}\sum_{j=1}^{\hat{M}(\boldsymbol{\lambda})}\frac{\alpha_j h_j}{\lambda_j}\right), \quad (8)$$

for $i = 1, \cdots, \hat{M}(\boldsymbol{\lambda})$.
Notice that it is always the case that $\hat{M}(\boldsymbol{\lambda}) \geq 1$ since at least one user should pass condition (7).

Formula (6) suggests that, besides the price, the best response of any user depends on the actions of the other users in the network. As suggested by [14], some discrete time iterative update algorithms converge to the NE point. For example, an algorithm where all the users update their power levels with probability 1 in each time slot can be shown to converge provided that condition $\frac{N-1}{L} < 1$ is satisfied. In computing the best response, the user does not need to communicate with any peer in the network. The quantity $\frac{\sum_{k=1,k\neq i}^{N} p_k h_k}{L}$ can be computed at the base station and provided to user i along with the price value λ_i. Therefore, assuming the quantity L to be of common knowledge, the user has all the information required to compute its best response in a decentralized fashion.

So far we have the NE point for the game of the users as a function of the imposed price vector $\boldsymbol{\lambda}$, as given by (8). We are striving to find an optimal price vector $\boldsymbol{\lambda}^*$ that solves the problem of the service provider (5). In other words, for a given $\boldsymbol{\lambda}$, $\hat{M}(\boldsymbol{\lambda})$ users satisfy condition (7), and therefore have positive power levels given by (8). The revenue for the service provider (1) can then be given by the formula

$$R(\boldsymbol{\lambda}, \boldsymbol{p}^*(\boldsymbol{\lambda})) = \frac{L}{L-1}\sum_{i=1}^{\hat{M}(\boldsymbol{\lambda})}\left(\alpha_i - \frac{1}{(L+\hat{M}(\boldsymbol{\lambda})-1)}\frac{\lambda_i}{h_i}\sum_{j=1}^{\hat{M}(\boldsymbol{\lambda})}\frac{\alpha_j h_j}{\lambda_j}\right). \quad (9)$$

Definition 2: Let $(\boldsymbol{p}^*, \boldsymbol{\lambda}^*)$ be a NE point for the Stackelberg game. Then the point is inner if $p_i^* > 0$ and $\lambda_i^* > 0$ for all $i = 1, \cdots, N$. Otherwise it is a boundary point.

In the following theorem we state our main result for a pricing policy for inner NE points:

Theorem 1 (Optimal Prices for Inner NE Points): Consider the Stackelberg game with N followers. Let the indexing of the users be done such that $\sqrt{\alpha_i} < \sqrt{\alpha_j} \Longrightarrow i > j$, with the ordering to be picked arbitrarily if $\sqrt{\alpha_i} = \sqrt{\alpha_j}$. If the following condition is satisfied for all $M \in \{1, \cdots, N\}$

$$\sqrt{\alpha_M} > \frac{1}{L+M-1}\sum_{j=1}^{M}\sqrt{\alpha_j}, \quad (10)$$

then the Stackelberg game admits an infinite number of inner NE points $(\boldsymbol{\lambda}^*, \boldsymbol{p}^*)$ such that

$$\frac{\lambda_i^*}{h_i\sqrt{\alpha_i}} = \frac{\lambda_j^*}{h_j\sqrt{\alpha_j}}, \quad \forall i,j = 1, \cdots, N, \quad (11)$$

and

$$p_i^* = \frac{L}{(L-1)}\frac{1}{\lambda_i^*}\left(\alpha_i - \frac{\sqrt{\alpha_i}}{(L+N-1)}\sum_{j=1}^{N}\sqrt{\alpha_j}\right). \quad (12)$$

Proof: The proof of the theorem is deferred to the Appendix. ■

The theorem characterizes optimal pricing for accommodating all the users in the network and shows that condition (10) is a sufficient condition. The formula of optimal prices (11) has an interesting proportional structure. It suggests charging more the users that have better channel conditions, i.e. higher h's, and who are more willing to pay for their utilities, i.e. higher α's.

Notice that if L is large enough, then condition (10) is satisfied for all $M \in \{1, \cdots, N\}$. Intuitively, by the formula of the SNR given by (2), the higher the spreading gain L is, the better the SNR for the user and the lesser the external effect due to the other users in the network. This way there will be more incentive for the users to have non-zero power levels.

It can be shown by substituting optimal prices from (11) in the revenue formula (9) that the revenue at any inner NE point is fixed and given by

$$R(\boldsymbol{\lambda}^*, \boldsymbol{p}^*(\boldsymbol{\lambda}^*)) = \frac{L}{L-1}\sum_{i=1}^{N}\left(\alpha_i - \frac{\sqrt{\alpha_i}}{(L+N-1)}\sum_{j=1}^{N}\sqrt{\alpha_j}\right). \quad (13)$$

Moreover, SNR values that are achieved at the base station are independent of the price. In particular, by using (8) in (2)

$$\hat{\gamma}_i = \frac{L\left(\sqrt{\alpha_i} - \frac{1}{(L+N-1)}\sum_{j=1}^{N}\sqrt{\alpha_j}\right)}{\sum_{k=1,k\neq i}^{N}\left(\sqrt{\alpha_k} - \frac{1}{(L+N-1)}\sum_{j=1}^{N}\sqrt{\alpha_j}\right)}. \quad (14)$$

The utilities for the users are also independent of the assigned optimal price and can be directly computed by the previous formula and equation (12).

Theorem 2: If condition (10) is satisfied for all $M \in \{1, \cdots, N\}$, then the revenue from any boundary point is always less than the revenue from inner NE points, that is, prices that give a boundary point are less profitable than prices that satisfy (11).

Proof: The proof of the theorem is omitted due to space constraints and is given in [15] ■

IV. Symmetric Users Case

In the case when condition (10) is not satisfied for all the users, optimality of prices in (11) is not guaranteed by Theorem 1. However, such problem requires more rigorous analysis and can be considered for an extended version of this paper.

In fact, it has been discussed in the previous section that for networks with large enough spreading factor L, condition (10) is always satisfied. The condition is also satisfied when all the users are symmetric in the sense that they have the same willingness to pay factor. In this case any solution for the Stackelberg game must be inner by Theorem 1. Besides being insightful, such cases are interesting since they arise in situations when cognitive radio users target specific applications offered by the wireless network, like the case for services offered by Mobile Virtual Network Operators MVNOs.

To shed light on some features of the symmetric users case, let $\alpha_i = \alpha_j = \alpha \ \forall i, j = 1, \cdots, N$. By Theorem 1 optimal prices satisfy

$$\frac{\lambda_i^*}{h_i} = \frac{\lambda_j^*}{h_j}, \quad \forall i, j = 1, \cdots, N.$$

It can be also shown that optimal power levels exercised by the users satisfy the following criteria

$$h_i p_i^* = h_j p_j^*, \quad i, j = 1, \cdots, N. \tag{15}$$

The optimal revenue for the service provider can be written in the following form using (13):

$$R(\boldsymbol{\lambda}^*, \boldsymbol{p}^*(\boldsymbol{\lambda}^*)) = \frac{\alpha N L}{L + N - 1}.$$

Now define the mapping $\rho(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ as

$$\rho(N) = \frac{\alpha N L}{L + N - 1}. \tag{16}$$

It is not hard to see that the mapping (16) is increasing and concave in the number of users N. Moreover

$$\lim_{N \longrightarrow \infty} \rho(N) = \alpha L.$$

The limit gives an upper bound on the revenue for the service provider. It shows that the more users the network accommodates, the better revenue the service provider gains up to a multiplicative value of the spreading gain L.

For the SNR that can be achieved in the symmetric users case, it can be shown from (14) to be

$$\hat{\gamma}_i = \frac{L}{N - 1}, \quad i = 1, \cdots, N. \tag{17}$$

Now, if γ_{min} is taken to be the minimum acceptable SNR at the base station, then it follows that N should be chosen such that

$$N \leq \frac{L}{\gamma_{min}} + 1.$$

The form gives an upper bound on the number of users that can be accommodated by the network to guarantee a certain threshold for the SNR.

V. Numerical Results

In this section we numerically verify the results in Theorem 1. For the sake of clarity of demonstration in the subsequent figures, we study a simple network with two users; $N = 2$. In this case the revenue for the service provider when the two users have positive power levels can be written using formula (9) as follows

$$R(\boldsymbol{\lambda}, \boldsymbol{p}^*(\boldsymbol{\lambda})) = \frac{L}{L^2 - 1} \left(L\alpha_1 + L\alpha_2 - \left(\frac{\lambda_1}{h_1} \frac{\alpha_2 h_2}{\lambda_2} + \frac{\lambda_2}{h_2} \frac{\alpha_1 h_1}{\lambda_1} \right) \right).$$

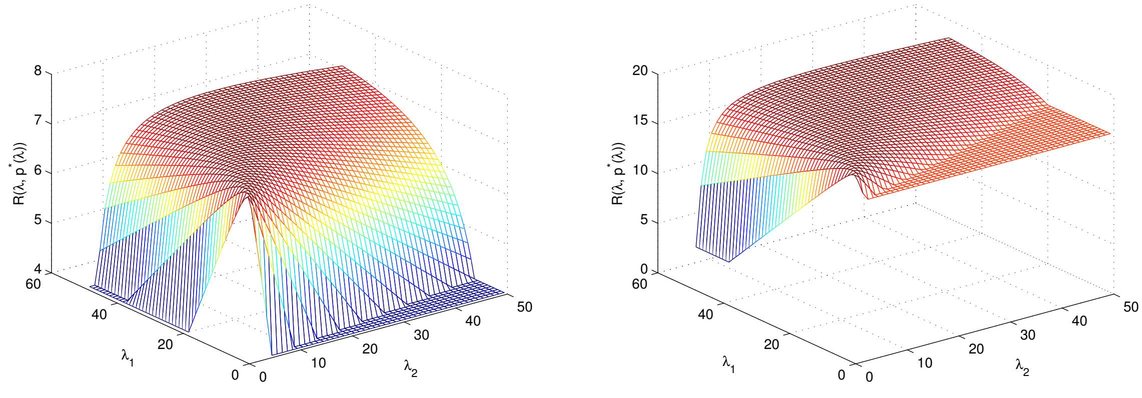
We first consider the symmetric users case where $\alpha_1 = \alpha_2$. We assume the gain of the channel for the first user to be twice that for the second user; i.e. $h_1 = 2h_2$. Theorem 1 suggests that an inner NE point can be achieved by charging the first user; i.e. the user with higher channel gain, double what the second user is charged. Namely, by (11) optimal prices satisfy the following line equation

$$\lambda_1^* = 2\lambda_2^*. \tag{18}$$

For each value (λ_1, λ_2) we test condition (7) for users 1 and 2. A user that passes the condition has positive power level given by (8) and 0 otherwise. The revenue for the service provider is directly computed by substitution in (1).

Figure 1(a) shows the revenue for the service provider for different prices imposed on the users. We use unit price increments up to 50 units. The values are computed for the following parameters: $L = 10$, $\alpha_1 = \alpha_2 = 4.0$, $h_1 = 0.5$, and $h_2 = 0.25$. The flat surface in the figure corresponds to price values where only one user passes condition (7), and the revenue in this case is fixed regardless of which user passes the test. The maximum revenue value is 7.27 and is obtained by exhaustive search on the computed values. It is verified to be achieved by prices that satisfy optimal price formula (18).

In another example we consider the other case where the users have the same channel gain but different willingness to pay factors. We adopt the case where $\alpha_1 = 4\alpha_2$. We use the parameters $L = 10$, $h_1 = h_2 = 0.5$, $\alpha_1 = 16$, and $\alpha_2 = 4$. The two users in this case pass condition (10), and therefore the optimal policy suggests charging the users according to (18). In Figure 1(b) we show the revenue for the service provider for different price values. The optimal revenue value is 18.59 and it is similarly verified to be achieved by prices that satisfy the form (18).



(a) (b)

Fig. 1. revenue for the service provider for different prices in the setup given in Section V for the following cases a) Users with similar willingness to pay factors $\alpha_1 = \alpha_2 = 4.0$, but different channel gains: $h_1 = 0.5$ and $h_2 = 0.25$. b) Users with similar channel gains $h_1 = h_2 = 0.5$, but different willingness to pay factors: $\alpha_1 = 16$ and $\alpha_2 = 4$. In both cases optimal revenue values are achieved by prices that satisfy (18).

VI. Conclusions and Future Work

We have studied pricing uplink power in wide-band cognitive radio networks for revenue maximization. We have formulated the problem as a Stackelberg game and presented an optimal pricing policy for inner NE points. The formula of optimal prices reveals that users with better channel conditions and more willingness to pay should be charged more. We have also studied properties for the optimal revenue and shown that for the case when users have the same willingness to pay the revenue is increasing and concave in the number of users in the network. Since this might lead to unacceptably small SNR values, we have also given an upper bound on the number of users that can be accommodated so that a minimum SNR is guaranteed.

Our future work involves investigating the problem when condition (10), which classifies users based on their willingness to pay for the service, is not necessarily satisfied. Optimality of prices in (11) is then not guaranteed, and a solution of the problem in this case hedges on more rigorous mathematical analysis.

Appendix

In this section we give a proof of Theorem 1.

Proof: Consider the objective revenue function (9) for any given vector $\boldsymbol{\lambda}$ such that $\hat{M}(\boldsymbol{\lambda}) = N$. First we show that a vector $\boldsymbol{\lambda}^*$ that satisfies (11) is a maximizer for the function. In particular, take the first order derivative with respect to $\lambda_i,\ i = 1, \cdots, N$

$$\frac{\partial R(\boldsymbol{\lambda}, \boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_i} = \frac{-1}{(L+N-1)}\left(\frac{1}{h_i}\sum_{j=1, j\neq i}^{N}\frac{\alpha_j h_j}{\lambda_j} - \frac{\alpha_i h_i}{\lambda_i^2}\sum_{k=1,k\neq i}^{N}\frac{\lambda_k}{h_k}\right).$$

The first order optimality condition; i.e. $\frac{\partial R(\boldsymbol{\lambda}, \boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_i} = 0,\ \forall i$ suggests then

$$\frac{\alpha_i h_i^2}{\lambda_i^2} = \frac{\sum_{j=1,j\neq i}^{N}\frac{\alpha_j h_j}{\lambda_j}}{\sum_{k=1,k\neq i}^{N}\frac{\lambda_k}{h_k}}, \quad i = 1, \cdots, N. \tag{19}$$

The previous system of equations admits an infinite number of solutions characterized by (11). To show this, notice that for any user $i = 1, \cdots, N$ the expression $\sum_{j=1,j\neq i}^{N}\frac{\alpha_j h_j}{\lambda_j}$ can be written in the equivalent form

$$\sum_{j=1,j\neq i}^{N}\frac{\alpha_j h_j}{\lambda_j} = \frac{\sum_{j=1,j\neq i}^{N}\alpha_j h_j\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right)}{\prod_{m=1,m\neq i}^{N}\lambda_m}.$$

Therefore equations (19) become

$$\frac{\alpha_i h_i^2}{\lambda_i^2} = \frac{\sum_{j=1,j\neq i}^{N}\alpha_j h_j\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right)}{\sum_{k=1,k\neq i}^{N}\frac{\lambda_k^2}{h_k}\left(\prod_{m=1,m\neq i,k}^{N}\lambda_m\right)}.$$

Simple manipulations then lead to

$$\frac{\lambda_i^2}{\alpha_i h_i^2}\sum_{j=1,j\neq i}^{N}\alpha_j h_j\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right) - \sum_{k=1,k\neq i}^{N}\frac{\lambda_k^2}{h_k}\left(\prod_{m=1,m\neq i,k}^{N}\lambda_m\right) = 0.$$

Summing up terms with the same indices and taking the product as a common factor result in

$$\sum_{j=1,j\neq i}^{N}\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right)\left(\frac{\lambda_i^2}{\alpha_i h_i^2}\alpha_j h_j - \frac{\lambda_j^2}{h_j}\right) = 0.$$

Now taking $\alpha_j h_j$ as a common factor in the second set of parenthesis results in

$$\sum_{j=1,j\neq i}^{N}\alpha_j h_j\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right)\left(\frac{\lambda_i^2}{\alpha_i h_i^2} - \frac{\lambda_j^2}{\alpha_j h_j^2}\right) = 0. \tag{20}$$

Notice that $\alpha_j h_j\left(\prod_{l=1,l\neq i,j}^{N}\lambda_l\right) > 0\ \forall i,j$ by the problem definition. Therefore, unless $\boldsymbol{\lambda}^*$ is chosen such that relation (11) is satisfied, the set of equations (20) cannot be satisfied.

To test the second order optimality condition, consider the $N \times N$ Hessian matrix given by

$$\boldsymbol{H}(\boldsymbol{\lambda}) = \begin{pmatrix} \frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_1^2} & \cdots & \frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_1 \partial \lambda_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_N \partial \lambda_1} & \cdots & \frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_N^2} \end{pmatrix},$$

where

$$\frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_i^2} = \frac{-1}{(L+N-1)} \left(\frac{2\alpha_i h_i}{\lambda_i^3} \sum_{k=1,k\neq i}^{N} \frac{\lambda_k}{h_k} \right)$$

and

$$\frac{\partial^2 R(\boldsymbol{\lambda},\boldsymbol{p}^*(\boldsymbol{\lambda}))}{\partial \lambda_i \partial \lambda_j} = \frac{1}{(L+N-1)} \left(\frac{\alpha_j h_j}{h_i \lambda_j^2} + \frac{\alpha_i h_i}{h_j \lambda_i^2} \right),$$

for $i,j = 1,\cdots,N$. Take also an arbitrary vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_N) \neq \boldsymbol{0}$ and notice that

$$\begin{aligned} \boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T = \frac{-1}{(L+N-1)} & \left(\sum_{i=1}^{N} x_i^2 \left(\frac{2\alpha_i h_i}{\lambda_i^{*3}} \sum_{k=1,k\neq i}^{N} \frac{\lambda_k^*}{h_k} \right) \right. \\ & \left. - \sum_{i=1}^{N} x_i \sum_{j=1,j\neq i}^{N} x_j \left(\frac{\alpha_i h_i}{h_j \lambda_i^{*2}} + \frac{\alpha_j h_j}{h_i \lambda_j^{*2}} \right) \right). \end{aligned}$$

But $\boldsymbol{\lambda}^*$ satisfies (11). Therefore, the equality can be written as

$$\boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T = \frac{-2}{(L+N-1)} \times \left(\sum_{i=1}^{N} \frac{x_i^2}{\lambda_i^{*2}} \sum_{k=1,k\neq i}^{N} \sqrt{\alpha_i \alpha_k} - \sum_{i=1}^{N} \frac{x_i \alpha_i h_i}{\lambda_i^{*2}} \sum_{j=1,j\neq i}^{N} \frac{x_j}{h_j} \right).$$

By simple manipulations the equality can be further taken to

$$\begin{aligned} & \boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T \\ &= \frac{-2}{(L+N-1)} \sum_{i=1}^{N} \frac{1}{\lambda_i^{*2}} \left(x_i^2 \sum_{k=1,k\neq i}^{N} \sqrt{\alpha_i \alpha_k} - x_i \alpha_i h_i \sum_{j=1,j\neq i}^{N} \frac{x_j}{h_j} \right) \\ &= \frac{-2}{(L+N-1)} \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \frac{1}{\lambda_i^{*2}} \left(x_i^2 \sqrt{\alpha_i \alpha_j} - x_i \alpha_i h_i \frac{x_j}{h_j} \right) \\ &= \frac{-2}{(L+N-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \left(\frac{x_i^2 \sqrt{\alpha_i \alpha_j}}{\lambda_i^{*2}} - \frac{x_i \alpha_i h_i}{\lambda_i^{*2}} \frac{x_j}{h_j} + \frac{x_j^2 \sqrt{\alpha_j \alpha_i}}{\lambda_j^{*2}} \right. \\ & \quad \left. - \frac{x_j \alpha_j h_j}{\lambda_j^{*2}} \frac{x_i}{h_i} \right). \end{aligned}$$

Now using (11) to write λ_j^* in terms of λ_i^* we get

$$\boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T = \frac{-2}{(L+N-1)} \times \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{\lambda_i^{*2}} \left(x_i^2 \sqrt{\alpha_i \alpha_j} - 2 x_i x_j \alpha_i \frac{h_i}{h_j} + x_j^2 \left(\frac{h_i}{h_j} \right)^2 \frac{\alpha_i^{\frac{3}{2}}}{\alpha_j^{\frac{1}{2}}} \right).$$

The previous equality can be finally written in the following form

$$\boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T = \frac{-2}{(L+N-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{\lambda_i^{*2}} \left(x_i (\alpha_i \alpha_j)^{\frac{1}{4}} - \frac{x_j h_i \alpha_i^{\frac{3}{4}}}{h_j \alpha_j^{\frac{1}{4}}} \right)^2.$$

Notice that

$$\boldsymbol{x}\boldsymbol{H}(\boldsymbol{\lambda}^*)\boldsymbol{x}^T \begin{cases} = 0 & \text{if } \boldsymbol{x} \text{ satisfies (11)} \\ < 0 & \text{otherwise.} \end{cases}$$

In fact, all the vectors that satisfy (11) follow a continuous line, where for a given $\boldsymbol{\lambda}^*$, any vector can be represented by $c\boldsymbol{\lambda}^*$ such that c is a scalar > 0. They all give in the same objective value; i.e. $R(\boldsymbol{\lambda}^*, \boldsymbol{p}^*(\boldsymbol{\lambda}^*)) = R(c\boldsymbol{\lambda}^*, \boldsymbol{p}^*(c\boldsymbol{\lambda}^*))$, as given later in the text by (13).

Finally, if condition (10) is satisfied for all $M \in \{1, \cdots, N\}$, then for a given $\boldsymbol{\lambda}^*$ condition (7) is satisfied for all M and the solution $(\boldsymbol{p}^*, \boldsymbol{\lambda}^*)$ is inner. Finally, $\boldsymbol{p}^*$ as given by (12) follows by direct substitution in (8). ■

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