A Standard Basis Free Algorithm for Computing the Tangent Cones of a Space Curve

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Abstract. We outline a method for computing the tangent cone of a space curve at any of its points. We rely on the theory of regular chains and Puiseux series expansions. Our approach is novel in that it explicitly constructs the tangent cone at arbitrary *and possibly irrational* points without using a standard basis.

Keywords: Computational algebraic geometry, tangent cone, regular chain, Puiseux series.

1 Introduction

Traditionally, standard bases, Groebner bases and cylindrical algebraic decomposition are the fundamental tools of computational algebraic geometry. The computer algebra systems CoCoA, Macaulay 2, Magma, Reduce, Singular have well-developed packages for computing standard bases or Groebner bases, on which they rely in order to provide powerful toolkits to algebraic geometers.

Recent progress in the theory of regular chains has exhibited efficient algorithms for doing local analysis on algebraic sets. One of the algorithmic strengths of the theory of regular chains is its *regularity test* procedure. In algebraic terms, this procedure decides whether a hypersurface contains at least one irreducible component of the zero set of the saturated ideal of a regular chain. Broadly speaking, this procedure separates the zeros of a regular chain that belong to a given hypersurface from those which do not. This regularity test permits to extend an algorithm working over a field into an algorithm working over a direct product of fields. Or, to phrase it in another way, it allows to extend an algorithm working at point into an algorithm working at a group of points.

Following that strategy, the authors of [8] have proposed an extension of Fulton's algorithm for computing the intersection multiplicity of two plane curves at the origin. To be precise, this paper extends Fulton's algorithm in two ways. First, thanks to the regularity test for regular chains, the construction is adapted such that it can work correctly at any point in the intersection of two plane curves, whether this point has rational coordiantes or not.

Secondly, an algorithmic criterion, see Theorem 1, is proposed for reducing intersection multiplicity computation in arbitrary dimension to the case of two plane curves. This algorithmic criterion requires to compute the tangent cone $TC_p(\mathscr{C})$ of a space curve $\mathscr{C}$ at one of its points p. In principle, this latter problem can be handled by means of standard basis (or Gröbner basis) computation. Available implementation (like those in MAGMA or SINGULAR) require that the point p is uniquely determined by the values of its coordinates. However, when decomposing a polynomial system, a point may be defined as one of the roots of a particular sub-system (typically a regular chain $\boldsymbol{h}$). Therefore it is desirable to be able to compute the tangent cones of $\mathscr{C}$ at any points defined by a given regular chain $\boldsymbol{h}$. Similarly, and as discussed in [8], it is desirable to be able compute the intersection multiplicity of a zero-dimensional algebraic set V at any points defined by a given regular chain $\boldsymbol{h}$. This type of tangent cone computation is addressed in the present paper.

Tangent cone computations can be approached at least in two ways. First, one can consider the formulation based on homogeneous components of least degree, see Definition 1. The original algorithm of Mora [9] follows this point of view. Secondly, one can consider the more "intutive" characterization based on limits of secants, see Lemma 1. This second approach, that we follow in this paper, requires to compute limits of algebraic functions. For this task, we take advantage of [2] where the authors show how to compute the limit points of the quasi-component of a regular chain. This type of calculation can be used for computing the Zariski closure of a constructible set In the present paper, it is used for computing tangent cones of space curves, thus providing an alternative to the standard approaches based on Groebner bases and standard bases, respectively.

The contributions of the prepsent paper are as follows

1. In Section 3, we present a proof of our algorithm criterion for reducing intersection multiplicity computation in arbitrary dimension to the plane case; this criterion was stated with no justification in [8].
2. In Section 4.1, with Lemma 2, under some assumption, we establsh a natural method for computing $TC_p(\mathscr{C})$; as limit of intersection of tangent spaces.
3. In Section 4.2, we relax the assumption of Section 4.1 and exhibit an algorithm for computing $TC_p(\mathscr{C})$.

This latter algorithm is implemented, in the `AlgebraicGeometryTools` subpackage [1] of the `RegularChains` library which is available at `www.regularchains.org`. Section 4.4 offers examples. However, an issue with MAPLE's `algcurves[puiseux]` command that we have no control over prohibits us from providing meaningful experimental results at this time. For those test cases which do not encounter error from the `algcurves[puiseux]` command we indeed calculate the correct tangent cone. We are currently re-implementing MAPLE's `algcurves[puiseux]` command and we will provide experimental results in a future report.

2 Preliminaries

Throughout this article, we denote by $\mathbb{K}$ a field with algebraic closure $\overline{\mathbb{K}}$, and by $\mathbb{A}^{n+1}(\overline{\mathbb{K}})$ the $(n+1)$-dimensional affine space over $\overline{\mathbb{K}}$, for some positive integer n. Let $\boldsymbol{x} := x_0, \ldots, x_n$ be $n+1$ variables ordered as $x_0 \succ \cdots \succ x_n$. We denote by $\mathbb{K}[\boldsymbol{x}]$ the corresponding polynomial ring. Let $\boldsymbol{h} \subset \mathbb{K}[\boldsymbol{x}]$ be a subset and $h \in \mathbb{K}[\boldsymbol{x}]$ be a polynomial. We say that h is *regular* modulo the ideal $\langle \boldsymbol{h} \rangle$ of $\mathbb{K}[\boldsymbol{x}]$ whenever h does not belong to any prime ideals associated with $\langle \boldsymbol{h} \rangle$, thus, whenever h is neither null nor a zero-divisor modulo $\langle \boldsymbol{h} \rangle$. The *algebraic set* of $\mathbb{A}^{n+1}(\overline{\mathbb{K}})$ consisting of the common zeros of the polynomials in $\boldsymbol{h}$ is written as $\mathbf{V}(\boldsymbol{h})$. For a subset $\mathbf{W} \subset \mathbb{A}^{n+1}(\overline{\mathbb{K}})$, we denote by $\mathbf{I}(\mathbf{W})$ the ideal of $\mathbb{K}[\boldsymbol{x}]$ generated by the polynomials vanishing at every point of $\mathbf{W}$. The ideal $\mathbf{I}(\mathbf{W})$ is radical and when $\overline{\mathbb{K}} = \mathbb{K}$ holds, Hilbert's Nullstellensatz states that $\sqrt{\langle \boldsymbol{h} \rangle} = \mathbf{I}(\mathbf{V}(\boldsymbol{h}))$.

In the next two sections, we review the main concepts used in this paper, namely tangent cones and regular chains. For the former, we restrict ourselves to tangent cones of a space curve and refer to [4] for details and the general[3] case. For the latter concept, we refer to [3], in particular for the specifications of the basic operations on regular chains.

2.1 Tangent Cone of a Space Curve

As above, let $\boldsymbol{h} \subset \mathbb{K}[\boldsymbol{x}]$. Define $\mathbf{V} := \mathbf{V}(\boldsymbol{h})$ and let $p := (p_0, \ldots, p_n) \in \mathbf{V}$ be a point. We denote by $\dim_p(\mathbf{V})$ the maximum dimension of an irreducible component C of $\mathbf{V}$ such that we have $p \in C$. Recall that the *tangent space* of $\mathbf{V}(\boldsymbol{h})$ at p is the algebraic set given by

$$T_p(\boldsymbol{h}) := \mathbf{V}(\ \mathbf{d}_p\,(f) : f \in \mathbf{I}(\mathbf{V}))$$

where $\mathbf{d}_p\,(f)$ is the *linear part* of f at p, that is, the affine form $\frac{\partial f}{\partial x_0}(p)(x_0 - p_0) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n)$. Note that $T_p(\boldsymbol{h})$ is a linear space. We say that $\mathbf{V}(\boldsymbol{h})$ is *smooth* at p whenever the dimension of $T_p(\boldsymbol{h})$ is $\dim_p(\mathbf{V})$ and *singular* otherwise. The *singular locus* of $\mathbf{V}(\boldsymbol{h})$, denoted by $\mathrm{sing}(\boldsymbol{h})$, is the set of the points $p \in \mathbf{V}(\boldsymbol{h})$ at which $\mathbf{V}(\boldsymbol{h})$ is singular.

Let $f \in \mathbb{K}[\boldsymbol{x}]$ be a polynomial of total degree d and $p := (p_0, \ldots, p_n) \in \mathbb{A}^{n+1}(\overline{\mathbb{K}})$ be a point such that $f(p) = 0$ holds. Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ be a $(n+1)$-tuple of non-negative integers. Denote: $(\boldsymbol{x} - p)^{\boldsymbol{\alpha}} := (x_0 - p_0)^{\alpha_0} \cdots (x_n - p_n)^{\alpha_n}$, where $|\alpha| = \alpha_0 + \cdots + \alpha_n$ is the total degree of $\boldsymbol{x} - p$. Since the polynomial $f \in \mathbb{K}[\boldsymbol{x}]$ has total degree d, it writes as a $\mathbb{K}$-linear combination of the form:

$$f = \sum_{|\boldsymbol{\alpha}|=0} c_{\boldsymbol{\alpha}} (\boldsymbol{x} - p)^{\boldsymbol{\alpha}} + \cdots + \sum_{|\boldsymbol{\alpha}|=d} c_{\boldsymbol{\alpha}} (\boldsymbol{x} - p)^{\boldsymbol{\alpha}}$$

[3] Note that in the book [3], and other classical algebraic geometry textbooks like [12], the tangent cone of an algebraic set at one of its points, is also an algebraic set. Two equivalent definitions appear in [3] and are recalled in Definition 1 and Lemma 1.

with all coefficients $c_{\boldsymbol{\alpha}}$ belonging to $\mathbb{K}$. Each summand $\mathrm{HC}_p(f;\, j) := \sum_{|\boldsymbol{\alpha}|=j} c_{\boldsymbol{\alpha}}(\boldsymbol{x}-p)^{\boldsymbol{\alpha}}$ is called the *homogeneous component in* $\boldsymbol{x}-p$ *of* f *in degree* j . Moreover, the *homogeneous component of least degree* of f in $\boldsymbol{x}-p$ is given by $\mathrm{HC}_p(f;\, \min) := \mathrm{HC}_p(f;\, j_{\min})$ where $j_{\min} = \min(j \in \mathbb{N} : \mathrm{HC}_p(f;\, j) \neq 0)$.

Definition 1 (Tangent Cone of a Curve). *Let $\mathscr{C} \subset \mathbb{A}^{n+1}(\overline{\mathbb{K}})$ be a curve and $p \in \mathscr{C}$ be a point. The* tangent cone *of $\mathscr{C}$ at a point p is the algebraic set denoted by $TC_p(\mathscr{C})$ and defined by $TC_p(\mathscr{C}) = \mathbf{V}(\mathrm{HC}_p(f;\, \min) : f \in \mathbf{I}(\mathscr{C}))$.*

One can show that $TC_p(\mathscr{C})$ consists of finitely many lines, all intersecting at p.

Fig. 1. This figure displays the typical "fish" curve, which is a planar curve given by $h = y^2 - x^2(x+1) \in \mathbb{Q}[x, y]$. Clearly, two tangent lines are needed to form a "linear approximation" of the curve at the origin. Elementary calculations show these two lines actually form the tangent cone of the fish curve at the origin.

If $\mathbf{I}(\mathscr{C})$ is generated by a single polynomial then computing $TC_p(\mathscr{C})$ is easy. Otherwise, this is a much harder computation. Let $\boldsymbol{h} \subset \mathbb{K}[\boldsymbol{x}]$ be such that $\mathbf{V}(\boldsymbol{h}) = \mathscr{C}$. As pointed out by Mora et al. in [10], one can compute $\langle\, \mathrm{HC}_p(f;\, \min) : f \in \mathbf{I}(\mathscr{C})\, \rangle$ by finding a graded Gröbner basis, say $\boldsymbol{G}$, of the *homogenization* of $\boldsymbol{h}$ (a process where an additional variable x_{n+1} is used to make every $h \in \boldsymbol{h}$ a homogeneous polynomial in $\mathbb{K}[\boldsymbol{x}][x_{n+1}]$). *Dehomogenizing* $\boldsymbol{G}$ by letting $x_{n+1} = 1$ produces the tangent cone of $\boldsymbol{h}$ [4, Chapter 9 §7 Proposition 4].

Tangent cones are intimately related to the notion of intersection multiplicity that we review below. As mentioned in the introduction, computing intersection multiplicities is the main motivation of the algorithm presented in this paper.

Definition 2. *Let $\boldsymbol{h} \subset \mathbb{K}[\boldsymbol{x}]$. The* intersection multiplicity *of p in $\mathbf{V}(\boldsymbol{h})$ is defined by $\mathrm{im}(p;\, \boldsymbol{h}) := \dim_{vec}(\mathscr{O}/\langle \boldsymbol{h} \rangle)$ where $\mathscr{O} := \{f/g : f, g \in \overline{\mathbb{K}}[\boldsymbol{x}],\, g(p) \neq 0\}$ is the* localization ring *of $\mathbb{K}[\boldsymbol{x}]$ at p and $\dim_{vec}(\mathscr{O}/\langle \boldsymbol{h} \rangle)$ is the dimension of $\mathscr{O}/\langle \boldsymbol{h} \rangle$ as a vector space over $\mathbb{K}$. Note by [5, Chapter 4. §2 Proposition 11] we may substitute the power series ring $\mathbb{K}[[\boldsymbol{x}-p]]$ for $\mathscr{O}$.*

Example 1. Let $\boldsymbol{x} = [x, y, z]$ and $\boldsymbol{h} = \{x,\, x - y^2 - z^2,\, y - z^3\} \subset \mathbb{K}[\boldsymbol{x}]$. Near the origin we know $x - y^2 - z^2 = -y^2 - z^2$ so $\boldsymbol{h} = \{x, y - z^3, z^2(z^4+1)\}$ and

$$\mathbb{K}[[\boldsymbol{x}]]/\langle \boldsymbol{h} \rangle = \mathbb{K}[[\boldsymbol{x}]]/\langle x,\, y - z^3,\, z^2 \rangle = \mathbb{K}[[\boldsymbol{x}]]/\langle x,\, y,\, z^2 \rangle = \{a + bz : a,\, b \in \mathbb{K}\}$$

implying $\mathrm{im}(\boldsymbol{0};\, \boldsymbol{h}) = 2$.

2.2 Regular Chain

Broadly speaking, a *regular chain* of $\mathbb{K}[\boldsymbol{x}]$ is a system of equations and inequations defined by polynomials in $\mathbb{K}[\boldsymbol{x}]$ such that each equation specifies, in an implicit manner, the possible values of one of the variables x_i as a function of the variables of least rank, namely $x_{i+1}, \ldots, x_n$. Regular chains are a convenient way to describe the solution set of a polynomial system. More precise statements follow.

Let $h \in \mathbb{K}[\boldsymbol{x}]$ be a non-constant polynomial. The *main variable* of h is the largest variable $x \in \boldsymbol{x}$ (for the ordering $x_0 \succ \cdots \succ x_n$) such that h has a positive degree in x. The *initial* of h, denoted $\mathrm{init}(h)$, is the *leading coefficient* of h w.r.t. its main variable. For instance the initial of $zx + t$ is x in $\mathbb{Q}[x \succ y \succ z \succ t]$ and 1 in $\mathbb{Q}[t \succ z \succ y \succ x]$.

Let $\boldsymbol{t} \subset \mathbb{K}[\boldsymbol{x}]$ consist of non-constant polynomials. Then, the set $\boldsymbol{t}$ is said *triangular* if any two polynomials in $\boldsymbol{t}$ have different main variables. When $\boldsymbol{t}$ is a triangular set, denoting by $\mathrm{I}_{\boldsymbol{t}}$ the product of the initials $\mathrm{init}(f)$ for $f \in \boldsymbol{t}$, we call *saturated ideal* of $\boldsymbol{t}$, written $\mathrm{sat}(\boldsymbol{t})$, the column ideal $\mathrm{sat}(\boldsymbol{t}) = \langle \boldsymbol{t} \rangle : \mathrm{I}_{\boldsymbol{t}}^{\infty}$ and we call *quasi-component* of $\boldsymbol{t}$ the basic constructible set $\mathbf{W}(\boldsymbol{t}) := \mathbf{V}(\boldsymbol{t}) \setminus \mathbf{V}(\mathrm{I}_{\boldsymbol{t}})$.

Definition 3 (Regular Chain). *The triangular set $\boldsymbol{t} \subset \mathbb{K}[\boldsymbol{x}]$ is a* regular chain *if either $\boldsymbol{t}$ is empty or the initial of f is regular modulo* $\mathrm{sat}(\boldsymbol{t} \setminus \{f\})$*, where f is the polynomial in $\boldsymbol{t}$ with largest main variable.*

Regular chains are used to decompose both algebraic sets and radical ideals, leading to two types of decompositions called respectively *Wu-Lazard* and *Kalkbrener* decompositions. More precisely, we have the following definition.

Finitely many regular chains $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_e \subset k[\boldsymbol{x}]$ form a Kalkbrener decomposition of $\sqrt{\langle \boldsymbol{h} \rangle}$ (resp. a Wu-Lazard decomposition of $\mathbf{V}(\boldsymbol{h})$) whenever we have $\sqrt{\langle \boldsymbol{h} \rangle} = \sqrt{\mathrm{sat}(\boldsymbol{t}_0)} \cap \cdots \cap \sqrt{\mathrm{sat}(\boldsymbol{t}_e)}$ (resp. $\mathbf{V}(\boldsymbol{h}) = \mathbf{W}(\boldsymbol{t}_0) \cup \cdots \cup \mathbf{W}(\boldsymbol{t}_e)$). These two types are different since the quasi-component of a regular chain $\boldsymbol{t}$ may not be an algebraic set. One should note that the Zariski closure of $\mathbf{W}(\boldsymbol{t})$ (that is, the intersection of all algebraic sets containing $\mathbf{W}(\boldsymbol{t})$) is the zero set (i.e. algebraic set) of $\mathrm{sat}(\boldsymbol{t})$. One should observe, however, that if $\mathrm{sat}(\boldsymbol{t})$ is zero-dimensional then the quasi-component $\mathbf{W}(\boldsymbol{t})$ and the algebraic set $\mathbf{V}(\boldsymbol{t})$ coincide. Practically efficient algorithms computing both types of decompositions appear in [3].

Regular chains enjoy important algorithmic properties. One of them is the ability to test whether a given polynomial $f \in \mathbb{K}[\boldsymbol{x}]$ is regular or not modulo the saturated ideal of a regular chain $\boldsymbol{t} \subset \mathbb{K}[\boldsymbol{x}]$. This allows us to specify an operation, called Regularize, as follows. The function call Regularize $(f, \boldsymbol{t})$ computes regular chains $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_e \subset \mathbb{K}[\boldsymbol{x}]$ such that $\sqrt{\mathrm{sat}(\boldsymbol{t})} = \sqrt{\mathrm{sat}(\boldsymbol{t}_0)} \cap \cdots \cap \sqrt{\mathrm{sat}(\boldsymbol{t}_e)}$ holds and for $i = 0, \ldots, e$, either f is zero modulo $\mathrm{sat}(\boldsymbol{t}_i)$ or f is regular modulo $\mathrm{sat}(\boldsymbol{t}_i)$. When $\mathrm{sat}(\boldsymbol{t})$ is zero-dimensional, one can give a simple geometrical interpretation to Regularize: this operation separates the points of $\mathbf{V}(\boldsymbol{t})$ belonging to $\mathbf{V}(f)$ from those which do not lye on $\mathbf{V}(f)$.

3 Computing Intersection Multiplicities in Higher Dimension

Our interest in a standard-basis free algorithm for computing tangent cones comes by way of an overall goal to compute intersection multiplicities in arbitrary dimension. As mentioned in the introduction, in a previous paper [8], relying on the book of Fulton [7] and the theory of regular chains, we derived an algorithm for computing intersection multiplicities of planar curves. We also sketched an algorithm criterion, see Theorem 1 below, for reducing the computation of intersection multiplicities in arbitrary dimension to computing intersection multiplicities in lower dimension. When applicable, successive uses of this criterion reduces intersection multiplicity computation in arbitrary dimension to the bivariate case.

Theorem 1. *For $\boldsymbol{h} = h_0, \ldots, h_{n-1}, h_n \in \mathbb{K}[\boldsymbol{x}]$ such that $\mathbf{V}(h_0, \ldots, h_{n-1}, h_n)$ is zero-dimensional, for $p \in \mathbf{V}(h_n)$, if the hyper-surface $\mathbf{V}(h_n)$ is not singular at p and if that the tangent space π of $\mathbf{V}(h_n)$ at p intersects transversally*[4] *the tangent cone of the curve $\mathbf{V}(h_0, \ldots, h_{n-1})$ at p, then we have*

$$\operatorname{im}(p;\, h_0, \ldots, h_{n-1},\, h_n) = \operatorname{im}(p;\, h_0, \ldots, h_{n-1},\, \pi)\,,$$

hence, there is a polynomial map which takes $\boldsymbol{h}$ to a lower dimensional subspace while leaving the intersection multiplicity of $\mathbf{V}(\boldsymbol{h})$ at p invariant.

Checking whether this criterion is applicable, requires to compute the tangent cone of the curve $\mathbf{V}(h_0, \ldots, h_{n-1})$ at p, which motivates the present paper. This algorithmic criterion was stated in [8] without justification, although the authors had a long and technical proof available in a technical report extending [8]. In the PhD thesis of the fourth author [13], a simpler proof was obtained and we present it below.

Proof. The theorem follows from results of [12, Chapter IV]; we reuse the same notation as in that reference when feasible.

Since p is an isolated point of $\mathbf{V}(\boldsymbol{h})$, any irreducible component of $\mathbf{V}(h_0, \ldots, h_{n-1})$ through p must have dimension one. By Lemma 2 in [12, Chapter IV.1.3] it follows $\overline{\mathscr{O}}$ is a one-dimensional local ring, where

$$\overline{\mathscr{O}} := \mathscr{O} / \langle h_0, \ldots, h_{n-1} \rangle\,.$$

Let $\mathscr{C}_0, \ldots, \mathscr{C}_r$ be the irreducible components of $\mathbf{V}(h_0, \ldots, h_{n-1})$ passing through p and let $\mathfrak{p}_0, \ldots, \mathfrak{p}_r$ be their respective defining (prime) ideals in $\mathscr{O}$. Our transversality assumption ensures h_n and π are both nonzero divisors in $\overline{\mathscr{O}}$ and consequently, since $\overline{\mathscr{O}}$ is a one-dimensional local ring, we use Equation (6) from [12,

[4] Two algebraic sets V_0 and V_1 in $\mathbb{A}^{n+1}(\overline{\mathbb{K}})$ *transversally intersect* at a point $p \in V_0 \cap V_1$ whenever their tangent cones intersect at $\{p\}$ *only* once or not at all. Note that if one of V_0 is a linear space, then it is its own tangent cone at p. Note also that, for a sake of clarity, we have restricted Definition 1 to tangent cones of curves, although tangent cones of algebraic sets of higher dimension are defined similarly, see cite[4].

Chapter IV.1.3] to deduce

$$\operatorname{im}(p;\, h_0, \ldots, h_{n-1},\, h_n) = \sum_{i=0}^{r} m_i \dim_{\text{vec}}(\overline{\mathscr{O}}/\langle\, \mathfrak{p}_i,\, h_n\,\rangle) \tag{1}$$

and

$$\operatorname{im}(p;\, h_0, \ldots, h_{n-1},\, \pi) = \sum_{i=0}^{r} m_i \dim_{\text{vec}}(\overline{\mathscr{O}}/\langle\, \mathfrak{p}_i,\, \pi\,\rangle) \tag{2}$$

for some constants $m_1, \ldots, m_r$ that we need not define more precisely.

Remark 1. In the original reference the dimensions above are written as lengths but [6, Example A.1.1] permits us to use the vector space dimension instead. This holds for all the dimensions written below as well.

Because $\langle h_0, \ldots, h_{n-1}\rangle \subset \mathfrak{p}_i$ for all i, we can rewrite (1) and (2) as (resp.) $\dim_{\text{vec}}(\mathscr{O}/\langle\, \mathfrak{p}_i,\, h_n\,\rangle)$ and $\dim_{\text{vec}}(\mathscr{O}/\langle\, \mathfrak{p}_i,\, \pi\,\rangle)$. Hence it suffices to prove, exploiting that $\langle\, h_0, \ldots, h_{n-1}\,\rangle$ has been replaced by a dimension one prime ideal, that

$$\dim_{\text{vec}}(\mathscr{O}/\langle\, \mathfrak{p}_i,\, h_n\,\rangle) = \dim_{\text{vec}}(\mathscr{O}/\langle\, \mathfrak{p}_i,\, \pi\,\rangle)$$

for all $i = 1, \ldots, r$ to conclude.

Fix i for the remainder of this proof. The prime ideal $\mathfrak{p}_i$ defines a curve $\mathscr{C} \subset \overline{\mathbb{K}}^{n+1}$. Let $\mathscr{C}' \subset \overline{\mathbb{K}}^{n'+1}$ be a normalization of $\mathscr{C}$ given by $\nu : \mathscr{C}' \to \mathscr{C}$; thus $\mathscr{C}'$ is non-singular. It follows from [12, Chapter IV.1.3.(9)] that

$$\dim_{\text{vec}}(\mathscr{O}/\langle\, \mathfrak{p}_i, h_n\,\rangle) = \sum_{\nu(p')=p} \dim_{\text{vec}}(\mathscr{O}_{\mathscr{C}',p'}/h_n^*),$$

when $\mathscr{O}_{\mathscr{C}',p'}$ is the local ring of $\mathscr{C}'$ at p' and h_n^* is the *pull-back* of h_n by ν. A similar expression holds for π.

Now fix p' in the fiber $\nu^{-1}(p)$. We prove

$$\dim_{\text{vec}}(\mathscr{O}_{\mathscr{C}',p'}/h_n^*) = \dim_{\text{vec}}(\mathscr{O}_{\mathscr{C}',p'}/\pi^*).$$

Without loss of generality shift to the origin, that is, assume $p = 0 \in \overline{\mathbb{K}}^{n+1}$ and $p' = 0 \in \overline{\mathbb{K}}^{n'+1}$ and also let t be a *uniformizer* for $\mathscr{C}'$ at p' (remember that $\mathscr{C}'$ is non-singular). Finally, write $\nu = (\nu_0, \ldots, \nu_n)$, with all ν_i in $\overline{\mathbb{K}}[\mathscr{C}']$.

Expanding $\nu = (\nu_0, \ldots, \nu_n)$ in power series at the origin permits us to view them as in $\overline{\mathbb{K}}[[t]]^{n+1}$. With this in mind, and without loss of generality, assume ν_0 has the smallest valuation among $\nu_0, \ldots, \nu_n$ (otherwise, do a change of coordinates in $\overline{\mathbb{K}}^{n+1}$). Call this valuation r, so that we can write, for all i:

$$\nu_i(t) = \nu_{i,\,r}\, t^r + \nu_{i,\,r+1}\, t^{r+1} + \cdots$$

It follows the component of the $TC_{\mathbf{0}}(\mathscr{C})$ corresponding to the image $\nu(\mathscr{C}')$ around p' is the limit of secants having directions

$$\left(\frac{\nu_0(t)}{\nu_0(t)}, \frac{\nu_1(t)}{\nu_0(t)}, \ldots, \frac{\nu_n(t)}{\nu_0(t)}\right).$$

This limit is a line with direction

$$\left(1, \frac{\nu_{1,\,r}}{\nu_{0,r}} \ldots, \frac{\nu_{n,\,r}}{\nu_{n,\,r}}\right),$$

or equivalently $(\nu_{1,\,r}, \ldots, \nu_{n,\,r})$. Because we assumed p is the origin, h_n has a writing

$$h_n(x_0, \ldots, x_n) = \pi + \text{higher order terms}$$

with $\pi = h_{n,0}\, x_0 + \cdots + h_{n,n}\, x_n$; the transversality assumption implies

$$h_{n,0}\, \nu_{0,r} + \cdots + h_{n,n}\, \nu_{n,r} \neq 0.$$

Using the local parameter t, the multiplicities

$$\dim_{\text{vec}}(\mathscr{O}_{\mathscr{C}',p'}/h_n^*) \quad \text{and} \quad \dim_{\text{vec}}(\mathscr{O}_{\mathscr{C}',p'}/\pi^*)$$

can be rewritten as the respective valuations in t of h_n^* and π^*, that is, of

$$h_n(\nu_0(t), \ldots, \nu_n(t)) \quad \text{and} \quad \pi(\nu_0(t), \ldots, \nu_n(t)).$$

The latter is easy to find; it reads

$$\pi(\nu_0(t), \ldots, \nu_n(t)) =$$
$$(h_{n,0}\, \nu_{0,r} + \cdots + h_{n,n}\, \nu_{n,r})t^r + (h_{n,0}\, \nu_{0,r+1} + \cdots + h_{n,n}\, \nu_{n,r+1})t^{r+1} + \cdots.$$

Due to the shape of h_n, the former is

$$h_n(\nu_0(t), \ldots, \nu_n(t)) = (h_{n,0}\, \nu_{0,r} + \cdots + h_{n,n}\, \nu_{n,r})t^r + \text{higher order terms.}$$

Since we know $h_{n,0}\, \nu_{0,r} + \cdots + h_{n,n}\, \nu_{n,r} \neq 0$, both expressions must have the same valuation r, so we are done. □

4 Computing Tangent Lines as Limits of Secants

From now on, the coefficient field $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers and the affine space $\mathbb{A}^{n+1}(\mathbb{C})$ is endowed with both Zariski topology and the Euclidean topology. While Zariski topology is coarser than the Euclidean topology, we have the following key result (Corollary 1 in Section I.10 of Mumford's book [11]): For an irreducible algebraic set $\mathbf{V}$ and a subset $U \subseteq \mathbf{V}$ open in the Zariski topology induced on $\mathbf{V}$, the closure of U in Zariski topology and the closure of U in the Euclidean topology are both equal to $\mathbf{V}$. It follows that, for a regular chain $\boldsymbol{t} \subset \mathbb{C}[\boldsymbol{x}]$ the closure of $\mathbf{W}(\boldsymbol{t})$ in Zariski topology and the closure of $\mathbf{W}(\boldsymbol{t})$ in the Euclidean topology are equal, thus both equal to $\mathbf{V}(\mathrm{sat}(\boldsymbol{t}))$. This result provides a bridge between techniques from algebra and techniques from analysis. The authors of [2] take advantage of Mumford's result to tackle the following problem: given a regular chain $\boldsymbol{t} \subset \mathbb{C}[\boldsymbol{x}]$, compute the (non-trivial) limit points of the quasi-component of $\boldsymbol{t}$, that is, the set $\lim(\mathbf{W}(\boldsymbol{t})) := \overline{\mathbf{W}(\boldsymbol{t})} \setminus \mathbf{W}(\boldsymbol{t})$.

In the present paper, we shall obtain the lines forming the tangent cone of a space curve at a point by means of a limit computation process. And in fact, this limit computation will reduce to computing $\lim(\mathbf{W}(\boldsymbol{t}))$ for some regular chain $\boldsymbol{t}$. To this end, we start by stating the principle of our method in Section 4.1. Then, we turn this principle into an actual algorithm in Section 4.2 via an alternative characterization of a tangent cone, based on *secants*.

4.1 An Algorithmic Principle

Let $\boldsymbol{h} = \{h_0, \cdots, h_{n-1}\} \subset \mathbb{C}[\boldsymbol{x}]$ be n polynomials such that $\mathscr{C} = \mathbf{V}(\boldsymbol{h})$ is a curve, that is, a one-dimensional algebraic set. Let $p \in \mathscr{C}$ be a point. The following proposition is well-known, see Theorem 6 in Chapter 9 of [4].

Lemma 1. *A line L through p lies in the tangent cone $TC_p(\mathscr{C})$ if and only if there exists a sequence $\{q_k : k \in \mathbb{N}\}$ of points on $\mathscr{C} \setminus \{p\}$ converging to p and such that the secant line L_k containing p and q_k becomes L when q_k approaches p.*

Under some mild assumption, we derive from Lemma 1 a method for computing $TC_p(\mathscr{C})$. We assume that for each $h \in \boldsymbol{h}$, the hyper-surface $\mathbf{V}(h)$ is non-singular at p. This assumption allows us to approach the lines of $TC_p(\mathscr{C})$ with the intersection of the tangent spaces $T_q(h_0), \ldots, T_q(h_{n-1})$ when $q \in \mathscr{C}$ is an sufficiently small neighborhood of p. A more precise description follows.

For each branch of a connected component $\mathscr{D}$ through p of $\mathscr{C} = \mathbf{V}(\boldsymbol{h})$ there exists a neighborhood B about p (in the Euclidean topology) such that $\mathbf{V}(h_0), \ldots, \mathbf{V}(h_{n-1})$ are all non-singular at each $q \in (B \cap \mathscr{D}) \setminus \{p\}$. Observe also that the singular locus $\mathrm{sing}(\mathscr{D})$ contains a *finite* number of points. It follows that we can take B small enough so that $B \cap \mathrm{sing}(\mathscr{D})$ is either empty or $\{p\}$. Define

$$v(q) := T_q(h_0) \cap \cdots \cap T_q(h_{n-1}),$$

where $T_q(h_i)$ is the tangent space of $\mathbf{V}(h_i)$ at p.

Lemma 2 states that we can obtain $TC_p(\mathscr{C})$ by finding the limits of $v(q)$ as q approaches p. Since $TC_p(\mathscr{C})$ is the union of all the $TC_p(\mathscr{D})$, this yields a method for computing $TC_p(\mathscr{C})$.

Lemma 2. *The collection of limits of lines $v(q)$ as q approaches p in $(B \cap \mathscr{D}) \setminus \{p\}$ gives the tangent cone of $\mathscr{D}$ at q. That is to say*

$$TC_p(\mathscr{D}) = \lim_{q \to p} v(q) = \lim_{q \to p} T_q(h_0) \cap \cdots \cap T_q(h_{n-1}).$$

Proof. There are two cases, either

1. $\mathscr{D}$ is *smooth* at p and $B \cap \operatorname{sing}(\mathscr{D}) = \emptyset$, or
2. $\mathscr{D}$ is *singular* at p and $B \cap \operatorname{sing}(\mathscr{D}) = \{p\}$.

Case 1. Assume $q \in B \cap \mathscr{D}$ is arbitrary and observe $\mathscr{D}$ is smooth within B and thereby the tangent cone of $\mathscr{D}$ is simply the tangent space (i.e. $TC_q(\mathscr{D}) = T_q(\mathscr{D})$).

Notice $T_q(\mathscr{D})$ is a sub-vector space of $v(q)$. Indeed, let $w \in T_q(\mathscr{D})$ be any tangent vector to $\mathscr{D}$ at q. As $\mathscr{D}$ is a curve in each $\mathbf{V}(h)$ for $h \in \boldsymbol{h}$ it follows w is a vector tangent to each $\mathbf{V}(h)$ as well. Correspondingly $w \in T_q(h)$ for any $h \in \boldsymbol{h}$ and thus $w \in v(q)$.

Finally, since $h_0, \ldots, h_{n-1}$ form a local complete intersection in B, we know $v(q)$ is a one-dimensional subspace of each $T_q(h_0)$. Since $w \in T_q(h)$ for each $h \in \boldsymbol{h}$, the vector w must span this subspace. Thus, for each $q \in B \cap \mathscr{D}$, we have

$$T_q(\mathscr{D}) = T_q(h_0) \cap \cdots \cap T_q(h_{n-1})\,.$$

Taking the limit of each side of the above equality, when q approaches p and using again the fact that $\mathscr{D}$ is smooth at $q = p$, we obtain the desired result, that is, $TC_p(\mathscr{D}) = \lim_{q \to p} v(q)\,$.

Case 2. Assume $\mathscr{D} \cap B - \{p\}$ is a finite union of smooth curves $\mathscr{D}_0, \ldots, \mathscr{D}_j$. These are the smooth branches of $\mathscr{D} \cap B$ meeting at the singular point p. Each j corresponds to a unique line

$$L_j = \lim_{q \to p} v(q) \subset T_p(\mathscr{D})$$

as q approaches p *along* $\mathscr{D}_j$.

By Lemma 1 the tangent cone $TC_p(\mathscr{D})$ is the collection of limits to p of secant lines through p in $\mathscr{D}$. Such lines given by secants along $\mathscr{D}_j$ must coincide with L_j. More precisely

$$L_0 \cup \cdots \cup L_j \subset TC_p(\mathscr{D})\,.$$

Because each $\mathscr{D}_j$ is smooth there is only one secant line for each j and thereby

$$L_0 \cup \cdots \cup L_j = TC_p(\mathscr{D})$$

as desired.

4.2 Algorithm

Under a smoothness assumption (which is potentially expensive to test) Lemma 2 states a principle for computing $TC_p(\mathscr{C})$. Let us now turn this principle into a precise algorithm and relax this smoothness assumption as well. To this end, we make use of Lemma1.

Let q be a point on the curve $\mathscr{C} = \mathbf{V}(\boldsymbol{h})$ with coordinates $\boldsymbol{x}$. Further let $\widehat{pq}$ be a unit vector in the direction of $\overline{pq}$ (i.e. the line through p and q). To exploit Lemma1 we must calculate the set

$$\left\{ \lim_{\substack{q \to p \\ q \neq p}} \widehat{pq} \right\},$$

which is indeed a set because $\mathscr{C}$ may have several branches through p yielding several lines in the tangent cone $TC_p(\mathscr{C})$.

Let $\boldsymbol{t} \subset \mathbb{C}[\boldsymbol{y}][\boldsymbol{x}]$ be a zero-dimensional regular chain encoding[5] the point p, that is, such that we have $\mathbf{V}(\boldsymbol{t}) = \{p\}$. Note that the introduction of $\boldsymbol{y}$ for the coordinates of p is necessary because the "moving point" q is already using $\boldsymbol{x}$ for its own coordinates. Consider the polynomial set

$$\boldsymbol{s} = \boldsymbol{t} \cup \boldsymbol{h}.$$

and observe that the ideal $\langle\, \boldsymbol{s} \,\rangle$ is one-dimensional in the polynomial ring $\mathbb{C}[x_{n-1} \succ \cdots \succ x_0 \succ y_{n-1} \succ \cdots \succ y_0]$. Let $\{\boldsymbol{t}_0, \ldots, \boldsymbol{t}_e\} \subset \mathbb{C}[\boldsymbol{y}][\boldsymbol{x}]$ be one-dimensional regular chains forming a Kalkbrener decomposition of $\sqrt{\langle \boldsymbol{s} \rangle}$. Thus we have

$$\mathbf{V}(\boldsymbol{s}) = \overline{\mathbf{W}(\boldsymbol{t}_0)} \cup \cdots \cup \overline{\mathbf{W}(\boldsymbol{t}_e)}.$$

Computing with the normal vector $\widehat{pq}$ is unnecessary and instead we divide the vector $\overrightarrow{pq}$ by $x_n - y_n$. Since the n-th coordinate of $\frac{\overrightarrow{pq}}{x_n - y_n}$ is 1, this vector remains non-zero when q approaches p. However, this trick enables a limit computation only when $x_n - y_n$ vanishes finitely many times in $\mathbf{V}(\boldsymbol{s})$. When this is the case, the lines of the tangent cone, that not contained in the hyperplane $y_n = x_n$, can be obtained via limits of meromorphic functions (namely Puiseux series expansions) by letting $x_n \to y_n$ and using the techniques of [2]. Moreover we are ensured there is an ordering of $\boldsymbol{x}$ for which $x_n - y_n$ is regular, as we argue below.

Since the tangent cone may have lines contained in the hyperplane $y_n = x_n$, additional computations are needed to capture them. There are essentially two options:

1. Perform a random linear change of the coordinates so as to assume that, generically, $y_n = x_n$ contains no lines of $TC_p(\mathscr{C})$.
2. Compute in turn the lines not contained in the hyperplane $y_i = x_i$ for $i = 0, \ldots, n$ and remove the duplicates; indeed no lines of the tangent cone can satisfy simultaneously $y_i = x_i$ for $i = 0, \ldots, n$.

Our experiments with theses two approaches have suggest that, although the second one seems computationally more expensive, it avoids the expression swell of the first one and is practically efficient.

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[5] In practice, we may use a zero-dimensional regular chain $\boldsymbol{t} \subset \mathbb{C}[\boldsymbol{y}][\boldsymbol{x}]$ such that $\{p\} \subseteq \mathbf{V}(\boldsymbol{t}) \subseteq \mathscr{C}$ holds. Then, the following discussion will bring the tangent cone at several points of $\mathscr{C}$ instead of p only.

From now on, we focus on computing the lines of the tangent cone *not* contained in the hyperplane $y_n = x_n$. Or, equivalently, we assume that the tangent cone transversally intersects the hyperplane $y_n = x_n$.

We note that, deciding whether $x_n - y_n$ vanishes finitely many times in $\mathbf{V}(\boldsymbol{s})$ *can*, be done algorithmically by testing whether $x_n - y_n$ is regular modulo the saturated ideal of each regular chain $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_e$. The operation Regularize described in Section 2 performs this task.

Consider now $\boldsymbol{t}_j$, that is, one of the regular chains $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_e$. Thanks to the specifications of Regularize, we may assume w.l.o.g. that either $x_n - y_n$ is regular modulo $\mathrm{sat}(\boldsymbol{t}_j)$ or that $x_n - y_n \equiv 0 \bmod \mathrm{sat}(\boldsymbol{t}_j)$ holds.

Consider the latter case first. If $x_n - y_n \equiv 0 \bmod \mathrm{sat}(\boldsymbol{t}_j)$ then $\overline{\mathbf{W}(\boldsymbol{t}_j)} \subseteq \mathbf{V}(x_n - y_n)$ permits us to try to divide each component of $\overline{pq}$ by $x_{n-1} - y_{n-1}$ instead of $x_n - y_n$. A key observation is that there is $d \in [0, n]$ such that $x_d - y_d \not\equiv 0 \bmod \mathrm{sat}(\boldsymbol{t}_j)$ necessarily holds. Indeed, if $x_i - y_i \equiv 0 \bmod \mathrm{sat}(\boldsymbol{t}_j)$ would hold for all $i \in [0, n]$ then $\overline{\mathbf{W}(\boldsymbol{t}_j)} \subset \mathbf{V}(x_0 - y_0) \cap \cdots \cap \mathbf{V}(x_n - y_n)$ would hold as well. Since the $\boldsymbol{y}$ coordinates are fixed by $\boldsymbol{t}$, the algebraic set $\overline{\mathbf{W}(\boldsymbol{t}_j)}$ would be zero-dimensional—a contradiction.

Hence, up to a variable renaming, we can assume that $x_n - y_n$ is regular modulo $\mathrm{sat}(\boldsymbol{t}_j)$. Therefore, the algebraic set $\mathbf{V}(x_n - y_n) \cap \overline{\mathbf{W}(\boldsymbol{t}_j)}$ is zero-dimensional, thus, each component of $\overline{pq}$ is divisible by $x_n - y_n$, when q is close enough to p, with $q \neq p$. Define

$$m_0 = \frac{x_0 - y_0}{x_n - y_n}, \ldots, m_n = \frac{x_n - y_n}{x_n - y_n}.$$

and regard $\boldsymbol{m} = m_0, \ldots, m_n$ as new variables, that we call *slopes*, for clear reasons. Observe that the vector of coordinates $(m_0, \ldots, m_n, 1)$ is a normal vector of the secant line $\overline{pq}$. Thus, our goal is to "solve for" $\boldsymbol{m}$ when x_n approaches y_n with $(y_0, \ldots, y_n, x_0, \ldots, x_n) \in \mathbf{W}(\boldsymbol{t}_j)$.

We turn this question into one computing the limit points of a one-dimensional regular chain, so as to use the algorithm of [2]. To this end, we extend the regular chain $\boldsymbol{t}_j$ to the regular chain $M_j \subset \mathbb{C}[\boldsymbol{m}][\boldsymbol{y}][\boldsymbol{x}]$ given by

$$M_j = \boldsymbol{t}_j \cup \begin{cases} m_0(x_0 - y_0) - (x_n - y_n) \\ \vdots \\ m_n(x_n - y_n) - (x_n - y_n) \end{cases} .$$

Note that M_j is one-dimensional in this extended space and computing $\lim(\mathbf{W}(M_j))$, using the algorithm of [2], solves for $\boldsymbol{m}$ when $x_n \to y_n$ with $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{W}(\boldsymbol{t}_0)$. Therefore and finally, the desired set $\{\lim_{q \to p, q \neq p} \ \widehat{pq}\}$ is obtained as the limit points of the quasi-components of $M_0, \ldots, M_n$.

Remark 2. Observe that the process described above determines the slopes $m_0, \ldots, m_n$ as roots of the top n polynomials of zero-dimensional regular chains in the variables $m_n \succ \cdots \succ m_0 \succ x_n \succ \cdots \succ x_0 \succ y_n \cdots \succ y_0$. Performing a change of variable ordering to $\boldsymbol{x} \succ \boldsymbol{m} \succ \boldsymbol{y}$ expresses $m_0, \ldots, m_{n-1}$ as functions of the coordinates of the point p only. We consider this a more desirable output.

4.3 Equations of Tangent Cones

In the previous section, we saw ho to compute the tangent cone $TC_p(\mathscr{C})$ in the form of the slopes of vectors defining the lines of $TC_p(\mathscr{C})$. Instead, one may prefer to obtain $TC_p(\mathscr{C})$ in the form of the equations of the the lines of $TC_p(\mathscr{C})$. We explain below how to achieve this. Let S be an arbitrary point with coordinates $(X_0, \ldots, X_n)$. This point belongs to one of the lines of the tangent cone (corresponding to the branches of the curve defined by $\overline{\mathbf{W}(\boldsymbol{t}_j)}$) if and only if the vectors

$$\frac{\overrightarrow{pq}}{x_n - y_n} = \begin{pmatrix} 1 \\ m_{n-1} \\ \vdots \\ m_0 \end{pmatrix} \quad \text{and} \quad \overline{pS} = \begin{pmatrix} X_n - y_n \\ X_{n-1} - y_{n-1} \\ \vdots \\ X_0 - y_0 \end{pmatrix}$$

are collinear. That is, if and only if we have the following relations

$$\begin{cases} X_n = m_n(x_n - y_n) + y_n \\ \quad\vdots \\ X_0 = m_0(x_n - y_n) + y_0. \end{cases} \tag{3}$$

Consider a regular chain (obtained with the process described in Remark 2) thus expressing the slopes $m_0, \ldots, m_{n-1}$ as functions of the coordinates $y_0, \ldots, y_n$ of p. Let us extend this regular chain with the relations from Equation (3), so as to obtain a one-dimensional regular chain in the variables $X_n \succ \cdots \succ X_0 \succ m_{n-1} \succ \cdots \succ m_0 \succ y_n \cdots \succ y_0$. Next, we eliminate the variables $m_0, \ldots, m_{n-1}$, with the above equations. This is, indeed, legal since the only point of a line of the tangent cone where the equation $x_n = y_n$ holds is p itself. Finally, this elimination process consists simply of substituting $\frac{X_i - y_i}{x_n - y_n}$ for m_i into the equations defining $m_0, \ldots, m_n$.

4.4 Examples

The following examples illustrates our technique for computing tangent cones as limits. We write tangent cones using unions to save vertical space and to separate slope from point.

Example 2. Consider calculating the tangent cone of the fish $h = y^2 - x^2(x+1)$ at the origin. The Puiseux expansions of h at $x = 0$ in T are given by

$$\begin{cases} y = -T - \frac{1}{2}T^2 + O(T^3) \\ x = T \end{cases} \quad \text{and} \quad \begin{cases} y = T + \frac{1}{2}T^2 + O(T^3) \\ x = T \end{cases}$$

and substituting these values into $ym - x$ produces

$$\left(-\tfrac{1}{2}T^2 - T\right)m - T \;\text{ and }\; \left(\tfrac{1}{2}T^2 + T\right)m - T.$$

Call these expressions M_0 and M_1 respectively.

To find the value of m at $T = 0$ we find the Puiseux series expansions for M_0 and M_1 at $T = 0$ in U; these are respectively.

$$\begin{cases} m = -1 + \frac{1}{2}\,U - \frac{1}{4}\,U^2 + O(U^3) \\ T = U \end{cases} \quad \text{and} \quad \begin{cases} m = 1 - \frac{1}{2}\,U + \frac{1}{4}\,U^2 + O(U^3) \\ T = U \end{cases}.$$

Taking $U \to 0$ in the above produces the (expected) slopes of 1 and -1.

Fig. 2. Limiting secants along $\mathbf{V}\left(x^2 + y^2 + z^2 - 1,\, x^2 - y^2 - z\right)$.

Example 3. Consider Figure 2, i.e. secants along the curve $\boldsymbol{h} = \{x^2 + y^2 + z^2 - 1,\, x^2 - y^2 - z\} \subset \mathbb{K}[x, y, z]$ limiting to a point given by a zero dimensional regular chain $\boldsymbol{t} = \left\langle x + y,\, 2y^2 - 1,\, z\right\rangle$.

$$TC_{\boldsymbol{t}}(\boldsymbol{h}) = \begin{cases} m_1 - 1 \\ m_2 \\ m_3 \end{cases} \cup \begin{cases} 2x^2 - 1 \\ 2y^2 - 1 \\ z \end{cases}$$

or alternatively (using equations of lines instead)

$$TC_{\boldsymbol{t}}(\boldsymbol{h}) = \left\{ z \pm \frac{4x}{\sqrt{2}} + 2,\, y - x \pm \frac{2}{\sqrt{2}} \right\}.$$

Notice the slope for *four* points are encoded here. In particular the points

$$\left\{ \left(\frac{1}{\pm\sqrt{2}}, \frac{1}{\pm\sqrt{2}}, 0 \right), \left(-\frac{1}{\pm\sqrt{2}}, \frac{1}{\mp\sqrt{2}}, 0 \right) \right\}$$

have slope given by the vector $\langle 1,\, 0,\, 0\rangle$.

Fig. 3. Secants along $\mathbf{V}\left(x^2 + y^2 + z^2 - 1\right) \cap \mathbf{V}\left(x^2 - y^2 - z(z - 1)\right)$ limiting to $(0,\, 0,\, 1)$.

Example 4. Consider Figure 3, i.e. secants along the curve $\boldsymbol{h} = \{x^2 + y^2 + z^2 - 1,\, x^2 - y^2 - z(z - 1)\} \subset k[x, y, z]$ limiting to $(0,\, 0,\, 1)$

$$TC_{(0,0,1)}(\boldsymbol{h}) = \begin{cases} m_1 + m_2 \\ 2m_2^2 - 6m_2 + 3 \\ m_3 \end{cases} \cup \begin{cases} x \\ y \\ z - 1 \end{cases}$$

or alternatively (using equations of lines instead)

$$TC_{(0,0,1)}(\boldsymbol{h}) = \left\{z - 1,\, y^2 - 3x^2\right\}.$$

Notice the values of the slopes here are in the algebraic closure of the coefficient ring. In particular, they are

$$\left\{\left(\tfrac{3}{2} + \sqrt{6},\, \tfrac{3}{2} + \sqrt{6},\, 0\right),\, \left(\tfrac{3}{2} - \sqrt{6},\, \tfrac{3}{2} - \sqrt{6},\, 0\right)\right\}.$$

5 Conclusion

We presented an alternative and Gröbner-free method for calculating the tangent cone of a space curve at any of its points. In essence, this is done by simulating a limit calculation along a curve using variable elimination. From this limit we can construct each line of the tangent cone by solving for the vector of instantaneous slope along each tangents corresponding secant lines. Finally, this slope vector can be converted into equations of lines.

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