A STATISTICAL MODEL WHICH COMBINES FEATURES OF FACTOR ANALYTIC AND ANALYSIS OF VARIANCE TECHNIQUES*

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This paper describes a method of matrix decomposition which retains the ability of factor analytic techniques to summarize data in terms of a relatively low number of coordinates; but at the same time, does not sacrifice the useful analysis of variance heuristic of partitioning data matrices into independent sources of variation which are relatively simple to interpret. The basic model is essentially a two-way analysis of variance model which requires that the matrix of interaction parameters be decomposed by using factor analytic techniques. Problems of judging statistical significance are discussed; and an illustrative example is presented.

Analysis of variance methods of decomposing matrices provide one of the most powerful methods presently available to aid one in understanding matrices of data. Another set of powerful tools which often help one to understand data is based on factor analytic techniques. Tukey [1962] has emphasized the potential importance of work showing relationships between these two techniques and in using each technique to complement the other. Literature which is directly relevant to this important problem is surprisingly scant. With the notable exceptions of work by Tukey [1962], Creasy [1957], and Burt [1947, 1966], investigators have usually used *either* factor analytic techniques *or* analysis of variance techniques, but have rarely used both in combination. In the present paper, relations between factor analytic and analysis of variance techniques are discussed and features of both techniques are combined to form a powerful method for decomposing two-way tables.

The basic purpose of factor analytic techniques is to reduce the dimensionality of the data by expressing it in terms of new coordinates. In addition, it is generally hoped that the new coordinates will describe meaningful "dimensions" of the original data. Various types of rotation of the coordinate axes are often employed in seeking such meaningful dimensions.

Standard analysis of variance methods partition data matrices into independent terms which are relatively simple to interpret; i.e., the grand mean, main effects which represent the degree to which row elements exert an effect over all columns and vice versa, and interaction terms which represent the degree to which the combined effect of a given row and column element is different from the sum of their individual main effects.

The method of matrix decomposition described in this paper retains the ability of factor analytic techniques to summarize data in terms of a relatively low number of coordinates; but at the same time, does not sacrifice the relative ease of interpretation which standard analysis of variance models afford. The basic model described here is essentially a two-way analysis of variance model which requires that the interaction parameters be decomposed by factor analytic techniques. Tukey [1962] has remarked that this type of procedure might provide a powerful tool for many types of data analysis. The present paper expands upon the basic idea of using factor analytic techniques to study structure in interaction, and works out the details for both a fixed model and a mixed model version of the technique. Approximate methods for judging the statistical significance of factors are discussed; and exact significance tests for judging "factor similarity" are presented. The paper is concluded with a numerical example.

1. The Factor Analytic Decomposition of a Matrix

Throughout this paper the terms "factor analytic techniques" and "factor model" are used in a generic sense to refer to matrix decomposition methods which involve solving for eigenvalues and eigenvectors. It is necessary to keep in mind that there are important differences between "factor model" in the present sense, and the more common "Factor Analysis model" which refers specifically to a decomposition method which involves inter-correlating variables and estimating communalities. This paper does not explicitly consider Factor Analysis models in the narrow sense; but rather, is concerned with the "factor analytic" decomposition described below.

Up to the point where we begin discussing tests of statistical significance, we shall treat the data matrix, X, as though its entries represent "true" population parameters, rather than being estimates of such parameters. Thus, initially we shall write the factor analytic and all other model equations without including terms for error.

Factor analytic techniques for rectangular matrices are based on the fact (see Horst, [1963]) that any two-way matrix, X, can be expressed as

(1.1) $$X = ADB'$$

where

$X = $ a $J \times K$ matrix with elements $\{x_{jk}(j=1, 2, \cdots, J; k=1, 2, \cdots, K)\}$,

A = a $J \times N$ orthonormal matrix (i.e., $A'A = I$, where I represents the identity matrix) with elements α_{jn} $(n = 1, 2, \cdots, N)$,

D = a $N \times N$ diagonal matrix with elements $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n \geq \cdots \geq d_N$,

B = a $K \times N$ orthonormal matrix (i.e., $B'B = I$) with elements β_{kn},

N = the rank of X, and if for convenience we let $J \leq K$ we have $N \leq J \leq K$.

The matrices A, D, and B of (1.1) can be obtained by solving first for the characteristic vectors, and characteristic roots of the $J \times J$ matrix XX'. The $J \times N$ matrix A then consists of the characteristic vectors, and the $N \times N$ diagonal matrix D consists of the square roots of the characteristic roots of XX'. The $K \times N$ matrix B can then be found by solving the equation,

$$(1.2a) \qquad\qquad B = X'AD^{-1}.$$

The above solution specifies that the matrices D and A be found by solving for the eigenvalues and eigenvectors of the matrix XX' and that then the matrix B be obtained from (1.2a). It is also possible to solve for the matrices D and B by finding the eigenvalues and eigenvectors of the matrix $X'X$ and then obtaining A from

$$(1.2b) \qquad\qquad A = XBD^{-1}.$$

Thus, for ease of calculation it is convenient to solve for the eigenvalue-and eigenvectors of whichever matrix, XX' or $X'X$, has the smaller dsi mensions.

One of the primary reasons that the factor analytic model provides a very powerful method for decomposing matrices is based on the Eckart–Young [1936] theorem which states that the matrix of rank r which provides the best estimate, in the least squares sense, of X is obtained by using (1.1) and simply retaining only the first r columns of the matrices A and B and using only the first r terms in D. In order to facilitate later discussion, (1.1) is now rewritten in expanded form as:

$$(1.3) \qquad x_{jk} = d_1\alpha_{j1}\beta_{k1} + d_2\alpha_{j2}\beta_{k2} + \cdots + d_n\alpha_{jn}\beta_{kn} + \cdots + d_N\alpha_{jN}\beta_{kN}.$$

Another important characteristic of the factor analytic decomposition (1.3) is that every set of terms $\{d_n\alpha_{jn}\beta_{kn}$ $(j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ is orthogonal to every other set of terms $\{d_{n'}\alpha_{jn'}\beta_{kn'}$ $(j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ that is,

$$(1.4) \qquad\qquad \sum_j \sum_k (d_n\alpha_{jn}\beta_{kn})(d_{n'}\alpha_{jn'}\beta_{kn'}) = 0,$$

for all values of n and n' where $n \neq n'$, and the symbol $\sum_j$ is used to signify $\sum_{j=1}^{J}$. The fact that the sets of terms in the factor model (1.3) are orthogonal

to each other can be proven easily by rearranging (1.4) to obtain

$$(1.5) \qquad d_n d_{n'} (\sum_j \alpha_{jn} \alpha_{jn'})(\sum_k \beta_{kn} \beta_{kn'}) = 0.$$

Writing $\sum_j \alpha_{jn} \alpha_{jn'}$ for all values of n and n' in matrix terms gives $A'A$, and $\sum_k \beta_{kn} \beta_{kn'}$ is written as $B'B$. Since A and B are orthonormal matrices we know, by definition, that $A'A = B'B = I$. Hence, for all values of n and n' where $n \neq n'$, (1.5) reduces to the product of a constant $(d_n d_{n'})$ times zero, which, of course, is zero. Thus, we see that the N sets of terms of the factor analytic model (1.3) are mutually orthogonal.

In later discussion it will be important that we know how the matrices A and B of the factor analytic decomposition differ depending on whether the matrix X is factored without prior modification, converted to a row or column-centered matrix (i.e., forcing all row or column sums, respectively, to equal zero), or is doubly-centered. We will now show that when a row-centered matrix, Z, is factored, the matrix B of the factor analytic model (1.1) is column-centered, and hence that $\sum_k \beta_{kn} = 0$ for all n. Premultiplying both sides of (1.2), which allows one to solve for B when given A, D, and $X = Z$, by a $1 \times K$ (row) vector of unities, U', gives

$$(1.6) \qquad (U'Z')AD^{-1} = U'B.$$

Since Z is row-centered, Z' is column-centered and $U'Z'$ is a $1 \times J$ (row) vector of zeroes. Clearly, the left side of (1.6) is a $1 \times N$ (row) vector of zeros and B must therefore be a column-centered matrix. It is also easy to show that when a column-centered matrix is factored, the matrix A is column-centered. It then follows readily that when a doubly-centered matrix is factored, the columns of both A and B sum to zero.

The Basic FANOVA Model

Since the model described here combines features of both factor analytic (FA) techniques and analysis of variance (ANOVA) techniques, we refer to it as the "FANOVA" (factor analysis of variance) model for decomposing two-way matrices. The standard analysis of variance model for decomposing a two-way table is,

$$(1.7) \qquad x_{jk} = \mu + R_j + C_k + \gamma_{jk},$$

where μ (the grand mean) is a constant, $\sum_j R_j = \sum_k C_k = 0$, and $\sum_j \gamma_{jk} = \sum_k \gamma_{jk} = 0$ for all j and k. The $\{R_j\}$ and $\{C_k\}$, respectively, represent the row and column main effects and the $\{\gamma_{jk}\}$ represent the interaction parameters. (The brace notation $\{\ \}$ denotes the *set* of quantities indicated: e.g., $\{\gamma_{jk}\}$ refers to the set consisting of the JK values, γ_{jk} with $j = 1, 2, \cdots, J$; $k = 1, 2, \cdots, K$.) Expressing the analysis of variance parameters, μ, $\{R_j\}$, $\{C_k\}$, and $\{\gamma_{jk}\}$, in terms of the $\{x_{jk}\}$ yields the equations:

(1.8) $$\mu = x_{..} ,$$

(1.9) $$R_j = x_{j.} - x_{..} ,$$

(1.10) $$C_k = x_{.k} - x_{..} ,$$

(1.11) $$\gamma_{jk} = x_{jk} - x_{j.} - x_{.k} + x_{..} ,$$

where a dot replacing a subscript indicates that an arithmetic mean has been taken over the entire range of the replaced subscript. (The dot notation is used frequently throughout this paper.)

The basic FANOVA model is essentially a two-way analysis of variance model which requires that the matrix of interaction parameters $\{\gamma_{jk}\}$ be expressed as the sum of several successive multiplicative contrasts such that each contrast is orthogonal to all previous contrasts and accounts for a maximum of the remaining variance of the $\{\gamma_{jk}\}$. A contrast among the interaction parameters $\{\gamma_{jk}\}$ is a linear function of the $\{\gamma_{jk}\}$

(1.12) $$\Psi = \sum_j \sum_k w_{jk}\gamma_{jk} = \sum_j \sum_k w_{jk}x_{jk} ,$$

where the $\{w_{jk}\}$ are (known) constants subject to the restrictions $\sum_j w_{jk} = \sum_k w_{jk} = 0$. A normalized interaction contrast meets the additional condition, $\sum_j \sum_k w_{jk}^2 = 1$. A multiplicative contrast is here defined as a contrast among the $\{\gamma_{jk}\}$ such that,

(1.13) $$w_{jk} = p_j q_k ,$$

where, of course, $\sum_j p_j = \sum_k q_k = 0$.

Since the matrix of $\{\gamma_{jk}\}$ is a doubly-centered matrix (and by convention $J \leq K$), it has a maximum rank of $(J - 1)$ and hence, when expressed in terms of the factor model, can be prefectly reproduced by

(1.14) $$\gamma_{jk} = \sum_n^{J-1} d_n\alpha_{jn}\beta_{kn} .$$

Since the matrix of $\{\gamma_{jk}\}$ is doubly-centered, the matrices A and B of the factor model decomposition will be column-centered. Thus, $\sum_j \alpha_{jn} = \sum_k \beta_{kn} = 0$ for all values of n; and therefore, the terms $\{\alpha_{jn}\beta_{kn}\ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ can be thought of as defining a multiplicative contrast among the interaction parameters $\{\gamma_{jk}\}$. Since this is true for all values of n, we refer to the $(J - 1)$ sets of terms, $\{d_n\alpha_{jn}\beta_{kn}\ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$, in (1.14) as *interaction factors*.

It is often convenient to express the factor resolution (1.14) of the $\{\gamma_{jk}\}$ as the product of two terms, rather than three. In matrix notation we write (1.14) as

(1.15) $$\Gamma = ADB'$$

where Γ is a $J \times K$ matrix with entries $\{\gamma_{ik}\}$, and the matrices A, D, and B are defined by (1.1). If we now define matrices P and Q' as

$$(1.16) \qquad\qquad P = AD^{1/2} \quad \text{and} \quad Q' = D^{1/2}B',$$

we can express the matrix Γ as

$$(1.17) \qquad\qquad\qquad \Gamma = PQ'.$$

The above definitions of the matrices P and Q are not unique, but it is usually sensible to use the definitions given by (1.16) because they keep the entries of P and Q at approximately the same order of magnitude. Making in the standard analysis of variance model (1.7) the substitution suggested by (1.17), and writing the matrix product PQ' in expanded form gives

$$(1.18) \qquad\qquad x_{ik} = \mu + R_i + C_k + \sum_{n}^{J-1} p_{in}q_{kn} \, .$$

Equation (1.18) expresses a model where the matrix of $\{\gamma_{ik}\}$ is completely factored. However, in applications of factor analytic techniques one is rarely interested in completely factoring a matrix. Rather, it is hoped that an adequate approximation to the matrix can be obtained by extracting a number of factors which is considerably less than the rank of the matrix. In order to take account of this fact we now define the symbol M to refer, not to the rank of a matrix in general, but to the number of factors actually retained in order to approximate the matrix being factored. Under these conditions we can express the initial interaction parameters as

$$(1.19) \qquad\qquad \gamma_{ik} = \sum_{m} p_{im}q_{km} + \phi_{ik} \, ,$$

where $(m = 1, 2, \cdots, M)$ and the $\{\phi_{ik}\}$ represent the parameters of the residual interaction, and meet the conditions $\sum_{i} \phi_{ik} = \sum_{k} \phi_{ik} = 0$ for all j and k. Taking the standard analysis of variance resolution of the matrix X, and substituting the right-hand side of (1.19) for the interaction parameters $\{\gamma_{ik}\}$ yields

$$(1.20) \qquad\qquad x_{ik} = \mu + R_i + C_k + \sum_{m} p_{im}q_{km} + \phi_{ik} \, ,$$

which represents the FANOVA model for decomposing a two-way table. The residual interaction parameters can be obtained by subtraction as in

$$\phi_{ik} = \gamma_{ik} - \sum_{m} p_{im}q_{km} \, ,$$

or, alternatively, by using the relation,

$$\phi_{ik} = \sum_{n=M+1}^{J-1} p_{in}q_{kn} \, .$$

One convenient scheme for presenting the results of a FANOVA analysis

is obtained by expressing (1.20) in matrix notation. Letting Φ represent a $J \times K$ matrix with entries $\{\phi_{jk}\}$, and defining the matrices R and C as

$$R = \begin{bmatrix} \sqrt{\mu} & R_1 & 1.0 \\ \sqrt{\mu} & R_2 & 1.0 \\ \vdots & \vdots & \vdots \\ \sqrt{\mu} & R_j & 1.0 \\ \vdots & \vdots & \vdots \\ \sqrt{\mu} & R_J & 1.0 \end{bmatrix} \qquad C = \begin{bmatrix} \sqrt{\mu} & 1.0 & C_1 \\ \sqrt{\mu} & 1.0 & C_2 \\ \vdots & \vdots & \vdots \\ \sqrt{\mu} & 1.0 & C_k \\ \vdots & \vdots & \vdots \\ \sqrt{\mu} & 1.0 & C_K \end{bmatrix}$$

we express (1.20) as

$$(1.21) \qquad \begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix} = \begin{bmatrix} R & P^* \end{bmatrix} \begin{bmatrix} C' \\ \hline Q^{*\prime} \end{bmatrix} + \begin{bmatrix} & & \\ & \Phi & \\ & & \end{bmatrix},$$

$$\quad J \times K \qquad J \times (3+M) \quad (3+M) \times K \qquad J \times K$$

where the $J \times M$ matrix P^* and the $K \times M$ matrix Q^* consist, respectively, of the first M columns of the matrices P and Q; and where the values written beneath each matrix represent its dimensions.

As mentioned earlier, the basic FANOVA model is essentially a two-way analysis of variance model which requires that factor analytic techniques be used in seeking structure in the matrix of interaction parameters $\{\gamma_{jk}\}$. We shall now discuss some of the relations between the FANOVA method of matrix decomposition and alternative methods of investigating the structure of a matrix of interaction parameters.

2. Alternative Methods for Decomposing Interaction Parameters

First we will consider the case where neither the row nor the column elements are quantitative or naturally ordered. The following procedures all involve partitioning the interaction parameters into a set of highly structured (or systematic) terms and a set of less structured (or unsystematic) residual terms, which are obtained by subtraction. Thus, in his test for nonadditivity in a row $\times$ column design with one observation per cell, Tukey [1949] suggests that the interaction parameters be decomposed into the following "one degree-of-freedom for non-additivity" and $(J-1)(K-1) - 1$ df for residual

$$(2.1) \qquad \gamma_{jk} = gR_jC_k + \theta_{jk},$$

where g is a constant, the $\{R_j\}$ and $\{C_k\}$ are, respectively, the row and column main effects; and the residuals $\{\theta_{jk}\}$ are obtained by subtraction

(i.e., $\theta_{ik} = \gamma_{ik} - gR_iC_k$). Mandel [1961] has proposed a model which is more highly structured than (2.1) and which includes (2.1) as a special case. Mandel's decomposition for the interaction parameters is given by

$$(2.2) \qquad\qquad \gamma_{ik} = gR_iC_k + \lambda_iC_k + \theta'_{ik} ,$$

where $\sum_i R_i\lambda_i = 0$ and, of course, $\sum_i \lambda_i = 0$; and the $\{\theta'_{ik}\}$ are the residuals. Solving for the constant g in (2.2) accounts for one df and solving for the $\{\lambda_i\}$ accounts for $(J - 2)$ df, leaving $(J - 1)(K - 2)$ df which are accounted for by the residual. Finally, Tukey's [1962] "basic vacuum cleaner" includes the systematic portion of Mandel's model and goes a step further, as shown by the decomposition,

$$(2.3) \qquad\qquad \gamma_{ik} = gR_iC_k + \lambda_iC_k + R_i\eta_k + \theta''_{ik} ,$$

where $\sum_k C_k\eta_k = 0$, $\sum_k \eta_k = 0$, and the $\{\theta''_{ik}\}$ are the residuals. It is of interest to note that the three highly structured sets of terms in (2.3) account for $(J + K - 3)$ df, which, as will be shown later, is also the number of df accounted for by the first interaction factor of the FANOVA model.

The decomposition methods suggested by (2.1), (2.2), and (2.3) all possess the limitation that they cannot provide useful information about the structure of the interaction parameters unless there are substantial differences between row means or between column means, or between both. The FANOVA model resolution of the $\{\gamma_{ik}\}$, on the other hand, is applicable irrespective of whether or not substantial main effects exist; and also allows for the possibility that the $\{\gamma_{ik}\}$ are completely independent of any existing row or column main effects. Another advantage of the FANOVA method of decomposing matrices is that it enables one to find up to $(J - 1)$ multiplicative contrasts such that each successive contrast accounts for a maximum of any remaining variation due to row by column interaction.

Tukey [1962] briefly mentions that one natural way to continue the basic vacuum cleaner is provided by factor analytic methods. Depending on such things as the subject-matter and the primary purpose of doing the analysis, it may indeed be advantageous to postpone application of factor analytic techniques until some or all of the interaction variation which is picked up by Tukey's vacuum cleaner is removed.

When the levels of factors in a fixed effects analysis of variance design are quantitative and can be considered as representing equal steps along some underlying continuum; users of analysis of variance often investigate the response surface further by partitioning main effects and interaction terms into orthogonal multiplicative contrasts (trends) such as linear $\times$ linear, linear $\times$ quadratic, quadratic $\times$ linear, etc. [e.g., Winer, 1962; Snedecor, 1956]. In general, this procedure requires $(J - 1)(K - 1)$ orthogonal multiplicative contrasts in order to perfectly reproduce the JK interaction parameters; whereas the FANOVA model accounts for the

interaction perfectly by specifying only $(J - 1)$ orthogonal multiplicative contrasts. For example, in a 3×4 design, six *a priori* contrasts are needed to account for all of the interaction variation; while the FANOVA model accounts for the interaction perfectly by specifying only two orthogonal multiplicative contrasts. Furthermore, in this example the first interaction factor of the FANOVA decomposition will necessarily account for *at least* 50% (and probably much more) of the interaction variation. Although the relative advantage of using the FANOVA decomposition becomes much greater as J $(J \leq K)$ increases and as the size of K relative to J increases; it seems clear that the advantages of using the FANOVA decomposition can be of practical significance even when dealing with small matrices.

Having discussed some of the advantages of the FANOVA model for decomposing two-way matrices, we now turn to a discussion of methods for judging the statistical significance of interaction factors and the residual interaction.

3. *Statistical Tests of Hypotheses in the Fixed FANOVA Model*

Our discussion up to this point has focused on applying the FANOVA model to a matrix, X, whose entries are equal to "true" population parameters. But in practical applications we, of course, do not know the values of the population parameters; but rather, use samples of observed data to estimate the entries in X. In this section we consider problems which arise in applying the FANOVA decomposition to data containing error.

The fixed FANOVA model

A two-way fixed FANOVA model, and its corresponding analysis of variance model, is one in which the levels of both ways of the design are determined by some systematic, non-random procedure. Letting y_{ijk} denote the ith observation $(i = 1, 2, \cdots, I)$ in the j, k cell, we make the following assumptions which are standard in the analysis of variance,

$$(3.1) \qquad\qquad y_{ijk} = x_{jk} + e_{ijk},$$

where the $\{e_{ijk}\}$ represent uncontrolled sources of variation which are independently and normally distributed with zero means and equal variance σ_e^2 for every j, k cell; and are statistically independent of the true cell means, $\{x_{jk}\}$. If for x_{jk} in (3.1) we substitute the standard analysis of variance decomposition of the $\{x_{jk}\}$ (1.7), we obtain the model equation:

$$(3.2) \qquad\qquad y_{ijk} = \mu + R_j + C_k + \gamma_{jk} + e_{ijk}.$$

Least squares estimates of μ, $\{R_j\}$, $\{C_k\}$ and the $\{\gamma_{jk}\}$ in (3.2) are given by

(3.3) $\hat{\mu} = y \ldots ,$

(3.4) $\hat{R}_i = y_{.i.} - y \ldots ,$

(3.5) $\hat{C}_k = y_{..k} - y \ldots ,$

(3.6) $\hat{\gamma}_{ik} = y_{.ik} - y_{.i.} - y_{..k} + y \ldots .$

Replacing the interaction parameters $\{\gamma_{ik}\}$ in (3.2) by their factor model decomposition (1.19), we obtain the following expression of the two-way fixed effects FANOVA model:

$$(3.7) \qquad y_{ijk} = \mu + R_i + C_k + \sum_m p_{im} q_{km} + \phi_{ik} + e_{ijk} ,$$

where all terms are defined and restricted as specified in (1.20) and (3.1). Since the row and column main effects in the FANOVA model are identical in all respects to the main effects in analysis of variance, we can use conventional methods for testing the hypotheses that all $\{R_i\}$ equal zero or that all $\{C_k\}$ equal zero.

Mean squares for interaction factors

Our first step toward developing a rough guide for judging whether or not the mth interaction factor accounts for a statistically significant amount of the variation in the $\{x_{ik}\}$ is to define mean squares for the interaction factors. We will first write the quantities which must be minimized in order to fulfill the conditions specified by the FANOVA model. Let $_f \hat{\phi}_{ik}$ represent the estimated residual interaction terms which result when $f[0 \leq f \leq (J - 1)]$ successive interaction factors have been extracted from the matrix of $\{\hat{\gamma}_{ik}\}$; that is

$$(3.8) \qquad _f \hat{\phi}_{ik} = \hat{\gamma}_{ik} - \sum_m^f \hat{d}_m \hat{\alpha}_{im} \hat{\beta}_{km} ,$$

where, it will be recalled, $\hat{d}_m \hat{\alpha}_{im} \hat{\beta}_{km}$ represents the contribution of the mth interaction factor to the j,kth cell, and can alternatively be written as $\hat{p}_{im} \hat{q}_{km}$. The FANOVA model requires that, for all the values of $f > 0$ ($f = 1, 2, \cdots , J - 1$), we minimize the $(J - 1)$ quantities

$$(3.9) \qquad E_f = \sum_i \sum_k (d_f \alpha_{if} \beta_{kf} - _{(f-1)} \hat{\phi}_{ik})^2 ,$$

where the $\{_{(f-1)} \hat{\phi}_{ik}\}$ are treated as fixed when solving for estimates of the $\{d_f \alpha_{if} \beta_{kf}\}$. Our work is simplified now by recalling that Eckart and Young [1936] proved that least squares estimates of the $\{\alpha_{if}\}$, $\{\beta_{kf}\}$, and $\{d_f\}$ which minimize each of the $(J - 1)$ quantities represented by (3.9) can be found by expressing the matrix of $\{\hat{\gamma}_{ik}\}$ in terms of the decomposition specified by the factor model. Thus we can now write

$$(3.10) \qquad \hat{\Gamma} = \hat{A} \hat{D} \hat{B}'$$

where $\hat{\Gamma}$ represents a $J \times K$ matrix with entries $\{\hat{\gamma}_{ik}\}$, and in general a caret, $\wedge$, indicates that all entries in the matrix are least squares estimates of population parameters.

We will now determine the number of linearly independent parameters which are fit in calculating each interaction factor. The mth interaction factor is subject to the restrictions

$$\sum_j \alpha_{jm} = 0 \qquad \sum_k \beta_{km} = 0,$$

and due to the fact that the A and B matrices of the factor model (1.1) are orthonormal, the mth interaction factor has two restrictions imposed upon it by the requirements that

$$\sum_j \alpha_{jm}^2 = 1 \qquad \sum_k \beta_{km}^2 = 1$$

and finally, also due to the fact that $A'A = B'B = I$, the mth interaction factor is subject to the following $[2(m-1)]$ orthogonality restrictions:

$$\sum_j \alpha_{j1}\alpha_{jm} = 0 \qquad \sum_k \beta_{k1}\beta_{km} = 0$$

$$\sum_j \alpha_{j2}\alpha_{jm} = 0 \qquad \sum_k \beta_{k2}\beta_{km} = 0$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\sum_j \alpha_{j(m-1)}\alpha_{jm} = 0 \qquad \sum_k \beta_{k(m-1)}\beta_{km} = 0.$$

In general then, the mth interaction factor of the FANOVA model has $[4 + 2(m-1)]$ or simplifying, $(2m + 2)$ linearly independent restrictions. Since the mth interaction factor is expressed by the J α_{jm} values + the K β_{km} values + the "regression" weight; d_m ; a total of $(J + K + 1)$ values are used to express the mth interaction factor. Thus, the mth interaction factor accounts for $[(J + K + 1) - (2m + 2)]$ or, simplifying, $(J + K - 1 - 2m)$ **df**. We now define the mean square for the mth interaction factor as

$$(3.11) \qquad \mathrm{MS}_{Fm} = \mathrm{SS}_{Fm}/(J + K - 1 - 2m),$$

where SS_{Fm} represents the sum of squares accounted for by the mth interaction factor and is obtained as described below.

Treating the factor weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ in the FANOVA model as fixed weights which define a normalized contrast Ψ_{Fm} among the interaction parameters, we estimate the "value" of the contrast in the conventional manner [e.g., Scheffé, 1959] by

$$(3.12) \qquad \hat{\Psi}_{Fm} = \sum_i \sum_k \hat{\alpha}_{jm}\hat{\beta}_{km}\hat{\gamma}_{jk} = \sum_i \sum_k \hat{\alpha}_{jm}\hat{\beta}_{km}y_{.jk} .$$

(Context should make it clear whether we are using the symbol Ψ_{Fm} to refer

to the value of a contrast or to the function which defines the contrast.) The sum of squares accounted for by the normalized contrast Ψ_{Fm} is given by

$$(3.13) \qquad\qquad SS_{\Psi_{Fm}} = I\hat{\Psi}_{Fm}^2.$$

We will now show that $\hat{d}_m^2 = \hat{\Psi}_{Fm}^2$; and therefore, in practice, the calculations described by (3.12) will not be carried out since the $\{\hat{d}_m^2\}$ are simply the eigenvalues which are obtained in the process of factoring the matrix $\hat{\Gamma}$. Beginning with the factor analytic decomposition of $\hat{\Gamma}$ (3.10), it can be readily shown that

$$(3.14) \qquad\qquad \hat{D} = \hat{A}'\hat{\Gamma}\hat{B}.$$

Expanding (3.14) gives

$$(3.15) \qquad \hat{d}_m = \sum_i \sum_k \hat{\alpha}_{im}\hat{\beta}_{km}\hat{\gamma}_{ik} = \sum_i \sum_k \hat{\alpha}_{ik}\hat{\beta}_{km}y_{.ik} ,$$

where it is important to remember that we are treating the $\{\hat{\alpha}_{im}\}$ and $\{\hat{\beta}_{km}\}$ as fixed. Since the right-hand sides of (3.12) and (3.15) are identical we see that

$$(3.16) \qquad \hat{d}_m = \hat{\Psi}_{Fm} = \sum_i \sum_k \hat{\alpha}_{im}\hat{\beta}_{km}y_{.ik} .$$

Thus, substitution in (3.13) gives

$$(3.17) \qquad\qquad SS_{Fm} = I\,\hat{d}_m^2 ,$$

and making in (3.11) the substitution suggested by (3.17) yields

$$(3.18) \qquad MS_{Fm} = I\,\hat{d}_m^2/(J + K - 1 - 2m).$$

A mean square for the residual interaction

Having defined plausible mean squares for the interaction factors of the FANOVA model, we now define a mean square for the residual interaction. The sum of squares accounted for by the residual interaction, Fres, depends on the number of interaction factors, M, retained in the model; and is obtained by subtraction in the following manner,

$$(3.19) \qquad SS_{Fres} = SS_{RC} - I\sum_m \hat{d}_m^2 = SS_{RC} - \sum_m SS_{Fm} ,$$

where SS_{RC} represents the sum of squares due to all row by column interaction and is defined as

$$(3.20) \qquad SS_{RC} = I\sum_i \sum_k \hat{\gamma}_{ik}^2 = I\sum_i \sum_k (y_{.ik} - y_{.i.} - y_{..k} + y_{...})^2.$$

Alternatively, when the $\{\gamma_{ik}\}$ have been completely factored, one may wish to compute the residual sums of squares by using the relation

$$(3.21) \qquad\qquad SS_{Fres} = I\sum_{n=M+1}^{J-1} \hat{d}_n^2$$

and checking to see that

(3.22) $$SS_{RC} = \sum_m SS_{Fm} + SS_{Fres} .$$

The number of linearly independent restrictions imposed on the residual interaction also depends on the value of M and is obtained by subtraction. As is well known, $(J - 1)(K - 1)$ **df** are accounted for in calculating SS_{RC} ; and since the mth factor extracted from the matrix of $\{\hat{\gamma}_{jk}\}$ accounts for $(J+K-1-2m)$ **df**, we see that $[(J-1)(K-1)-\sum_m (J+K-1-2m)]$ or simplifying, $(J - 1 - M)(K - 1 - M)$ **df** are accounted for by the residual interaction. Thus we define the mean square for the residual interaction as

(3.23) $$MS_{Fres} = SS_{Fres}/(J - 1 - M)(K - 1 - M).$$

A rough index for judging the significance of interaction factors and the residual interation

A central problem in data analysis is that of finding simple and heuristically useful methods of summarizing variation in experimental data [e.g., Green and Tukey, 1960; Tukey, 1962]. It is in this spirit that we present some rough guides for judging the stability of factor weights and describe some quantities which provide useful summary descriptions of $R \times C$ variation in the data.

One useful measure for summarizing variation of the data is simply the proportion of $R \times C$ interaction variation accounted for by the mth interaction factor (i.e., SS_{Fm}/SS_{RC}). In describing the results of a FANOVA analysis it is, of course, also useful to calculate the "mean squares" given by (3.20) and (3.23), since they provide a measure of how much variation *per estimated parameter* we have accounted for. One might say that a mean square tells us how much sum of squares we have "bought" for each **df** "spent."

Now we shall consider a rough index to aid us in judging the statistical significance of interaction factors. Recall that SS_{Fm} equals $I \hat{d}_m^2$, and that

(3.24) $$\hat{d}_m = \sum_i \sum_k \hat{\alpha}_{im}\hat{\beta}_{km}y_{.jk} .$$

If the values $\{\hat{\alpha}_{im}\hat{\beta}_{km} (j = 1, 2, \cdots , J; k = 1, 2, \cdots , K)\}$ were constants which had been chosen before observing the $\{\hat{\gamma}_{jk}\}$, we could treat them as weights in an *a priori* contrast; and, for all m, could test the hypothesis that $d_m = 0$ (i.e., the hypothesis that the mth interaction factor accounts for none of the variation in the $\{x_{jk}\}$) by conventional analysis of variance methods [e.g., Winer 1962]. But of course, the FANOVA model does use *a posteriori* information about the $\{\hat{\gamma}_{jk}\}$; and it is for this reason that we assign $(J + K - 1 - 2m)$ **df** (rather than one **df**) to SS_{Fm} . Matters are

further complicated by the fact that the $\{\hat{d}_m\}$ are not linear functions of the observations $\{y_{ijk}\}$.

However, asymptotically the $\{\hat{\alpha}_{jm}\}$ and $\{\hat{\beta}_{km}\}$ are constants which equal the $\{\alpha_{jm}\}$ and $\{\beta_{km}\}$, respectively; and thus, asymptotically the $\{\hat{d}_m\}$ are linear functions of the observations $\{y_{ijk}\}$. When we let

$$(3.25) \qquad \Delta a_{jm} = \alpha_{jm} - \hat{\alpha}_{jm} \quad \text{and} \quad \Delta b_{km} = \beta_{km} - \hat{\beta}_{km} \, ,$$

it can be shown that the probability that $\Delta a_{jm} > \epsilon$ and $\Delta b_{km} > \epsilon$, for any value of $\epsilon > 0$, approaches zero as the number of observations, I, upon which the data is based increases, i.e., $\text{limit}_{I \to \infty} \Delta a_{jm} = 0$, and $\text{limit}_{I \to \infty} \Delta b_{km} = 0$. Thus, asymptotically $\hat{d}_m$ is a linear function of the observations; and consequently, the asymptotic distributional properties of SS_{Fm} $(m = 1, 2, \cdots , M)$ and SS_{Fres} can be determined by well known procedures for dealing with linear functions of normally distributed observations. Hence, under the hypothesis that $d_m = 0$ (and under the assumptions of the fixed FANOVA model) the $(M + 1)$ sums of squares $I \, \hat{d}_1^2, I \, \hat{d}_2^2, \cdots , I \, \hat{d}_m^2, \cdots , I \, \hat{d}_M^2$ and $(\text{SS}_{RC} - I \sum_m \hat{d}_m^2)$, following division by σ_e^2, are asymptotically distributed as χ^2 with $(J + K - 3), (J + K - 5), \cdots , (J + K - 1 - 2m), \cdots ,$ $(J + K - 1 - 2M)$ and $(J - 1 - M)(K - 1 - M)$ df, respectively. Asymptotically, it also follows that the $M + 1$ quantities $\{\text{SS}_{Fm} \ (m = 1, 2, \cdots , M)\}$ and SS_{Fres} are all statistically independent of each other; and that the expected mean square for the mth interaction factor is

$$(3.26) \qquad E(\text{MS}_{Fm}) = I\sigma_{Fm}^2 + \sigma_e^2 \, ,$$

where

$$(3.27) \qquad \sigma_{Fm}^2 = d_m^2 / (J + K - 1 - 2m)$$

and $\sigma_e^2 = E(\text{MS}_e)$ and

$$(3.28) \qquad \text{MS}_e = SS_e / JK(I - 1) = \sum_i \sum_j \sum_k (y_{ijk} - y_{.jk})^2 / JK(I - 1).$$

The quantity SS_e / σ_e^2 is, of course, distributed as χ^2 with $JK(I - 1)$ df. A statement analogous to (3.26) can also be made regarding MS_{Fres} .

From the above facts we conclude that asymptotically (i.e., $I \to \infty$) the hypothesis that $\sigma_{Fm}^2 = 0$, and the equivalent hypothesis that $d_m = 0$, is rejected at the α level of probability if

$$(3.29) \qquad \text{MS}_{Fm} / \text{MS}_e \geq F_{\alpha ; (J+K-1-2m), JK(I-1)} \, ,$$

where the right side of the equation refers to the upper-α point of the F-distribution with $(J + K - 1 - 2m)$ and $JK(I - 1)$ df. Use of the criterion described by (3.29) is suggested as a rough-and-ready aid to one's intuition in judging whether or not the mth interaction factor accounts for a statistically significant amount of the total variation. It is important to remember that although the ratio $\text{MS}_{Fm} / \text{MS}_e$ is asymptotically distributed as F, *its exact*

distribution is unknown. The accuracy of the F-distribution as an approximation to the unknown, true distribution depends upon the magnitude of the $\{\Delta a_{im}\}$ and $\{\Delta b_{km}\}$. In practice the criterion (3.29) will be reasonably accurate when the values $\{\hat{\alpha}_{im}\hat{\beta}_{km}\}$ are reasonably stable; i.e., when further increases in I do not greatly affect the values of the $\{\hat{\alpha}_{im}\}$ and $\{\hat{\beta}_{km}\}$. Thus, it is possible for 15 or 20 observations on data with very small error variation to yield reasonably stable values of $\{\hat{\alpha}_{im}\}$ and $\{\hat{\beta}_{km}\}$; while a much greater number of observations on highly variable data may yield markedly unstable values of $\{\hat{\alpha}_{im}\}$ and $\{\hat{\beta}_{km}\}$.

Residual interaction. The residual interaction sum of squares, SS_{Fres} , provides a measure of the amount of interaction variation remaining after removing that variance accounted for by M interaction factors. It is of course possible to extract enough interaction factors to make SS_{Fres} negligible or even equal to zero. However, it seems very unwise to extract an interaction factor which accounts for an obviously non-significant and trivial amount of variation. Such a factor, by definition, does not describe systematic variation in the data. Thus, even in cases where the residual interaction is obviously substantial, any interaction factor which does not itself account for a substantial amount of variation should not be retained.

Asymptotically, the quantity SS_{Fres}/σ_e^2 is statistically independent of the interaction factor mean squares and is distributed as χ^2 with $(J - 1 - M)$ $\cdot (K - 1 - M)$ **df.** Thus, as a rough guide, we may reject the hypothesis that all $\phi_{jk} = 0$ if

$$(3.30) \qquad MS_{Fres}/MS_e \geq F_{\alpha; (J-1-M)(K-1-M), JK(I-1)} \cdot$$

Note that, of course, the same comments made regarding use of criterion (3.29) also apply here.

Table 1 outlines the computations for the two-way fixed effects FANOVA model.

Additional guides for judging significance in the fixed FANOVA model

In this section we describe tests for finding lower bounds (conservative test) and upper bounds (liberal test) for p-values applying to tests of the hypothesis that $d_m = 0$. Tests of the hypothesis that all the Fres parameters $\{\phi_{jk}\}$ equal zero can be developed by directly analogous methods, but will not be explicitly presented.

A conservative test. The significance test described in this paragraph is conservative in the sense that it will yield too few "significant" decisions (i.e., high Type II error) but keeps the probability of making a Type I error $\leq \alpha$. Scheffé's [1959] method for judging all possible contrasts in a fixed model analysis of variance provides a conservative test for judging the significance of the M interaction factors and the residual interaction factor in the fixed effects FANOVA model. Using Scheffé's method, the

TABLE 1

Computational Formulas for the Two-way Fixed Effects
FANOVA Model

Source	df	Sums of Squares
Rows (R)	$J-1$	$IK \sum_j y^2_{.j.} - IJK y^2_{...}$
Cols (C)	$K-1$	$IJ \sum_k y^2_{..k} - IJK y^2_{...}$
RxC	$(J-1)(K-1)$	$I \sum_j \sum_k y^2_{.jk} - IJK y^2_{...} - SS_R - SS_C$
F1	$J+K-3$	$I\hat{d}^2_1$
F2	$J+K-5$	$I\hat{d}^2_2$
.	.	.
.	.	.
.	.	.
Fm	$J+K-1-2\underline{m}$	$I\hat{d}^2_m$
.	.	.
.	.	.
.	.	.
FM	$J+K-1-2\underline{M}$	$I\hat{d}^2_M$
Fres	$(J-1-M)(K-1-M)$	$SS_{RC} - \sum_m SS_{Fm}$
Error	$JK(I-1)$	$\sum_i \sum_j \sum_k y^2_{ijk} - I \sum_j \sum_k y^2_{.jk}$

probability that all contrasts tested will be significant is $\geq 1 - \alpha$; irrespective of the number of contrasts estimated and irrespective of whether the contrasts are selected *a priori* or are chosen after examining the data, as in the FANOVA model. Applying Scheffé's method to the problem of judging the significance of the mth interaction factor in the fixed FANOVA model, we conclude with probability $\geq 1 - \alpha$ that $d_m \neq 0$ if

$$(3.31) \qquad \frac{SS_{Fm}/1}{SS_e/JK(I-1)} \geq (J-1)(K-1)F_{\alpha;(J-1)(K-1),JK(I-1)} .$$

It is useful to note that the approximate test described earlier (3.29) and the conservative test (3.31) differ only in that they assign different **df** to

SS_{Fm} . It is easy to see that the number of df assigned to SS_{Fm} by the approximate test (i.e., $J + K - 1 - 2m$) cascades in steps of two df as the value of m increases; whereas Scheffé's method (3.31), in effect, assigns a constant number of df, $(J - 1)(K - 1)$, to SS_{Fm} irrespective of the value of m.

A liberal test. The significance test presented below is "liberal" in the sense that is has high probability of making Type I errors (i.e., rejecting the null hypothesis when it is true) but has a very low probability of making a Type II error. Although the liberal test will overestimate the number of statistically significant interaction factors, it is sometimes helpful to know the maximum number of statistically significant factors. The liberal test is based on the fact that an *a priori* hypothesis always has greater power than the corresponding *a posteriori* hypothesis. The liberal test ignores the fact that *a posteriori* information is used in selecting the best interaction factors, and treats the obtained factor weights $\{\hat{\alpha}_{im}\hat{\beta}_{km}\}$ as though they had been known *a priori*. Thus, the liberal significance test for the mth interaction factor is obtained simply by using conventional methods for testing the significance of *a priori* contrasts [e.g., Winer, 1962]. Thus, in the case of the fixed FANOVA model, we reject the hypothesis that $d_m = 0$ if

$$(3.32) \qquad \frac{SS_{Fm}/1}{MS_e} \geq F_{\alpha;1,JK(I-1)} .$$

In using the FANOVA model it is often useful to apply the asymptotic, the conservative, and the liberal tests in combination. First, the conservative and liberal tests, respectively, enable one to find the lower and upper bounds on the p-values applying to given interaction factors; and the asymptotic test can then be used to aid one in judging the significance of those factors which are neither accepted by the conservative test nor rejected by the liberal test.

An "exact test." As mentioned earlier, conventional tests which are exact are available for judging the significance of *a priori* contrasts. It is valid to treat contrast weights as *a priori* so long as the basis on which they are selected does not use information about the specific set of parameter estimates to which they are to be applied. Thus, in addition to using data from previous experiments as a basis for defining *a priori* weights $\{\hat{\alpha}_{im}\hat{\beta}_{km}\}$ which define an "*a priori* interaction factor," it is also valid to use a randomly selected subset of data from a single experiment to define *a priori* interaction factors. Given I observations in a two-way fixed model, some proportion, p, of the observations are randomly selected from each cell to define, say, set V of data; and the remaining $(1 - p)I$ observations per cell define set W of data. The data from, say, set V are then decomposed according to the FANOVA model. Since V and W comprise two statistically independent sets of data, the interaction factors (and main effects) found for set V of data can be used to define contrasts which are *a priori* with

respect to set W data. Set W data is then analyzed by using conventional methods for judging significance of a *priori* contrasts (obtained from set V) in a fixed model analysis of variance.

4. *Judging "Factor Similarity" in the Fixed Effects FANOVA Model*

Consider a three-way $(G \times R \times C)$ fixed effects analysis of variance model with y_{itik} representing the ith $(i = 1, 2, \cdots, I)$ observation at the tth $(t = 1, 2, \cdots, T)$ level of G, jth level of R, and kth level of C. Assume that we have the FANOVA decomposition for the $R \times C$ "summary table" of data obtained by averaging over observations and over levels of G. The problem of judging factor similarity, as defined here, arises when we wish to judge whether or not an interaction factor accounts for an equal amount of variation at each level of G. Say, for example, that our dependent variable is performance score on some tasks; and the levels of R and C are three different diagonastic categories and four different drugs, respectively. Having averaged over levels of G, say a stress vs. no stress manipulation, and having obtained the FANOVA decomposition of the diagnostic category by drug summary table; we want to test whether each component of the FANOVA decomposition accounts for an equal amount of variation of the category by drug profile of results *within* both the stress and no stress groups. The significance tests described in the following section are all exact; and require only the conventional assumptions for the three-way fixed effects analysis of variance model.

Fit of the over-all interaction factor at each level of G

Letting x_{tik} represent the population value which is estimated by $y_{.tik}$, we first obtain the least squares estimate of the overall $R \times C$ profile of scores $\{x_{.ik}\}$ by calculating

$$(4.1) \qquad\qquad \hat{x}_{.ik} = y_{..ik} ;$$

and then obtain the FANOVA model decomposition of the $\{y_{..ik}\}$. We then approach the problem of judging factor similarity by testing whether regression of the T subtables of $R \times C$ means $\{y_{.tik} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ on the mth set of estimated interaction factor weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ leads to significantly different regression weights $\{\hat{d}_{tm} \ (t = 1, 2, \cdots, T)\}$ for different levels of G.

The significance test described below is based on the fact that the parameter estimates associated with $R \times C$ and $G \times R \times C$ variation are statistically independent quantities. This means that information about $R \times C$ variation tells us nothing about $G \times R \times C$ variation. Therefore, we can treat the M sets of estimated interaction factor weights $\{\hat{\alpha}_{jk}\hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ and the residual interaction parameter estimates $\{\hat{\phi}_{jk}\}$ as $(M + 1)$ mutually orthogonal contrasts which are a *priori*

with respect to $G \times R \times C$ variation. Thus, it is valid to test the hypothesis that our estimate of the mth interaction factor accounts for different amounts of variation at each level of G by simply making the same assumptions and using the same computational procedures as are used to test the hypothesis that, say, a (linear R) $\times$ (quadratic C) contrast accounts for equal amounts of $R \times C$ variation within each level of G. The basic idea expressed in this paragraph should be remembered since it is central not only to the significance test described in this section, but also to tests which are described later in the paper.

Following conventional procedures for testing the significance of *a priori* interaction contrasts [see Winer, 1962], we obtain least squares estimates of the T regression weights $\{d_{tm} \ (t = 1, 2, \cdots, T)\}$ by calculating

$$(4.2) \qquad \hat{d}_{tm} = \sum_j \sum_k \hat{\alpha}_{jm} \hat{\beta}_{km} y_{.tjk} ,$$

where of course, the $\{\hat{\alpha}_{jm} \ (j = 1, 2, \cdots, J)\}$ and $\{\hat{\beta}_{km} \ (k = 1, 2, \cdots, K)\}$ are treated as fixed weights which define an interaction contrast. We then compute

$$(4.3) \qquad \mathrm{SS}_{GFm} = I \sum_t (\hat{d}_{tm} - \hat{d}_{.m})^2 = I \sum_t \hat{d}_{tm}^2 - \mathrm{SS}_{Fm} ,$$

where

$$(4.4) \qquad \mathrm{SS}_{Fm} = IT(\sum_j \sum_k \hat{\alpha}_{jm} \hat{\beta}_{km} y_{..jk})^2 = IT \hat{d}_{.m}^2 .$$

Since $(T - 1)$ independent parameters are estimated in computing SS_{GFm}, we define

$$(4.5) \qquad \mathrm{MS}_{GFm} = \mathrm{SS}_{GFm}/(T - 1).$$

Under the usual assumptions of the analysis of variance model the quantity $\mathrm{SS}_{GFm}/\sigma_e^2$ is distributed as non-central χ^2 with $(T - 1)$ df. Hence, for any given value of m, we reject the hypothesis that all $\{d_{tm} \ (t = 1, 2, \cdots, T)\}$ are equal if

$$(4.6) \qquad \mathrm{MS}_{GFm}/\mathrm{MS}_e \geq F_{\alpha : T-1, TJK(I-1)} .$$

It is important to recognize that (4.6) provides an exact test of the hypothesis that the contrast defined by the *estimates* $\{\hat{\alpha}_{jm} \hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ accounts for an equal amount of variation of each level of G. However, strictly speaking, (4.6) is not a test of whether the *population parameters* $\{\alpha_{jm} \beta_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ define a contrast which accounts for an equal amount of variation at each level of G; and it is reasonable to treat it as such only to the extent that the estimates $\{\hat{\alpha}_{jm} \hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ reflect the "true" pattern of factor weights in the population.

Similarly, a test of the hypothesis that the estimated $R \times C$ residual interaction, Fres, accounts for an equal amount of variation at each level of G is obtained by treating the Fres parameter estimates $\{\hat{\phi}_{ik}\}$ as defining a fixed contrast which is *a priori* with respect to $G \times R \times C$ interaction variation. Hence we compute

$$(4.7) \qquad \mathrm{MS}_{GFres} = \mathrm{SS}_{GFres}/(T-1)$$

where

$$(4.8) \qquad \mathrm{SS}_{GFres} = I \sum_{t} (\sum_{j} \sum_{k} \hat{\phi}_{im} y_{.tik})^2 / \sum_{j} \sum_{k} \hat{\phi}_{ik}^2 - \mathrm{SS}_{Fres} ,$$

and

$$(4.9) \qquad \mathrm{SS}_{Fres} = \mathrm{SS}_{RC} - IT \sum_{m} \hat{d}_m^2 .$$

The significance of variation due to $G \times$ Fres interaction is tested against the usual error term, MS_e.

The SS accounted for by the remaining $G \times R \times C$ interaction is given by

$$(4.10) \qquad \mathrm{SS}_{GRC[res]} = \mathrm{SS}_{GRC} - \sum_{m} \mathrm{SS}_{GFm} - \mathrm{SS}_{GFres} ,$$

where SS_{GRC} represents the SS due to the overall $G \times R \times C$ interaction. The terms on the right side of (4.10) are mutually orthogonal and, respectively, account for $(T-1)(J-1)(K-1)$, $(T-1)M$, and $(T-1)$ df. Hence we define

$$(4.11) \qquad \mathrm{MS}_{GRC[res]} = \mathrm{SS}_{GRC[res]}/\lambda,$$

where $\lambda = [(J-1)(K-1) - (M+1)](T-1)$. The hypothesis that no $G \times R \times C$ interaction remains after removing variation due to $G \times Fm$ (for all m) and $G \times$ Fres interaction is rejected if

$$(4.12) \qquad \mathrm{MS}_{GRC[res]}/\mathrm{MS}_e \geq F_{\alpha;\lambda, TJK(I-1)} .$$

Table 2 summarizes the computations used in partitioning the overall $G \times R \times C$ variation in order to judge factor similarity.

The methods used above for judging factor similarity can of course also be applied to the problem of judging the similarity of main effect profiles for different groups of Ss. Space limitations require that a comparison of the above technique and that of simply treating the overall $R \times C$ interaction as a measure of profile similarity be left for a later time. We now turn to a discussion of the three-way repeated-measurements FANOVA model.

5. *The Three-way Repeated-Measurements FANOVA Model*

In this section we consider a design in which repeated measurements which vary along two dimensions ($R \times C$) are taken on several subjects (S).

TABLE 2

Partitioning of GxRxC Variation in Order to
Judge Factor Similarity

Source	df	Sum of Squares
GxF1	$(T-1)$	$I \sum_t \hat{d}^2_{t1} - SS_{F1}$
GxF2	$(T-1)$	$I \sum_t \hat{d}^2_{t2} - SS_{F2}$
.	.	.
.	.	.
.	.	.
GxFm	$(T-1)$	$I \sum_t \hat{d}^2_{tm} - SS_{Fm}$
.	.	.
.	.	.
.	.	.
GxFM	$(T-1)$	$I \sum_t \hat{d}^2_{tM} - SS_{FM}$
GxFres	$(T-1)$	$I \sum_t (\sum_j \sum_k \hat{\phi}_{jk} y_{.tjk})^2 / \sum_j \sum_k \hat{\phi}^2_{jk} - SS_{Fres}$
GRC[res]	λ*	$SS_{GRC} - \sum_m SS_{GFm} - SS_{GFres}$

* $\lambda = [(J-1)(K-1) - (M+1)](T-1)$

We also allow for one or more independent replications (Rep) within each cell. We write this design as Rep $(S \times R \times C)$ where replications and subjects are treated as random and the levels of R and C are treated as fixed. The analysis of variance model for this design is

$$(5.1) \quad y_{hijk} = \mu + S_i + R_j + C_k + \gamma_{jk} + \theta_{ij} + \pi_{ik} + \tau_{ijk} + e_{hijk},$$

where y_{hijk} denotes the observation on the hth $(h = 1, 2, \cdots, H)$ independent replication of the ith subject's score on the j, kth measurement. We make the following assumptions:

a. $\displaystyle\sum_j R_j = \sum_k C_k = \sum_j \gamma_{jk} = \sum_k \gamma_{jk}$

$$= \sum_i \theta_{ij} = \sum_k \pi_{ik} = \sum_j \tau_{ijk} = \sum_k \tau_{ijk} = 0$$

b. The $\{S_i\}$, $\{\theta_{ij}\}$, $\{\pi_{ik}\}$, and $\{\tau_{ijk}\}$ are jointly normal with zero means.

c. The $\{e_{hijk}\}$ are independently and normally distributed with zero mean and variance σ_s^2 ; and are independent of the $\{S_i\}$, $\{\theta_{ij}\}$, $\{\pi_{ik}\}$, and $\{\tau_{ijk}\}$.

The analysis of variance model (5.1) implies the following decomposition of the "true" measurements $\{x_{jk}\}$ averaged over the population of subjects and replications,

$$(5.2) \qquad\qquad x_{jk} = \mu + R_j + C_k + \gamma_{jk} \; .$$

A least squares estimate of x_{jk} is given by

$$(5.3) \qquad\qquad \hat{x}_{jk} = y_{..jk}.$$

Applying the FANOVA model decomposition to the matrix of $\{y_{..jk}\}$ yields least squares estimates of the grand mean, the row and column main effects, and of the M interaction factors and the residual interaction. It is in the spirit of trying to provide some heuristically useful ways of summarizing the results of a FANOVA decomposition of three-way mixed model data, that the following quantities and indices are presented.

Interaction factors in the repeated-measurements FANOVA model

The sums of squares and mean squares for the M interaction factors and the residual interaction, Fres, for the three-way repeated-measurement FANOVA model are directly analogous to the corresponding quantities for the fixed FANOVA model and are given by

$$(5.4) \qquad\qquad \mathrm{SS}_{Fm} = HI\hat{d}_m^2 \; ,$$

$$(5.5) \qquad\qquad \mathrm{SS}_{Fres} = \mathrm{SS}_{RC} - \sum_m \mathrm{SS}_{Fm} \; ,$$

and

$$(5.6) \qquad\qquad \mathrm{MS}_{Fm} = \mathrm{SS}_{Fm}/(J + K - 1 - 2m),$$

$$(5.7) \qquad\qquad \mathrm{MS}_{Fres} = \mathrm{SS}_{Fres}/(J - 1 - M)(K - 1 - M) \; .$$

The above quantities summarize variation in the two-way table of data which is obtained by averaging over replications and subjects. The following two paragraphs describe measures which summarize variation due to individual differences in the size of the contribution of various interaction factors to the scores of individual subjects.

Variation due to individual differences can be investigated by treating the factor weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, 2, \cdots , J; \ k = 1, 2, \cdots K)\}$ as fixed weights which define a contrast; and computing for each interaction factor the I quantities

$$(5.8) \qquad\qquad \hat{d}_{im} = \sum_j \sum_k \hat{\alpha}_{jm}\hat{\beta}_{km} y_{.ijk} \; .$$

The larger the value of $\{\hat{d}_{im}\}$ for a given subject, the larger the *absolute contribution* of the mth factor in determining his $R \times C$ table of scores $\{y_{\cdot ijk} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$. Thus, the ith subject's $R \times C$ table of measurements as estimated solely on the basis of (a) his "score," $\hat{d}_{im}$, on the mth interaction factor and (b) the *overall* factor profile defined by the weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$, can be obtained by

$$(5.9) \qquad\qquad y_{\cdot ijk(m)} = \hat{d}_{im}\hat{\alpha}_{jm}\hat{\beta}_{km} .$$

For the mth interaction factor, the sum of squares of deviations of subjects' individual regression weights $\{\hat{d}_{im} \ (i = 1, 2, \cdots, I)\}$ around the average regression weight $\hat{d}_{\cdot m}$ is

$$(5.10) \qquad \mathrm{SS}_{SFm} = H \sum_i (\hat{d}_{im} - \hat{d}_{\cdot m})^2 = H \sum_i \hat{d}_{im}^2 - \mathrm{SS}_{Fm} .$$

Since calculations of SS_{SFm} involves fitting only $(I - 1)$ linearly independent parameters, we define

$$(5.11) \qquad\qquad \mathrm{MS}_{SFm} = \mathrm{SS}_{SFm}/(I - 1).$$

Similarly, we compute variation due to individual differences in the absolute size of the contribution of the contrast defined by the estimated Fres parameters $\{\hat{\phi}_{jk}\}$ by

$$(5.12) \qquad \mathrm{SS}_{SFres} = H \sum_i \left(\sum_j \sum_k \hat{\phi}_{jk} y_{\cdot ijk}\right)^2 \Big/ \sum_j \sum_k \hat{\phi}_{jk}^2 - \mathrm{SS}_{Fres} .$$

As in the case of the interaction factors, only $(I - 1)$ linearly independent parameters are fit in calculating SS_{SFres} and we define

$$(5.13) \qquad\qquad \mathrm{MS}_{SFres} = \mathrm{SS}_{SFres}/(I - 1).$$

At this point it seems natural to describe a quantity which we will refer to as the F/R ratio (Fixed/Random) and which is obtained by

$$(5.14) \qquad\qquad (\mathrm{F/R})_{Fm} = \mathrm{MS}_{Fm}/\mathrm{MS}_{SFm}.$$

A similar ratio can, of course, be computed for the residual interaction and for main effects. The calculations made in obtaining the F/R ratio are similar to those used in computing an F-ratio for judging the significance of say, a linear $\times$ quadratic trend; but, we emphasize that the distribution of the F/R ratio is not known. Although no attempt to discuss the distribution properties of the F/R ratio will be made here, some of its heuristic usefulness should become more clear when it is discussed in the context of an illustrative example which is presented later. The following presentation of a liberal and conservative test of the hypothesis that $d_m = 0$ may also help to suggest "interpretations" of the F/R ratio.

A conservative test and a liberal test

Conservative test. A conservative test of significance of interaction factors and the residual interaction in the three-way mixed FANOVA model is obtained by a straight-forward extension of Scheffé's [1959] method for judging all possible contrasts among main effects of the fixed way of a two-way mixed model analysis of variance. We can think of the interaction parameter estimates $\{\hat{\gamma}_{jk}\}$ as being obtained by taking the mean over the range of i of the quantities $\{\hat{\gamma}_{ijk}\}$ which are given by

$$(5.15) \qquad \hat{\gamma}_{ijk} = y_{.ijk} - y_{.i.k} - y_{.ij.} + y_{.i..}.$$

Since only $(J - 1)(K - 1)$ parameters are estimated in calculating the JK quantities, $\{\hat{\gamma}_{jk}\}$, it is possible to write $(J - 1)(K - 1) = V$ new values, $\{\hat{z}_v \; (v = 1, 2, \cdots, V)\}$ which retain all the information in the $\{\hat{\gamma}_{jk}\}$. Similarly, it is possible to express the I sets of $\{\hat{\gamma}_{ijk} \; (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ in terms of new values, $\{\hat{z}_{iv}\}$, such that

$$(5.16) \qquad \hat{z}_{.v} = \hat{z}_v .$$

Under the assumptions of the mixed analysis of variance model, the I vectors $(z_{i1}, z_{i2}, \cdots, z_{iv}, \cdots, z_{iV})$ are independently and normally distributed with an arbitrary pattern of means and with a variance-covariance matrix of arbitrary form. From this point it is simple to extend Scheffé's [1959, pp. 271–74] method to obtain the present test which allows us to reject, at the $\leq \alpha$ level, the hypothesis that $d_m = 0$ if

$$(5.17) \qquad \frac{SS_{Fm}/1}{MS_{SFm}} \geq \frac{(I-1)(J-1)(K-1)}{I-(J-1)(K-1)} \; F_{\alpha;(J-1)(K-1),I-(J-1)(K-1)} .$$

Clearly, the test (5.17) can be used only when $I > (J - 1)(K - 1)$. The inequality (5.17) can also be written as

$$(5.18) \qquad C_{Fm} = \frac{SS_{Fm}/(J-1)(K-1)}{SS_{SFm}/I-(J-1)(K-1)} \geq F_{\alpha;(J-1)(K-1),I-(J-1)(K-1)} .$$

Although the test given by (5.17) and (5.18) is *extremely* conservative in the sense that it has very high probability of making Type II errors; it may well be of practical use when I (the number of observations) is large relative to $(J - 1)(K - 1)$. In most applications it is probably wise to let α equal .15 or .20 when using this test. A conservative significance test for the residual interaction (or any other contrast among the $\{\gamma_{jk}\}$ is obtained by substituting the appropriate "fixed" and "random" SS for SS_{Fm} and SS_{SFm} in (5.17) or (5.18).

Liberal test. As in the fixed effects FANOVA model, a liberal test of the hypothesis that $d_m = 0$ is obtained simply by treating the interaction factor weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \; (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ as though they

defined an *a priori* contrast. Thus, we reject the hypothesis that $d_m = 0$ if

(5.19)
$$L_{Fm} = \frac{\text{SS}_{Fm}/1}{\text{MS}_{SFm}} \geq F_{\alpha;1,I-1} .$$

It is sometimes convenient to compute L_{Fm} by using the relation

(5.20)
$$L_{Fm} = (J + K - 1 - 2m)(\text{F/R})_{Fm} .$$

It is interesting to note that the ratios C_{Fm} , $(\text{F/R})_{Fm}$, and L_{Fm} all differ essentially in that they assign different df to SS_{Fm} and/or SS_{SFm} ; and that

(5.21)
$$C_{Fm} < (\text{F/R})_{Fm} < L_{Fm} .$$

An "exact test"

As in the fixed FANOVA model, it is a simple matter to obtain an exact significance test of the various FANOVA components of variance if we have some *a priori* basis for predicting what the precise form of the components will be. Thus, instead of trying to test the significance of inter-action factors obtained by factoring interaction parameter estimates obtained by using the entire set of subjects, the I subjects (observations) can be randomly divided into two groups, say, V and W. Since this results in two independent sets of data, it is now possible to use the FANOVA decomposition of, say, set V of data to define contrasts which are *a priori* with respect to set W data. The estimated main effects $\{\hat{R}_i\}$ and $\{\hat{C}_k\}$, the M sets of estimated interaction factor weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km}$ $(j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ and the residual interaction parameter estimates $\{\hat{\phi}_{jk}\}$ can all be used as contrasts. Conventional methods can then be used to judge the significance of the contrasts (obtained from set V data) when applied to set W data.

A significance test for $S \times Fm$ interactions when $H > 1$

As discussed above, the quantity SS_{SFm} provides a measure of variation due to individual differences in the absolute size of the contribution of the *mth estimated* interaction factor to subjects' measurements. Thus, if the profile of estimated weights of the *mth* interaction factor accounted for an equal amount of variation in each subject's $R \times C$ table of measurements, it would be true that $\hat{d}_{1m} = \hat{d}_{2m} = \cdots = \hat{d}_{im} = \cdots = \hat{d}_{Im}$ and therefore SS_{SFm} would equal zero. When the $\{\hat{\alpha}_{jm}\hat{\beta}_{km}$ $(j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ are treated as fixed, it can readily be shown that

(5.22)
$$E(\text{MS}_{SFm}) = H\sigma^2_{SFm} + \sigma^2_e ,$$

where $\sigma^2_e = E(\text{MS}_e)$ and

(5.23)
$$\sigma^2_{SFm} = \sum_i (d_{im} - d_{.m})^2/(I - 1).$$

Since variation due to $R \times C$ interaction is statistically independent of variation due to $S \times R \times C$ interaction, we can treat all sets of $\{\hat{\alpha}_{jm}\hat{\beta}_{km}$ $(j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ as fixed weights which define contrasts which are *a priori* with respect to $S \times R \times C$ variation. Thus, under the usual assumptions for judging the significance of orthogonal *a priori* contrasts, we reject the hypothesis that $\sigma^2_{SFm} = 0$ if

(5.24) $\mathrm{MS}_{SFm}/\mathrm{MS}_e \geq F_{\alpha;I-1,IJK(H-1)}$.

Similarly the hypothesis that $\sigma^2_{SFres} = 0$ is rejected if

(5.25) $\mathrm{MS}_{SFres}/\mathrm{MS}_e \geq F_{\alpha;I-1,IJK(H-1)}$.

We postpone consideration of significance tests for $S \times Fm$ interactions when $H = 1$ until after the following discussion of SRC [res] interaction.

The SRC[res] interaction

The $S \times R \times C$ variation remaining after variation due to all SS_{SFm} and SS_{Fres} has been removed is given by

(5.26) $\mathrm{SS}_{SRC[res]} = \mathrm{SS}_{SRC} - \sum_m \mathrm{SS}_{SFm} - \mathrm{SS}_{FSres}$

where SS_{SRC} represents the SS due to the overall $S \times R \times C$ interaction. The df assigned to the terms on the right side of (5.26) are $(I-1)(J-1)(K-1)$, $(I-1)M$, and $I-1$, respectively; and it is easy to show that $\lambda = [(J-1)(K-1) - (M+1)](I-1)$ df should be assigned to $\mathrm{SS}_{SRC[res]}$. Hence we define

(5.27) $\mathrm{MS}_{SRC[res]} = \mathrm{SS}_{SRC[res]}/\lambda.$

Under the hypothesis that the true variation due to SRC[res] equals zero, i.e., $\sigma^2_{SRC[res]} = 0$, the quantity $\mathrm{SS}_{SRC[res]}$ is distributed as $\sigma^2_e \chi^2$ with λ **df**, and we reject the hypothesis that $\sigma^2_{SRC[res]} = 0$ if

(5.28) $\mathrm{MS}_{SRC[res]}/\mathrm{MS}_e \geq F_{\alpha;\lambda,IJK(H-1)}$.

If $H = 1$, we cannot test the hypothesis that $\sigma^2_{SRC[res]} = 0$.

Having described a significance test we now turn to a brief discussion concerning the "interpretation" of the SRC[res] interaction. The overall $S \times R \times C$ interaction provides a measure of the degree to which the profiles of subjects' $R \times C$ means $\{y_{.ijk} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ differ from each other. Thus, if all subjects' $R \times C$ mean profiles are perfectly parallel, $\mathrm{SS}_{SRC} = 0$. As indicated by (5.26), the interaction of subjects with the M interaction factors and the residual interaction of subjects with the M interaction factors and the residual interaction factor accounts for only a part of the total $S \times R \times C$ interaction. The $M + 1$ interaction contrasts employed in the FANOVA model are selected *a posteriori* to exhaust all

the $R \times C$ sum of squares; but since these same contrasts are *a priori* with respect to $S \times R \times C$ interaction, they behave in the same manner as any other $(M + 1)$ mutually orthogonal *a priori* contrasts, such as linear $\times$ linear, quadratic $\times$ linear, etc. Thus, the FANOVA model partitions the $S \times R \times C$ interaction into two major parts; with one part consisting of $M + 1$ components representing individual profile differences with respect to interaction factors and the residual interaction, and the remaining major part representing individual profile differences with respect to $R \times C$ contrasts which are orthogonal to the interaction factors and the residual interaction.

Significance of $S \times Fm$ interactions when $H = 1$

When only one replication within each i, j, k cell is available one may wish to assume that $\sigma^2_{SRC[\text{res}]} = 0$ and reject the hypothesis that $\sigma^2_{SFm} = 0$ if

$$(5.29) \qquad \text{MS}_{SFm}/\text{MS}_{SRC[\text{res}]} \geq F_{\alpha;I-1,\lambda} \, ,$$

where $\lambda = [(J - 1)(K - 1) - (M + 1)](I - 1)$. Similarly, one may reject the hypothesis that $\sigma^2_{S\ \text{Fres}} = 0$ if

$$(5.30) \qquad \text{MS}_{S\ \text{Fres}}/\text{MS}_{SRC[\text{res}]} \geq F_{\alpha;I-1,\lambda} \, .$$

Since $E(\text{MS}_{SRC[\text{res}]}) = H\sigma^2_{SRC[\text{res}]} + \sigma^2_e$, it is easy to see that the tests described by (5.29) and (5.30) will err on the conservative side if $\sigma^2_{SRC[\text{res}]} > 0$.

Table 3 presents a summary of the computations used in decomposing $R \times C$ and $S \times R \times C$ sources of variation according to the requirements of the three-way repeated-measurements FANOVA model.

6. Judging "Factor Similarity" in Repeated-measurement Designs

Described in this section is a test for judging "factor similarity" over different groups (G) of subjects in a design which consists of fixed repeated measurements $(R \times C)$ on randomly selected subjects (S) who are nested in the fixed groups (see Winer [1962] for a discussion of this design). For convenience we assume that there is only one replication per cell, and that each group contains an equal number of subjects. Thus, in this design there are T groups of I subjects; and JK measurements are obtained on each of the IT subjects. Many learning experiments are of this type; with levels of G being different experimental treatments, levels of R being different stimulus characteristics, and levels of C being different blocks of learning trials. The population $R \times C$ profile of measurements in the tth group is defined by the set $\{x_{tjk} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$, and is estimated by

$$(6.1) \qquad \hat{x}_{tjk} = y_{.tjk} \, .$$

Assume that the overall $R \times C$ profile $\{y_{..jk} \ (j = 1, 2, \cdots, J; k =$

TABLE 3

Computations for Decomposing RxC and SxRxC
Variation in the Three-way Repeated-Measurements FANOVA Model

Source	df	Sums of Squares
RxC	$(J-1)(K-1)$	
F1	$J+K-3$	$HI\hat{d}_1^2$
F2	$J+K-5$	$HI\hat{d}_2^2$
.	.	.
.	.	.
.	.	.
Fm	$J+K-1-2m$	$HI\hat{d}_m^2$
.	.	.
.	.	.
.	.	.
FM	$J+K-1-2M$	$HI\hat{d}_M^2$
Fres	$(J-1-M)(K-1-M)$	$SS_{RC} - \sum_m SS_{Fm}$
SxRxC	$(I-1)(J-1)(K-1)$	
SxF1	$I-1$	$H \sum_i (\sum_j \sum_k \hat{\alpha}_{j1}\hat{\beta}_{k1}y_{.ijk})^2 - SS_{F1}$
SxF2	$I-1$	$H \sum_i (\sum_j \sum_k \hat{\alpha}_{j2}\hat{\beta}_{k2}y_{.ijk})^2 - SS_{F2}$
.	.	.
.	.	.
.	.	.
SxFm	$I-1$	$H \sum_i (\sum_j \sum_k \hat{\alpha}_{jm}\hat{\beta}_{km}y_{.ijk})^2 - SS_{Fm}$
.	.	.
.	.	.
.	.	.
SxFM	$I-1$	$H \sum_i (\sum_j \sum_k \hat{\alpha}_{jM}\hat{\beta}_{kM}y_{.ijk})^2 - SS_{FM}$
SxFres	$I-1$	$H \sum_i (\sum_j \sum_k \hat{\phi}_{jk}y_{.ijk})^2 / \sum_j \sum_k \hat{\phi}_{jk}^2 - SS_{Fres}$
SRC[res]	λ*	$SS_{SRC} - \sum_m SS_{SFm} - SS_{SFres}$
Error	$IJK(H-1)$	$\sum_h \sum_i \sum_j \sum_k y_{hijk}^2 - H \sum_i \sum_j \sum_k y_{.ijk}^2$

* $\lambda = [(J-1)(K-1) - (M+1)](I-1)$

1, 2, $\cdots$, K)} has been computed and then decomposed into the FANOVA model components:

$$(6.2) \qquad y_{..jk} = \hat{\mu} + \hat{R}_j + \hat{C}_k + \sum_m \hat{d}_m \hat{\alpha}_{jm} \hat{\beta}_{km} + \hat{\phi}_{jk} .$$

By a method directly analogous to that used in the fixed FANOVA model, we then approach the problem of judging "factor similarity" by testing, for each of the M interaction factors, the hypothesis that the contrast defined by the {$\hat{\alpha}_{jm}\hat{\beta}_{km}$ ($j = 1, 2, \cdots , J; k = 1, 2, \cdots , K$)} accounts for an equal amount of variation in each of the T population group profiles {x_{tjk} ($j = 1, 2, \cdots , J; k = 1, 2, \cdots , K$)}. Of course, we also can ask a similar question concerning the residual interaction and the main effects.

The parameter estimates associated with the analysis of variance sources, $R \times C$, $G \times R \times C$, and $S(G) \times R \times C$ [i.e., (S within G) $\times R \times C$] lie in three mutually orthogonal spaces are statistically independent. This means that information about variation due to $R \times C$ interaction tells us nothing about variation due to $G \times R \times C$ and $S(G) \times R \times C$ variation. Hence, it follows that the interaction factors (and the residual interaction) which are estimated on the basis of information about $R \times C$ variation, define contrasts which are *a priori* with respect to variation due to the $G \times R \times C$ and $S(G) \times R \times C$ sources. A test of whether the $G \times Fm$ term is significant is presented here as a test of "factor similarity." The significance of $G \times Fm$ variation is tested by conventional methods [e.g., Winer, 1962] for judging the significance of, say, a $G \times$ (linear R) $\times$ (quadratic C) term in an analysis of variance design of the type now being discussed.

Following conventional methods, scores for the IT subjects on the mth estimated interaction factor are computed from

$$(6.3) \qquad \hat{d}_{itm} = \sum_j \sum_k \hat{\alpha}_{jm} \hat{\beta}_{km} y_{itjk} .$$

For each value of m (i.e., for each *a priori* contrast defined by the M interaction factors) we now have scores for each of the IT subjects. Under the assumption that the variance of the normally distributed {d_{itm} ($i = 1, 2, \cdots , I$)} is the same for all T groups, we can do a one-way analysis of variance on each of the M sets of data {$\hat{d}_{itm}$ ($i = 1, 2, \cdots , I; t = 1, 2, \cdots , T$)}. The group main effect obtained will actually be the $G \times Fm$ effect, and the error variation will actually be variation due to $S(G) \times Fm$, i.e., individual differences (of subjects nested in groups) in the size of the contribution of the mth estimated interaction factor to subjects' scores. Thus,

$$(6.4) \qquad \mathrm{SS}_{GFm} = I \sum_t (\hat{d}_{.tm} - \hat{d}_{..m})^2,$$

and since $\mathrm{SS}_{Fm} = IT\hat{d}_{..m}^2$,

$$(6.5) \qquad \mathrm{SS}_{GFm} = I \sum_t \hat{d}_{.tm}^2 - \mathrm{SS}_{Fm} .$$

The SS for the error term is

$$(6.6) \quad SS_{S(G)Fm} = \sum_i \sum_t (\hat{d}_{itm} - \hat{d}_{.tm})^2 = \sum_i \sum_t \hat{d}_{itm}^2 - SS_{GFm} - SS_{Fm} .$$

The mean squares are

$$(6.7) \qquad\qquad MS_{GFm} = SS_{GFm}/(T - 1),$$

and

$$(6.8) \qquad\qquad MS_{S(G)Fm} = SS_{S(G)Fm}/(I - 1)T.$$

Under the usual assumptions for judging the significance of orthogonal *a priori* contrasts, we reject the hypothesis that the mth estimated interaction factor contributes equally to subjects' measurements at each level of G if

$$(6.9) \qquad\qquad MS_{GFm}/MS_{S(G)Fm} \geq F_{\alpha; T-1, (I-1)T} .$$

We emphasize that, as in the fixed model the test of "factor similarity" suggested by (6.9) is an exact test of the hypothesis that the contrast defined by the *estimates* $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K)\}$ accounts for an equal amount of variation at each level of G.

$G \times Fres$ *variation*. A significance test for variation due to the $G \times$ Fres source is obtained by computing the IT scores

$$(6.10) \qquad\qquad \hat{w}_{it} = \sum_i \sum_k \hat{\phi}_{ik} y_{itjk} ,$$

and doing a one-way analysis of variance on the $\{\hat{w}_{it}\}$. However, since $\sum_i \sum_k \hat{\phi}_{ik}^2 \neq 1$, it is necessary to divide the SS obtained by $\sum_i \sum_k \hat{\phi}_{ik}^2$ in order to make the values commensurate with the other SS obtained in the overall analysis. Thus,

$$(6.11) \qquad SS_{GFres} = [I \sum_t \hat{w}_{.t}^2 / \sum_i \sum_k \hat{\phi}_{ik}^2] - SS_{Fres} ,$$

$$(6.12) \qquad SS_{S(G)Fres} = [\sum_i \sum_t \hat{w}_{it}^2 / \sum_i \sum_k \hat{\phi}_{ik}^2] - SS_{GFres} - SS_{Fres} ,$$

where

$$(6.13) \qquad\qquad SS_{Fres} = IT\hat{w}_{..}^2 = SS_{RC} - \sum_m SS_{Fm} .$$

The hypothesis that the contrast defined by the $\{\hat{\phi}_{ik}\}$ accounts for an equal amount of variation at each level of G is rejected if

$$(6.14) \qquad \frac{SS_{GFres}/(T - 1)}{SS_{S(G)Fres}/(I - 1)T} \geq F_{\alpha; T-1, (I-1)T} .$$

$GRC[res]$ *variation*. Described in this paragraph is a test of whether a

significant amount of $G \times R \times C$ interaction variation remains after variation due to $G \times Fm$ (for all m) and $G \times Fres$ interaction has been removed. Variation due to GRC[res] interaction is obtained by computing

$$(6.15) \qquad SS_{GRC[res]} = SS_{GRC} - \sum_{m} SS_{GFm} - SS_{GFres} ,$$

and variation due to individual differences in the contribution of GRC[res] variation is given by

$$(6.16) \qquad SS_{S(G)RC[res]} = SS_{S(G)RC} - \sum_{m} SS_{S(G)Fm} - SS_{S(G)Fres} .$$

The mean squares are

$$(6.17) \qquad MS_{GRC[res]} = SS_{GRC[res]}/\delta,$$

where $\delta = (T - 1)[(J - 1)(K - 1) - (M + 1)]$, and

$$(6.18) \qquad MS_{S(G)RC[res]} = SS_{S(G)RC[res]}/\nu,$$

where $\nu = T(I - 1)[(J - 1)(K - 1) - (M + 1)]$. The hypothesis that no $G \times R \times C$ interaction remains after removing variation due to $G \times Fm$ (for all m) and $G \times Fres$ interaction is rejected if

$$(6.19) \qquad MS_{GRC[res]}/MS_{S(G)RC[res]} \geq F_{\alpha;\delta,\nu} .$$

The test given by (6.19) is a conventional one and assumes that the variation in $SS_{S(G)RC[res]}$ is homogeneous.

Table 4 presents a summary of the computational formulas used when judging "factor similarity" over different groups of subjects in a repeated-measurements FANOVA model. Work by Gollob [1965] illustrates use of the above tests of "factor similarity."

Methods directly analogous to those presented above for judging factor similarity can be used to test whether main effect profiles, or $R \times C$ interaction profiles are the same over T different groups. We will not discuss this possibility further here; but emphasize that the resulting significance tests taken singly require only that the "contrast scores" assigned to subjects in each group meet the usual one-way analysis of variance assumptions; and are valid irrespective of the form of the population variance-covariance matrices associated with the original measurements.

7. Illustrative Example

In order to emphasize and clarify central features of both the fixed and the mixed FANOVA model a detailed illustrative example is now presented. The basic FANOVA decomposition of a two-way table of data is the same for both the fixed and mixed FANOVA model. In the case of judging the statistical significance of various components of the decomposition, however, the fixed and mixed models differ in several important ways;

with the procedures for the mixed model being considerably more complex. It is for this reason that we present a detailed example of a *mixed* model analysis. Gollob [1965] has described the results of applying a mixed effects FANOVA model to the problem of predicting the evaluative (good-bad) rating of a sentence subject as it is described by the total sentence. All sentences fit the sentence frame: The *adjective* man *verb noun*. For example: The vicious man harms criminals. The kind man likes alcoholics. All possible combinations of eight adjectives, six verbs, and four nouns were used to construct 192 sentences. Twenty-four *S*s rated the man described by each stimulus sentence on an 11-point good-bad scale. We will discuss the FANOVA decomposition of the three-way table of data resulting after averaging over the adjectives (evil, cruel, cynical, uncouth, uninteresting, friendly, considerate, kind). The verbs and objects used are presented in Table 5. In the resulting table we let y_{ijk} represent the ith S's mean rating (over adjectives) of men described by sentences containing the jth verb and kth adjective. Least squares estimates of the grand mean μ, verb main effects $\{V_j\}$, object main effects $\{O_k\}$, and interaction parameters $\{\gamma_{jk}\}$ were obtained by substituting $y_{\cdot jk}$ for x_{jk} in

TABLE 4

Computational Formulas for Judging "Factor

Similarity" in the Repeated-measurement FANOVA Model

Source	df	Sums of Squares
GxRxC	$(T-1)(J-1)(K-1)$	
GxF1	$(T-1)$	$I \sum_t \hat{d}^2_{\cdot t1} - SS^*_{F1}$
GxF2	$(T-1)$	$I \sum_t \hat{d}^2_{\cdot t2} - SS_{F2}$
.	.	.
.	.	.
.	.	.
GxFm	$(T-1)$	$I \sum_t \hat{d}^2_{\cdot tm} - SS_{Fm}$
.	.	.
.	.	.
.	.	.
GxFM	$(T-1)$	$I \sum_t \hat{d}^2_{\cdot tM} - SS_{FM}$
GxFres	$(T-1)$	$I \sum_t (\sum_j \sum_k \hat{\phi}_{jk} y_{\cdot tjk})^2 / \sum_j \sum_k \hat{\phi}^2_{jk} - SS_{Fres}$
GRC[res]	$\delta^\dagger$	$SS_{GRC} - \sum_m SS_{GFm} - SS_{GFres}$

S(G)xRxC	(I-1)(T)(J-1)(K-1)	
S(G)xF1	(I-1)T	$\sum_i \sum_t \hat{d}_{it1}^2 - SS_{GF1} - SS_{F1}$
S(G)xF2	(I-1)T	$\sum_i \sum_t \hat{d}_{it2}^2 - SS_{GF2} - SS_{F2}$
.	.	.
.	.	.
.	.	.
S(G)xFm	(I-1)T	$\sum_i \sum_t \hat{d}_{itm}^2 - SS_{GFm} - SS_{Fm}$
.	.	.
.	.	.
.	.	.
S(G)xFM	(I-1)T	$\sum_i \sum_t \hat{d}_{itM}^2 - SS_{GFM} - SS_{FM}$
S(G)xFres	(I-1)T	$\sum_i \sum_t (\sum_j \sum_k \hat{\phi}_{jk} y_{itjk})^2 / \sum_j \sum_k \hat{\phi}_{jk}^2 - SS_{GFres} - SS_{Fres}$
S(G)RC[res]	ν ‡	$SS_{S(G)RC} - \sum_m SS_{S(G)Fm} - SS_{S(G)Fres}$

* $SS_{Fm} = IT\hat{d}_m^2$ where $\hat{d}_m^2$ represents the $\underline{m}$th eigenvalue obtained in using the factor model to decompose the matrix of $\{\hat{\gamma}_{jk}\}$.

 $SS_{Fres} = SS_{RC} - \sum_m SS_{Fm}$.

† $\delta = (T-1)[(J-1)(K-1)-(M+1)]$

‡ $\nu = T(I-1)[(J-1)(K-1)-(M+1)]$

formulas (1.8) to (1.11). The $\{y_{.ik}\}$ and the parameter estimates are presented in Table 5. An analysis of variance summary table for this data is presented in Table 6. The verb and object main effects and the verb $\times$ object interaction are all highly significant. Using percentages based on variance component estimates, [see Hays, 1963; pp. 406–7, 438], we find that the verb main effect accounts for 58%, the object main effect for 2%, and the verb $\times$ object interaction accounts for 40% of the "between predicates" variation [e.g., for the verb main effect: $\hat{\sigma}_v^2/(\hat{\sigma}_v^2 + \hat{\sigma}_0^2 + \hat{\sigma}_{v0}^2) = (1.52)/(1.52 + .05 + 1.03) = .58$].

To obtain the FANOVA decomposition of the $\{y_{.ik}\}$ we now must express the 6 $\times$ 4 matrix $\hat{\Gamma}$, of interaction parameter estimates $\{\hat{\gamma}_{ik}\}$, in terms of the factor model (1.1). Since $J > K$ we solve for matrices $\hat{D}^2$ and $\hat{B}$ by finding the eigenvectors $\{\hat{\beta}_{km}\}$ and eigenvalues $\{\hat{d}_m^2\}$ of the 4 $\times$ 4 matrix $\hat{\Gamma}'\hat{\Gamma}$ and then use (1.2b) to obtain the matrix $\hat{A}$. The matrices $\hat{A}$, $\hat{D}$, and $\hat{B}$ are presented in Table 7.

TABLE 5

Observed Means, Fitted Values, and Interaction Parameter Estimates*

The three entries in the j,kth cell are the
    (a) mean rating (over observations and adjectives) for the predicate
    (b) value fitted by summing the grand mean and main effects, and
    (c) interaction parameter estimates.
The main effects are given in brackets along the left and upper margins.

grand mean $\hat{\mu} = -.43$	[.22] physicans	[.16] colleagues	[-.11] alcoholics	[-.27] criminals
[1.51] helps	1.77 (1.31) .46	1.42 (1.24) .18	1.88 (.98) .90	-.72 (.82) -1.55
[1.30] befriends	1.22 (1.09) .13	1.10 (1.02) .08	1.32 (.75) .57	-.18 (.60) -.78
[.27] praises	1.22 (.06) 1.16	.95 (.00) .96	-1.00 (-.27) -.72	-1.82 (-.43) -1.39
[-.53] criticizes	-1.14 (-.73) -.41	-1.03 (-.80) -.23	-1.26 (-1.07) -.19	-.40 (-1.22) .82
[-1.02] frustrates	-1.95 (-1.22) -.73	-1.83 (-1.29) -.54	-1.95 (-1.56) -.39	-.04 (-1.71) 1.67
[-1.54] hates	-2.37 (-1.75) -.62	-2.25 (-1.81) -.44	-2.25 (-2.08) -.17	-1.00 (-2.23) 1.23

* The slight inaccuracies which appear in this table (and in the following
  tables) are due to rounding error.

Table 3 summarizes the computations used in partitioning variation due to $R \times C$ (Verb $\times$ Object) and $S \times R \times C$ ($S \times$ Verb $\times$ Object) interaction in a three-way mixed-effects FANOVA model. Thus the sums of squares accounted for by the mth interaction factor is given by

$$(7.1) \qquad SS_{Fm} = I\hat{d}_m^2,$$

where the $\{\hat{d}_m^2\}$ are the eigenvalues of $\hat{\Gamma}'\hat{\Gamma}$ and have been obtained in the process of factoring the table of interaction parameter estimates, $\hat{\Gamma}$. Since the number of Ss, I, equals 24, and from Table 7 we see that $\hat{d}_1^2 = (3.703)^2 = 13.711$; we compute the SS accounted for by the first interaction factor by $SS_{F1} = (24)(13.711) = 329.064$. The mth interaction factor is assigned

TABLE 6

Analysis of Variance of Mean Ratings

Source	df	SS	MS	F
Predicates	23	1146.70		
V	5	741.11	148.22	56.06*
O	3	22.56	7.52	14.60*
VxO	15	383.02	25.53	30.54*
Individual Differences	552	721.91		
S	23	93.72	4.07	
SxV	115	304.03	2.64	
SxO	69	35.54	.51	
SxVxO	345	288.62	.84	
Total	575			

* $p < .001$ by Greenhouse and Geisser conservative test (see Winer, 1962). This test allows the relevant variance-covariance matrices to be of arbitrary form. The test rejects the null hypothesis if the usually computed ratios exceed the tabled value given for the F-distribution with 1 and $(I-1)df$.

TABLE 7

Factor Analytic Decomposition ($\hat{I} = \hat{A}\hat{D}\hat{B}'$) of the Interaction
Parameter Estimates

	The Matrix $\hat{A}$				The Matrix $\hat{D}$				The Matrix $\hat{B}'$			
	$F1\{\hat{a}_{j1}\}$	$F2\{\hat{a}_{j2}\}$	$F3\{\hat{a}_{j3}\}$		d_1	d_2	d_3			phys	coll	alc	crim
helps	.453	.533	.485	F1	3.703	0	0	$F1\{\hat{\beta}_{k1}\}$	.419	.296	.133	-.848
befriends	.222	.357	-.534	F2	0	1.497	0	$F2\{\hat{\beta}_{k2}\}$	.336	-.362	.855	-.158
praises	.501	-.758	.051	F3	0	0	.084	$F3\{\hat{\beta}_{k3}\}$	.679	-.729	-.027	.077
criticizes	-.259	-.047	-.482									
frustrates	-.523	-.105	.492									
hates	-.394	.020	-.016									

$(J + K - 1 - 2m)$ df, so the first factor is assigned $(6 + 4 - 1 - 2) = 7$ df. Each successive factor is assigned two df less than the preceding factor. The MS for the first interaction factor is $329.064/7 = 47.010$. The results of making the preceding computations for all three interaction factors are presented in the top half of Table 8.

The SS accounted for by the $S \times Fm$ interaction is computed by thinking of the weights $\{\hat{\alpha}_{jm}\hat{\beta}_{km} \ (j = 1, \cdots, J; \ k = 1, \cdots, K)\}$ as defining a contrast; and applying the standard formula for normalized contrasts,

$$(7.2) \qquad SS_{SFm} = \sum_i (\sum_j \sum_k \hat{\alpha}_{jm}\hat{\beta}_{km}y_{ijk})^2 - SS_{Fm} = \sum_i \hat{d}_{im}^2 - SS_{Fm} .$$

In calculating the $\{\hat{d}_{im}\}$ for the mth interaction factor it is useful to obtain

$$(7.3) \qquad\qquad\qquad \hat{w}_{jkm} = \hat{\alpha}_{jm}\hat{\beta}_{km} ,$$

where $\hat{\alpha}_{jm}$ and $\hat{\beta}_{km}$ are the entries in the mth columns of the matrices $\hat{A}$ and $\hat{B}$, respectively. The I values $\{\hat{d}_{im} \ (i = 1, \cdots, I)\}$ are then computed by

$$(7.4) \qquad\qquad\qquad \hat{d}_{im} = \sum_j \sum_k \hat{w}_{jkm}y_{ijk} ,$$

and are substituted in (7.2) to obtain SS_{SFm}. A convenient check on the computation of the $\{\hat{d}_{im}\}$ is provided by the relation $\hat{d}_{.m} = \hat{d}_m$. The SS and MS for $S \times F1$, $S \times F2$, and $S \times F3$ are presented in Table 8.

TABLE 8

Summary Table of Variation Accounted for
Interaction Factors

Source	df	Sums of Squares	Mean Squares	(F/R)$_{Fm}$	F-Ratio
VxO	15	383.016	25.535		
F1	7	329.064(85.92)[†]	47.010	9.41	
F2	5	53.784(14.04)	10.757	13.70	
F3	3	.169(.04)	.052	.24	
SxVxO	345	288.621	.836		
SxF1	23	114.892(39.81)[‡]	4.995		9.14*
SxF2	23	18.046(6.25)	.785		1.44
SxF3	23	4.969(1.72)	.216		.40
SVO[res]	276	150.714(52.22)	.546		

† Percentage of VxO interaction accounted for by the mth interaction factor.

‡ Percentage of SxVxO variation accounted for by "source."

* $p \leq .0001$

Having computed the relevant mean squares, we will now decide how many interaction factors we must retain in order to adequately account for the verb $\times$ object interaction. Using the liberal test (5.19) we obtain ratios of $(7)(9.41) = 65.87$ and $(5)(13.70) = 68.50$, respectively, for $F1$ and $F2$. These ratios are impressively large when considered in the light of the fact that mean ratings of the type analyzed in this example are known to be very stable over different groups of subjects [Gollob, 1965]. In general, the more reasonable it is to assume that different groups of subjects would give highly similar mean ratings $\{y_{.jk}\}$, the more reasonable it is to think of the liberal test as being an approximation to the "exact test" which actually applies weights obtained from one set of data to a different set of data. Note also that although $F2$ accounts for only 14% of $V \times O$ variation, individual differences in the size of its contribution are so slight that $F2$ is judged more statistically significant than $F1$ which accounts for 85% of the $V \times O$ variation. For the third factor the liberal test yields a ratio of $(3)(.24) = .72$ which is obviously not statistically significant. In addition, Table 8 shows that the third interaction factor accounts for only .04% of the $V \times O$ variation; and Table 11 shows that its largest possible contribution to any predicate is only .03. Furthermore, Table 8 shows that variation due to $S \times F3$ is small and nonsignificant; and therefore, $F3$ is not an important factor in describing individual differences. It is clear that the third interaction factor is of neither statistical nor "practical" significance; and therefore, we will not consider it further.

The conservative test described by (5.18) assigns $(J-1)(K-1) = 15$ **df** to all SS_{F_m} and $I - (J-1)(K-1) = 9$ df to all SS_{SF_m}. The resulting ratios for $F1$ and $F2$, respectively, are $C_{F1} = 1.74$ and $C_{F2} = 1.79$ which reach significance only at the .25 level. This test is very conservative indeed!

The $(F/R)_{F_m}$ ratios for the three interaction factors are presented in Table 8. Although no such evidence is now available, experience with using the FANOVA model may show that under some conditions it is a useful procedure to look up tabled p-values as though the F/R ratios were distributed as F_α with $(J + K - 1 - 2m)$ and $(I - 1)$ **df**. Or perhaps some empirically determined cut-off point such as a F/R ratio of say, 3 or 4 required for "significance" will be found useful for some types of subject-matter. Under the assumption that we will at least be somewhere in the ball-park, we pretend that the F/R ratios are distributed as $F_{\alpha;J+K-1-2m,I-1}$ and find that for both the first and second factor $p < .0001$.

On the basis of the above considerations, we tentatively decide that the first two interaction factors are necessary and sufficient to adequately account for the $V \times O$ variation. Thus, $M = 2$ and $SS_{F_{res}}$, which would ordinarily be found by subtraction $(SS_{VO} - \sum_m SS_{F_m})$, is identical with $F3$. The SS due to SVO[res] interaction is given in Table 8.

Since we have only one replication per cell we use $MS_{SVO[res]}$ as an

error term in judging the significance of the $S \times Fm$ "sources" of variation. The results, which are presented in Table 8, show that $S \times F1$ interaction (which accounts for 40% of $S \times V \times O$ variation) is highly significant and that $S \times F2$ and $S \times F3$ interactions are slight and non-significant. Thus, a major portion of individual differences in the profiles of $V \times O$ interaction parameter estimates for Ss can be accounted for simply by assigning each S a "score" which determines the size of the contribution of $F1$ to his ratings.

Qualitative features of the FANOVA decomposition.

Keeping the grand mean separate for convenience, Table 9 presents the FANOVA decomposition of the data. Table 10 shows (a) the size of the contribution of $F1$ to each j, k cell, and (b) the residual in each j, k cell. Table 10 also presents the percentage of SS_{F1} which is contributed to given rows and columns of the table. (Thus, 20.52% of the SS accounted for by the first interaction factor is in the four cells defined by combinations of "helps" with each of the four objects.) Table 11 presents the corresponding data for the second factor. It is helpful to note that the proportion of mth

TABLE 9*

FANOVA Decomposition for Predicate Means

	Verb	Obj	F1	F2		phys	coll	alc	crim
helps	1.51	1.00	.87	.65	Verb	1.00	1.00	1.00	1.00
befriends	1.30	1.00	.43	.44	Obj	.22	.16	-.11	-.27
praises	.27	1.00	.96	-.93	F1	.81	.57	.26	-1.63
criticizes	-.53	1.00	-.50	-.06	F2	-.41	-.44	1.05	-.19
frustrates	-1.02	1.00	-1.01	-.13					
hates	-1.54	1.00	-.76	.02	$\sum$				
%VxO Var.	-----	----	85.92	14.04	99.96				
%"Predi-cate" Var.†	58.46	1.87	34.08	5.57	99.98				

(grand mean = -.43)

* This table is read as follows: Consider the predicate "praises criminals" (obtained mean rating = -1.82). The 1x4 (row) vector associated with "praises" times the 4x1 (column) vector associated with "criminals" yields (.27)(1.00) +(1.00)(-.27)+(.96)(-1.63)+(-.93)(-.19) = .270-.270-1.565+.177 = -1.39; and adding the grand mean (-.43) gives a "predicted" rating of -1.82.

† The "% 'Predicate' Var" for the mth interaction factor was computed by multiplying the proportion of VxO sums of squares accounted for by the mth factor times the percentage of predicate variation (estimated on the basis of variance components) due to VxO interaction.

TABLE **10**

The First Interaction Factor

The two entries in the j,kth cell are
  (a)   the contribution of the first interaction
        factor, $\hat{d}_1\hat{\alpha}_{j1}\hat{\beta}_{k1}$, and
  (b)   the first factor residual.

The "weights" for verbs $\{\hat{\alpha}_{j1}\}$ and objects $\{\hat{\beta}_{k1}\}$
are bracketed in the margins, and $\hat{d}_1 = 3.703$.

$[\hat{d}_1 = 3.703]$	[.419] phys	[.296] coll	[.133] alc	[-.848] crim	% SS$_{F1}$ in row j
[.453] help	(.70) −.24	(.50) −.32	(.22) .68	(−1.42) −.12	20.5
[.222] befrnd	(.34) −.21	(.24) −.16	(.11) .46	(−.70) −.09	4.9
[.501] praise	(.78) .38	(.55) .41	(.25) −.97	(−1.57) .18	25.1
[−.259] crit	(−.40) .00	(−.28) .05	(−.13) −.06	(.81) .01	6.7
[−.523] frust	(−.81) .08	(−.57) .03	(−.26) −.13	(1.64) .03	27.4
[−.394] hate	(−.61) −.01	(−.43) −.01	(−.19) .02	(1.24) −.01	15.5
% SS$_{F1}$ in col k	17.6	8.8	1.8	71.9	100.1

factor variation contributed by the *j, k*th cell is

(7.5) $$(\hat{d}_m\hat{\alpha}_{jm}\hat{\beta}_{km})^2/\hat{d}_m^2 = \hat{\alpha}_{jm}^2\hat{\beta}_{km}^2 ,$$

where it will be recalled that $\hat{d}_m \hat{\alpha}_{jm} \hat{\beta}_{km}$ represents the contribution of the mth factor to the j, kth cell. The proportion of Fm variation contributed by the cells in the jth row is

$$(7.6) \qquad\qquad \sum_k \hat{\alpha}^2_{jm} \hat{\beta}^2_{km} = \hat{\alpha}^2_{jm} ,$$

and similarly, the proportion of Fm variation contributed by the kth column is $\hat{\beta}^2_{km}$.

The Verb and Object main effects. As shown in Table 9 the verb main effects account for 58% of the total variation due to predicates, are highly significant, and distinguish sharply between "good and bad" verbs. The large verb main effect suggests that when other information is averaged out, a man who, say, hates tends to be thought of as a "hater;" and similarly, a man who, say, helps tends to be thought of as a "helper." The object main effect in both sets of predicates also clearly separates "good and bad;" and although it accounts for less than 2% of the variation due to predicates, is still highly significant. Thus, when other information is averaged out, it seems that Ss rate a man negatively (or positively) simply for being associated with negatively (or positively) evaluated objects.

The first interaction factor. Upon examining Table 9, one readily notes that the first interaction factor accounts for the bulk, about 86%, of the Verb × Object interation. The verb and object weights of the first interaction factor clearly suggest the distinction between "good and bad" verbs and objects. Note, however, that the object "alcoholics" is assigned a small, but positive weight. In general, $F1$ contributes a positive evaluation when the verb and object have evaluative connotations of the same sign; and contributes a negative evaluation when the verb and object have evaluative connotations of opposite sign. For example, $F1$ contributes a large positive evaluation to the good-good and bad-bad predicates, "helps (+) physicians (+)" and "hates (−) criminals (−);" and contributes a large negative evaluation to the good-bad and bad-good predicates, "helps (+) criminals (−)" and "hates (−) physicians (+)." This result is, of course, what would be predicted by "balance" theory [e.g., Heider, 1967; Abelson, 1963].

The second interaction factor. The second interaction factor seems to describe an aspect of meaning which is more subtle than the simple good-bad dimension of the first factor. The second interaction factor assigns all the negative verbs weights near zero, and distinguishes between the positive verbs by assigning "praises" a large negative weight and assigning "helps" and "befriends" moderately sized positive weights. Notice also that the object "alcoholic" is assigned a large positive weight, while "physician" and "colleague" have moderate sized negative weights, and the weight assigned to "criminal" is near zero. The net effect of the weight assignments is that the second interaction factor contributes a highly negative component,

relative to the other predicates, to "praise alcoholics;" and it seems reasonable to hypothesize that there is a sense in which Ss think of alcoholics as people

TABLE 11

The Second Interaction Factor

The two entries in the j,kth cell are
  (a) the contribution of the second interaction factor, $\hat{d}_2\hat{\alpha}_{j2}\hat{\beta}_{k2}$, and
  (b) the second factor residual.

The "weights" for verbs $\{\hat{\alpha}_{j2}\}$ and objects $\{\hat{\beta}_{k2}\}$ are bracketed in the margins, and $\hat{d}_2 = 1.497$.

$[\hat{d}_2 = 1.497]$	[.336] phys	[-.362] coll	[.855] alc	[-.158] crim	% SS_{F2} in row j
[.533] help	(-.27) .03	(-.29) -.03	(.68) .00	(-.13) .00	28.4
[.357] befrnd	(-.18) -.03	(-.19) .03	(.46) .00	(-.08) .00	12.8
[-.758] praise	(.38) .00	(.41) .00	(-.97) .00	(.18) .00	57.5
[-.047] crit	(.02) -.02	(.03) .03	(-.06) .00	(.01) .00	.2
[-.105] frust	(.05) .03	(.06) -.03	(-.14) .00	(.03) .00	1.1
[.020] hate	(-.01) .00	(-.01) .00	(.03) .00	(.00) .00	0.0
% SS_{F2} in row k	11.3	13.1	73.1	2.5	100.0

who are relatively less deserving of praise than are colleagues or physicians. Simultaneously, the second factor suggests that Ss consider colleagues and physicians as people who, relative to alcoholics, are less in need of help and friends. This interpretation seems intuitively reasonable, and although the second interaction factor accounts for less than 6% of the predicate variation, it attained an even higher level of statistical significance than did the first interaction factor which accounts for 34% of the predicate variation. This, of course, is due to the fact that there was little individual difference in the importance of the second interaction factor. Thus, the aspect of meaning which it suggests seems to be particularly stable across Ss. In this light, it is not surprising that we were able to arrive at a reasonable interpretation of the factor. It is also of interest to note that the variance accounted for by $F2$ is concentrated primarily in those cells which define predicates whose object is alcoholics. The role of $F2$ is further clarified by noting that $F1$ accounts for only about 13% of the variation due to inter-action of all verbs with alcoholic; whereas $F2$ accounts for 86% of the variation due to interaction of all verbs with alcoholic. Thus, the second factor is the *primary* determiner of interaction between the verbs of this study used in combination with alcoholics as objects. One advantage of the present method of analysis is that this point is brought out clearly. Although this completes our discussion of the present illustrative example of a mixed FANOVA analysis; more details and additional data are presented in Gollob [1965].

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