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# A statistical study of the evolution of the orbits of long-period comets

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**Summary**. Since orbital periods of long-period comets are much longer than those of perturbing planets (Jupiter and Saturn), energy perturbations of a comet can be regarded as a random variable which obeys a Gaussian or double exponential distribution. Starting with a given initial energy, energy states of a large number of comets are followed by a Monte Carlo method until they are lost to the solar system by dynamical expulsion or by physical disintegration. First, the lifetime distribution of comets is so computed and it is shown that with reasonable values of $\kappa^2$ (rate of physical disintegration), more than 90 per cent of comets are expelled from the solar system within 3 Myr. Secondly, the number of perihelion passages before escape or disintegration is calculated analytically by Samuelson's method and compared with the Monte Carlo result. It is shown that the probability of comets remaining in the solar system is aymptotically proportional to $N^{-1/2}$, a result first obtained numerically by Everhart. It is further shown that if $\kappa^2 = 0.02$, more than 90 per cent of comets are lost in 40 revolutions or so. Thirdly, and most important, the energy distribution among the cometary population is computed and it is shown that within 3 or 6 Myr, the energy distribution of comets still left in the system takes a form which is almost independent of the initial distribution, and comes to have a strong concentration in the interval $0 < 1/a < 5 \times 10^{-5}\,\text{AU}^{-1}$. Comparison with the observed distribution shows that if the comets were formed or captured at an earlier epoch, that must have taken place at 3.6 ~ 9 Myr BP.

## 1 Introduction

Among nearly 600 comets contained in Marsden's (1972) cometary catalogue, some 500 are classified as parabolic or nearly parabolic comets. Their semimajor axes ($a$) range from $10^2$ to $10^5$ AU (positive $a$ corresponds to elliptic orbit) so that their binding energy with respect to the solar system barycentre is comparable to planetary perturbations. Thus some long-period comets whose original incoming orbits are definitely elliptic are transformed into hyperbolic orbits by planetary perturbations (mainly due to Jupiter and Saturn). Marsden, Sekanina & Everhart (1978) recently computed original and future values of $1/a$ for 200 comets and it has been shown that some 50 of them will eventually escape from the solar system with negative values of binding energy. It is thus important to investigate theoretically the effect which planetary perturbation will have upon the population of long-period comets. Since four to five comets are discovered every year and their average period is some $10^4$ yr or greater, the number of comets associated with the solar system at the present epoch must be at least a few million, so that the problem can be tackled from a statistical point of view, instead of following orbits of individual comets.

The first theoretical approach to the problem was made by van Woerkom (1948) who derived a differential equation for the cometary population on the assumption that the energy perturbation is small compared with the energy. However, since the two are comparable, a more satisfactory approach would be that of a random walk in an energy space, as formulated and solved by Hammersley (1961) and by Lyttleton & Hammersley (1964). Assuming that a comet is formed with a given energy value, it is subjected to random energy changes until the binding energy acquires a negative value and the comet is lost to the solar system. They computed, by a Monte-Carlo method, the lifetime distribution of long-period comets.

On the other hand, it is also important to know how the planetary perturbations will modify the distribution of energy in the cometary population. As first noted by Oort (1950), when the distribution of $1/a$ values is plotted, it shows a marked concentration in the range $0 < 1/a < 10^{-4}\,\mathrm{AU}^{-1}$. Oort regards these comets as *new comets* whose orbits were deflected into the planetary region by stellar perturbations. In this respect, the planetary perturbation alone may be capable of explaining the observed tendency of the $1/a$ distribution.

The problem is important in the interpretation of the $1/a$ distribution. If it is *assumed* that comets as members of the solar system are a permanent feature, the Oort interpretation is inescapable. However, there is a possibility that comets are a transient phenomenon. For instance, the Lyttleton (1953) theory assumes that comets are formed from time to time as the Sun passes through dense interstellar dust clouds. Again, comets may form in dense clouds (McCrea 1975) and are then captured by the solar system into nearly parabolic orbits (Yabushita & Hasegawa 1978). If the latter possibility is assumed, it is important to know whether the presently observed $1/a$ distribution is explicable in terms of planetary perturbations alone, since there is no longer a continuous supply of comets from the Oort cloud of comets. On the other hand, Weissman (1978) simulated the population dynamics of comets assuming that there is a continuous supply from the Oort's clouds of comets, and obtained fairly reasonable agreement with observation.

The object of the present paper is to investigate the effect of planetary perturbations upon the population of long-period comets. We will consider three features of the problem. First, the distribution of lifetimes of comets in the presence of physical disintegration will be investigated using Hammersley's method. Secondly, the distribution of the number of perihelion passages before escape will be calculated theoretically and compared with the Monte Carlo result. Thirdly, and most important, we will investigate how the distribution of binding energies of comets varies with time.

As to the second problem, Everhart (1976) has shown by following orbits of hypothetical comets that the fraction of comets remaining is proportional to $N^{-1/2}$, where $N$ is the number of perihelion passages. That this must be so will be shown analytically in the present paper. As to the third problem, Everhart followed evolution of orbits which initially have $a$ value of $4 \sim 5$ AU and obtained the $1/a$ distribution as a function of $N$. He was not able to obtain a strong concentration at very small values of $1/a$. However, in the present paper it will be shown that if the $1/a$ distribution is calculated as a function of time and not as a function of $N$, the $1/a$ distribution of observable comets will take a form which is close to that observed, whatever the initial orbit, provided the initial $a$ is greater than some 1000 AU.

## 2 Magnitude of planetary perturbations

In a discussion of planetary perturbations of long-period comets it is first required to know the energy perturbation which a comet undergoes at each passage through the planetary region. Since comets are assumed to be subjected to random energy change (energy per unit mass is $1/2a$, but $1/a$ may be simply called energy), the distribution of the change of $1/a$ (or $\delta(1/a)$) should be known beforehand.

Strictly speaking, the energy perturbation $[\delta(1/a)]$ is a function of $\Omega$ (longitude of ascending node), $\omega$ (argument of perihelion), $i$ (inclination) and $q$ (perihelion distance), even if it is assumed that the times of perihelion passage are assumed uniformly distributed. For uniformly distributed $\Omega$ and $\omega$, and fixed $q$, the magnitude of the perturbation depends upon $i$ (Yabushita 1972b). For fixed $q$ and $i$ the distribution is nearly Gaussian, but there is a dip near $\delta(1/a) = 0$ (Everhart 1968). Again, the distribution has a long tail which is proportional to $[\delta(1/a)]^{-3}$. When these complications are fully taken into account, simulation of a random walk process in cometary energy space becomes complicated. There is, furthermore, a consideration that the escape of comets from the solar system occurs as a result of several perturbations which can be regarded as almost independent. According to the central limit theorem of probability, the sum of a large number ($N$) of random variables sampled from a population whose mean is zero is asymptotically a Gaussian distribution which depends only upon $N$ and the standard deviation ($\sigma$). Thus it may be expected that the adoption of a simple form of the $\delta(1/a)$ distribution will give a reasonable result on cometary survival, provided that a reasonable value of the standard deviation is adopted. It may be noted that the distribution is not strictly symmetrical with respect to $\delta(1/a) = 0$ (Everhart 1969); the asymmetry arises from very close encounters of comets with Jupiter (or Saturn). Since we are not concerned with such close encounters, but with accumulation of ordinary perturbations, the distribution will be assumed symmetrical throughout the present paper. For these reasons, we will adopt the following simplified forms: (1) Gaussian distribution and (2) double exponential distribution;

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/\sigma^2), \qquad \text{Gaussian}, \tag{2.1}$$

$$P(x) = \frac{1}{\sqrt{2}\sigma} \exp(-\sqrt{2}|x|/\sigma), \qquad \text{double exponential}, \tag{2.2}$$

where $P\,\delta x$ is the probability that the change of $1/a$ lies in $(x, x + \delta x)$, and $\sigma$ is the standard deviation. In each case the mean of the distribution is zero.

Even if it is assumed that comets approach from every direction in the celestial sphere with equal probability, the standard deviation, $\sigma$, of $\delta(1/a)$ is, strictly speaking, a function of $q$, the perihelion distance. However, we are here concerned with comets which are or eventually become observable so that $q$ is less than 3 or 4 AU. The standard deviations calculated by various authors are given below:

| | | |
|---|---|---|
| $\sigma = 78 \times 10^{-5}$ AU$^{-1}$ | $q = 1$ AU | van Woerkom (1948) |
| $= 128 \times 10^{-5}$ | $= 4.5$ | |
| $\sigma = 72 \times 10^{-5}$ AU$^{-1}$ | $q = 1$ AU | Kerr (1961) |
| $= 42 \times 10^{-5}$ | $= 4.5$ | |
| $\sigma = 74 \times 10^{-5}$ AU$^{-1}$ | $q = 1.166$ AU | Yabushita (1972a) |
| $= 52 \times 10^{-5}$ | $= 2.94$ | |
| $= 46 \times 10^{-5}$ | $= 5.2$ | |

$\sigma = 76 \times 10^{-5}$ AU$^{-1}$ from an analysis of the orbit of Halley's comet, Kendall (1961)

$\sigma = 45.5 \times 10^{-5}$ AU$^{-1}$ mean of nearly 400 comets, Everhart & Raghavan (1970).

It is seen that apart from van Woerkom's value, $\sigma$ decreases with $q$. If $\sigma$ is taken as $66.6 \times 10^{-5}$ AU$^{-1}$, the corresponding value of $a$ is 1500 AU, and this appears to be a convenient choice. In the following, we will adopt $\sigma = 66.6 \times 10^{-5}$ AU throughout. Then, if a distribution law of the form (2.1) or (2.2) with unit variance is adopted, the unit of semimajor axis is 1500 AU and the unit of time is 58094 yr. This may be compared with Hammersley who adopted $\sigma = 75 \times 10^{-5}$ AU$^{-1}$, and the corresponding unit of time = 50000 yr.

In the following, the normal distribution will be assumed throughout, except in Section 4 where the distribution of $N$, the number of perihelion passages, is calculated. This is because the $N$ distribution can be obtained analytically when the double exponential distribution is adopted.

## 3 Distribution of the lifetime of comets

In this section, we calculate the probability that a comet which formed with $1/a = x_0$ is bound to the solar system for a period greater than $T$. In doing so, we adopt the method used by Hammersley. A cometary orbit is subjected to random changes of $1/a$ according to the distribution law (2.1). After $n$ such changes, the values of $1/a$ takes the value

$$x_1 = x_0(\text{given}) + \xi_1$$
$$x_n = x_{n-1} + \xi_n \quad n = 1, 2, \ldots \tag{3.1}$$

where $\xi_1, \ldots, \xi_n$ are normal random numbers with unit variance. The interval of time that a comet is bound to the system is

$$z = x_1^{-3/2} + x_2^{-3/2} + \ldots x_n^{3/2}. \tag{3.2}$$

such that $x_1, x_2, \ldots, x_n > 0$, $x_{n+1} \leqslant 0$. It is required to find the probability that $z > T$ as a function of $T$. Note that $z$ is the time since the first perihelion passage, in contrast to Hammersley's formulation where time is measured since cometary formation at aphelia. Hammersley adopted a Monte Carlo method to find the distribution of $z$, and it is followed here. He calculated the $z$ distribution only for the case where there is no disintegration of comets owing to meteor stream formation and evaporation due to solar radiation. We have included the effect by adopting a probability $\kappa^2$ with which a comet ceases to exist at each approach to perihelion. ($\kappa^2$ is greater for the first than for subsequent perihelion passages. This remark is due to E. Everhart. For simplicity, $\kappa^2$ will be assumed to remain constant.) Again, if the $1/a$ value becomes large, the comet is no longer long periodic, but short periodic. For short-period comets, perturbation by the planets cannot be regarded as random. For instance, the orbital period of Halley's comet does not change randomly; it changes in such a way that the average period of 77 yr is maintained (Kiang 1972; Yeomans 1977). We have therefore included the effect of diffusion into short-period orbits by terminating the random walk when $x$ exceeds 50. This value of $x$ corresponds to $a = 30$ AU. This choice is admittedly rather arbitrary. Later, it will be shown that this does not affect the form of the distribution of lifetimes when a small but finite value of $\kappa^2$ is included.

In Figs 1, 2 and 3, the probability that $z$ exceeds $T$ is plotted. These curves closely resemble the ones calculated by Hammersley and reproduced in Lyttleton & Hammersley

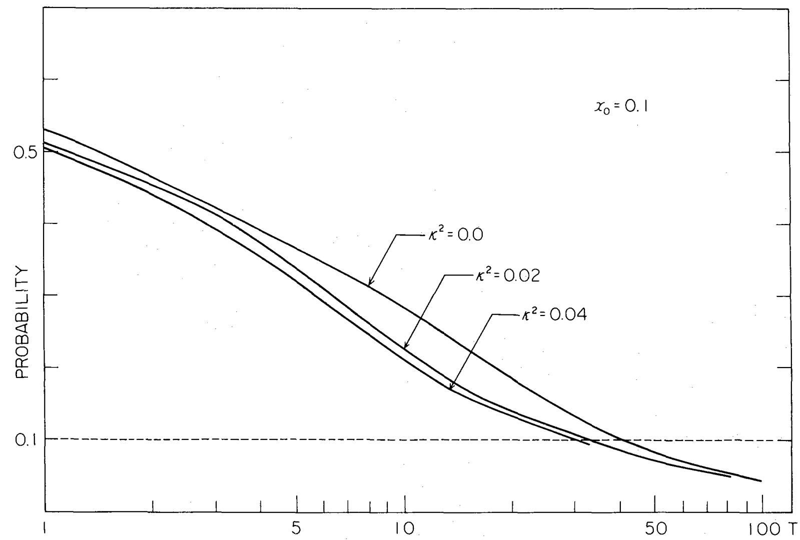


**Figure 1.** The probability that a comet with initial energy $x_0 = 0.1$ is left in the solar system after $T$ units of time. One unit of $T$ corresponds to $5.8 \times 10^4$ yr.

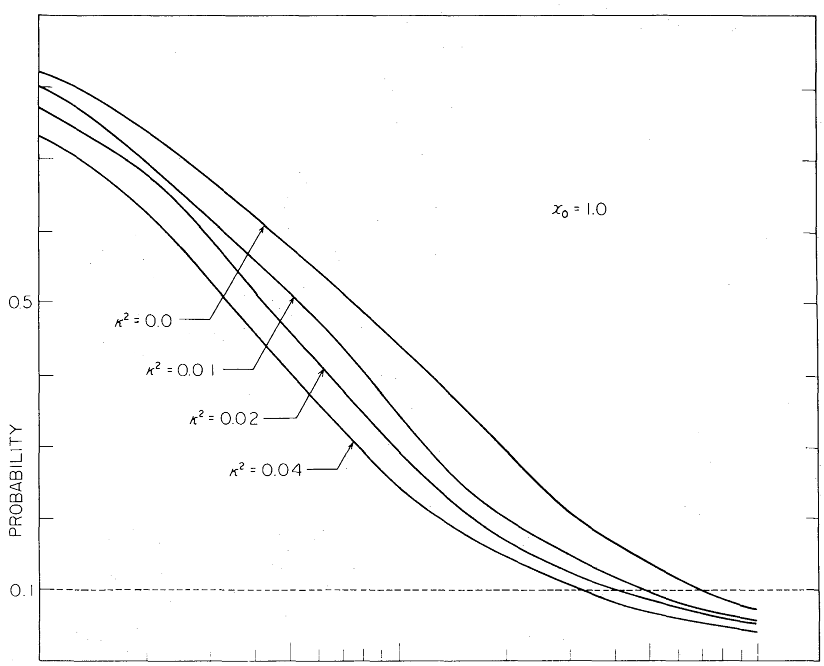


**Figure 2.** Probability distribution of lifetimes of long-period comets.

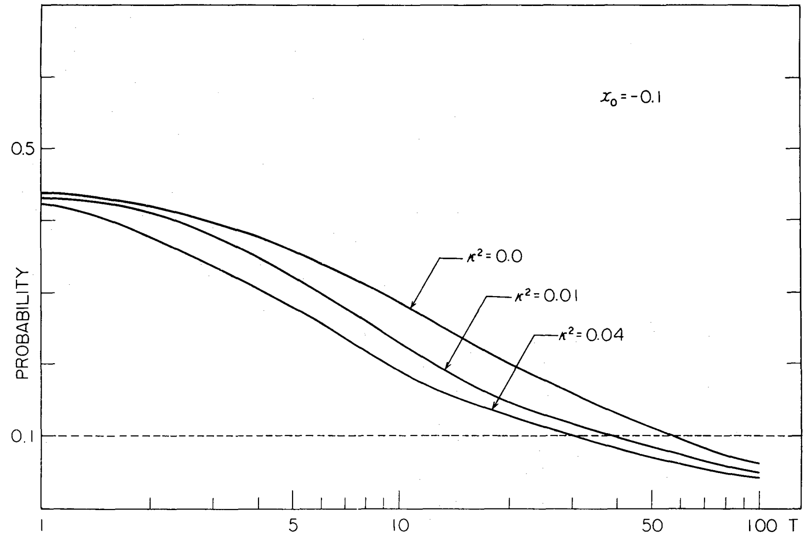


**Figure 3.** Probability distribution of lifetimes of long-period comets. The initial orbit is weakly hyperbolic.

(1964). The effect of a small but non-zero value of $\kappa^2$ (rate of physical disintegration) is rather large for small values of $T$ (smaller than 50, say).

The case $x_0 = -0.1$ was also computed in order to see what would happen if new comets encountered the solar system with slightly hyperbolic velocity at great distances from it. The value $x_0 = -1$ corresponds to the velocity of 0.769 km/s at great distances, while $x_0 = -0.1$ corresponds to 0.243 km/s, when the value 1500 AU is adopted as the unit of distance. Even for $x_0 = -0.1$, 46 per cent of comets are captured at first passage through planetary region, and transformed into elliptic orbits. As Fig. 3 shows, the behaviour of the curves are not much different from the cases where initial orbits are elliptic.

It may be readily noted that after 50 units of time, nearly 90 per cent of comets will be lost to the solar system. In physical units, this time is roughly $3 \times 10^6$ yr, which is much less than the age of the solar system. This point had earlier been emphasized by Lyttleton & Hammersley.

Finally, some remarks should be made with respect to the diffusion into short-period orbits. The percentages of comets which diffuse into short-period orbits ($a \leqslant 30$ AU) are as follows:

3.2 per cent $\kappa^2 = 0$ $x_0 = 1.0$

1.7 per cent $\kappa^2 = 0$ $x_0 = 0.1$

1.7 per cent $\kappa^2 = 0$ $x_0 = -0.1$.

When a small but non-zero value of $\kappa^2$ is adopted (0.02 or greater), no case of diffusion into short-period comets has been found among 1000 samplings carried out for each value of $x_0$. Thus, it appears that observable short-period comets are unlikely to result from the accumulation of small perturbations at each passage through the planetary region. This confirms one of Everhart's (1976) findings.

## 4 Distribution of the number of orbits

A comet may be said to have made $N$ orbits if it has passed through perihelion $N$ times before leaving the solar system. Hammersley obtained the distribution of $N$ as a byproduct of the Monte Carlo calculation of the distribution of cometary lifetimes. However, as will be shown shortly, it is possible to obtain *analytically* the distribution of $N$ if it is assumed that the distribution of $\delta(1/a)$ is a double-exponential function (2.2) and that the initial value of $1/a$ is zero. It is also possible to obtain the $N$ distribution by a Monte Carlo method. A comparison of the two results may be used to assess the accuracy of the Monte Carlo estimate, apart from an intrinsic interest in the $N$ distribution.

The method to be adopted here was originally proposed by Samuelson (1948). Let $F_n(x)$ be the distribution of $1/a$ values after $n$ perihelion passages; it is related to the distribution $F_{n-1}(x)$ at the previous step by the equation

$$F_n(x) = \int_0^\infty P(x-y)\,F_{n-1}(y)\,dy, \quad n = 1, 2, 3, \ldots \tag{4.1}$$

where $P(x-y)\,\delta x\,\delta y$ is the probability that a comet with a $1/a$ value in the range $(y, y+\delta y)$ is perturbed to $(x, x+\delta x)$. Multiplying each equation by $\lambda^{n-1}$ and adding, one obtains the equation

$$\sum_{n=1}^{\infty} \lambda^{n-1} F_n(x) = \int_0^\infty P(x-y)\,F_0(y)\,dy + \lambda \int_0^\infty P(x-y) \sum_{n=2}^{\infty} \lambda^{n-2} F_{n-1}(y)\,dy \tag{4.2}$$

or

$$F(\lambda, x) = G(x) + \lambda \int_0^\infty P(x-y)\,F(\lambda, y)\,dy \tag{4.3}$$

where

$$G(x) = \int_0^\infty P(x-y)\,F_0(y)\,dy, \quad F(\lambda, y) = \sum_{n=1}^{\infty} \lambda^{n-1} F_n(x). \tag{4.4}$$

This is an integral equation derived by Samuelson in relation to stochastic processes. When $P$ is a double exponential function (2.2) with unit variance,

$$P(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|),$$

equation (4.4) takes the form

$$F(\lambda, x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|) + \frac{\lambda}{\sqrt{2}} \int_0^\infty F(\lambda, y) \exp(-\sqrt{2}|x-y|)\,dy, \tag{4.5}$$

if $F_0(y) = \delta(y-\epsilon)$, $0 < \epsilon \ll 1$.

When a solution to the above equation has been found, $F_n(x)$ can be calculated from (4.4) to give

$$F_n(x) = \frac{1}{(n-1)!} \left. \frac{d^{n-1} F(x, \lambda)}{d\lambda^{n-1}} \right|_{\lambda=0}. \tag{4.6}$$

Furthermore, the probability that a comet is still bound to the solar system after $N$ orbits is simply given by

$$\int_0^\infty F_N(x)\,dx.$$

In the Appendix, it is shown that equation (4.5) can be solved analytically to give

$$P_N \equiv \int_0^\infty F_N(x)\,dx = \frac{(2N-1)!!}{(2N)!!}, \quad N = 1, 2, 3, \ldots \tag{4.7}$$

when there is no physical disintegration ($\kappa^2 = 0$). When physical disintegration is allowed for, the above expression should be replaced by

$$P_N = \frac{(2N-1)!!}{(2N)!!}(1-\kappa^2)^N = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(N+\tfrac{1}{2})}{\Gamma(N+1)}(1-\kappa^2)^N. \tag{4.8}$$

By making use of Stirling's formula for the asymptotic behaviour of the gamma function $\Gamma(x)$, it is readily found that

$$P_N \sim \sqrt{\frac{1}{\pi N}} \exp(-\kappa^2 N), \quad N \gg 1. \tag{4.9}$$

Equation (4.8) gives the required $N$ distribution.

It is not difficult to calculate the expected value of the number of orbits which a comet will describe before leaving the solar system. Since $P_N$ is the probability of more than $N$ orbits, the probability that a comet will leave the system after $N$ orbits is

$$P_N - P_{N+1} = \frac{(2N-1)!!}{(2N+2)!!}[1+\kappa^2(2N+1)](1-\kappa^2)^N.$$

Therefore, the expected value, $E(N)$ of $N$ is given by

$$E(N) = \sum_{N=1}^{\infty} (p_N - p_{N+1})\,N.$$

However, owing to the well-known expansions that

$$\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} x^n = (1+x)^{-1/2} - 1,$$

$$\sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!}(-1)^n x^n = 1-(1+x)^{1/2},$$

$E(N)$ can easily be calculated to give

$$E(N) = \frac{1-\kappa}{2\kappa(1+\kappa)} + \frac{\kappa}{2}\left(2\kappa - 2 + \frac{1-\kappa^2}{\kappa^2}\right) \sim \frac{1}{\kappa} \qquad 0 < \kappa \ll 1.$$

As $\kappa \to 0$, $E(N) \to \infty$ as might be expected. Thus, there is an essential difference between the case with $\kappa^2 = 0$ and the case of a non-zero rate of disintegration. In the former case, the expected value of number of orbits is infinite, while in the latter, $E(N)$ is finite. That the expectation value $E(N)$ is infinite is consistent with the general first passage theory of stochastic processes (see, e.g. Bartlett 1966).

To compare the analytical expression (4.7) with a Monte Carlo result, we carried through a numerical experiment which consists of 1000 samplings, and the result is shown in Fig. 4. In the numerical work, we took into consideration that the effect of diffusion into short-period orbits should be excluded. Thus, instead of terminating the random walk when $x$ exceeds 50, we terminated the walk when $x \geqslant 200$.

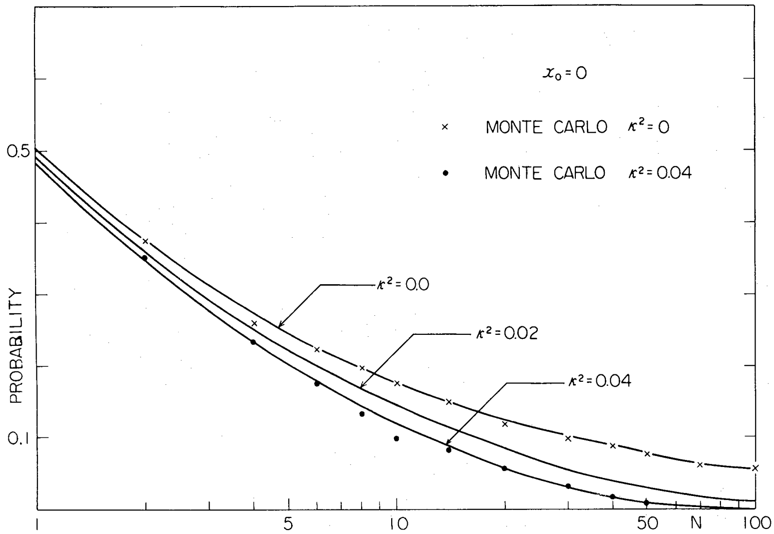


**Figure 4.** The curves give a theoretical value ($P_N = (1/\sqrt{\pi})\ \Gamma(N + ½) \times (1 - \kappa^2)^N/N!$) of the probability that a comet makes $N$ or more than $N$ perihelion passages. Crosses and dots are Monte Carlo results.

As far as one can see, the experimental result agrees with the theoretical value to within a few per cent. We may therefore have some confidence in the Monte Carlo result given in Section 3.

In order to see the effect of physical disintegration, we have plotted theoretical values of the probability of making $N$ or more than $N$ orbits starting with the initial value $x_0 = 0$. In Figs 5–7, similar values obtained by Monte Carlo work are given. One notes that the tail of the $N$ distribution is almost independent of the $x_0$ value, a feature similar to the distribution of $T$.

As mentioned in the introduction, Everhart has empirically found that the distribution of $N$ is proportional to $N^{-1/2}$ when physical disintegration is not taken into account. He computed initially parabolic as well as weakly elliptic cometary orbits and empirically

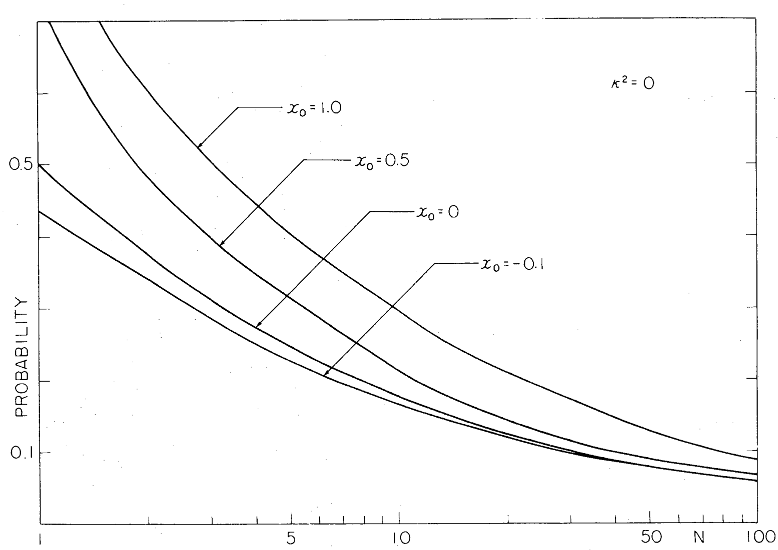


**Figure 5.** Probability distribution of $N$. Except for the case $x_0 = 0$, the curves give results of the Monte Carlo computation.

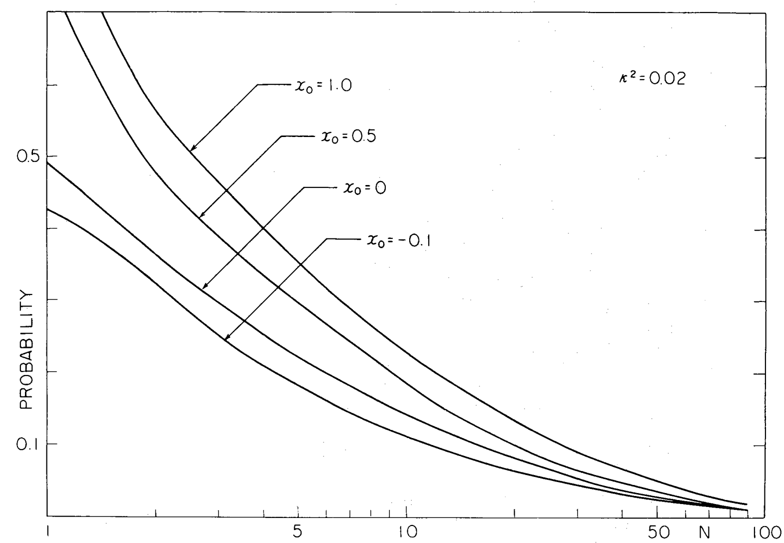


**Figure 6.** Probability distribution of $N$. Except for the case $x_0 = 0$, the curves give results of the Monte Carlo computation.

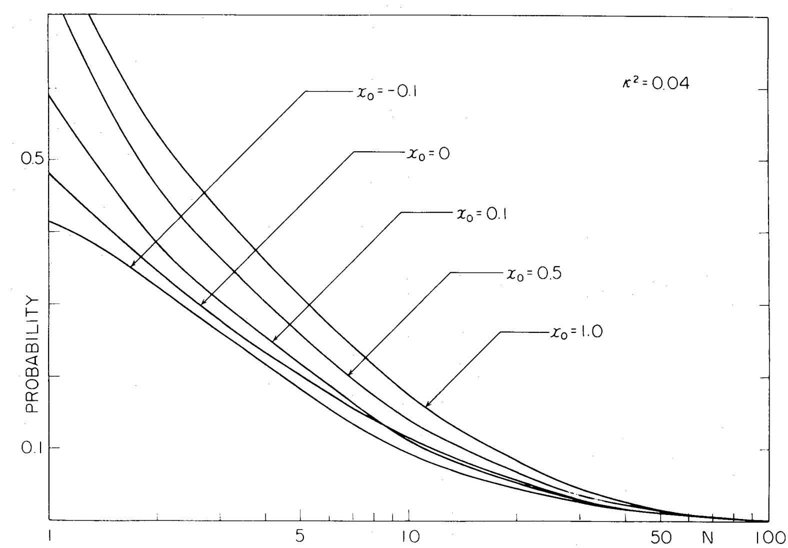


**Figure 7.** Probability distribution of $N$. Except for the case $x_0 = 0$, the curves give results of the Monte Carlo computation.

obtained the $N^{-1/2}$ dependence. Our rigorous analysis based on double exponential distribution confirms his finding. Presumably, the $N^{-1/2}$ dependence will not depend critically upon the functional form of the distribution of energy perturbation.

## 5 Histogram distribution of binding energy

As early as 1920, Russell (1920) noted that $1/a$ values of long-period comets are not distributed uniformly but that many comets have very small $1/a$ values. Oort (1950) then used 19 incoming (original) values of $1/a$ and confirmed the earlier, somewhat qualitative, finding of Russell. In doing so, accurate original values of $1/a$ are needed. And so far, the most up to date values are those calculated by Marsden *et al.* (1978). They calculated original and

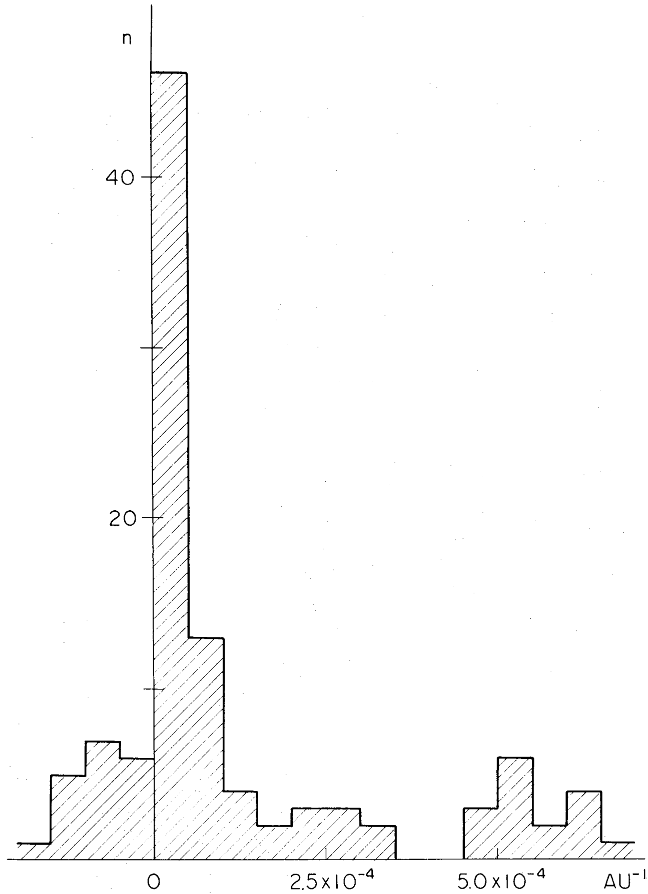


**Figure 8.** Histogram distribution of incoming (original) values of $1/a$ of observed long-period comets. $1/a$ values are taken from Marsden *et al.* (1978).

future values of 200 comets. In Fig. 8, we give a histogram distribution of $1/a$ values as calculated by them. One notes that the distribution has a strong concentration in the interval $0 < 1/a < 10^{-4}\,\text{AU}^{-1}$, and particularly in the range $0 < 1/a < 5 \times 10^{-5}\,\text{AU}^{-1}$. Oort interpreted those comets as *new* comets which had not previously passed through planetary region and which were deflected into small perihelion distance orbits by stellar perturbations.

However, it is conceivable that planetary perturbation alone may affect the distribution of $1/a$ values (which we will henceforth call the *energy distribution*). Indeed, even if comets are assumed to have a large energy value, say $10^{-3}\,\text{AU}^{-1}$, initially, the energy spectrum is expected to change quickly owing to planetary perturbations in 50 or 100 units of time. One can anticipate that the energy distribution will come to have a strong concentration near zero by the following semi-quantitative arguments.

If the average $1/a$ value remained almost constant throughout the lifetime of the cometary population, the rate of expulsion ($\lambda$) of comets per revolution would remain almost constant. Thus, after $N$ revolutions, the proportion of comets still bound to the Sun would be proportional to $\exp(-\lambda N)$, which is contrary to our finding that it is proportional to $N^{-1/2}$. Again, Hammersley showed that the tail of the distribution of $T$ is proportional to $T^{-2/3}$, irrespective of the initial energy. Under the same assumption as above, the proportion of comets remaining in the system would be proportional to $\exp(-\lambda T A^{-1})$, where $A$ is the average period, which is contrary to Hammersley's result. Hence, one would expect that as $T$ increases, the average period would also increase, which in turn is equivalent to the decrease of the average value of $1/a$.

In order to verify numerically the above argument, we carried through a numerical experiment, which may be described as follows. First, fix a value of $T$, and specify an initial value, $x_0$, of energy. Compute $Z$ by the equation

$$Z = x_1^{-3/2} + x_2^{-3/2} + \ldots + x_n^{-3/2}$$

for $n = 1, 2, \ldots$. There are two possibilities. The computed value of $Z$ may exceed $T$ for the first time. In this case, record the final value of energy, $x_n$. A comet may be lost to the system by dynamical expulsion or physical disintegration before $Z$ exceeds $T$, in which case one proceeds to the next random sampling. In this way, one is capable of obtaining the energy distribution in the cometary population at time $T$. By varying the value of $T$, it is possible to see how the distribution varies with $T$. In the numerical experiment, the number of comets which survived for more than $T$ units of time has been fixed at 1000.

In Figs 9–11, we give the distribution of energy values of comets which pass through perihelion in a fixed interval of time. This is obtained by multiplying the number of comets with a given value of $1/a$ by $a^{-3/2}$ which is precisely the orbital period. The figures give the case where $x_0 = 1$ and $\kappa^2 = 0.02$. This corresponds to an initial semimajor axis, $a = 1500$ AU. One easily notes that, owing to dynamical perturbations, the energy distribution changes rapidly and, as $T$ increases, a concentration at a small $1/a$ value appears. At $T = 50$, nearly

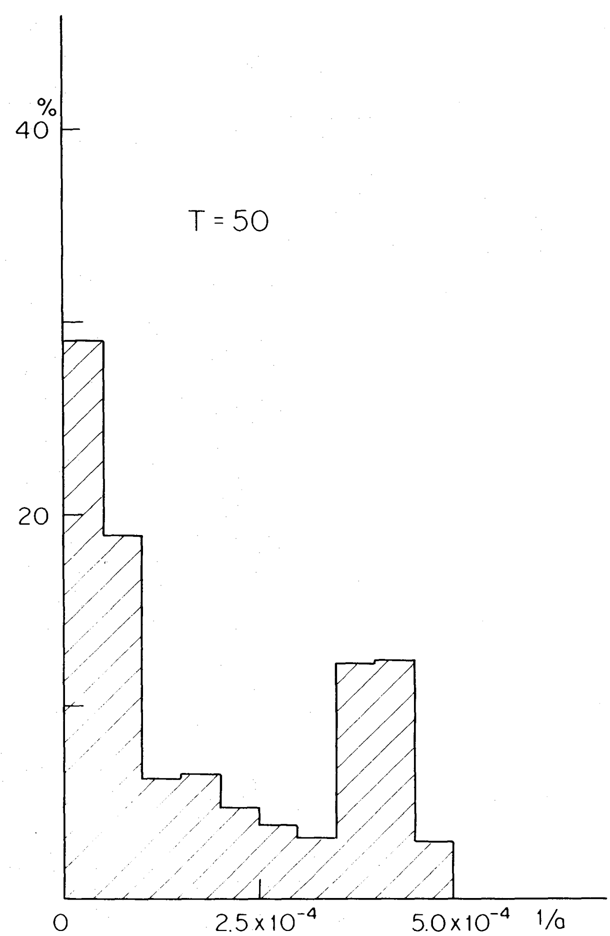


**Figure 9.** Histogram distribution of $1/a$ values of observable comets as computed by the Monte Carlo method. $x_0 = 1.0$ and $\kappa^2 = 0.02$. $T = 50$.

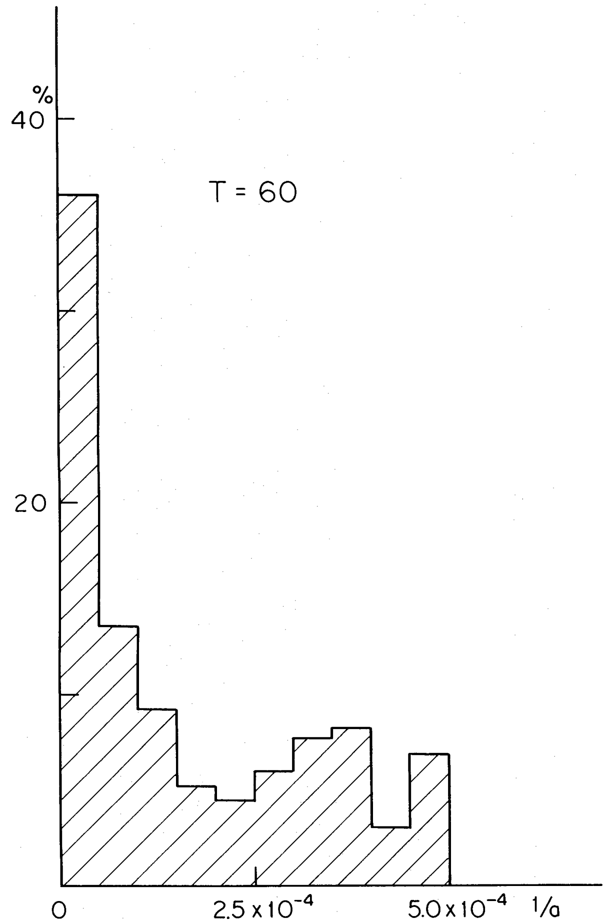


**Figure 10.** Histogram distribution of $1/a$ values of observable comets as computed by the Monte Carlo method. $x_0 = 1.0$ and $\kappa^2 = 0.02$. $T = 60$.

25 per cent of observable comets have $1/a$ values in the interval $(0, 5 \times 10^{-5}\,\text{AU}^{-1})$, and at $T = 100$, almost 55 per cent of them are found in the same interval.

In order to investigate how the above tendency is modified when a different initial value of $1/a$ is adopted, we considered three different cases, and in order to save space the results are given in the following tables, instead of appealing to graphs. One notes that as the initial value $x_0$ is decreased the above tendency becomes even more remarkable.

Although we are not concerned with any specific theory on the origin of comets, it is nevertheless worthwhile to point out how the present computation may have a bearing on cometary cosmogony. In Lyttleton's accretion hypothesis, initial comets have semimajor axes of some $10^3$ AU, so that the initial value of $1/a$ in our units is comparable to unity. On the other hand, Oort's *new comets* have initially $a \simeq 10^4$ AU or greater, so that his theory corresponds to $x_0 = 0.1$. The case $x_0 = -0.1$ corresponds to initially weakly hyperbolic comets (Yabushita & Hasegawa 1978).

By comparing the three tables with each other, it is immediately seen that the energy distribution at $T = 50$ or 100 is almost independent of the initial energy; Indeed, even for the most extreme case $x_0 = -0.1$, the energy distribution at $T = 50$ or 100 is hardly distinguishable from the cases $x_0 = 0.1$ or 0.5.

The result thus shows that the distribution of $1/a$ values of comets varies rapidly with time and could be explained in terms of planetary perturbations alone.

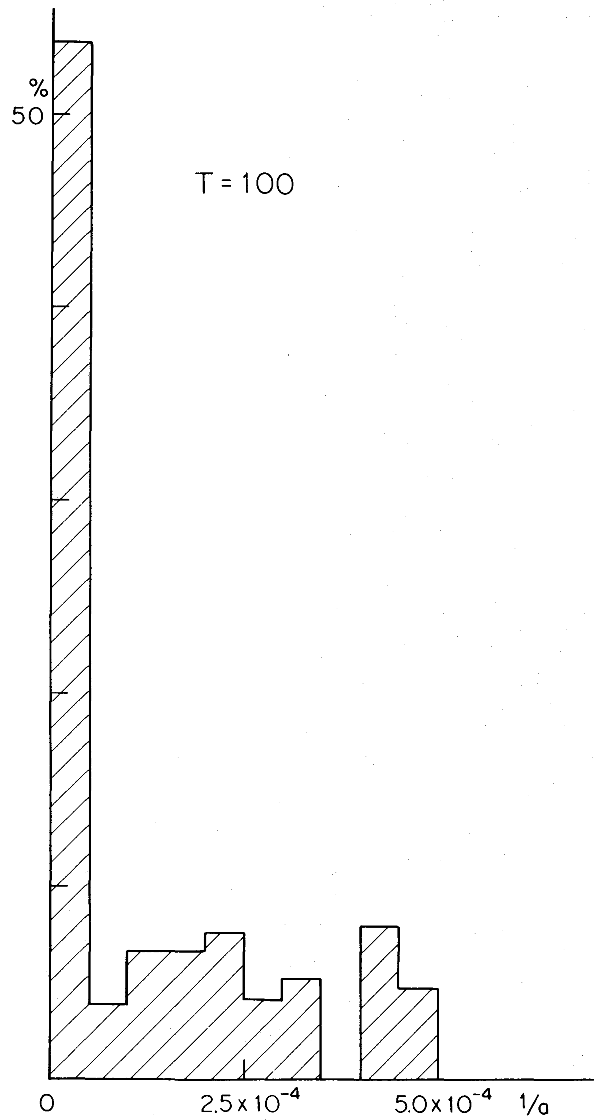


**Figure 11.** Histogram distribution of $1/a$ values of observable comets as computed by the Monte Carlo method. $x_0 = 1.0$ and $\kappa^2 = 0.02$. $T = 100$.

## 6 Conclusions and discussions

From the results presented in Section 3–5, it is possible to derive some conclusions which may have a bearing on the cosmogony of comets. First, the dynamical expulsion of comets from the solar system takes place so rapidly that after $3–6 \times 10^6$ yr after their formation, only 10 per cent or less of them are left, which confirms the earlier findings of Lyttleton & Hammersley. Secondly, the distribution of the number of perihelion passage, $N$, is asymptotically proportional to $N^{-1/2}$ for initially parabolic comets, which is in accord with the empirical result of Everhart. When physical disintegration is allowed for, it is proportional to $N^{-1/2} \exp(-\kappa^2 N)$. Since $\kappa^2$ is non-zero for actual comets, comets which make more than 40–50 revolutions are extremely rare (less than 10 per cent, say). Thirdly, and most important, energy distribution among observable comets varies rapidly and after 50 units of time ($3 \times 10^6$ yr), no trace of initial energy can be found. Indeed, whatever the initial energy, the $1/a$ distributions after 50 units are almost the same, as has been shown in Section 5. This result is implicitly contained in the work of Lyttleton & Hammersley (1964).

**Table 1.** Distribution of $1/a$ values of long-period comets (Monte Carlo result), $\kappa^2 = 0.02$, $x_0$(initial) = 0.5.

| $1/a$ AU$^{-1}$ ($\times 10^{-5}$) | $T = 5$ | $T = 10$ | $T = 50$ | $T = 100$ |
|---|---|---|---|---|
| 0–5 | 108 | 185 | 829 (24.6%) | 942 (55.1%) |
| 5–10 | 111 | 199 | 85 (13.1%) | 20 (6.1%) |
| 10–15 | 120 | 219 | 27 (8.9%) | 4 (2.6%) |
| 15–20 | 127 | 73 | 6 (3.3%) | 7 (7.6%) |
| 20–25 | 108 | 49 | 10 (8.0%) | 5 (7.9%) |
| 25–30 | 43 | 29 | 7 (7.6%) | 3 (6.4%) |
| 30–35 | 48 | 23 | 2 (2.8%) | 0 (0%) |
| 35–40 | 30 | 14 | 4 (6.9%) | 3 (10.2%) |
| 40–45 | 18 | 13 | 6 (12.5%) | 1 (4.1%) |
| 45–50 | 20 | 13 | 5 (12.3%) | 0 (0%) |
| 50–55 | 23 | 13 | 1 | 0 |
| 55–60 | 18 | 10 | 3 | 0 |
| 60–65 | 12 | 6 | 0 | 0 |
| 65–70 | 15 | 13 | 1 | 3 |
| 70–75 | 11 | 12 | 0 | 1 |
| 75–80 | 9 | 7 | 3 | 0 |
| 80–85 | 5 | 7 | 2 | 2 |
| 85–90 | 14 | 5 | 0 | 2 |
| 90–95 | 4 | 5 | 1 | 0 |
| 95–100 | 10 | 3 | 0 | 1 |

Figures in brackets give the fraction of comets in each energy interval calculated from the Monte Carlo result, allowing for the period.

**Table 2.** Distribution of $1/a$ values of long-period comets, $\kappa^2 = 0.02, x_0 = 0.1$.

| $1/a$ AU$^{-1}$ ($\times 10^{-5}$) | $T = 5$ | $T = 10$ | $T = 50$ | $T = 100$ |
|---|---|---|---|---|
| 0–5 | 136 | 205 | 844 (30.8%) | 930 (48.3%) |
| 5–10 | 131 | 259 | 70 (13.3%) | 22 (5.9%) |
| 10–15 | 120 | 202 | 27 (11.0%) | 9 (5.2%) |
| 15–20 | 130 | 71 | 13 (8.8%) | 8 (7.7%) |
| 20–25 | 107 | 38 | 7 (6.9%) | 4 (5.6%) |
| 25–30 | 44 | 22 | 1 (1.3%) | 5 (9.5%) |
| 30–35 | 32 | 19 | 4 (1.7%) | 4 (9.7%) |
| 35–40 | 17 | 18 | 3 (6.4%) | 0 (0%) |
| 40–45 | 26 | 14 | 3 (7.7%) | 1 (3.6%) |
| 45–50 | 15 | 13 | 4 (12.0%) | 1 (4.3%) |
| 50–55 | 8 | 7 | 1 | 0 |
| 55–60 | 16 | 6 | 4 | 2 |
| 60–65 | 7 | 8 | 1 | 2 |
| 65–70 | 16 | 4 | 1 | 0 |
| 70–75 | 7 | 3 | 2 | 0 |
| 75–80 | 15 | 9 | 0 | 2 |
| 80–85 | 11 | 9 | 0 | 1 |
| 85–90 | 10 | 5 | 1 | 0 |
| 90–95 | 4 | 2 | 0 | 0 |
| 95–100 | 4 | 5 | 2 | 1 |

Figures in brackets give the fraction of comets in each energy interval calculated from the Monte Carlo result, allowing for the period.

**Table 3.** Distribution of $1/a$ values of long-period comets. The initial orbit is weakly hyperbolic, $x_0 = -0.1$, $\kappa^2 = 0.02$. $x_0 = -0.1$ corresponds to a hyperbolic velocity 243 m/s at a great distance.

| $1/a$ AU$^{-1}$ ($\times 10^{-5}$) | $T = 5$ | $T = 10$ | $T = 50$ | $T = 100$ |
|---|---|---|---|---|
| 0–5 | 119 | 226 | 827 (30.5%) | 954 (65.0%) |
| 5–10 | 124 | 231 | 91 (17.5%) | 18 (6.3%) |
| 10–15 | 135 | 218 | 23 (9.5%) | 9 (6.8%) |
| 15–20 | 147 | 68 | 10 (6.8%) | 3 (3.8%) |
| 20–25 | 113 | 36 | 9 (8.9%) | 2 (3.7%) |
| 25–30 | 45 | 23 | 5 (6.7%) | 1 (2.5%) |
| 30–35 | 20 | 14 | 1 (1.7%) | 1 (3.2%) |
| 35–40 | 18 | 10 | 2 (4.3%) | 1 (3.9%) |
| 40–45 | 27 | 12 | 3 (7.8%) | 1(4.8%) |
| 45–50 | 16 | 12 | 2 (6.1%) | 0 (0%) |
| 50–55 | 20 | 9 | 3 | 1 |
| 55–60 | 9 | 13 | 3 | 0 |
| 60–65 | 12 | 6 | 0 | 0 |
| 65–70 | 15 | 5 | 3 | 0 |
| 70–75 | 12 | 5 | 1 | 0 |
| 75–80 | 9 | 8 | 0 | 0 |
| 80–85 | 5 | 6 | 0 | 0 |
| 85–90 | 7 | 4 | 1 | 1 |
| 90–95 | 7 | 4 | 0 | 1 |
| 95–100 | 6 | 1 | 1 | 1 |

Figures in brackets give the fraction of comets in each energy interval calculated from the Monte Carlo result, allowing for the period.

**Table 4.** Theoretical (Monte Carlo) result ($x_0 = 1$) and the observed distribution of $1/a$. Figures in brackets give the fraction of comets in each energy interval calculated from the Monte Carlo result, allowing for the period.

| $1/a$ (AU$^{-1}$) ($\times 10^{-5}$) | $10^{7.5} a^{-3/2}$ | $T = 50$ | $T = 60$ | $T = 100$ | $T = 150$ | Observed distribution |
|---|---|---|---|---|---|---|
| 0–5 | 3.95 | 818 (28.9%) | 875 (36.0%) | 943 (53.6%) | 964 (67.5%) | 46 (60.5%) |
| 5–10 | 20.5 | 103 (18.8%) | 63 (13.5%) | 13 (3.8%) | 13 (4.7%) | 13 (17.1%) |
| 10–15 | 44.2 | 16 (6.3%) | 20 (9.2%) | 10 (6.3%) | 3 (2.3%) | 4 (9.3%) |
| 15–20 | 73.2 | 10 (6.5%) | 7 (5.3%) | 6 (6.3%) | 3 (3.8%) | 2 (2.6%) |
| 20–25 | 106 | 5 (4.7%) | 4 (4.4%) | 5 (7.6%) | 3 (5.6%) | 3 (3.9%) |
| 25–30 | 144 | 3 (3.8%) | 4 (6.0%) | 2 (4.1%) | 1 (2.6%) | 3 (3.9%) |
| 30–35 | 185 | 2 (3.3%) | 4 (7.7%) | 2 (5.3%) | 2 (6.6%) | 2 (2.6%) |
| 35–40 | 196 | 7 (12.2%) | 4 (8.2%) | 0 (0%) | 2 (6.9%) | 0 (0%) |
| 40–45 | 277 | 5 (12.3%) | 1 (2.9%) | 2 (7.9%) | 0 (0%) | 0 (0%) |
| 45–50 | 327 | 1 (2.9%) | 2 (6.8%) | 1 (4.7%) | 0 (0%) | 3 (3.9%) |
| 50–55 | 380 | 1 | 0 | 1 | 0 | 6 |
| 55–60 | 436 | 0 | 2 | 1 | 0 | 2 |
| 60–65 | 494 | 2 | 0 | 1 | 1 | 4 |
| 65–70 | 554 | 2 | 0 | 1 | 0 | 1 |
| 70–75 | 617 | 3 | 2 | 0 | 0 | 1 |

In calculating the percentage of comets in each energy interval, correction is made for the period. Those with $1/a > 50 \times 10^{-5}$ AU$^{-1}$ are excluded because data are scarce.

Everhart made a similar analysis but he plotted $1/a$ values against $N$, and was not able to obtain a distribution such as shown in Fig. 11. In contrast to his result, our work shows that whatever the initial value of $1/a$, after some time the distribution comes to have a strong concentration near $1/a = 0$.

Current interpretation of the peak near $1/a = 0$ of the observed distribution of binding energies is that it indicates the existence of a large number of comets, which are unobservable owing to large perihelion distances, and that such comets are deflected into the observable region of the solar system by the perturbation of passing stars. If it is assumed that the solar system always has had comets and that the cometary population is now in a steady state, then the current interpretation is inescapable. Weissman (1978) carried out a simulation of the process and obtained a distribution of $1/a$ values which is close to that observed. One must, however, keep in mind that scientific observations of comets spans only 200 yr, and there is no rigorous basis for the assumption of a steady state. It is equally possible to argue that comets are a transient phenomenon. If so, it will be important to see whether the observed distribution of $1/a$ can be accounted for by planetary perturbation alone, assuming that comets were injected into the solar system at an earlier epoch. The result presented in Section 5 shows that the distribution of $1/a$ as observed at present is also consistent with comets as a transient phenomenon of the solar system. The result that the distribution of $1/a$ values (corrected for period) is hardly dependent upon the initial value after 50 or 100 units of time makes it possible to specify on the above assumption the time when comets first became members of the solar system. By comparing Fig. 8, which gives the observed distribution, with Tables 1–4, which give the theoretical distribution, it may be concluded that the time is approximately 100 units (or 5.8 Myr) before present. In any case, it is unlikely that $T < 50$ or $T > 150$. A definite answer as to which of the two assumptions is correct may be got by a successful future mission to a comet (or comets).

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## Appendix

Here it will be shown that the integral equation (4.5) admits a solution which has a simple analytical form. To do so, assume a solution of the form

$$F(\lambda, x) = A \exp(kx)$$

where $A$ and $k$ are constants. By inserting the above form of $F(\lambda, x)$ into equation (4.5), it is easily found that

$$A \exp(kx) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}x) + \frac{\lambda A}{\sqrt{2}} \left\{ \frac{1}{k+\sqrt{2}} [\exp(kx) - \exp(-\sqrt{2}x)] - \frac{\exp(kx)}{k-\sqrt{2}} \right\}, \quad x > 0$$

provided that $k < \sqrt{2}$. This equation is satisfied if

$$A = \frac{\lambda A}{\sqrt{2}(k+\sqrt{2})} - \frac{\lambda A}{\sqrt{2}(k-\sqrt{2})}, \qquad \frac{\lambda A}{k+\sqrt{2}} = 1.$$

These equations can be solved to give

$$k = -\sqrt{2(1-\lambda)}, \quad A = \frac{\sqrt{2}}{\sqrt{1-\lambda}+1}.$$

The solution $F(\lambda, x)$ is therefore given by

$$F(\lambda, x) = \sqrt{2}\, \frac{1-\sqrt{1-\lambda}}{\lambda} \exp\left[-\sqrt{2(1-\lambda)}\, x\right].$$

We note that

$$\int_0^\infty \sum_{r=1}^\infty \lambda_{r-1} F_r(x)\, dx = \int_0^\infty F(\lambda, x)\, dx = \frac{1}{\lambda}\left(\frac{1}{\sqrt{1-\lambda}} - 1\right) = \sum_{r=1}^\infty \frac{(2r-1)!!}{(2r)!!} \lambda^{r-1}.$$

Thus, we get the following integral,

$$\int_0^\infty F_r(x)\, dx = \frac{(2r-1)!!}{(2r)!!}$$

which is equation (4.7).