Research Article

Konrad Schrempf*

A standard form in (some) free fields: How to construct minimal linear representations

Abstract: We describe a standard form for the elements in the universal field of fractions of free associative algebras (over a *commutative* field). It is a special version of the normal form provided by Cohn and Reutenauer and enables the use of linear algebra techniques for the construction of minimal linear representations (in standard form) for the sum and the product of two elements (given in a standard form). This completes "minimal" arithmetic in free fields since "minimal" constructions for the inverse are already known. The applications are wide: linear algebra (over the free field), rational identities, computing the left gcd of two non-commutative polynomials, etc.

Keywords: free associative algebra, minimal linear representation, admissible linear system, left greatest common divisor, non-commutative polynomials

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Introduction

The embedding of the integers into the field of rational numbers is easy, whereas that of non-commutative rings (into skew fields) is not that straightforward even in special cases [1]. After Ore's construction, it took almost 40 years for many contributors to develop a more general theory [2, Chapter 7]. The embedding of the *free associative algebra* (over a *commutative* field) into a ring of quotients (non-commutative localization) is even classified as "ugly" [3].

On the other hand, there are many parallels between the ring of integers and the free associative algebra, for example, both have a *distributive factor lattice* [4] or [5, Section 3.5]. And, starting from the *normal form* (minimal linear representation) of Cohn and Reutenauer [6] of an element in the *universal field of fractions* (of the free associative algebra), we formulate a *standard form* which can be seen as a "generalized" fraction. As a reminiscence to the work with "classical" fractions (we learn at school), we call them briefly "free fractions" [7].

For an introduction to free fields, we recommend [8, Section 9.3] and with respect to linear representations in particular [9]. For details we refer to [2, Chapter 7] or [10, Section 6.4]. Since this work is only one part in a series, further information and references can be found in [11] (linear word problem, minimal inverse), [12] (polynomial factorization) and [13] (general factorization theory).

The main idea of the *standard form* is simple: instead of viewing the system matrix of a linear representation as a single "block" we transform it into a block upper triangular form with smaller diagonal blocks. If these *pivot blocks* are small enough (called "refined"), *linear* techniques can be used to eliminate *all* superfluous block rows or columns, that is, solving "local" word problems, in the "sum" or the "product" of two linear representations, eventually yielding a *minimal* linear representation. This can be accomplished with complexity $O(dn^5)$ for an

* **Corresponding author: Konrad Schrempf,** University of Vienna, Faculty for Mathematics, Oskar-Morgenstern-Platz 1, 1090, Wien, Austria, e-mail: math@versibilitas.at
ORCID: Konrad Schrempf, 0000-0001-8509-009X

alphabet with d letters and a linear representation of dimension n with pivot blocks of size $\sqrt{n}$. For the refinement of pivot blocks, we need to solve – at least in general – systems of polynomial equations. However, *linear* techniques can be used in some cases to "break" big pivot blocks into smaller ones (see Remark 5.8).

Since almost all here is rather elementary, it should be noted that it needs some efforts to dig deep enough into the theory in the background (in particular that of Cohn) to really understand what is going on. Despite the main results (mentioned in the following), there is only one non-trivial observation (formulated in Theorem 4.16): a given *refined* linear representation is minimal if *none* of the *linear* systems of equations for block row or column minimization has a solution. (In the general non-refined case nothing can be said about minimality if there is no more "linear" minimization step possible.)

In other words, while the *normal form* [6] "linearizes" the word problem in free fields [11, Section 2], the *standard form* "linearizes" (the minimization of) the sum and the product (of two elements).

Section 1 provides the basic setup, and Section 2 summarizes several (different) constructions of linear representations for the rational operations (sum, product and inverse). In a first reading, only the first two propositions are important. In Section 3, we develop the notation to be able to formulate a *standard form* in Definition 3.8. The main result is then Theorem 4.16 (or Algorithm 4.14) for the minimization in Section 4. And finally, in Section 5, some applications are mentioned. Example 5.1 can also serve as an introduction for the work (by hand) with linear representations.

The intention of this paper is to be independent of the other three papers in this series (about the free field) as far as possible and leave it to the reader, for example, to interpret a standard form of the inverse of a polynomial as its factorization (into irreducible elements). Although the idea for minimizing "polynomial" linear representations is similar to that in Section 4, [12, Algorithm 32] is only a very special case of Algorithm 4.14 and the "refinement" in the former case is trivial.

Beside the algebraic approach presented here, there are analytical methods for solving the *word problem* (or testing *rational identities* like in Example 5.1) in polynomial time by plugging in "sufficiently large" matrices [14,15]. Closely related to linear representations are *realizations* which can be "cut down" by plugging in operators [16] or "reduced" by plugging in matrices [17]. Yet another point of view is from *invariant subspaces* [18,19].

Once the rich structure of Cohn and Reutenauer's *normal form* becomes "visible", a lot can be done, transforming the rather abstract free field into an *applied*. One can use non-commutative "rational functions" just like rational numbers, respectively, "free fractions" just like "classical" fractions.

1 Preliminaries

We represent elements (in free fields) by *admissible linear systems* (Definition 1.9), which are just a special form of *linear representations* (Definition 1.4). Rational operations (scalar multiplication, addition, multiplication and inverse) can be easily formulated in terms of linear representations (Proposition 2.2).

Notation. The set of the natural numbers is denoted by $\mathbb{N} = \{1, 2,...\}$. Zero entries in matrices are usually replaced by (lower) dots to emphasize the structure of the non-zero entries unless they result from transformations where there were possibly non-zero entries before. We denote by I_n the identity matrix and Σ_n the permutation matrix that reverses the order of rows/columns (of size n), respectively, I and Σ if the size is clear from the context.

Let $\mathbb{K}$ be a *commutative* field, $\bar{\mathbb{K}}$ its algebraic closure and $X = \{x_1, x_2,..., x_d\}$ be a *finite* (non-empty) alphabet. $\mathbb{K}\langle X \rangle$ denotes the *free associative algebra* (or *free $\mathbb{K}$-algebra*) and $\mathbb{F} = \mathbb{K}(\langle X \rangle)$ its *universal field of fractions* (or "free field") [9,10]. An element in $\mathbb{K}\langle X \rangle$ is called (non-commutative or nc) *polynomial*. In our examples, the alphabet is usually $X = \{x, y, z\}$. Including the algebra of *nc rational series* we have the following chain of inclusions:

$$\mathbb{K} \subsetneqq \mathbb{K}\langle X \rangle \subsetneqq \mathbb{K}^{\mathrm{rat}}\langle\langle X \rangle\rangle \subsetneqq \mathbb{K}(\langle X \rangle) =: \mathbb{F}.$$

The *free monoid* X^* generated by X is the set of all *finite words* $x_{i_1} x_{i_2} \cdots x_{i_n}$ with $i_k \in \{1, 2,..., d\}$. An element of the alphabet is called *letter*, one of the free monoid *words*. The multiplication on X^* is the *concatenation* of

words, that is, $(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) = x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}$, with neutral element 1, the *empty word*. For detailed introductions see [20, Chapter 1] or [21, Section I.1].

Definition 1.1. (Inner Rank, Full Matrix, Hollow Matrix [5,9]) Given a matrix $A \in \mathbb{K}\langle X \rangle^{n \times n}$, the *inner rank* of A is the smallest number $m \in \mathbb{N}$ such that there exists a factorization $A = TU$ with $T \in \mathbb{K}\langle X \rangle^{n \times m}$ and $U \in \mathbb{K}\langle X \rangle^{m \times n}$. The matrix A is called *full* if $m = n$, *non-full* otherwise. It is called *hollow* if it contains a zero submatrix of size $k \times l$ with $k + l > n$.

Definition 1.2. (Associated and Stably Associated Matrices [10]) Two matrices A and B over $\mathbb{K}\langle X \rangle$ (of the same size) are called *associated* over a subring $R \subseteq \mathbb{K}\langle X \rangle$ if there exist invertible matrices P, Q over R such that $A = PBQ$. A and B (not necessarily of the same size) are called *stably associated* if $A \oplus I_p$ and $B \oplus I_q$ are associated for some unit matrices I_p and I_q. Here by $C \oplus D$ we denote the diagonal sum $\begin{bmatrix} C & . \\ . & D \end{bmatrix}$.

Lemma 1.3. [10, Corollary 6.3.6] *A linear square matrix over $\mathbb{K}\langle X \rangle$ which is not full is associated over $\mathbb{K}$ with a linear hollow matrix.*

Definition 1.4. (Linear Representations, Dimension, Rank [6,9]) Let $f \in \mathbb{F}$. A *linear representation* of f is a triple $\pi_f = (u, A, v)$ with $u \in \mathbb{K}^{1 \times n}$, full $A = A_0 \otimes 1 + A_1 \otimes x_1 + \cdots + A_d \otimes x_d$, that is, A is invertible over $\mathbb{F}$, $\forall \ell \, A_\ell \in \mathbb{K}^{n \times n}, v \in \mathbb{K}^{n \times 1}$ and $f = uA^{-1}v$. The *dimension* of π_f is $\dim(u, A, v) = n$. It is called *minimal* if A has the smallest possible dimension among all linear representations of f. The "empty" representation $\pi = (,,)$ is the minimal one of $0 \in \mathbb{F}$ with $\dim \pi = 0$. Let $f \in \mathbb{F}$ and π be a *minimal* linear representation of f. Then the rank of f is defined as rank $f = \dim \pi$.

Remark. Cohn and Reutenauer defined linear representations slightly more generally, namely, $f = c + uA^{-1}v$ with possibly non-zero $c \in \mathbb{K}$ and call it *pure* when $c = 0$. Two linear representations are called *equivalent* if they represent the same element [9]. Two (pure) linear representations (u, A, v) and $(\tilde{u}, \tilde{A}, \tilde{v})$ of dimension n are called *isomorphic* if there exist invertible matrices $P, Q \in \mathbb{K}^{n \times n}$ such that $u = \tilde{u}Q$, $A = P\tilde{A}Q$ and $v = P\tilde{v}$ [9].

Theorem 1.5. [9, Theorem 1.4] *If $\pi' = (u', A', v')$ and $\pi'' = (u'', A'', v'')$ are equivalent (pure) linear representations, of which the first is minimal, then the second is isomorphic to a representation $\pi = (u, A, v)$ which has the block decomposition*

$$u = \begin{bmatrix} . & u' & * \end{bmatrix}, \quad A = \begin{bmatrix} * & * & * \\ . & A' & * \\ . & . & * \end{bmatrix} \quad and \quad v = \begin{bmatrix} * \\ v' \\ . \end{bmatrix}.$$

Remark. In principle, given π'' of dimension n, one can look for invertible matrices P, Q such that $P\pi''Q = (u''Q, PA''Q, Pv'')$ has the form of π to minimize π''. However, for an alphabet with d letters and a lower left block of zeros of size $k \times (n - k)$ we would get $(d + 1)k(n - k) + k$ polynomial equations with at most quadratic terms and two equations of degree n (to ensure invertibility of the transformation matrices) in $2n^2$ *commuting* unknowns. This is already rather challenging for $n = 5$. The goal is therefore to use linear techniques as far as possible.

Definition 1.6. (Left and Right Families [6]) Let $\pi = (u, A, v)$ be a linear representation of $f \in \mathbb{F}$ of dimension n. The families $(s_1, s_2, \ldots, s_n) \subseteq \mathbb{F}$ with $s_i = (A^{-1}v)_i$ and $(t_1, t_2, \ldots, t_n) \subseteq \mathbb{F}$ with $t_j = (uA^{-1})_j$ are called *left family* and *right family*, respectively. $L(\pi) = \text{span}\{s_1, s_2, \ldots, s_n\}$ and $R(\pi) = \text{span}\{t_1, t_2, \ldots, t_n\}$ denote their linear spans (over $\mathbb{K}$).

Proposition 1.7. [6, Proposition 4.7] *A representation $\pi = (u, A, v)$ of an element $f \in \mathbb{F}$ is minimal if and only if both the left family and the right family are $\mathbb{K}$-linearly independent. In this case, $L(\pi)$ and $R(\pi)$ depend only on f.*

Definition 1.8. (Element Types [13]) An element $f \in \mathbb{F}$ is called of type $(1, *)$ (respectively, $(0, *)$) if $1 \in R(f)$, that is, $1 \in R(\pi)$ for some *minimal* linear representation π of f (respectively, $1 \notin R(f)$). It is called *of type* $(*, 1)$ (respectively, $(*, 0)$) if $1 \in L(f)$ (respectively, $1 \notin L(f)$). Both subtypes can be combined.

Remark. The following definition is a special case of Cohn's more general *admissible systems* [2, Section 7] and the slightly more general *linear representations* [6].

Definition 1.9. (Admissible Linear Systems [11]) A linear representation $\mathcal{A} = (u, A, v)$ of $f \in \mathbb{F}$ is called *admissible linear system* (ALS) for f if $u = e_1 = [1, 0, \ldots, 0]$. The element f is then the first component of the (unique) solution vector $s = A^{-1}v$. An ALS is also written as $As = v$, or if $v = [0, \ldots, 0, \lambda]$ as $(1, A, \lambda)$. Given a linear representation $\mathcal{A} = (u, A, v)$ of dimension n of $f \in \mathbb{F}$ and invertible matrices $P, Q \in \mathbb{K}^{n \times n}$, the transformed $P\mathcal{A}Q = (uQ, PAQ, Pv)$ is again a linear representation (of f). If $\mathcal{A}$ is an ALS, the transformation (P, Q) is called *admissible* if the first row of Q is $e_1 = [1, 0, \ldots, 0]$.

Remark. The left family $(A^{-1}v)_i$ (respectively, the right family $(uA^{-1})_j$) and the solution vector s of $As = v$ (respectively, t of $u = tA$) are used synonymously.

Transformations can be done by elementary row- and column operations, explained in detail in [11, Remark 1.12]. For further remarks and connections to the related concepts of linearization and realization see [11, Section 1].

For elements in the free associative algebra $\mathbb{K}\langle X \rangle$ a special form (with an upper unitriangular system matrix) can be used. It plays a crucial role in the factorization of polynomials because it allows us to formulate a minimal polynomial multiplication (Proposition 2.10) and *upper unitriangular* transformation matrices (invertible by definition) suffice to find all possible factors (up to trivial units). For details we refer to [12, Section 2].

Remark 1.10. While it was intended in [12] to derive a "better" standard form including knowledge about the factorization it turned out later that this is not that easy in the general case because of the necessity to distinguish several cases in the multiplication [13, Section 5]. The term "pre-standard ALS" (for polynomials) is now replaced by *polynomial ALS* (which is just a special form of a *refined* ALS, Section 3). And a *minimal* polynomial ALS is already in a *standard form*. Particularly to avoid confusion with special transformation matrices for the factorization, everything is put into a uniform context in [22]. For an overview see also [7, Figure 1]. There are only some minor changes in the formulation of results and proofs necessary, for example, to construct the product in [12, Lemma 39] using $\frac{\lambda_2}{\lambda}\mathcal{A}_1$ and $\frac{\lambda}{\lambda_2}\mathcal{A}_2$.

Definition 1.11. (Polynomial ALS and Transformation [12]) An ALS $\mathcal{A} = (u, A, v)$ of dimension n with system matrix $A = (a_{ij})$ for a non-zero polynomial $0 \neq p \in \mathbb{K}\langle X \rangle$ is called *polynomial*, if
(1) $v = [0, \ldots, 0, \lambda]^{\mathsf{T}}$ for some $\lambda \in \mathbb{K}$ and
(2) $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i > j$, that is, A is upper triangular.

An admissible transformation (P, Q) for an ALS $\mathcal{A}$ is called *polynomial* if it has the form

$$(P, Q) = \left(\begin{bmatrix} 1 & \alpha_{1,2} & \cdots & \alpha_{1,n-1} & \alpha_{1,n} \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & \alpha_{n-2,n-1} & \alpha_{n-2,n} \\ & & & 1 & \alpha_{n-1,n} \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & \beta_{2,3} & \cdots & \beta_{2,n} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & \beta_{n-1,n} \\ & & & & 1 \end{bmatrix} \right).$$

If additionally $\alpha_{1,n} = \alpha_{2,n} = \ldots = \alpha_{n-1,n} = 0$, then (P, Q) is called *polynomial factorization transformation*.

Definition 1.12. Let $M = M_1 \otimes x_1 + \cdots + M_d \otimes x_d$ with $M_i \in \mathbb{K}^{n \times n}$ for some $n \in \mathbb{N}$. An element in $\mathbb{F}$ is called *regular* if it has a linear representation (u, A, v) with $A = I - M$, that is, $A_0 = I$ in Definition 1.4, or equivalently, if A_0 is regular (invertible).

2 Rational operations

Usually, we want to construct *minimal* admissible linear systems (out of minimal ones), that is, perform "minimal" rational operations. Minimal scalar multiplication is trivial. In some special cases, *minimal multiplication* or even *minimal addition* (if two elements are disjoint [9]) can be formulated (Proposition 2.10 or [13, Theorem 5.2] and [13, Proposition 3.5]). For the *minimal inverse*, we have to distinguish four cases, which are summarized in Theorem 2.11. In general, we have to *minimize* admissible linear systems. This will be the main goal of the following sections.

Proposition 2.1. (Minimal Monomial [11, Proposition 4.1]) *Let $k \in \mathbb{N}$ and $f = x_{i_1} x_{i_2} \dots x_{i_k}$ be a monomial in $\mathbb{K}\langle X \rangle \subseteq \mathbb{K}(\langle X \rangle)$. Then*

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . & \cdots & . \end{bmatrix}, \begin{bmatrix} 1 & -x_{i_1} & & & \\ & 1 & -x_{i_2} & & \\ & & \ddots & \ddots & \\ & & & 1 & -x_{i_k} \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ \vdots \\ . \\ 1 \end{bmatrix} \right)$$

is a minimal polynomial ALS of dimension $\dim(\mathcal{A}) = k + 1$.

Proposition 2.2. (Rational Operations [9]) *Let $0 \neq f, g \in \mathbb{F}$ be given by the admissible linear systems $\mathcal{A}_f = (u_f, A_f, v_f)$ and $\mathcal{A}_g = (u_g, A_g, v_g)$, respectively, and let $0 \neq \mu \in \mathbb{K}$. Then admissible linear systems for the rational operations can be obtained as follows:*
The scalar multiplication μf is given by

$$\mu \mathcal{A}_f = (u_f, A_f, \mu v_f).$$

The sum $f + g$ is given by

$$\mathcal{A}_f + \mathcal{A}_g = \left(\begin{bmatrix} u_f & . \end{bmatrix}, \begin{bmatrix} A_f & -A_f u_f^\top u_g \\ . & A_g \end{bmatrix}, \begin{bmatrix} v_f \\ v_g \end{bmatrix} \right).$$

The product fg is given by

$$\mathcal{A}_f \cdot \mathcal{A}_g = \left(\begin{bmatrix} u_f & . \end{bmatrix}, \begin{bmatrix} A_f & -v_f u_g \\ . & A_g \end{bmatrix}, \begin{bmatrix} . \\ v_g \end{bmatrix} \right).$$

And the inverse f^{-1} is given by

$$\mathcal{A}_f^{-1} = \left(\begin{bmatrix} 1 & . \end{bmatrix}, \begin{bmatrix} -v_f & A_f \\ . & u_f \end{bmatrix}, \begin{bmatrix} . \\ 1 \end{bmatrix} \right).$$

Since we need alternative constructions (to that in Proposition 2.2) for the product, we state them here in Propositions 2.6 and 2.7. Before, we need some technical results from [11] and [12]. However, these are rearranged such that similarities become more obvious and the flexibility in applications is increased. In particular, one can prove Lemma 2.4 by applying Lemma 2.3, see [13, Lemma 3.8].

Lemma 2.3. [12, Lemma 25]. *Let $\mathcal{A} = (u, A, v)$ be an ALS of dimension $n \geq 1$ with $\mathbb{K}$-linearly independent left family $s = A^{-1} v$ and $B = B_0 \otimes 1 + B_1 \otimes x_1 + \cdots + B_d \otimes x_d$ with $B_\ell \in \mathbb{K}^{m \times n}$, such that $Bs = 0$. Then there exists a (unique) $T \in \mathbb{K}^{m \times n}$ such that $B = TA$.*

Lemma 2.4. (For Type $(*, 1)$ [11, Lemma 4.11]) *Let $\mathcal{A} = (u, A, v)$ be a minimal ALS with $\dim \mathcal{A} = n \geq 2$ and $1 \in L(\mathcal{A})$. Then there exists an admissible transformation (P, Q) such that the last row of PAQ is $[0, \dots, 0, 1]$ and $Pv = [0, \dots, 0, \lambda]^\top$ for some $\lambda \in \mathbb{K}$.*

Lemma 2.5. (For Type $(1, *)$ [11, Lemma 4.12]) *Let $\mathcal{A} = (u, A, v)$ be a minimal ALS with* $\dim \mathcal{A} = n \geq 2$ *and* $1 \in R(\mathcal{A})$. *Then there exists an admissible transformation (P, Q) such that the first column of PAQ is* $[1, 0, \ldots, 0]^\top$ *and $Pv = [0, \ldots, 0, \lambda]^\top$ for some $\lambda \in \mathbb{K}$.*

Remark. If g is of type $(*, 1)$, then, by Lemma 2.4, *each* minimal ALS for g can be transformed into one with the last row of the form $[0, \ldots, 0, 1]$. If g is of type $(1, *)$, then, by Lemma 2.5, *each* minimal ALS for g can be transformed into one with the first column of the form $[1, 0, \ldots, 0]^\top$. This can be done by *linear* techniques, see the remark before [11, Theorem 4.13].

Remark. Since $p \in \mathbb{K}\langle X \rangle$ is of type $(1, 1)$, both constructions can be used for the minimal polynomial multiplication (Proposition 2.10). The alternative proof in [13] relies in particular on Lemma 2.9. One could call the multiplication from Proposition 2.2 type $(*, *)$. For a discussion of *minimality* of different types of multiplication we refer to [13].

Proposition 2.6. (Multiplication Type $(1, *)$ [13, Proposition 3.11]) *Let $f, g \in \mathbb{F} \setminus \mathbb{K}$ be given by the admissible linear systems $\mathcal{A}_f = (u_f, A_f, v_f) = (1, A_f, \lambda_f)$ of dimension n_f of the form*

$$\mathcal{A}_f = \left(\begin{bmatrix} 1 & . & . \end{bmatrix}, \begin{bmatrix} a & b' & b \\ a' & B & b'' \\ . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ \lambda_f \end{bmatrix} \right)$$

and $\mathcal{A}_g = (u_g, A_g, v_g) = (1, A_g, \lambda_g)$ of dimension n_g, respectively. Then an ALS for fg of dimension $n = n_f + n_g - 1$ is given by

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . & . \end{bmatrix}, \begin{bmatrix} a & b' & \lambda_f b u_g \\ a' & B & \lambda_f b'' u_g \\ . & . & A_g \end{bmatrix}, \begin{bmatrix} . \\ . \\ v_g \end{bmatrix} \right).$$

Proposition 2.7. (Multiplication Type $(*, 1)$ [13, Proposition 3.12]) *Let $f, g \in \mathbb{F} \setminus \mathbb{K}$ be given by the admissible linear systems $\mathcal{A}_f = (u_f, A_f, v_f) = (1, A_f, \lambda_f)$ of dimension n_f and $\mathcal{A}_g = (u_g, A_g, v_g) = (1, A_g, \lambda_g)$ of dimension n_g of the form*

$$\mathcal{A}_g = \left(\begin{bmatrix} 1 & . & . \end{bmatrix}, \begin{bmatrix} 1 & b' & b \\ . & B & b'' \\ . & c' & c \end{bmatrix}, \begin{bmatrix} . \\ . \\ \lambda_g \end{bmatrix} \right),$$

respectively. Then an ALS for fg of dimension $n = n_f + n_g - 1$ is given by

$$\mathcal{A} = \left(\begin{bmatrix} u_f & . & . \end{bmatrix}, \begin{bmatrix} A_f & e_{n_f} \lambda_f b' & e_{n_f} \lambda_f b \\ . & B & b'' \\ . & c' & c \end{bmatrix}, \begin{bmatrix} . \\ . \\ \lambda_g \end{bmatrix} \right).$$

Remark. Note that the transformation in the following lemma is *not* necessarily admissible. However, except for $n = 2$ (which can be treated by permuting the last two elements in the left family), it can be chosen such that it is *admissible*. The proof is similar to that of [12, Lemma 2.4].

Lemma 2.8. [13, Lemma 3.15] *Let $\mathcal{A} = (u, A, v)$ be an ALS of dimension $n \geq 2$ with $v = [0, \ldots, 0, \lambda]^\top$ and $\mathbb{K}$-linearly dependent left family $s = A^{-1}v$. Let $m \in \{2, 3, \ldots, n\}$ be the minimal index such that the left subfamily $\underline{s} = (A^{-1}v)_{i=m}^n$ is $\mathbb{K}$-linearly independent. Let $A = (a_{ij})$ and assume that $a_{ii} = 1$ for $1 \leq i \leq m$ and $a_{ij} = 0$ for $j < i \leq m$ (upper triangular $m \times m$ block) and $a_{ij} = 0$ for $j \leq m < i$ (lower left zero block of size $(n - m) \times m$). Then there exists matrices $T, U \in \mathbb{K}^{1 \times (n+1-m)}$ such that*

$$U + (a_{m-1,j})_{j=m}^n - T(a_{ij})_{i,j=m}^n = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \quad \text{and} \quad T(v_i)_{i=m}^n = 0.$$

Lemma 2.9. [13, Lemma 3.16]. *Let $p \in \mathbb{K}\langle X\rangle \backslash \mathbb{K}$ and $g \in \mathbb{F} \backslash \mathbb{K}$ be given by the minimal admissible linear systems $\mathcal{A}_p = (u_p, A_p, v_p)$ and $\mathcal{A}_g = (u_g, A_g, v_g)$ of dimensions n_p and n_g, respectively, with $1 \in R(g)$. Then the left family of the admissible linear systems $\mathcal{A} = (u, A, v)$ for pg of dimension $n = n_p + n_g - 1$ from Proposition 2.7 is $\mathbb{K}$-linearly independent.*

Proposition 2.10. (Minimal Polynomial Multiplication [12, Proposition 26]) *Let $0 \neq p, q \in \mathbb{K}\langle X\rangle$ be given by the minimal polynomial admissible linear systems $\mathcal{A}_p = (1, A_p, \lambda_p)$ and $\mathcal{A}_q = (1, A_q, \lambda_q)$ of dimension $n_p, n_q \geq 2$, respectively. Then the ALS $\mathcal{A}$ from Proposition 2.6 for pq is minimal of dimension $n = n_p + n_q - 1$.*

Theorem 2.11. (Minimal Inverse [11, Theorem 4.13]) *Let $f \in \mathbb{F} \backslash \mathbb{K}$ be given by the minimal admissible linear system $\mathcal{A} = (u, A, v)$ of dimension n. Then a minimal ALS for f^{-1} is given in the following way:*
f of type $(1, 1)$ yields f^{-1} of type $(0, 0)$ with $\dim(\mathcal{A}') = n - 1$:

$$\mathcal{A}' = \left(1, \begin{bmatrix} -\lambda\Sigma b'' & -\Sigma B\Sigma \\ -\lambda b & -b'\Sigma \end{bmatrix}, 1\right) \quad \text{for} \quad \mathcal{A} = \left(1, \begin{bmatrix} 1 & b' & b \\ . & B & b'' \\ . & . & 1 \end{bmatrix}, \lambda\right).$$

f of type $(1, 0)$ yields f^{-1} of type $(1, 0)$ with $\dim(\mathcal{A}') = n$:

$$\mathcal{A}' = \left(1, \begin{bmatrix} 1 & -\frac{1}{\lambda}c & -\frac{1}{\lambda}c'\Sigma \\ . & -\Sigma b'' & -\Sigma B\Sigma \\ . & -b & -b'\Sigma \end{bmatrix}, 1\right) \quad \text{for} \quad \mathcal{A} = \left(1, \begin{bmatrix} 1 & b' & b \\ . & B & b'' \\ . & c' & c \end{bmatrix}, \lambda\right).$$

f of type $(0, 1)$ yields f^{-1} of type $(0, 1)$ with $\dim(\mathcal{A}') = n$:

$$\mathcal{A}' = \left(1, \begin{bmatrix} -\lambda\Sigma b'' & -\Sigma B\Sigma & -\Sigma a' \\ -\lambda b & -b'\Sigma & -a \\ . & . & 1 \end{bmatrix}, 1\right) \quad \text{for} \quad \mathcal{A} = \left(1, \begin{bmatrix} a & b' & b \\ a' & B & b'' \\ . & . & 1 \end{bmatrix}, \lambda\right).$$

f of type $(0, 0)$ yields f^{-1} of type $(1, 1)$ with $\dim(\mathcal{A}') = n + 1$:

$$\mathcal{A}' = \left(1, \begin{bmatrix} \Sigma v & -\Sigma A\Sigma \\ . & u\Sigma \end{bmatrix}, 1\right).$$

(Recall that the permutation matrix Σ reverses the order of rows/columns.)

Corollary 2.12. *Let $p \in \mathbb{K}\langle X\rangle$ with $\operatorname{rank} p = n \geq 2$. Then $\operatorname{rank}(p^{-1}) = n - 1$.*

Corollary 2.13. *Let $0 \neq f \in \mathbb{F}$. Then $f \in \mathbb{K}$ if and only if $\operatorname{rank}(f) = \operatorname{rank}(f^{-1}) = 1$.*

Remark. Note that $n \geq 2$ for type $(1, 1), (1, 0)$ and $(0, 1)$. The block B is always square of size $n - 2$. For $n = 2$, the system matrix of $\mathcal{A}$ is

$$\begin{bmatrix} 1 & b \\ . & 1 \end{bmatrix} \text{ for type } (1, 1), \quad \begin{bmatrix} 1 & b \\ . & c \end{bmatrix} \text{ for type } (1, 0) \quad \text{and} \quad \begin{bmatrix} a & b \\ . & 1 \end{bmatrix} \text{ for type } (0, 1).$$

3 A standard form

After providing a "language" to be able to formulate "operations" on the system matrix of an admissible linear system, we can define a *standard form* (Definition 3.8). A standard ALS will be *minimal* and (has) *refined* (pivot blocks). In the following section, we minimize a refined ALS. This is somewhat technical to describe but simple (we need to solve *linear* systems of equations). At the end of this section, we illustrate how to refine pivot blocks. To the contrary, this is easy to describe but in general very difficult to accomplish (we need to solve *polynomial* systems of equations). Since the latter is closely related to factorization, we refer to [12] and [13] for further information.

Definition 3.1. (Pivot Blocks, Block Transformation) Let $\mathcal{A} = (u, A, v)$ be an admissible linear system and denote $A = (A_{ij})_{i,j=1}^m$ the block decomposition (with square diagonal blocks A_{ii}) with *maximal m* such that $A_{ij} = 0$ for $i > j$. The diagonal blocks A_{ii} are called *pivot blocks*, the number m is denoted by $\#_{\mathrm{pb}}(\mathcal{A}) = \#_{\mathrm{pb}}(A)$. The *dimension* (or *size*) of a pivot block A_{ii} for $i \in \{1, 2,\ldots, m\}$ is $\dim_i(\mathcal{A})$. For $i < 1$ or $i > m$ let $\dim_i(\mathcal{A}) = 0$. A (admissible) transformation (P, Q) is called (admissible) *block transformation* (for $\mathcal{A}$) if $P_{ij} = Q_{ij} = 0$ for $i > j$ (with respect to the block structure of $\mathcal{A}$).

Notation. Let $\mathcal{A} = (u, A, v)$ be an ALS with m pivot blocks of size $n_i = \dim_i(\mathcal{A})$. We denote by $n_{i:j} = \dim_{i:j}(\mathcal{A}) = n_i + n_{i+1} + \cdots + n_j$ the sum of the sizes of the pivot blocks A_{ii} to A_{jj} (with the convention $n_{i:j} = 0$ for $j < i$). For a given system, the identity matrix of size $n_{i:j}$ is denoted by $I_{i:j}$. If (P, Q) is an admissible transformation for $\mathcal{A}$, then $(PAQ)_{ij}$ denotes the (to the block decomposition of A corresponding) block (i, j) of size $n_i \times n_j$ in PAQ, $(Pv)_i$ that of size $n_i \times 1$ in Pv and $(uQ)_j$ that of size $1 \times n_j$ in uQ.

Notation. Components in the left family $s = A^{-1}v$ are (as usual) denoted by s_i. The jth component for $1 \le j \le \dim_i(\mathcal{A})$ of the ith block for $1 \le i \le \#_{\mathrm{pb}}(\mathcal{A})$ is denoted by $s_{i(j)}$. A subfamily of s with respect to the pivot block k is denoted by $s_{\underline{k}}$, $s_{i:j} = (s_i, s_{i+1},\ldots, s_j)$. Analogous is used for the right family $t = uA^{-1}$.

Notation. A "grouping" of pivot blocks $\{i, i + 1,\ldots, j\}$ of the system matrix is denoted by $A_{i:j,i:j}$. If it is clear from the context where the block ends (respectively, starts), we write $A_{i:,i:}$ (respectively, $A_{:j,:j}$), in particular with respect to a given pivot block. For example, $A_{1:,1:}$, $A_{k,k}$ and $A_{:m,:m}$.

Definition 3.2. (Admissible Pivot Block Transformation) Let $\mathcal{A} = (u, A, v)$ be an ALS with $m = \#_{\mathrm{pb}}(\mathcal{A})$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. An admissible transformation (P, Q) of the form $(I_{1:k-1} \oplus \bar{T} \oplus I_{k+1:m}, I_{1:k-1} \oplus \bar{U} \oplus I_{k+1:m})$ with $\bar{T}, \bar{U} \in \mathbb{K}^{n_k \times n_k}$ is called *admissible for pivot block k* or (admissible) *k-th pivot block transformation*.

Definition 3.3. (Refined Pivot Block and Refined ALS) Let $\mathcal{A} = (u, A, v)$ be an ALS with $m = \#_{\mathrm{pb}}(\mathcal{A})$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. A pivot block A_{kk} (for $1 \le k \le m$) is called *refined* if there does not exist an admissible pivot block transformation $(P, Q)_k$ such that $(PAQ)_{kk}$ has a lower left block of zeros of size $i \times (n_k - i)$ for an $i \in \{1, 2,\ldots, n_k - 1\}$. The admissible linear system $\mathcal{A}$ is called *refined* if all pivot blocks are refined.

Remarks. That the "form" of a refined ALS is not unique, it will be illustrated in the following example. Moreover, a refined ALS is not necessarily refined over $\bar{\mathbb{K}}$: Let

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . \end{bmatrix}, \begin{bmatrix} 1 & x \\ 2x & 1 \end{bmatrix}, \begin{bmatrix} . \\ 1 \end{bmatrix} \right).$$

Adding $\sqrt{2}$-times row 1 to row 2 and subtracting $\sqrt{2}$-times column 2 of column 1 yields

$$\begin{bmatrix} 1 - \sqrt{2}x & x \\ 0 & 1 + \sqrt{2}x \end{bmatrix} s = \begin{bmatrix} . \\ 1 \end{bmatrix}.$$

See also [12, Example 37].

Example 3.4. Let $f = (y^{-1} - x)^{-1}$ be given by the minimal ALS

$$\mathcal{A}_f = \left(\begin{bmatrix} 1 & . \end{bmatrix}, \begin{bmatrix} 1 & -y \\ -x & 1 \end{bmatrix}, \begin{bmatrix} . \\ 1 \end{bmatrix} \right).$$

For $f + 3z$, we have the (minimal) ALS

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} 1 & -y & -1 & . \\ -x & 1 & x & . \\ . & . & 1 & -z \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ 1 \\ . \\ 3 \end{bmatrix} \right).$$

Since $1 \in R(\mathcal{A})$ and $\mathcal{A}$ is constructed by the addition in Proposition 2.2, it can easily be transformed – in a controlled way – into another ALS with *refined* pivot block structure. First, we add column 3 to column 1,

$$\mathcal{A}' = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} 0 & -y & -1 & . \\ 0 & 1 & x & . \\ 1 & . & 1 & -z \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ 1 \\ . \\ 3 \end{bmatrix} \right),$$

and then we exchange rows 1 and 3:

$$\mathcal{A}'' = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} 1 & . & 1 & -z \\ . & 1 & x & . \\ . & -y & -1 & . \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ 1 \\ . \\ 3 \end{bmatrix} \right).$$

Definition 3.5. (Block Decomposition of an ALS, Block Row and Column Transformation) Let $\mathcal{A} = (u, A, v)$ be an ALS of dimension n with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. For $1 \leq k \leq m$, the *block decomposition* with respect to the pivot block k is the system

$$\mathcal{A}^{[\underline{k}]} = \left(\begin{bmatrix} u_{\underline{1}:} & . & . \end{bmatrix}, \begin{bmatrix} A_{1:,1:} & A_{1:,k} & A_{1:,:m} \\ . & A_{k,k} & A_{k,:m} \\ . & . & A_{:m,:m} \end{bmatrix}, \begin{bmatrix} v_{\underline{1}:} \\ v_{\underline{k}} \\ v_{:\underline{3}} \end{bmatrix} \right)$$

with (square) diagonal blocks $A_{1:,1:}$, $A_{k,k}$ and $A_{:m,:m}$ of size $n_{1:k-1}$, n_k respectively $n_{k+1:m}$. ($\underline{k}$ is used here to emphasize that k is a block index.) By $\mathcal{A}^{[-\underline{k}]}$ we denote the ALS $\mathcal{A}^{[\underline{k}]}$ without block row/column k of dimension $n - n_k$ (not necessarily equivalent to $\mathcal{A}^{[\underline{k}]}$):

$$\mathcal{A}^{[-\underline{k}]} = \left(\begin{bmatrix} u_{\underline{1}} & . \end{bmatrix}, \begin{bmatrix} A_{1:,1:} & A_{1:,:m} \\ . & A_{:m,:m} \end{bmatrix}, \begin{bmatrix} v_{\underline{1}} \\ v_{\underline{3}} \end{bmatrix} \right).$$

An admissible transformation $(P, Q)_{\underline{k}} = (P(\bar{T}, T), Q(\bar{U}, U))_{\underline{k}}$ of the form

$$(P, Q)_{\underline{k}} = \left(\begin{bmatrix} I_{1:k-1} & . & . \\ . & \bar{T} & T \\ . & . & I_{k+1:m} \end{bmatrix}, \begin{bmatrix} I_{1:k-1} & . & . \\ . & \bar{U} & U \\ . & . & I_{k+1:m} \end{bmatrix} \right) \tag{3.6}$$

is called kth *block row transformation* for $\mathcal{A}^{[\underline{k}]}$, one of the form $(P(\bar{T}, T), Q(\bar{U}, U))^{\underline{k}}$,

$$(P, Q)^{\underline{k}} = \left(\begin{bmatrix} I_{1:k-1} & T & . \\ . & \bar{T} & . \\ . & . & I_{k+1:m} \end{bmatrix}, \begin{bmatrix} I_{1:k-1} & U & . \\ . & \bar{U} & . \\ . & . & I_{k+1:m} \end{bmatrix} \right) \tag{3.7}$$

is called kth *block column transformation* for $\mathcal{A}^{[\underline{k}]}$. For $\bar{T} = \bar{U} = I_{n_k}$ we write also $P(T)$, respectively, $Q(U)$ and call the block transformation *particular*.

Definition 3.8. (Standard Admissible Linear System) A *minimal* and *refined* ALS $\mathcal{A} = (u, A, v) = (1, A, \lambda)$, that is, $v = [0, \ldots, 0, \lambda]$, is called *standard*.

Remark. For a polynomial p given by a *standard* ALS $\mathcal{A}$ (of dimension $n \geq 2$), the minimal inverse of $\mathcal{A}$ (of dimension $n - 1$) is refined if and only if $\mathcal{A}$ is obtained by the minimal polynomial multiplication of its irreducible factors q_i in $p = q_1 q_2 \cdots q_m$. For a detailed discussion about the factorization of polynomials (in free associative algebras) we refer to [12]. One of the simplest non-trivial examples is

$$p = x(1 - yx) = x - xyx = (1 - xy)x.$$

With respect to the general factorization theory it is open to show that the free field is a "similarity unique factorization domain" [13].

Example 3.9. (Factorization versus Refinement) Later, in Example 5.1 (Hua's identity), we need to refine a pivot block. Although the necessary transformations are obvious, the procedure should be illustrated in a

systematic way. But first we have a look on how this 2×2 block appears, namely, by inverting the element given by the ALS

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . & . \end{bmatrix}, \begin{bmatrix} x & 0 & 1 \\ . & 1 & y \\ . & x & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ . \\ -1 \end{bmatrix} \right).$$

If we exchange columns 2 and 3, it is immediate that this is the "product" of the admissible linear systems

$$\mathcal{A}_1 = \left([1], [x], [-1] \right) \text{ and } \mathcal{A}_2 = \left(\begin{bmatrix} 1 & . \end{bmatrix}, \begin{bmatrix} y & 1 \\ 1 & x \end{bmatrix}, \begin{bmatrix} . \\ -1 \end{bmatrix} \right).$$

Applying the minimal inverse on

$$\mathcal{A}' = \left(\begin{bmatrix} 1 & . & . \end{bmatrix}, \begin{bmatrix} x & 1 & 0 \\ . & y & 1 \\ . & 1 & x \end{bmatrix}, \begin{bmatrix} 0 \\ . \\ -1 \end{bmatrix} \right),$$

we get a *refined* (and minimal) ALS, namely,

$$\mathcal{A}'' = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} -1 & -x & -1 & . \\ . & -1 & -y & . \\ . & . & -1 & -x \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix} \right).$$

The factorization here is really simple.

Example 3.10. (Pivot Block Refinement) Now we focus on the refinement of the second pivot block in the ALS

$$\mathcal{A} = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} 1 & 1 & x & . \\ . & y & 1 & . \\ . & 1 & 0 & x \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix} \right).$$

(This is the ALS (5.3) with the first three rows scaled by -1.) We are looking for an admissible transformation (P, Q) of the form

$$(P, Q) = \left(\begin{bmatrix} 1 & . & . & . \\ . & \alpha_{2,2} & \alpha_{2,3} & . \\ . & \alpha_{3,2} & \alpha_{3,3} & . \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} 1 & . & . & . \\ . & \beta_{2,2} & \beta_{2,3} & . \\ . & \beta_{3,2} & \beta_{3,3} & . \\ . & . & . & 1 \end{bmatrix} \right).$$

In particular, these matrices P and Q have to be *invertible*, that is, we need the conditions $\det(P) \neq 0$ and $\det(Q) \neq 0$. To create a lower left 1×1 block of zeros in $(PAQ)_{2,2}$, we need to solve the following *polynomial* system of equations (with *commuting* unknowns α_{ij} and β_{ij}):

$$\begin{aligned} \alpha_{2,2}\alpha_{3,3} - \alpha_{2,3}\alpha_{3,2} &= 1, \\ \beta_{2,2}\beta_{3,3} - \beta_{2,3}\beta_{3,2} &= 1, \\ \alpha_{3,2}\beta_{3,2} + \alpha_{3,3}\beta_{2,2} &= 0 \text{ for } 1 \text{ and} \\ \alpha_{3,2}\beta_{2,2} &= 0 \text{ for } y. \end{aligned}$$

We obtain the last two equations by multiplication of the transformation blocks with the corresponding coefficient matrices of the pivot blocks (irrelevant equations are marked with "$*$" on the right hand side)

$$\begin{bmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} \beta_{2,2} & \beta_{2,3} \\ \beta_{3,2} & \beta_{3,3} \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \text{ for } 1 \text{ and}$$

$$\begin{bmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} \begin{bmatrix} 1 & . \\ . & . \end{bmatrix} \begin{bmatrix} \beta_{2,2} & \beta_{2,3} \\ \beta_{3,2} & \beta_{3,3} \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \text{ for } y.$$

To solve this system of polynomial equations, Gröbner-Shirshov bases (Bokut and Kolesnikov wrote the nice survey [23] about these bases going back to the 1960s) can be used. For detailed discussions, we refer to [24] or [25]. The basic idea comes from [9, Theorem 4.1] and is also used in [12, Proposition 42] and [13, Section 5]. For further remarks on the refinement of pivot blocks see also [7, Section 4.4].

4 Minimizing a refined ALS

First of all, we derive the *left*, respectively, *right* block minimization equations. For that we consider an admissible linear system $\mathcal{A} = (u, A, v)$ of dimension n with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. For $1 \leq k < m$, we transform this system using the block row transformation $(P, Q) = (P(\bar{T}, T), Q(\bar{U}, U))_k$, namely,

$$
PAQ = \begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{T} & T \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix} \begin{bmatrix} A_{1,:1} & A_{1:,k} & A_{1:,:m} \\ \cdot & A_{k,k} & A_{k,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{U} & U \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{1,:1} & A_{1:,k} & A_{1:,:m} \\ \cdot & \bar{T}A_{k,k} & \bar{T}A_{k,:m} + TA_{:m,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{U} & U \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{1,:1} & A_{1:,k}\bar{U} & A_{1:,k}U + A_{1:,:m} \\ \cdot & \bar{T}A_{k,k}\bar{U} & \bar{T}A_{k,k}U + \bar{T}A_{k,:m} + TA_{:m,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix} \quad \text{and}
$$

$$
Pv = \begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{T} & T \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix} \begin{bmatrix} v_{1:} \\ v_{\underline{k}} \\ v_{:\underline{m}} \end{bmatrix} = \begin{bmatrix} v_{1:} \\ \bar{T}v_{\underline{k}} + Tv_{:\underline{m}} \\ v_{:\underline{m}} \end{bmatrix}.
$$

Now we can read off a *sufficient* condition for $(Q^{-1}s)_{\underline{k}} = 0^{n_k \times 1}$, namely, the existence of matrices T, $U \in \mathbb{K}^{n_k \times n_{k+1:m}}$ and invertible matrices $\bar{T}$, $\bar{U} \in \mathbb{K}^{n_k \times n_k}$ such that

$$
\bar{T}A_{k,k}U + \bar{T}A_{k,:m} + TA_{:m,:m} = 0^{n_k \times n_{k+1:m}} \text{ and } \bar{T}v_{\underline{k}} + Tv_{:\underline{m}} = 0^{n_k \times 1}.
$$

Since $\bar{T}$ is invertible (as a diagonal block of an invertible matrix P), this condition is equivalent to the existence of matrices T', $U \in \mathbb{K}^{n_k \times n_{k+1:m}}$ such that

$$
A_{k,k}U + A_{k,:m} + \underbrace{\bar{T}^{-1}T}_{=:T'}A_{:m,:m} = 0^{n_k \times n_{k+1:m}} \text{ and } v_{\underline{k}} + \underbrace{\bar{T}^{-1}T}_{=:T'}v_{:\underline{m}} = 0^{n_k \times 1}. \tag{4.1}
$$

Applying the block column transformation $(P, Q) = (P(\bar{T}, T), Q(\bar{U}, U))^{\underline{k}}$, we obtain

$$
PAQ = \begin{bmatrix} I_{1:k-1} & T & \cdot \\ \cdot & \bar{T} & \cdot \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix} \begin{bmatrix} A_{1,:1} & A_{1:,k} & A_{1:,:m} \\ \cdot & A_{k,k} & A_{k,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} I_{1:k-1} & U & \cdot \\ \cdot & \bar{U} & \cdot \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{1,:1} & A_{1:,k} + TA_{k,k} & A_{1:,:m} + TA_{k,:m} \\ \cdot & \bar{T}A_{k,k} & \bar{T}A_{k,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} I_{1:k-1} & U & \cdot \\ \cdot & \bar{U} & \cdot \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{1,:1} & A_{1:,1:}U + A_{1:,k}\bar{U} + TA_{k,k}\bar{U} & A_{1:,:m} + TA_{k,:m} \\ \cdot & \bar{T}A_{k,k}\bar{U} & \bar{T}A_{k,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix}
$$

and therefore a *sufficient* condition for $(tP^{-1})_{\underline{k}} = 0^{1 \times n_k}$, namely, the existence of matrices T, $U' \in \mathbb{K}^{n_{1:k-1} \times n_k}$ such that

$$
A_{1,:1}\underbrace{U\bar{U}^{-1}}_{=:U'} + A_{1:,k} + TA_{k,k} = 0^{n_{1:k-1} \times n_k}. \tag{4.2}
$$

Remark. A variant of the *linear* system of Eq. (4.1) also appears in [11, Lemma 2.3] and [11, Theorem 2.4] (linear word problem).

Remark 4.3. (Extended ALS) In some cases, it is necessary to use an *extended* ALS to be able to apply all necessary *left* minimization steps, for example, for $f^{-1}f$ if f is of type $(1, 1)$. Let $\mathcal{A} = (u, A, v) = (1, A, \lambda)$ be an ALS with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks and $k = 1$. The "extended" block decomposition is then (the block row $A_{1:,1:}$ vanishes)

$$\mathcal{A}^{[k]} = \left(\begin{bmatrix} 1 & | & . & . \end{bmatrix}, \begin{bmatrix} 1 & A_{0,k} & . \\ \hline . & A_{k,k} & A_{k,:m} \\ . & . & A_{:m,:m} \end{bmatrix}, \begin{bmatrix} . \\ . \\ \underline{v_{:m}} \end{bmatrix} \right)$$

with $A_{0,k} = [-1, 0,\ldots, 0]$. The first row in $\mathcal{A}^{[1]}$ is only changed indirectly (via admissible column operations) and stays scalar. Therefore, it can be removed easily (if necessary). This is illustrated in Example 4.5.

Notation. Given an admissible linear system $\mathcal{A}$, by $\tilde{\mathcal{A}} = \mathcal{A}^{[+0]}$ we denote the (to $\mathcal{A}$ equivalent) *extended* ALS. Conversely, $\tilde{\mathcal{A}}^{[-0]} = (\mathcal{A}^{[+0]})^{[-0]} = \mathcal{A}$. The additional row and column are indexed by 0. If $\tilde{\mathcal{A}}$ is transformed admissibly, $\tilde{\mathcal{A}}^{[-0]}$ is an ALS.

Definition 4.4. (Minimization Equations and Transformations) Let $\mathcal{A} = (u, A, v)$ be an ALS of dimension n with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. For $k \in \{1, 2,\ldots, m - 1\}$, Eq. (4.1),

$$A_{k,k}U + A_{k,:m} + TA_{:m,:m} = 0^{n_k \times n_{k+1:m}} \text{ and } v_{\underline{k}} + Tv_{\underline{:m}} = 0^{n_k \times 1}$$

with respect to the block decomposition $\mathcal{A}^{[k]}$ and the particular block row transformation $(P(T), Q(U))_{\underline{k}}$ are called *left block minimization equations*. They are denoted by $\mathcal{L}_{\underline{k}} = \mathcal{L}_{\underline{k}}(\mathcal{A})$. A solution by the block row pair (T, U) is denoted by $\mathcal{L}_{\underline{k}}(T, U) = 0$. For $k \in \{2, 3,\ldots, m\}$, Eq. (4.2),

$$A_{1:,1:}U + A_{1:,k} + TA_{k,k} = 0^{n_{1:k-1} \times n_k}$$

with respect to the block decomposition $\mathcal{A}^{[k]}$ and the particular block column transformation $(P(T), Q(U))^{\underline{k}}$ are called *right block minimization equations*. They are denoted by $\mathcal{R}_{\underline{k}} = \mathcal{R}_{\underline{k}}(\mathcal{A})$. A solution by the block column pair (T, U) is denoted by $\mathcal{R}_{\underline{k}}(T, U) = 0$.

In the following example, we have a close look on the role of the factorization and how to avoid the use of possibly non-linear techniques. All the steps are explained in detail and correspond (with exception of the solution of the linear systems of equations) to that in the following algorithm.

Example 4.5. For $f = x^{-1}(1 - xy)^{-1}$ and $g = x$, we consider $h = fg = (1 - yx)^{-1}$ given by the (non-minimal) ALS (constructed by Proposition 2.7)

$$\mathcal{A} = (u, A, v) = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} x & 1 & . & . \\ . & y & -1 & . \\ . & -1 & x & -x \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix} \right),$$

whose pivot blocks are *refined*. Here there exists an admissible transformation (with $T = 0$, $U = 1$ and *invertible* blocks $\bar{T}, \bar{U} \in \mathbb{K}^{3 \times 3}$)

$$(P, Q) = \left(\begin{bmatrix} 1 & 0 & 0 & . \\ 0 & 1 & 0 & . \\ 1 & 0 & 1 & T \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & . \\ 0 & 1 & 0 & . \\ -1 & 0 & 1 & U \\ . & . & . & 1 \end{bmatrix} \right),$$

which yields the ALS

$$P\mathcal{A}Q = \mathcal{A}' = \left(\begin{bmatrix} 1 & . & . & . \end{bmatrix}, \begin{bmatrix} x & 1 & . & . \\ 1 & y & -1 & -1 \\ 0 & 0 & x & 0 \\ . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix} \right)$$

in which one can eliminate row 3 and column 3 (and – after an appropriate row operation – also the last row and column).

However, minimization can be accomplished much easier: first we observe that the left subfamily $s_{2:3}$ (of $\mathcal{A}$) is $\mathbb{K}$-linearly independent. Also, the right subfamily $t_{1:2}$ is $\mathbb{K}$-linearly independent. For the left family with respect to the first pivot block, we consider the extended ALS (see also Remark 4.3)

$$\left[\begin{array}{c|cccc} 1 & -1 & . & . & . \\ \hline . & x & 1 & . & . \\ . & . & y & -1 & . \\ . & . & -1 & x & -x \\ . & . & . & . & 1 \end{array}\right] s = \left[\begin{array}{c} . \\ \hline . \\ . \\ . \\ 1 \end{array}\right]$$

of $\mathcal{A}$, where the upper row and the left column are indexed by zero. Now we add row 3 to row 1, subtract column 1 from column 3 and add column 1 to column 4 (this transformation can be found by solving a linear system of equations):

$$\left[\begin{array}{c|cccc} 1 & -1 & . & 1 & -1 \\ \hline . & x & 0 & 0 & 0 \\ . & . & y & -1 & . \\ . & . & -1 & x & -x \\ . & . & . & . & 1 \end{array}\right] s = \left[\begin{array}{c} . \\ \hline 0 \\ . \\ . \\ 1 \end{array}\right].$$

Now we can remove row 1 and column 1:

$$\left[\begin{array}{c|ccc} 1 & . & 1 & -1 \\ \hline . & y & -1 & . \\ . & -1 & x & -x \\ . & . & . & 1 \end{array}\right] s = \left[\begin{array}{c} . \\ \hline . \\ . \\ 1 \end{array}\right]. \tag{4.6}$$

Before we do the last (right) minimization step, we transform the extended ALS back into a "normal" by exchanging columns 1 and 2, scaling the (new) column 1 by –1 and subtract it from column 3:

$$\left[\begin{array}{c|ccc} 1 & -1 & . & 0 \\ \hline . & 1 & y & -1 \\ . & -x & -1 & 0 \\ . & . & . & 1 \end{array}\right] s = \left[\begin{array}{c} . \\ \hline . \\ . \\ 1 \end{array}\right].$$

(If necessary right minimization steps can be performed until one reaches a pivot block with the corresponding non-zero entry in row 0.) Now we can remove row 0 and column 0 again. The last step to a minimal ALS for $fg = (1 - yx)^{-1}$ is trivial:

$$\left[\begin{array}{ccc} 1 & y & 0 \\ -x & -1 & 0 \\ . & . & 1 \end{array}\right] s = \left[\begin{array}{c} 1 \\ . \\ 1 \end{array}\right].$$

After removing the last row and column, we exchange the two rows to get a *standard* ALS.

Now at least one question should have appeared: *How* can one prove – for a given block index k – the $\mathbb{K}$-linear independence of the left (respectively, right) subfamily $s_{k:m}$ (respectively, $t_{1:k}$) in general, assuming that $s_{k+1:m}$ (respectively, $t_{1:k-1}$) is $\mathbb{K}$-linearly independent? For an answer some preparation is necessary.

Lemma 4.7. [9, Lemma 1.2]. *Let $f \in \mathbb{F}$ given by the linear representation $\pi_f = (u, A, v)$ of dimension n. Then $f = 0$ if and only if there exist invertible matrices $P, Q \in \mathbb{K}^{n \times n}$ such that*

$$P\pi_f Q = \left(\begin{bmatrix} \tilde{u}_1 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_{1,1} & 0 \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} \end{bmatrix}, \begin{bmatrix} 0 \\ \tilde{v}_2 \end{bmatrix} \right)$$

for square matrices $\tilde{A}_{1,1}$ and $\tilde{A}_{2,2}$.

Theorem 4.8. (Left Block Minimization) *Let $\mathcal{A} = (u, A, v) = (1, A, \lambda)$ be an ALS of dimension n with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. Let $k \in \{1, 2, \ldots, m-1\}$ such that the left subfamily $s_{\underline{k+1:m}}$ with respect to the block decomposition $\mathcal{A}^{[k]}$ is $\mathbb{K}$-linearly independent while $s_{\underline{k:m}}$ is $\mathbb{K}$-linearly dependent. Then there exists a block row transformation $(P, Q) = (P(\bar{T}, T), Q(\bar{U}, U))_k$, such that $\tilde{\mathcal{A}} = P\mathcal{A}Q$ has the form*

$$
\begin{bmatrix}
A_{1:,1:} & \tilde{A}_{1:,k'} & \tilde{A}_{1:,k''} & \tilde{A}_{1:,:m} \\
\cdot & \tilde{A}_{k',k'} & 0 & 0 \\
\cdot & \tilde{A}_{k'',k'} & \tilde{A}_{k'',k''} & \tilde{A}_{k'',:m} \\
\cdot & \cdot & \cdot & A_{:m,:m}
\end{bmatrix}
\begin{bmatrix}
\tilde{s}_{\underline{1:}} \\
0 \\
\tilde{s}_{\underline{k''}} \\
s_{\underline{:m}}
\end{bmatrix}
=
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot \\
v_{\underline{:m}}
\end{bmatrix}.
\tag{4.9}
$$

If the pivot block $A_{k,k}$ is refined, then there exists a particular block row transformation $(P, Q) = (P(T), Q(U))_k$, such that the left block minimization equations

$$
A_{k,k}U + A_{k,:m} + TA_{:m,:m} = 0^{n_k \times n_{k+1:m}} \quad \text{and} \quad v_{\underline{k}} + Tv_{\underline{:m}} = 0^{n_k \times 1}
$$

are fulfilled.

Proof. We refer to the block decomposition

$$
\mathcal{A}^{[k]} = \left(\begin{bmatrix} u_{\underline{1:}} & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} A_{1:,1:} & A_{1:,k} & A_{1:,:m} \\ \cdot & A_{k,k} & A_{k,:m} \\ \cdot & \cdot & A_{:m,:m} \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \\ v_{\underline{:m}} \end{bmatrix} \right).
$$

Due to the $\mathbb{K}$-linear independence of the left subfamily $s_{\underline{k+1:m}}$, there exists an invertible matrix $\tilde{Q}$ with blocks $\bar{U}^\circ \in \mathbb{K}^{n_k \times n_k}$ and $U^\circ \in \mathbb{K}^{n_k \times n_{k+1:m}}$, such that $(\tilde{Q}^{-1}s)_{n_{1:k-1}+1} = 0$, that is, the first component in $s_{\underline{k}}$ can be eliminated. Let

$$
A' = \begin{bmatrix} A_{k,k} & A_{k,:m} \\ \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} \bar{U}^\circ & U^\circ \\ \cdot & I_{k+1:m} \end{bmatrix} \quad \text{and} \quad v' = \begin{bmatrix} \cdot \\ v_{\underline{:m}} \end{bmatrix}.
$$

Then $\mathcal{A}' = (u', A', v')$ is an ALS for $0 \in \mathbb{F}$ and we can apply Lemma 4.7 to get a transformation

$$
(P', Q') = \left(\begin{bmatrix} \bar{T} & T \\ T_{:m,k} & T_{:m,:m} \end{bmatrix}, \begin{bmatrix} \bar{U}' & U' \\ U_{:m,k} & U_{:m,:m} \end{bmatrix} \right)
$$

such that $P'A'Q'$ has the respective upper right block of zeros and – without loss of generality – $P'v' = v'$. Clearly, we can choose $U_{:m,:m} = I_{k+1:m}$. And since $s_{\underline{:m}}$ is $\mathbb{K}$-linearly independent, the block of zeros in

$$
P'A' = \begin{bmatrix} \bar{T} & T \\ T_{:m,k} & T_{:m,:m} \end{bmatrix} \begin{bmatrix} A'_{k,k} & A'_{k,:m} \\ \cdot & A_{:m,:m} \end{bmatrix} = \begin{bmatrix} \bar{T}A'_{k,k} & \bar{T}A'_{k,:m} + TA_{:m,:m} \\ T_{:m,k}A'_{k,k} & T_{:m,k}A'_{k,:m} + T_{:m,:m}A_{:m,:m} \end{bmatrix}
$$

is *independent* of $T_{:m,k}$ and $T_{:m,:m}$, thus we can choose $T_{:m,k} = 0$ and $T_{:m,:m} = I_{k+1:m}$. Now it is obvious that the columns in the lower left block of

$$
\begin{aligned}
P'A'Q' &= \begin{bmatrix} \bar{T}A'_{k,k} & \bar{T}A'_{k,:m} + TA_{:m,:m} \\ \cdot & A_{:m,:m} \end{bmatrix} \begin{bmatrix} \bar{U}' & U' \\ U_{:m,k} & I \end{bmatrix} \\
&= \begin{bmatrix} \bar{T}A'_{k,k}\bar{U}' + (\bar{T}A'_{k,:m} + TA_{:m,:m})U_{:m,k} & \bar{T}A'_{k,k}U' + \bar{T}A'_{k,:m} + TA_{:m,:m} \\ A_{:m,:m}U_{:m,k} & A_{:m,:m} \end{bmatrix}
\end{aligned}
$$

are linear combinations of the columns of $A_{:m,:m}$ and therefore we can assume that $U_{:m,k} = 0$. Now let $\bar{U} = \bar{U}^\circ \bar{U}'$ and $U = \bar{U}^\circ U' + U^\circ$. Then $P\mathcal{A}Q$ has the desired form (4.9) for the (for $k > 1$ admissible) block row transformation

$$
(P, Q) = \left(\begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{T} & T \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix}, \begin{bmatrix} I_{1:k-1} & \cdot & \cdot \\ \cdot & \bar{U} & U \\ \cdot & \cdot & I_{k+1:m} \end{bmatrix} \right).
$$

For the second part, we first have to show that *each* component in $s_{\underline{k}}$ can be eliminated by a linear combination of components of $s_{\underline{k+1:m}}$, that is, $n_{k''} = 0$. We assume to the contrary that $n_{k''} > 0$. But then – by (4.9) – $\bar{T} A_{k,k} \bar{U}$ would have an upper right block of zeros of size $(n_k - n_{k''}) \times n_{k''}$ and therefore (after an appropriate permutation) a lower left, contradicting the assumption on a refined pivot block. Hence, there exists a matrix $U \in \mathbb{K}^{n_k \times n_{k+1:m}}$ such that $s_{\underline{k}} - U[s_{\underline{k+1}}, \ldots, s_{\underline{m}}] = 0$. By assumption $v_k = 0$. Now we can apply – as in Lemma 2.8 – Lemma 2.3 with the ALS $(1, A_{:m,:m}, \lambda)$ and $B = -A_{k,k} U - A_{k,:m}$ (and $s_{\underline{:m}}$). Thus, there exists a matrix $T \in \mathbb{K}^{n_k \times n_{k+1:m}}$ fulfilling $A_{k,k} U + A_{k,:m} + T A_{:m,:m} = 0$. Since the last column of T is zero, we also have $T v_{\underline{:m}} = 0$. With $\bar{T} = \bar{U} = I_{1:n_k}$ the transformation (P, Q) is the appropriate particular block row transformation. $\square$

Remark. For the proof of the second part of the theorem, one can use alternatively Lemma 4.7, which is more powerful but with respect to the use of linear techniques not that obvious.

Remark. Note that the left subfamily $(\tilde{s}_{\underline{k''}}, s_{\underline{:m}})$ is not necessarily $\mathbb{K}$-linearly independent. If necessary, one can apply the theorem again after removing block row and column k'.

Remark. For $k = 1$, if necessary, one must use an extended ALS, see Remark 4.3.

Remark 4.10. Assuming $\mathbb{K}$-linear independence of the left subfamily $s_{\underline{k+1:m}}$ and a refined pivot block $A_{k,k}$, the second part of the previous theorem means nothing less than the possibility to check $\mathbb{K}$-linear (in-) dependence of the left subfamily $s_{\underline{k:m}}$ by *linear* techniques!

Theorem 4.11. (*Right Block Minimization*) *Let $\mathcal{A} = (u, A, v) = (1, A, \lambda)$ be an ALS of dimension n with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks of size $n_i = \dim_i(\mathcal{A})$. Let $k \in \{2, 3, \ldots, m\}$ such that the right subfamily $t_{1:k-1}$ with respect to the block decomposition $\mathcal{A}^{[k]}$ is $\mathbb{K}$-linearly independent, while $t_{1:k}$ is $\mathbb{K}$-linearly dependent. Then there exists a block column transformation $(P, Q) = (P(\bar{T}, T), Q(\bar{U}, U))^{\underline{k}}$, such that $\tilde{\mathcal{A}} = P \mathcal{A} Q$ has the form*

$$\begin{bmatrix} u_{\underline{1}} & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} t_{\underline{1}} & \tilde{t}_{\underline{2'}} & 0 & \tilde{t}_{\underline{3}} \end{bmatrix} \begin{bmatrix} A_{1,1} & \tilde{A}_{1,2'} & 0 & \tilde{A}_{1,3} \\ \cdot & \tilde{A}_{2',2'} & 0 & \tilde{A}_{2',3} \\ \cdot & \tilde{A}_{2'',2'} & \tilde{A}_{2'',2''} & \tilde{A}_{2'',3} \\ \cdot & \cdot & \cdot & A_{3,3} \end{bmatrix}. \tag{4.12}$$

If the pivot block $A_{k,k}$ is refined, then there exists a particular block column transformation $(P, Q) = (P(T), Q(U))^{\underline{k}}$, such that the right block minimization equations

$$A_{1:,1:} U + A_{1:,k} + T A_{k,k} = 0^{n_{1:k-1} \times n_k}$$

are fulfilled.

If one uses alternating left and right block minimization steps for the minimization, that is, applying Theorems 4.8 and 4.11, one has to take care that the $\mathbb{K}$-linear independence of the respective other subfamily is guaranteed. This is illustrated in the following example.

Example 4.13. Let $\mathcal{A} = (u, A, v) = (1, A, \lambda)$ be an ALS with $m = 5$ pivot blocks. For $k' = 2$, we assume that the left subfamily $s_{\underline{k+1:m}}$ is $\mathbb{K}$-linearly independent and we assume further that there exists a particular block row transformation (P, Q), such that the left block minimization equations are fulfilled, that is, $P \mathcal{A} Q$ has the form

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \tilde{A}_{1,3} & \tilde{A}_{1,4} & \tilde{A}_{1,5} \\ \cdot & A_{2,2} & 0 & 0 & 0 \\ \cdot & \cdot & A_{3,3} & A_{3,4} & A_{3,5} \\ \cdot & \cdot & \cdot & A_{4,4} & A_{4,5} \\ \cdot & \cdot & \cdot & \cdot & A_{5,5} \end{bmatrix} \begin{bmatrix} \tilde{s}_{\underline{1}} \\ 0 \\ s_{\underline{3}} \\ s_{\underline{4}} \\ s_{\underline{5}} \end{bmatrix} = \begin{bmatrix} \cdot \\ 0 \\ \cdot \\ \cdot \\ v_{\underline{5}} \end{bmatrix}.$$

If the right subfamily $t_{1:3}$ is $\mathbb{K}$-linearly independent, this is *not* necessarily the case for the right subfamily $t'_{1:3}$ of the smaller ALS $\mathcal{A}' = (P\mathcal{A}Q)^{[-\underline{k}']}$. That is, one has to apply Theorem 4.11 on $\mathcal{A}'$ with $k = 3$ to check that "again."

Algorithm 4.14. (Minimizing a refined ALS)

Input: $\mathcal{A} = (u, A, v) = (1, A, \lambda)$ *refined ALS (for an element f) with* $m = \#_{pb}(\mathcal{A}) \geq 2$ *pivot blocks*

    of size $n_i = \dim_i(\mathcal{A})$ *and* $\mathbb{K}$*-linearly independent subfamilies* $s_{\underline{m}}$ *and* $t_{\underline{1}}$.

Output: $\mathcal{A}' = (, ,)$, *if* $f = 0$, *or*

    a minimal *refined ALS* $\mathcal{A}' = (u', A', v') = (1, A', \lambda')$, *if* $f \neq 0$.

  1: $k := 2$
  2: **while** $k \leq \#_{pb}(\mathcal{A})$ **do**
  3:    $m := \#_{pb}(\mathcal{A})$
  4:    $k' := m + 1 - k$

                             lin. independent
     Is the left subfamily $(s_{\underline{k'}}, \overbrace{s_{\underline{k'+1}}, \ldots, s_{\underline{m}}})$ $\mathbb{K}$-linearly dependent?
  5:   **if** $\exists\, T, U \in \mathbb{K}^{n_k \times n_{k+1:m}}$ admissible $: \mathcal{L}_{\underline{k'}}(\mathcal{A}) = \mathcal{L}_{\underline{k'}}(T, U) = 0$ **then**
  6:     **if** $k' = 1$ **then**
  7:       **return** $(, ,)$
        endif
  8:     $\mathcal{A} := (P(T)\mathcal{A}Q(U))^{[-\underline{k}']}$
  9:     **if** $k > \max\{2, \frac{m+1}{2}\}$ **then**
 10:       $k := k - 1$
        endif
 11:    **continue**
     endif
 12:   **if** $k' = 1$ and $\exists\, T, U \in \mathbb{K}^{n_k \times n_{k+1:m}} : \mathcal{L}_{\underline{k'}}(\mathcal{A}^{[+0]}) = \mathcal{L}_{\underline{k'}}(T, U) = 0$ **then**
 13:     $\tilde{\mathcal{A}} := (P(T)\mathcal{A}^{[+0]}Q(U))^{[-\underline{k}']}$
 14:     $\mathcal{A} := \tilde{\mathcal{A}}^{[-0]}$
 15:     **if** $k > \max\{2, \frac{m+1}{2}\}$ **then**
 16:       $k := k - 1$
        endif
 17:    **continue**
     endif

                            lin. independent
     Is the right subfamily $(\overbrace{t_{\underline{1}}, \ldots, t_{\underline{k-1}}}, t_{\underline{k}})$ $\mathbb{K}$-linearly dependent?
 18:    **if** $\exists\, T, U \in \mathbb{K}^{n_{k-1:m} \times n_k}$ admissible $: \mathcal{R}_{\underline{k}}(\mathcal{A}) = \mathcal{R}_{\underline{k}}(T, U) = 0$ **then**
 19:      $\mathcal{A} := (P(T)\mathcal{A}Q(U))^{[-\underline{k}]}$
 20:      **if** $k > \max\{2, \frac{m+1}{2}\}$ **then**
 21:        $k := k - 1$
       endif
 22:     **continue**
     endif
 23:    $k := k + 1$
    done
 24:   **return** $P\mathcal{A}$, with P, such that $Pv = [0, \ldots, 0, \lambda']^\top$

Proof. The admissible linear system $\mathcal{A}$ represents $f = 0$ if and only if $s_1 = (A^{-1}v)_1 = 0$. Since all systems are equivalent to $\mathcal{A}$, this case is recognized for $k' = 1$ because by Theorem 4.8 there is an *admissible*

transformation such that the first left block minimization equation is fulfilled. Now assume $f \neq 0$. We have to show that both the left family s' and the right family t' of $\mathcal{A}' = (u', A', v')$ are $\mathbb{K}$-linearly independent, respectively. Let $m' = \#_{\mathrm{pb}}(\mathcal{A}')$ and for $k \in \{1, 2, \ldots, m'\}$ denote by

$$s'_{(k)} = (s'_{\underline{m'+1-k}}, s'_{\underline{m'+2-k}}, \ldots, s'_{\underline{m'}}) \ \text{ and } \ t'_{(k)} = (t'_{\underline{1}}, t'_{\underline{2}}, \ldots, t'_{\underline{k}})$$

the left and the right subfamily, respectively. By assumption $s'_{(1)}$ and $t'_{(1)}$ are $\mathbb{K}$-linearly independent, respectively. The loop starts with $k = 2$. Only if both $s'_{(k)}$ and $t'_{(k)}$ are $\mathbb{K}$-linearly independent, respectively, k is incremented. Otherwise, a left (Theorem 4.8) or a right (Theorem 4.11) minimization step was successful and the dimension of the current ALS is strictly smaller than that of the previous. Hence, since k is bounded from below, the algorithm stops in a finite number of steps. (How row 0 and column 0 are removed from the extended ALS in line 14 is illustrated in Example 4.5.) All transformations are such that $\mathcal{A}'$ is a refined ALS (and therefore in a *standard form*). For $\#_{\mathrm{pb}}(\mathcal{A}') = 1$ a priori only the left (or the right) family is $\mathbb{K}$-linearly independent (by assumption). But if that were not the case for the respective other family, then the assumption on refined pivot blocks would be contradicted by Theorem 4.11 (respectively, Theorem 4.8). □

Remark. Concerning details with respect to the complexity of such an algorithm we refer to [12, Remark 33]. Let d be the number of letters in our alphabet X. For $m = n$ pivot blocks and $k < n$ we have $2(k - 1)$ unknowns. By Gaussian elimination one gets complexity $O(dn^3)$ for solving a linear system for a minimization step, see [25, Section 2.3]. To build such a system and working on a linear matrix pencil $\begin{bmatrix} 0 & u \\ v & A \end{bmatrix}$ with $d + 1$ square coefficient matrices of size $n + 1$ (transformations, etc.) has complexity $O(dn^2)$. So we get overall (minimization) complexity $O(dn^4)$. The algorithm of Cardon and Crochemore [27] has complexity $O(dn^3)$ but works only for regular elements, that is, rational formal power series. For $\dim_i(\mathcal{A}) \approx \sqrt{n}$, we get complexity $O(dn^5)$ and for the word problem [11] with $m = 2$ we get complexity $O(dn^6)$.

Remark. It is clear that one can adapt the algorithm slightly if the input ALS is constructed by Proposition 2.2 out of two minimal admissible linear systems (for the sum and the product) in the standard form.

Remark. The solution of the word problem for two elements given by *minimal* admissible linear systems is *independent* of their refinement. If Algorithm 4.14 is applied to an ALS of which it is not known if it is refined, in some cases it is possible to check if the ALS $\mathcal{A}'$ is minimal, for example, if $\dim(\mathcal{A}') = \#_{\mathrm{pb}}(\mathcal{A}')$. If the pivot blocks are bigger but the right upper structure is "finer" one can instead – for $f \neq 0$ – try to minimize the inverse $(\mathcal{A}')^{-1}$. In concrete situations, there might be other possibilities to reach minimality.

Remark 4.15. Apart from minimization, this algorithm can be used to check if f is a left factor of an element fg, which is relevant for the minimal factor multiplication [13, Theorem 5.2]. And one can check if two elements are *disjoint* which is important for the *primary decomposition* [9, Theorem 2.3].

Another aspect of Algorithm 4.14 (respectively, Theorems 4.8 and 4.11) becomes visible immediately with Proposition 1.7 and Remark 4.10. The importance of the following theorem becomes clear if one needs to check $\mathbb{K}$-linear (in-)dependence of an arbitrary family $(f_1, f_2, \ldots, f_n)$ over the free field and it is not possible (anymore) to take a representation as formal power series (with coefficients over $\mathbb{K}$).

Theorem 4.16. ("Linear" Characterization of Minimality) *A refined admissible linear system $\mathcal{A} = (1, A, \lambda)$ for an element in the free field $\mathbb{F}$ with $m = \#_{\mathrm{pb}}(\mathcal{A}) \geq 2$ pivot blocks and $\lambda \neq 0$ is minimal if and only if neither the left block minimization equations $\mathcal{L}_{\underline{k}}(\mathcal{A})$ for $k \in \{1, 2, \ldots, m - 1\}$ nor the right block minimization equations $\mathcal{R}^{\underline{k}}(\mathcal{A})$ for $k \in \{2, 3, \ldots, m\}$ admit a solution.*

Proof. From the existence of a solution non-minimality follows immediately since in this case – after the appropriate transformation – rows and columns can be removed. And for non-minimality Proposition 1.7 implies that either the left or the right family is $\mathbb{K}$-linearly dependent. Without loss of generality assume

that it is the left $s = (s_1, s_2, \ldots, s_m)$ with minimal $k \in \{1, 2, \ldots, m - 1\}$ such that the left subfamily $(s_{k+1}, \ldots, s_m)$ is $\mathbb{K}$-linearly *in*dependent. Since the pivot blocks are refined, Theorem 4.8 implies the existence of a particular block row transformation $(P, Q)_k$ and therefore a solution of the k-th left block minimization equations. $\qquad\square$

5 Applications

Since the focus of this work is mainly minimization and one dedicated to "minimal" rational operations – collecting all techniques for practical application – is already available [7], only two applications are illustrated in the following examples. For other applications see also Remark 4.15.

Example 5.1. [Hua's Identity, 28] We have:

$$x - (x^{-1} + (y^{-1} - x)^{-1})^{-1} = xyx. \tag{5.2}$$

Proof. Minimal admissible linear systems for y^{-1} and x are

$$[y]s = [1] \quad \text{and} \quad \begin{bmatrix} 1 & -x \\ . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ 1 \end{bmatrix},$$

respectively. The ALS for the difference $y^{-1} - x$,

$$\begin{bmatrix} y & -y & . \\ . & 1 & -x \\ . & . & 1 \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ -1 \end{bmatrix}, \quad s = \begin{bmatrix} y^{-1} - x \\ -x \\ -1 \end{bmatrix}, \quad t = [y^{-1} \quad -1 \quad y^{-1} - x],$$

is minimal because the left family s is $\mathbb{K}$-linearly independent and the right family t is $\mathbb{K}$-linearly independent (Proposition 1.7). Clearly, we have $1 \in R(y^{-1} - x)$. Thus, by Lemma 2.5, there exists an admissible transformation

$$(P, Q) = \left(\begin{bmatrix} . & 1 & . \\ 1 & . & 1 \\ . & . & 1 \end{bmatrix}, \begin{bmatrix} 1 & . & . \\ 1 & 1 & . \\ . & . & 1 \end{bmatrix} \right),$$

which yields the ALS

$$\begin{bmatrix} 1 & 1 & -x \\ . & -y & 1 \\ . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ -1 \end{bmatrix}.$$

Now we can apply the inverse of type $(1, 1)$:

$$\begin{bmatrix} 1 & y \\ -x & -1 \end{bmatrix} s = \begin{bmatrix} . \\ 1 \end{bmatrix}, \quad s = \begin{bmatrix} (y^{-1} - x)^{-1} \\ -(1 - xy)^{-1} \end{bmatrix}.$$

This system represents a regular element $(y^{-1} - x)^{-1} = (1 - yx)^{-1}y$, and therefore can be transformed into a regular ALS (Definition 1.12) by scaling row 2 by -1. Then we add x^{-1} "from the left":

$$\begin{bmatrix} x & -x & . \\ . & 1 & y \\ . & x & 1 \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ -1 \end{bmatrix}, \quad s = \begin{bmatrix} x^{-1} + (y^{-1} - x)^{-1} \\ (y^{-1} - x)^{-1} \\ -(1 - xy)^{-1} \end{bmatrix}.$$

This system is minimal and – after adding row 3 to row 1 (to eliminate the non-zero entry in the right hand side) – we apply the (minimal) inverse of type $(0, 0)$:

$$
\begin{bmatrix}
-1 & -1 & -x & . \\
. & -y & -1 & . \\
. & -1 & 0 & -x \\
. & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ 1
\end{bmatrix}. \tag{5.3}
$$

Now we multiply row 1 and columns 2 and 3 by -1 and exchange columns 2 and 3 to get the following system:

$$
\begin{bmatrix}
1 & -x & -1 & . \\
. & 1 & y & . \\
. & . & 1 & -x \\
. & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ 1
\end{bmatrix},
\qquad
s =
\begin{bmatrix}
x - xyx \\
-yx \\
x \\
1
\end{bmatrix}.
$$

The next step would be a scaling by -1 and the addition of x (by Proposition 2.2). With two minimization steps we would reach again minimality. Alternatively, we can add a *linear* term to a polynomial (in a polynomial ALS) – depending on the entry v_n in the right hand side – directly in the upper right entry of the system matrix:

$$
\begin{bmatrix}
1 & -x & -1 & x \\
. & 1 & y & . \\
. & . & 1 & -x \\
. & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ 1
\end{bmatrix},
\qquad
s =
\begin{bmatrix}
-xyx \\
-yx \\
x \\
1
\end{bmatrix}.
\qquad \square
$$

Remarks. The transformation of the ALS (5.3) is a simple case of the *refinement* of a pivot block and is discussed in detail in Section 3. Hua's identity is also an example in [6]. It is worth to compare both approaches.

Example 5.4. (Left GCD) Given two polynomials $p, q \in \mathbb{K}\langle X \rangle \backslash \mathbb{K}$, one can compute the *left* (respectively, *right*) *greatest common divisor* of p and q by minimizing an admissible linear system for $p^{-1}q$ (respectively, pq^{-1}). This is now illustrated in the following example. Let $p = yx(1 - yx)z = yxz - yxyxz$ and $q = y(1 - xy)y = y^2 - yxy^2$. We want to find $h = \mathrm{lgcd}(p, q)$. An ALS for $p^{-1}q$ (constructed out of minimal admissible linear systems for p^{-1} and q by Proposition 2.6) is

$$
\begin{bmatrix}
z & 1 & . & . & . & . & . & . & . \\
. & x & -1 & . & . & . & . & . & . \\
. & -1 & y & -1 & . & . & . & . & . \\
. & . & . & x & -1 & . & . & . & . \\
. & . & . & . & y & -y & . & . & . \\
. & . & . & . & . & 1 & -x & 1 & . \\
. & . & . & . & . & . & 1 & -y & . \\
. & . & . & . & . & . & . & 1 & y \\
. & . & . & . & . & . & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ 1
\end{bmatrix}.
$$

Clearly, this system is *refined*. How to refine an ALS is discussed in Section 3. Note that there is a close connection to the factorization of p, for details see [12]. As a first step, we add column 5 to column 6 and row 6 to row 4,

$$
\begin{bmatrix}
z & 1 & . & . & . & 0 & . & . & . \\
. & x & -1 & . & . & 0 & . & . & . \\
. & -1 & y & -1 & . & 0 & . & . & . \\
. & . & . & x & -1 & 0 & -x & 1 & . \\
. & . & . & . & y & 0 & 0 & 0 & 0 \\
. & . & . & . & . & 1 & -x & 1 & . \\
. & . & . & . & . & . & 1 & -y & . \\
. & . & . & . & . & . & . & 1 & y \\
. & . & . & . & . & . & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ 1
\end{bmatrix},
$$

remove rows and columns 5 and 6,

$$
\begin{bmatrix}
z & 1 & . & . & . & . & . \\
. & x & -1 & . & . & . & . \\
. & -1 & y & -1 & . & . & . \\
. & . & . & x & -x & 1 & . \\
. & . & . & . & 1 & -y & . \\
. & . & . & . & . & 1 & y \\
. & . & . & . & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ . \\ . \\ . \\ 1
\end{bmatrix}
\tag{5.5}
$$

and remember the first (left) divisor $h_1 = y$ we have eliminated. (In the next step with a bigger block one can see immediately how to "read" a divisor directly from the ALS.) Now there are two ways to proceed: If it is not possible to create a zero block in the "L"-form (like before), one can try to change the upper pivot block structure to create a "double-L" zero block. Here, this is possible by subtracting column 4 from column 2 and adding row 2 to row 4. (For details on *similarity unique factorization* in this context see [12].) Afterward, we apply the (admissible) transformation

$$
(P, Q) =
\left(
\begin{bmatrix}
1 & . & . & . & . & . & . \\
. & 1 & . & . & . & 1 & . \\
. & . & 1 & . & 1 & . & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & 1
\end{bmatrix},
\begin{bmatrix}
1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . \\
. & . & 1 & . & . & 1 & . \\
. & . & . & 1 & 1 & . & . \\
. & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & 1
\end{bmatrix}
\right)
$$

to get the ALS

$$
\begin{bmatrix}
z & 1 & . & . & 0 & 0 & . \\
. & x & -1 & . & 0 & 0 & y \\
. & . & y & -1 & 0 & 0 & 0 \\
. & . & -1 & x & 0 & 0 & 0 \\
. & . & . & . & 1 & -y & . \\
. & . & . & . & . & 1 & y \\
. & . & . & . & . & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ . \\ . \\ . \\ . \\ 1
\end{bmatrix}.
\tag{5.6}
$$

How to get this transformation, (P, Q) is described in principle in Section 4. One can look directly for a "double-L" block transformation. In the third pivot block of (5.6), one can see immediately that a further (common) left factor is $h_2 = 1 - xy$ because the second equation reads $xs_2 - h_2^{-1} = 0$. We have eliminated $(1 - xy)^{-1}(1 - xy)$. Recall that a minimal ALS for h_2 is

$$
\begin{bmatrix}
1 & -x & 1 \\
. & 1 & -y \\
. & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ -1
\end{bmatrix},
$$

hence $\mathrm{rank}(h_2) = 3$ and (by Theorem 2.11) $\mathrm{rank}(h_2^{-1}) = 2$. Or, more general, for a (left) factor h_i with $\mathrm{rank}(h_i) = n_i \geq 2$ we can construct (by Proposition 2.6) an ALS of dimension $2(n_i - 1)$. After removing rows and columns $\{3, 4, 5, 6\}$ in the ALS (5.6), we obtain for $p^{-1}q$ the (in this case *minimal*) ALS (5.7)

$$
\begin{bmatrix}
z & 1 & . \\
. & x & y \\
. & . & 1
\end{bmatrix}
s =
\begin{bmatrix}
. \\ . \\ 1
\end{bmatrix}.
\tag{5.7}
$$

Hence, $h = h_1 h_2 = y(1 - xy) = \mathrm{lgcd}(p, q)$. The second possibility – starting from ALS (5.5) – is to do a right minimization step with respect to column 5, then one left with respect to rows 2 and 3 and finally a right (minimization step). Again one obtains the ALS (5.7) (up to admissible scaling of rows and columns) where the right factor y of q remains. Therefore, $\mathrm{lgcd}(p, q) = y - yxy$. For further details concerning the minimal polynomial multiplication (Proposition 2.10) we refer to [12].

Remark. It can happen that – after there is no more "L"-minimization step possible – the ALS is *not* minimal, that is, an additional "single" left or right minimization step can be carried out. More details on that are part of the general factorization theory [13]. Here it suffices to take a closer look at the ALS (5.7): *both* right factors of p (here xz), respectively, q (here y) can still be "read" directly. (This would not be possible any more if one left or right step would be carried out.)

Remark 5.8. For a *refined* ALS for p^{-1}, an ALS of dimension n for $p^{-1}q$ with "factors" of rank $\sqrt{n}$ and an alphabet with d letters, the complexity for computing the left (or right) gcd is roughly $O(dn^5)$. Although in general refinement is difficult because of the necessity to solve systems of polynomial equations (over a not necessarily algebraically closed field), especially for polynomials *linear* techniques are very useful for the factorization (and therefore for the refinement of the inverse). As an example, we take the polynomial $p = (1 - xy)(2 + yx)(3 - yz)(2 - zy)(1 - xz)(3 + zx)x$ of rank $n = 14$ which has already 64 terms. To get the first left (irreducible) factor (of rank 3), we just need to create an upper right block of zeros (in the system matrix) of size 2×11, which can be accomplished by either using columns 2–3 and rows 4–13 *or* column 2 and rows 3–13 (for elimination) [12]. Both cases result in *a linear* system of equations because column and row transformations do not "overlap." Solving one of these $2(n - 2)$ systems has (at most) complexity $O(dn^6)$, hence in total we have $O(dn^7)$. Checking irreducibility of polynomials (using Gröbner bases) works practically up to rank 12 [29].

6 Epilogue

This work is the last in a series for the development of tools for the work with linear representations (for elements in the free field), especially for the implementation in computer algebra systems. A "practical" guide giving an overview and an introduction is [22] (in German, with remarks on the implementation) and [7].

The main idea is as simple as in the usage of "classical" fractions (for elements in the field of rational numbers): *calculating*, *factorizing* and *minimizing* (or *cancelling*), for example

$$\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} = \frac{2 \cdot 3}{2 \cdot 2 \cdot 3} = \frac{1}{2}$$

or

$$\frac{1}{2} + \frac{3}{2} = \frac{4}{2} = \frac{2 \cdot 2}{2} = 2.$$

At some point, one stops this loop and uses the fraction (with *coprime* numerator and denominator). Clearly, one could simplify things by remembering the factorization of the numerator (for the product) and the denominator (for the sum and the product). In our case of the free field, linear representations (or admissible linear systems) are just "free fractions." However, to understand how the transition from using nc rational expressions (to represent elements in the free field) to (minimal) admissible linear systems in the *standard form* affects the capabilities of thinking about (free) nc algebra, one needs to go to the meta level [30].

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