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A Statistical Model for the Dilute Ferromagnet*)

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A statistical model for a kind of dilute ferromagnetism is presented. In this model, a sublattice is a mixture of magnetic ions and non-magnetic ions, and the other sublattice is occupied exclusively by magnetic ions. Assuming the Ising type exchange interaction between magnetic ions, some exact results for the system are obtained. The Curie temperature decreases as the concentration of magnetic ions decreases and reaches the absolute zero of temperature at a critical concentration. At concentrations higher than the critical value, the specific heat remains finite and has a cusp with vertical tangent at the Curie temperature. The critical concentrations for several types of lattices are also given.

§ 1. Introduction

After a brilliant work of Onsager,[1] there has been considerable progress in the problem of two-dimensional Ising model. The spontaneous magnetization was derived by Yang[2] for a square lattice, and the results have been extended to several kinds of two-dimensional lattices. For the susceptibility, however, an exact calculation is succesful only for Fisher's[3] model of antiferromagnetism of decorated square lattice.

We want to show one example which permits an exact calculation. A mixture of ferromagnetic substance and non-ferromagnetic substance exhibits ferromagnetism when the concentration of ferromagnetic substance exceeds a certain value, called a critical concentration. This is the problem of dilute ferromagnetism. As regards this problem there have appeared several kinds of approximate theories,[4] but we cannot solve it exactly even for a two-dimensional Ising lattice.

The model which exhibits some features of dilute ferromagnetism and permits an exact calculation is as follows.

Let us divide the whole lattice points of a crystal into two sublattices penetrating with each other. They are not necessarily equivalent. Every lattice point of one of the sublattices (called the M sublattice) is always occupied by a magnetic ion, and every lattice point of the other sublattice (called the D sublattice) is occupied by either a magnetic ion or a non-magnetic ion. A lattice

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point of the M sublattice is surrounded by lattice points of the D sublattice and vice versa. A magnetic ion is represented by an Ising spin variable which can attain the value $+1$ or -1.

Thus, to every lattice point of the M sublattice, we can attribute a spin variable $\mu_i(\mu_i=1$ or $-1)$. To every lattice point of the D sublattice, we can give a variable σ_j which can attain the value 0, $+1$ or -1. $\sigma_j=0$ corresponds to the occupation of a lattice point by a non-magnetic ion, and $\sigma_j=1$ and $\sigma_j=-1$ correspond to the two spin states of a magnetic ion, if it occupies the lattice point.

The interaction energy between an ion on the M sublattice and an ion on the D sublattice is assumed to be

$$-(J/2)\mu_i\sigma_j,$$

if they are neighboring. Thus, if both ions are magnetic ions, their interaction is of the Ising type, and non-magnetic ions are considered as if they were holes. On the basis of this model, we shall consider the dilute ferromagnetism for several two-dimensional lattices and derive the thermodynamic properties of them, in the following three sections.

In § 2, several kinds of decorated lattices are considered, where the decorated lattice points are the D sublattices and the corner points are the M sublattices.

In § 3, a honeycomb lattice which has two equivalent sublattices, one as the D sublattice and the other as the M sublattice, is considered, and also a decorated honeycomb lattice which has the decorated lattice points as the M sublattice and the corner points as the D sublattice. A diced lattice, in which lattice points with three neighbors are the D sublattice and the other lattice points are the M sublattice, is also considered.

In § 4, two kinds of multiply decorated lattices are considered. The main techniques employed throughout the present paper are the so-called "extended iteration process" and "extended star-triangular transformation". These transformations enable us to transform the grand partition function of a dilute ferromagnet to the partition function of a Ising ferromagnet.

§ 2. Decorated lattices

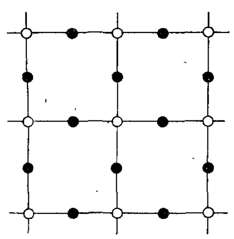

Fig. 1. Decorated square lattice.
○ : M sublattice
● : D sublattice

First we consider the case of a decorated square lattice. As the M sublattice, we take an assembly of the edge points, and as the D sublattice, we take the lattice points at the middle of the sides. By introducing a parameter ξ which is the chemical potential for the magnetic ions on the D sublattice divided by kT, where k is the Boltzmann constant and T is the absolute temperature, and by putting $J/2kT=L$, the grand partition function of the system can be written as

$$\Xi(\xi, L) = \sum_{\sigma_j=0}^{\pm 1} \cdots \sum_{\mu_i=\pm 1} \exp(L \sum_{<ij>} \mu_i \sigma_j + \xi \sum_j \sigma_j^2). \quad (2)$$

If the summation over σ_j is carried out first (the extended iteration process shown in Fig. 2), we have

$$\sum_{\sigma_j=0}^{\pm 1} \exp\{L\sigma_j(\mu_i + \mu_k) + \xi\sigma_j^2\} = A \exp(K\mu_i \mu_k), \quad (3)$$

where

$$A^2 = (1+2\, e^{\xi} \cosh 2L)(1+2e^{\xi}),$$
$$e^{2K} = (1+2\, e^{\xi} \cosh 2L)/(1+2e^{\xi}). \quad (4)$$

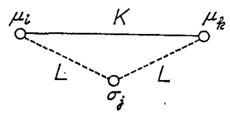

Fig. 2. Extended iteration process.

Therefore

$$\Xi(\xi, L) = A^{2N} Z_s(K), \quad (5)$$

where N is the number of the edge points, and $Z_s(K)$ is the partition function for the square lattice.

The mean number n of magnetic ions on the D sublattice is

$$n = \frac{\partial \ln \Xi}{\partial \xi} = 2N \frac{\partial \ln A}{\partial \xi} + \frac{\partial \ln Z_s(K)}{\partial K} \frac{\partial K}{\partial \xi}. \quad (6)$$

Introducing the notations p and ε by

$$n/2N = p, \qquad 1/2N \cdot \partial \ln Z_s/\partial K = \langle \mu\mu' \rangle \equiv \varepsilon \quad (7)$$

which represent the concentration of magnetic ions on the D sublattice and the nearest neighbor spin correlation respectively, we have from (6)

$$p = \frac{1-e^{-2K}}{2(\cosh 2L - 1)} \{\cosh 2L(1+\varepsilon) + e^{2K}(1-\varepsilon)\}. \quad (8)$$

Solving for $\cosh 2L$, we have

$$\cosh 2L = \frac{2p + (e^{2K}-1)(1-\varepsilon)}{2p - (1-e^{-2K})(1+\varepsilon)} \quad (9)$$

which is reduced to the formula for the ordinary Ising lattice (iteration process) when $p=1$,

$$\cosh 2L = e^{2K}. \quad (10)$$

Corresponding to the critical point K_c for the square lattice, we can determine

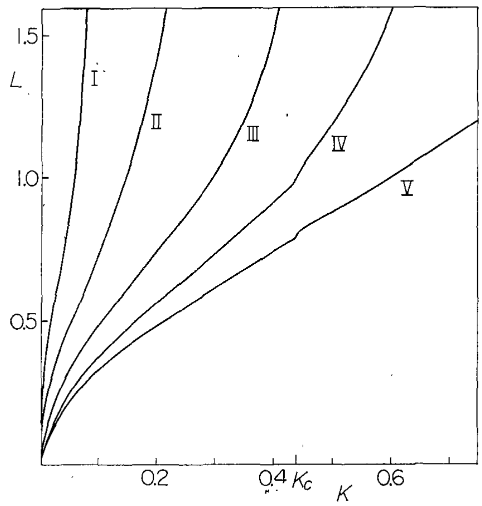

Fig. 3. Relation between K and L.
(I) $p=0.1$, (II) $p=0.25$, (III) $p=0.5$, (IV) $p=0.75$, (V) $p=0.95$.

the critical point L_c for the dilute ferromagnet. Using the data on the square lattice[1)]

$$\exp(-2K_c)=\sqrt{2}-1, \quad \varepsilon_c=\sqrt{2}/2 \qquad (11)$$

in (9), we have

$$\cosh 2L_c=1+\sqrt{2}/(2p-1). \qquad (12)$$

As p decreases from 1, L_c increases (i.e. $T_c \equiv J/2kL_c$ decreases) untill L_c becomes infinite (i.e. T_c becomes zero) when $p=1/2$. This value 1/2 for p is called the critical concentraton and designated by p_c. For p smaller than p_c, L becomes infinite for K smaller than K_c and K cannot attain K_c. Therefore, there occurs no phase change.

The same reasoning may be applied to several kinds of decorated lattices. In every case, the formula for p_c is given by

$$p_c=(1-\exp(-2K_c))(1+\varepsilon_c)/2, \qquad (13)$$

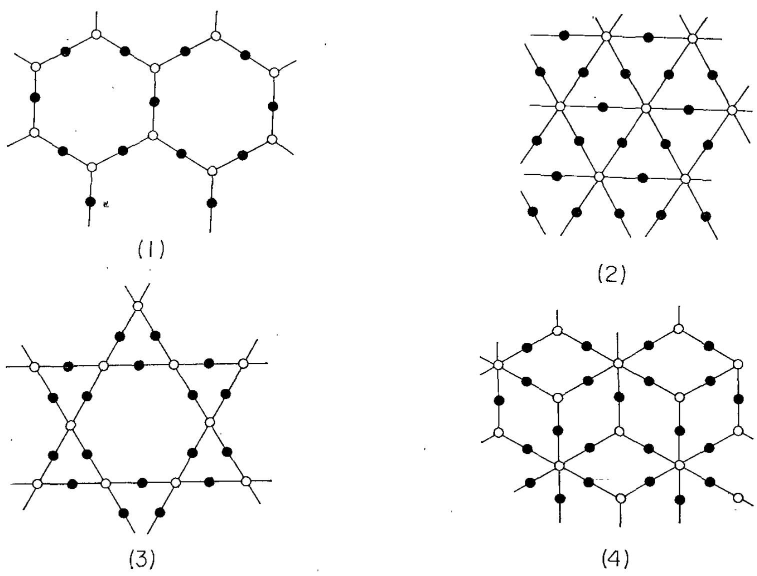

Fig. 4. Decorated lattices.
(1) Honeycomb (2) Triangular (3) Kagomé (4) Diced
○ : M sublattice ● : D sublattice

which is obtained by equating the denominator of (9) to zero at K_c.

It is interesting to see that the sum of the critical concentrations for two decorated lattices, whose original lattices are dual to each other (e.g. the honeycomb and the triangular lattices, the Kagomé and the diced lattices), is unity. Therefore, the critical concentration for a decorated square lattice becomes 1/2

Table I. The critical concentrations p_c for decorated lattices and the critical data for original lattices.

	Sq.	Hon.	Tri.	Kag.	Dice.	Diam.	S.C.	B.C.	F.C.
$\exp(-2K_c)$	$\sqrt{2}-1$	$2-\sqrt{3}$	$1/\sqrt{3}$	0.3933	0.4354	0.477	0.641	0.727	0.815
ε_c	$\sqrt{2}/2$	$4\sqrt{3}/9$	2/3	0.7440	0.6684	0.57	0.357	0.268	0.244
p_c	0.5	0.6478*)	0.3522	0.5290	0.4710	0.410	0.243	0.172	0.114

*) The value in the letter I is erroneous.

because of the self-duality of the square lattice. This theorem may be proved as follows. The partition function $Z(K)$ for a lattice and the partition function $Z^*(K^*)$ for the dual lattice are connected by the well-known Kramers-Wannier relation[5)]

$$Z(K)=2^{N-1-s/2}(\sinh 2K)^{s/2}Z^*(K^*), \qquad (14)$$

where N is the number of vertices of the former lattice and K and K^* are connected by the dual relations

$$\sinh 2K\ \sinh 2K^*=1,\ \cosh 2K\ \tanh 2K^*=\cosh 2K^*\ \tanh 2K=1$$

$$e^{-2K^*}=\tanh K,\ e^{-2K}=\tanh K^*. \qquad (15)$$

The bond number s is common between the two lattices. Putting

$$1/s\cdot\partial\ln Z(K)/\partial K=\varepsilon, \qquad 1/s\cdot\partial\ln Z^*(K^*)/\partial K^*=\varepsilon^* \qquad (16)$$

and differetiating the logarithm of (14) with respect to K, we have

$$\varepsilon=\coth 2K-\varepsilon^*/\sinh 2K, \qquad (17)$$

where we have used the relation

$$dK^*/dK=-1/\sinh 2K=-\sinh 2K^*, \qquad (18)$$

which is obtained from (15).

On the other hand, the critical concentration p_c^* for the decorated lattice of the dual lattice is

$$p_c^*=(1-\exp(-2K_c^*))(1+\varepsilon_c^*)/2. \qquad (19)$$

By (13), (15) and (19), we get

$$2(p_c+p_c^*-1)=\{1-\exp(-2K_c)\}(\varepsilon_c+\varepsilon_c^*/\sinh 2K_c-\coth 2K_c). \qquad (20)$$

Since the right-hand side of (20) is zero from (17), we have completed the proof.

The critical concentrations for decorated lattices are shown in the Table I. The critical data on the three-dimensional lattices shown there are those obtained by the approximation methods.[5),6)]

The internal energy per bond for the dilute ferromagnet is given by $-J/2\times\langle\sigma\mu\rangle$, where $\langle\sigma\mu\rangle$ means the nearest neighbor spin correlation. Partially differentiating the grand partition function (5) with respect to L, we have for the decorated square lattice

$$\langle\sigma\mu\rangle = 1/4N\cdot\partial \ln \Xi/\partial L = 1/2\cdot\{\partial \ln A/\partial L + \varepsilon\partial K/\partial L)$$
$$= (1-e^{-2K})(1+\varepsilon)\sinh 2L/2(\cosh 2L-1). \tag{21}$$

At the absolute zero ($L\to\infty$), we have, from (9), $\langle\sigma\mu\rangle = p$, as expected. The specific heat per bond $C(\equiv kL^2 d\langle\sigma\mu\rangle/dL)$ is given by

$$C = kL^2\left[\frac{1}{2}\coth 2L\left\{(1-e^{-2K})\frac{d\varepsilon}{dK} + 2e^{-2K}(1+\varepsilon)\right\}\frac{dK}{dL} - \frac{(1-e^{-2K})(1+\varepsilon)}{\cosh 2L-1}\right], \tag{22}$$

where

$$dL/dK = \coth 2L[1 + \{(1-p)(\cosh 2K-1)d\varepsilon/dK - 2p(p-1)\}$$
$$\times\{2p(p-1) - (1-\varepsilon^2)(\cosh 2K-1) + 2p(\cosh 2K - \varepsilon\sinh 2K)\}^{-1}]. \tag{23}$$

From Onsager's solution for the square lattice, we have

$$\varepsilon = \coth 2K(\pi/2 + k'\boldsymbol{K}(k))/\pi \tag{24}$$

$$d\varepsilon/dK = \coth^2 2K\{2\boldsymbol{K}(k) - 2\boldsymbol{E}(k) - (1-k')(\pi/2 + k'\boldsymbol{K}(k)\}/\pi,$$

where

$$k = 2\sinh 2K/\cosh^2 2K,\quad k' = \pm(1-k^2)^{1/2} = 2\tanh^2 2K - 1,$$

$$\boldsymbol{K}(k) = \int_0^{\pi/2}(1-k^2\sin^2\varphi)^{-1/2}d\varphi,$$

$$\boldsymbol{E}(k) = \int_0^{\pi/2}(1-k^2\sin^2\varphi)^{1/2}d\varphi. \tag{25}$$

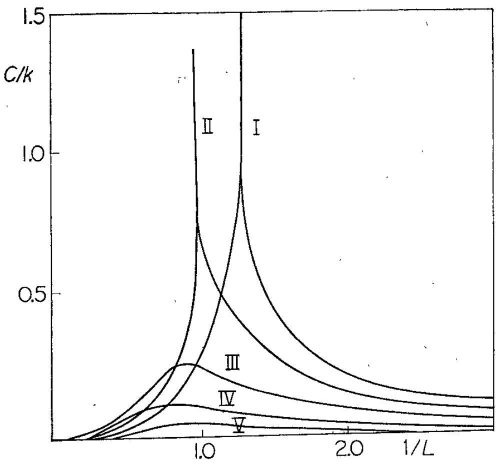

Fig. 5. Specific heat per bond for decorated square lattice.
(I) $p=0.95$, (II) $p=0.75$, (III) $p=0.5$, (IV) $p=0.25$, (V) $p=0.1$.

As is well known, at the critical point $K = K_c = -(1/2)\ln(\sqrt{2}-1)$, ε is finite but $d\varepsilon/dK$ becomes logarithmically infinite. Accordingly, by (22), the $C/k-1/L$ curve for the dilute ferromagnet has a cusp with vertical tangent at the critical point L_c when $1/2<p<1$. The value of C/k at the critical point L_c is given by

$$(C/k)_c = L_c^2(p-1/2)\{\sqrt{2}(3p-2)+1\}/(1-p), \tag{26}$$

which becomes infinity as p approaches 1.

Table II. The specific heat per bond at the critical temperature for each value of p.

p	0.5	0.55	0.6	2/3	0.7	0.75	0.8	0.85	0.9	0.95	0.99	1
$1/L_c$	0	0.586	0.720	0.854	0.912	0.991	1.063	1.130	1.192	1.252	1.297	1.308
C/k	0	0.163	0.345	0.684	0.914	1.378	2.077	3.248	5.595	12.64	69.04	∞

§ 3. Honeycomb lattice and diced lattice

Let us divide a honeycomb lattice into two equivalent sublattices: the M sublattice and the D sublattice. In this case, by using the extended star-triangular transformation (Fig. 6), we get

$$\sum_{\sigma=0}^{\pm 1}\exp\{L\sigma(\mu_1+\mu_2+\mu_3)+\xi\sigma^2\}=A\exp\{K(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)\}, \tag{27}$$

where

$$A^4=(1+2e^{\xi}\cosh 3L)(1+2e^{\xi}\cosh L)^3,$$
$$e^{4K}=(1+2e^{\xi}\cosh 3L)/(1+2e^{\xi}\cosh L). \tag{28}$$

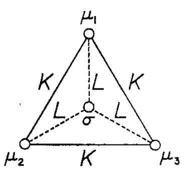

Fig. 6. Extended star-triangular transformation.

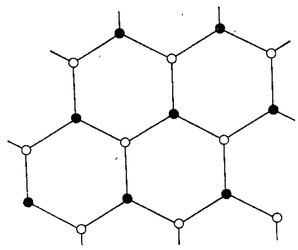

Fig. 7. Semi-dilute honeycomb lattice.
○ : M sublattice ● : D sublattice

Thus, the relation between the grand partition function $\Xi_h(\xi, L)$ for a semi-dilute honeycomb lattice (shown in Fig. 7) and the partition function $Z_t(K)$ for a triangular lattice is

$$\Xi_h(\xi, L)=A^{N/2}Z_t(K), \tag{29}$$

where N is the total number of lattice points for the honeycomb lattice. The mean number n of magnetic ions on the D sublattice is

$$n=\frac{\partial \ln \Xi_h}{\partial \xi}=\frac{N}{2}\frac{\partial \ln A}{\partial \xi}+\frac{\partial \ln Z_t}{\partial K}\frac{\partial K}{\partial \xi}. \tag{30}$$

The concentration p of the magnetic ions on the D sublattice and the nearest neighbor spin correlation $\langle\mu\mu'\rangle\equiv\varepsilon$ are given by

$$p=n/(N/2), \quad \varepsilon=\frac{\partial \ln Z_t}{\partial K}\Big/\frac{3N}{2}. \tag{31}$$

Then Eq. (30) becomes

$$p=\frac{\partial \ln A}{\partial \xi}+3\varepsilon\frac{\partial K}{\partial \xi}. \tag{32}$$

By (28), this becomes

$$p=\frac{1-e^{-4K}}{8(\cosh 2L-1)}\{(2\cosh 2L-1)(1+3\varepsilon)+3e^{4K}(1-\varepsilon)\}, \tag{33}$$

that is

$$2\cosh 2L-1=\frac{4p+3(e^{4K}-1)(1-\varepsilon)}{4p-(1-e^{-4K})(1+3\varepsilon)}. \tag{34}$$

The critical concentration p_c is given by

$$p_c=(1-\exp(-4K_c))(1+3\varepsilon_c)/4, \tag{35}$$

where K_c and ε_c are the critical values of K and ε, respectively. Using the critical data $\exp 4K_c=3$, $\varepsilon_c=2/3$ for the triangular lattice, we obtain

$$\cosh 2L_c=1+1/(2p-1), \tag{36}$$

which determines the critical concentration p_c to be 1/2.

The same reasoning can be applied to the decorated honeycomb lattice in which the vertical points are regarded as the D sublattice points and also to the diced lattice, as shown in Fig. 8, (1) and (2). In the former case, Eq. (35) is

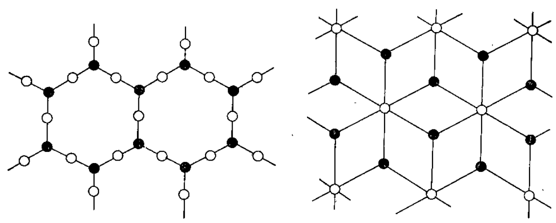

Fig. 8. (1) Decorated honeycomb lattice (2) Diced lattice.
○ : M sublattice ● : D sublattice

valid if we use the critical data on the Kagomé lattice $(\exp(4K_c)=3+2\sqrt{3}$, $\varepsilon_c=(1+2\sqrt{3})/6)$. For the diced lattice, however, as the bond parameter for the triangular lattice formed by the extended star-triangular transformation is $2K$, we have

$$\Xi_d(\xi, L)=A^{N/3}Z_t(2K), \tag{37}$$

where N is the number of vertices of the diced lattice. Putting

$$p=(N/3)^{-1}\partial \ln \Xi_d/\partial\xi,\ \varepsilon=(N/2)^{-1}\partial \ln Z_t(2K)/\partial K, \tag{38}$$

we have the same formulas as $(32)\sim(35)$.

In this case, however, the critical data are given by

$$\exp\{-4(2K_c)\}=1/3,\ \varepsilon_c=2/3, \tag{39}$$

which results from the double bonds of the triangular lattice.

Table III. The critical concentrations p_c for lattices in Figs. 7 and 8 (1), (2).

	Honey.	Dec. Honey.	Diced
p_c	0.500	0.683*)	0.317

The internal energy per bond for the semi-dilute honeycomb lattice is given by $-J/2\langle\sigma\mu\rangle$, where $\langle\sigma\mu\rangle$ is the nearest neighbor spin correlation given by

$$\begin{aligned}\langle\sigma\mu\rangle &= (3N/2)^{-1}\partial \ln \Xi_h/\partial L = 1/3\cdot\partial \ln A/\partial L+\varepsilon\partial K/\partial L \\ &= (1-e^{-4K})\{(2\cosh 2L+1)(1+3\varepsilon)+e^{4K}(1-\varepsilon)\}/8\sinh 2L. \end{aligned}\tag{40}$$

At the absolute zero of temperature ($L\to\infty$), we have $\langle\sigma\mu\rangle=p$ as expected. In Fig. 9, $\langle\sigma\mu\rangle/p$ is plotted against $1/L$ for several values of p. The specific heat per bond C is given by

$$\begin{aligned}C=&-\frac{kL^2(1-e^{-4K})}{2(\cosh 2L-1)}\left\{1+3\varepsilon+\frac{\cosh 2L\, e^{4K}}{(2\cosh 2L+1)}(1+\varepsilon)\right\}+\frac{kL^2}{2\sinh 2L}\\ &\times\left[(2\cosh 2L+1)\left\{e^{-4K}(1+3\varepsilon)+\frac{3}{4}(1-e^{-4K})\frac{d\epsilon}{dK}\right\}+e^{4K}(1-\varepsilon)\right.\\ &\left.\qquad -\frac{1}{4}(e^{4K}-1)\frac{d\varepsilon}{dK}\right]\frac{dK}{dL},\end{aligned}\tag{41}$$

where

$$\begin{aligned}dL/dK=&(2\cosh 2L-1)(\sinh 2L)^{-1}[1+\{3(\cosh 4K-1)(1-p)d\varepsilon/dK\\ &-8p(p-1)\}\{8p(p-1)-3(\cosh 4K-1)(1-\varepsilon)(1+3\varepsilon)+2p(e^{-4K}+3e^{4K})\\ &-12p\varepsilon\sinh 4K\}^{-1}].\end{aligned}\tag{42}$$

The nearest neighbor spin correlation $\varepsilon\equiv\langle\mu\mu'\rangle$ for the triangular lattice**) is given by

$$\varepsilon=\frac{2}{3\pi}\coth 2K\left(\frac{\pi}{2}+\frac{e^{4K}(e^{4K}-3)\boldsymbol{K}(k)}{(e^{4K}-1)(y^2-3+2\sqrt{3+2y})^{1/2}}\right), \tag{43}$$

where

$$k^2=\frac{4\sqrt{3+2y}}{y^2-3+2\sqrt{3+2y}},\quad y=\frac{e^{8K}+3}{2(e^{4K}-1)}. \tag{44}$$

*) The value in the letter I is erroneous.

**) The expressions presented here are brought to those given by Houtappel[7)] by a Landen transformation.

As K approaches $K_c=(1/4)\ln 3$, ε remains finite, but $d\varepsilon/dK$ becomes infinite as $\ln(K-K_c)$. Thus at $L=L_c$ corresponding to $K=K_c$, the specific heat C remains finite and has a cusp with vertical tangent.

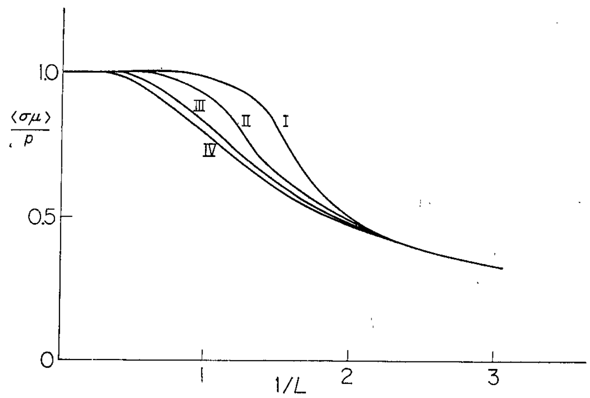

Fig. 9. Plott of $\langle\sigma\mu\rangle/p$ vs. $1/L$.
(I) $p=1$, (II) $p=0.75$, (III) $p=0.5$, (IV) $p=0.25$.

§ 4. Critical concentrations for multiply-decorated lattices

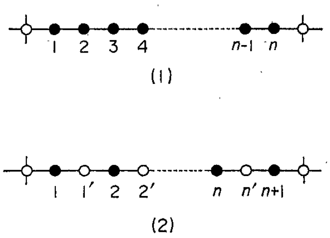

Fig. 10. A part of multiply decorated lattices.
○ ; M sublattice
● ; D sublattice

Finally, we consider the critical concentrations for some multiply decorated lattices. For a n-ply decorated lattice (shown in Fig. 10, (1)) in which there are n D sublattice points on every bond of original lattice, the critical concentration is independent of n. This will be proved as follows. By the extended iteration process for the n-ple decorations

$$\sum_{\sigma_n=0}^{\pm 1}\cdots\sum_{\sigma_1=0}^{\pm 1}\exp\{L(\mu\sigma_1+\sigma_1\sigma_2+\cdots+\sigma_n\mu')+\xi(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)\}=A\exp(K\mu\mu'), \quad (45)$$

we have

$$Ae^K=1+e^{\xi}G_1(L)+e^{2\xi}G_2(L)+\cdots\cdots+e^{n\xi}G_n(L),$$
$$Ae^{-K}=1+e^{\xi}F_1(L)+e^{2\xi}F_2(L)+\cdots\cdots+e^{n\xi}F_n(L), \quad (46)$$

where $G_m(L)$ and $F_m(L)$ are the partition functions for m magnetic ions distributed on n linear sites provided that μ and μ' are fixed at the value 1 or -1. The highest powers of e^L contained in $G_m(L)$ or $F_m(L)$ is e^{mL} when $m<n$. It is easily obtained that

$$G_n(L)=\{(e^L+e^{-L})^{n+1}+(e^L-e^{-L})^{n+1}\}/2,$$
$$F_n(L)=\{(e^L+e^{-L})^{n+1}-((e^L-e^{-L})^{n+1}\}/2. \quad (47)$$

As before, we can obtain the equations

$$e^{2K}=\{1+e^{\xi}G_1(L)+\cdots+e^{n\xi}G_n(L)\}/\{1+e^{\xi}F_1(L)+\cdots+e^{n\xi}F_n(L)\}, \tag{48}$$

$$p=\frac{1}{2n}\left\{\frac{e^{\xi}G_1(L)+\cdots+ne^{n\xi}G_n(L)}{1+e^{\xi}G_1(L)+\cdots+e^{n\xi}G_n(L)}(1+\varepsilon)+\frac{e^{\xi}F_1(L)+\cdots+ne^{n\xi}F_n(L)}{1+e^{\xi}F_1(L)+\cdots+e^{n\xi}F_n(L)}(1-\varepsilon)\right\}, \tag{49}$$

where p denotes the concentration of magnetic ions on the D sublattice and ε denotes the nearest neighbor spin correlation for the original lattice. Using the critical data K_c and ε_c on the original lattice, we can get the critical concentration p_c by making $L\to\infty$. As $\exp(2K_c)$ is finite, we must make $\xi\to-\infty$ in the following manner:

$$\exp\{n\xi+(n+1)L\}\to M \quad \text{(a finite value)},$$

$$\exp(m\xi+m'L)\to 0 \quad (m'\leqq m).$$

Thus we have, from (48) and (49),

$$\exp(2K_c)=1+M,$$

$$p_c=\{M/(2+2M)\}(1+\varepsilon_c)=\{1-\exp(-2K_c)\}(1+\varepsilon_c)/2, \tag{50}$$

which is equivalent to (13).

Next we consider a decorated lattice (shown in Fig. 10, (2)), in which there are n M sublattice points and $n+1$ D sublattice points on every bond of the original lattice. To know the critical concentration, we can use the formula (13) in which K_c and ε_c are replaced by L_c and ε_c', i.e. those for the n-ply decorated Ising[8] lattice given by

$$(\coth L_c)^{n+1}=\coth K_c$$

$$\varepsilon_c'=\frac{1}{2}\left\{\frac{\coth^n L_c+\coth L_c}{\coth K_c+1}(1+\varepsilon_c)+\frac{\coth^n L_c-\coth L_c}{\coth K_c-1}(1-\varepsilon_c)\right\};$$

where K_c and ε_c are the critical data on the original lattice.

Table IV gives p_c for several values of n. $n=0$ corresponds to the case discussed in § 2.

Table IV. Critical concentrations for the lattices in Fig. 10, (2).

n	0	1	2	3	∞
Sq.	0.5000	0.6957	0.7826	0.8312	1
Tri.	0.3522	0.5731	0.6855	0.7517	1
Hon.	0.6478	0.7998	0.8605	0.8930	1

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References

1) L. Onsager, Phys. Rev. **65** (1944), 117.
2) C. N. Yang, Phys. Rev. **85** (1952), 1332.
3) M. E. Fisher, Proc. Roy. Soc. A. **252** (1960), 66.
4) R. Brout, Phys. Rev. **115** (1959), 824.
G. S. Rushbrooke and D. J. Morgan, Mol. Phys. **4** (1961), 1.
G. S. Rushbrooke, J. Math. Phys. **5** (1964), 1106.
R. Abe, Prog. Theor. Phys. **31** (1964), 412.
5) C. Domb, Adv. in Phys. (Phil. Mag. Supp.) **9** (1960), 149.
6) J. W. Essam and M. F. Sykes, Physica **29** (1963), 378.
7) R. M. F. Houtappel, Physica **16** (1950), 341.
8) I. Syozi, Rev. Kobe Univ. of Mercantile Marine **2** (1955), 21.