A Steady-State Boussinesq-Stefan Problem with Continuous Extraction (*).

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Summary. – *One establishes an existence result for the weak solution to a steady-state strongly coupled system between a nonlinear two phases heat equation with convection and the Navier-Stokes equation in the liquid phase. The two phases Rayleigh-Bénard problem is included as the particular case corresponding to a zero extraction velocity.*

0. – Introduction.

It is known that even in steady solidification problems the natural convection in the bulk liquid plays often an important role in industrial processes for producing different materials. A typical example is the vertical Bridgman crystal growth system for producing single-crystal semiconductor materials, which have been recently considered in [CB] from a numerical point of view. Another important case is the continuous casting of metal ingots, where convection from pouring momentum has a continuing influence on the solidification behavior. In particular, that movement of liquid alters the shape of the solid-liquid interface (see [F], page 225). From the mathematical point of view this type of Stefan problems with convection, in steady-state and in a model case without continuous extraction, have been considered in [CDK], [CD] by means of weak solutions. In this work we conjugate their approach to include convection with the two phase model of [R] for the stationary continuous casting problem, in order to obtain an existence result. The presence of the continuous extraction generates additional nonlinearities in the heat equation, particularly the jump condition of the heat flux accross the free boundary. Our model also includes the possibility of a flow in the liquid phase whose viscosity is temperature dependent, and general nonlinear lateral coolings, including climatization processes as considered in [DL] and [CR]. Since we use an « a priori » L^∞-estimate for the temperature, we allow nonlinearities without growth conditions, extending the results of [CDK] and [CD].

In section 1 we introduce the equations of this coupled Boussinesq-Stefan system, with mixed thermal boundary conditions, first in its classical form, and afterwards

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in the weak solutions formulation. In section 2 we state our assumptions and we prove the existence of a weak solution. This is based on an approximating problem which is solved in section 4 by means of the Schauder fixed point theorem. In an intermediate step we study, in section 3, the existence, uniqueness and continuous dependence of the solution to an auxiliary mixed boundary value problem for the temperature, with Lipschitz nonlinearities in the divergence term. Finally, in section 5, we obtain the two-phase Rayleigh-Bénard problem with Dirichlet thermal boundary condition by letting the extraction velocity go to zero with an appropriate family of lateral coolings.

1. – Mathematical formulation.

The solidification problems we shall consider are based on melt and solid stratified in a cylindrical configuration $\Omega = \Gamma\times]0, l[$ of $\mathbb{R}^n$ ($\Gamma =]0, a[$ for $n = 2$ or Γ is an open bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Gamma$ for $n = 3$). We introduce $\Gamma_i = \Gamma\times\{i\}$, $i = 0, l$, $\Gamma_D = \Gamma_0 \cup \Gamma_l$ and $\Gamma_N = \partial\Gamma\times]0, l[$, we denote $X = (x, y, z)$ and the gradient by $\nabla = (\partial_x, \partial_y, \partial_z)$, so that $\Delta = \nabla\cdot\nabla$.

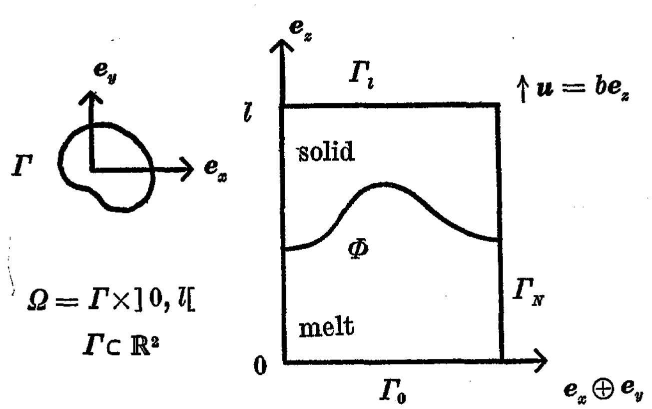

Fig. 1. Ingot or crystal geometry in $\mathbb{R}^3$.

We suppose the interface $\Phi = \{X \in \Omega: z = \varphi(x, y)\}$ in steady-state and the dimensionless pull rate of the crystal, or of the ingot, constant, so that the extraction velocity is given by $\boldsymbol{u} = b\boldsymbol{e}_z$ with $b > 0$ and $\boldsymbol{e}_z = (0, 1)$ if $n = 2$ or $\boldsymbol{e}_z = (0, 0, 1)$ if $n = 3$.

The temperature $T = T(X)$ verifies the stationary heat equation with convection

$$c(T)\boldsymbol{v}\cdot\nabla T = \nabla\cdot(k(T)\nabla T) \quad \text{in } \Omega\setminus\Phi\,, \tag{1}$$

where c and k are positive (bounded and discontinuous) functions representing, essentially, the specific heat and the thermal conductivity and $\boldsymbol{v}$ is the velocity.

Denoting by T_* the melting temperature at the interface, we renormalize the temperature by $\Theta = K(T) = \int_T^{T_*} k(\tau)\, d\tau$ and the equation (1) becomes at the solid region $\{\Theta > 0\}$ and at the liquid bulk $\{\Theta < 0\}$

$$\boldsymbol{v}\cdot\nabla f(\Theta) = \Delta\Theta \quad \text{in } \Omega\setminus\Phi = \{\Theta > 0\} \cup \{\Theta < 0\} \tag{2}$$

where $f = C\circ K^{-1}$ and $C(T) = \int_T^{T_*} c(\tau)\, d\tau$. At the interface we have the renormalized equilibrium melting temperature $\Theta = 0$ and the balance of heat fluxes, the Stefan condition,

$$-[\nabla\Theta]_-^+\cdot\boldsymbol{\nu} = -\lambda\boldsymbol{u}\cdot\boldsymbol{\nu} = \lambda b \quad \text{on } \Phi = \{\Theta = 0\}, \tag{3}$$

where $\lambda > 0$ is essentially the latent heat, $\boldsymbol{\nu} = (\partial_x\varphi, \partial_y\varphi, -1)$ is a normal vector to Φ and $[\ ]_-^+$ denotes the jump across Φ.

The velocity field $\boldsymbol{v} = \boldsymbol{v}(X)$ in the solid region, on the melt-solid interface and on the remainder boundary of the bulk liquid is assumed equal to the extraction velocity

$$\boldsymbol{v} = \boldsymbol{u} = b\boldsymbol{e}_z \quad \text{in } \{\Theta > 0\} \cup \Phi \cup \partial\{\Theta < 0\}. \tag{4}$$

Inside the bulk liquid $\boldsymbol{v}$ verifies the incompressibility condition

$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \{\Theta < 0\}, \tag{5}$$

and the steady-state momentum balance equation

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \boldsymbol{B}(\Theta) + \nabla\cdot S \quad \text{in } \{\Theta < 0\}, \tag{6}$$

where $\boldsymbol{B}$ represents the buoyance forces, p the pressure and S the viscous stress-tensor, which expression we assume given by the following constitutive law (where $D\boldsymbol{v} = \frac{1}{2}[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T]$)

$$S = 2\mu(\Theta)\, D\boldsymbol{v} = \mu(\Theta)[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T]. \tag{7}$$

Here we have followed the Oberbeck-Boussinesq approximation which is based on the main hypothesis that the variability of density (assumed equal to one for simplicity) due to changes in the temperature can be neglected. This simplification is also relevant to the interface conditions (3) and (4), in order to ensure no slip tangential to the solid part and incorporation of melt into the solid at a rate equal to the growth one. However we shall take into account the variation of viscosity μ with the temperature.

To complete this Boussinesq-Stefan problem we must prescribe the thermal boundary conditions, say

$$\Theta = \Theta_D \quad \text{on } \Gamma_D \,, \tag{8}$$

where Θ_D is given such that $\Theta_D|_{\Gamma_1} > 0$ and $\Theta_D|_{\Gamma_0} < 0$, and a nonlinear lateral cooling in the form

$$-\,\partial\Theta/\partial n = g(X, \Theta) \quad \text{on } \Gamma_N \,, \tag{9}$$

where, for each $X \in \Gamma_N$, $g(X, \cdot)$ is a monotone increasing function in Θ, eventually multivalued, (i.e. whose graph is a continuous curve in $\mathbb{R}^2$), includind the case of lateral climatization or ambiguous cooling as in [DL].

The strongly coupled problem (2)-(9) have been stated in its classical formulation, that is, by assuming the unknowns $\Theta, \boldsymbol{v}$ and the free boundary Φ smooth enough. However since we don't know if there exist classical solutions we need a weak formulation following the ideas of [CDK] and [R].

Multiplying by a smooth function ζ vanishing on Γ_D, the equation (2) with the interface condition (3) and the boundary condition (9) leads to (with $\boldsymbol{n} = \boldsymbol{\nu}/|\boldsymbol{\nu}|$)

$$\int_\Omega \{\nabla\Theta\cdot\nabla\zeta + [\boldsymbol{v}\cdot\nabla f(\Theta)]\zeta\} + \int_{\Gamma_N} g(\Theta)\zeta = \int_\Phi \zeta[\nabla\Theta]^+_-\cdot\boldsymbol{n} = -\int_\Phi \zeta\lambda b/|\boldsymbol{\nu}| \,. \tag{10}$$

On one hand, letting $\chi_{\{\Theta>0\}}$ denote the characteristic function of the solid phase, one has

$$\int_\Omega \chi_{\{\Theta>0\}}\,\partial_z\zeta = \int_{\{\Theta>0\}} \partial_z\zeta = -\int_\Phi \zeta/|\boldsymbol{\nu}| \,;$$

on the other hand, introducing the translated velocity

$$\boldsymbol{w} = \boldsymbol{v} - b\boldsymbol{e}_z \,, \tag{11}$$

which by (5) and the continuity assumption across Φ (4) is also a solenoidal field in Ω, one obtains from (10)

$$\int_\Omega [\nabla\Theta - f(\Theta)(\boldsymbol{w} + b\boldsymbol{e}_z) - \lambda b\chi_{\{\Theta>0\}}\boldsymbol{e}_z]\cdot\nabla\zeta + \int_{\Gamma_N} g(\Theta)\zeta = 0 \,, \tag{12}$$

for any smooth function ζ verifying $\zeta|_{\Gamma_D} = 0$. From (6), integrating by parts in $\{\Theta < 0\}$, it follows

$$\int_{\{\Phi<0\}} \{2\mu(\Theta)\, D\boldsymbol{w} : D\boldsymbol{\psi} - \boldsymbol{w}\cdot[(\boldsymbol{w} + b\boldsymbol{e}_z)\cdot\nabla\boldsymbol{\psi}] - \boldsymbol{B}(\Theta)\cdot\boldsymbol{\psi}\} = 0 \tag{13}$$

for any smooth solenoidal vector field ψ with compact support in $\{\Theta < 0\}$, using the usual notation: for the inner product of matrices.

Introduce, for any open bounded set $\mathcal{O} \subset \mathbb{R}^n$, the Hilbert space $V(\mathcal{O})$ which is the completion of the set of smooth divergence free vectors ($\nabla \cdot \psi = 0$) with compact support in $\mathcal{O}$ with respect to the norm associated with the usual inner product of $[H_0^1(\mathcal{O})]^n$:

$$((\boldsymbol{v}, \psi)) = \int_{\mathcal{O}} \nabla \boldsymbol{v} : \nabla \psi \,, \qquad \text{so } \|\boldsymbol{v}\|_{V(\mathcal{O})} = ((\boldsymbol{v}, \boldsymbol{v}))^{\frac{1}{2}} \,.$$

Now we can state the definition of a weak solution of our Boussinesq-Stefan problem, following the ideas of [CDK] and [R]:

DEFINITION. – We say that $(\Theta, g, \chi, \boldsymbol{w})$ is a weak solution of problem (2)-(9) with $\boldsymbol{v}$ given by (11), if

(14) $\quad \Theta \in C^0(\Omega) \cap H^1(\Omega)\,, \qquad \Theta = \Theta_D \quad \text{on } \Gamma_D;$

(15) $\quad g \in L^q(\Gamma_N)\,, \qquad q > n - 1\,, \qquad g(X) \in G(X, \Theta(X)) \quad \text{for a.e. } X \in \Gamma_N;$

(16) $\quad \chi \in L^\infty(\Omega)\,, \qquad 0 \leqslant \chi_{\{\Theta > 0\}} \leqslant \chi \leqslant 1 - \chi_{\{\Theta < 0\}} \leqslant 1 \quad \text{a.e. in } \Omega;$

(17) $\quad \boldsymbol{w} \in V(\Omega)\,, \qquad \boldsymbol{w} = 0 \quad \text{a.e. in } \{\Theta > 0\} \equiv \{x \in \Omega : \Theta(X) > 0\};$

(18) $$\int_{\Omega} \{\nabla\Theta - f(\Theta)[\boldsymbol{w} + b\boldsymbol{e}_z] - \lambda b \chi \boldsymbol{e}_z\} \cdot \nabla\zeta + \int_{\Gamma_N} g\zeta = 0 \quad \text{and}$$

(19) $$\int_{\{\Theta < 0\}} \{2\mu(\Theta)\, D\boldsymbol{w} : D\psi - \boldsymbol{w} \cdot [(\boldsymbol{w} + b\boldsymbol{e}_z) \cdot \nabla\psi]\} = \int_{\{\Theta < 0\}} \boldsymbol{B}(\Theta) \cdot \psi \,,$$

respectively, for all $\zeta \in H^1(\Omega)$: $\zeta|_{\Gamma_D} = 0$ and all $\psi \in V(\{\Theta < 0\})$.

We observe that, being $\partial\Omega$ Lipschitz, one has $V(\Theta) \hookrightarrow [L^4(\Omega)]^n$ for $n = 2, 3$ (see [L] or [T], for instance) and the nonlinear term in (19) makes sense.

2. – Existence of a weak solution.

For our existence result we shall assume that

(20) $\quad f : \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{B} : \mathbb{R} \to \mathbb{R}^n$ are continuous functions, $(n = 2, 3)$;

(21) $\quad \mu : \mathbb{R} \to \mathbb{R}$ is a strictly positive continuous function: $\mu(\tau) \geqslant \mu_0 > 0$; and, for some fixed constant $M > 0$, we suppose:

(22) $\quad \Theta_D \in H^{\frac{1}{2}}(\Gamma_D)$: $-M \leqslant \Theta_D < 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad 0 < \Theta_D \leqslant M \quad \text{on } \Gamma_l;$

(23) $\quad G_X(\cdot) = G(X, \cdot)$ is a maximal monotone graph in $\mathbb{R}^2$, for all $X \in \Gamma_N$, which verifies, for each $X \in \Gamma_N$, the following conditions

(24) $[-M, M] \subset D(G_X) \equiv \{\tau \in \mathbb{R}: G(X, \tau) \neq \emptyset\}$;

(25) $G_X(M) \subset [0, +\infty[\,, \quad G_X(-M) \subset]-\infty, 0]\,,$ and

(26) « $X \to g^0(X, \tau)$ » $\in L^q(\Gamma_N)$, for all $|\tau| \leqslant M$, with $q = 2 = n$ or $q > 2$ if $n = 3$.

Here we have denoted $g^0(X, \tau) = \text{proj}_{G_X(\tau)}\, 0$, i.e., the smallest number is absolute value of $G_X(\tau)$. Note that the assumption (25) is equivalent to the following sign condition

$$g^0(X, \tau)\,\tau \geqslant 0 \quad \text{for } |\tau| \geqslant M\,. \tag{25'}$$

THEOREM 1. – If the assumptions (20)-(26) hold, then there exists a weak solution of the Boussinesq-Stefan problem (2)-(9) with $\boldsymbol{v}$ given by (11), and such that $|\Theta| \leqslant M$ in $\bar{\Omega}$.

PROOF. – We shall obtain a weak solution by approximating the problem (14)-(19) with solutions to the one parameter ($\varepsilon > 0$) penalized and regularized problem:

PROBLEM (BS)$_\varepsilon$. – Find $\Theta_\varepsilon \in C^0(\bar{\Omega}) \cap H^1(\Omega)$, $\Theta_\varepsilon|_{\Gamma_D} = \tilde{\Theta}_\varepsilon$, $\boldsymbol{w}_\varepsilon \in V(\Omega)$ such that

$$\int_\Omega \{\nabla\Theta_\varepsilon - f_\varepsilon(\Theta_\varepsilon)[\boldsymbol{w}_\varepsilon + b\boldsymbol{e}_z] - \lambda b \chi_\varepsilon(\Theta_\varepsilon)\boldsymbol{e}_z\}\cdot\nabla\zeta + \int_{\Gamma_N} g_\varepsilon(\Theta_\varepsilon)\,\zeta = 0\,, \tag{27}$$

$$\int_\Omega \{2\mu(\Theta_\varepsilon)\, D\boldsymbol{w}_\varepsilon : D\boldsymbol{\psi} - \boldsymbol{w}_\varepsilon[(\boldsymbol{w}_\varepsilon + b\boldsymbol{e}_z)\cdot\nabla\boldsymbol{\psi}]\} + \frac{1}{\varepsilon}\int_\Omega \chi_\varepsilon(\Theta_\varepsilon)\,\boldsymbol{w}_\varepsilon\cdot\boldsymbol{\psi} = \int_\Omega \boldsymbol{B}(\Theta_\varepsilon)\cdot\boldsymbol{\psi} \tag{28}$$

respectively, for any $\zeta \in H^1(\Omega)$: $\zeta|_{\Gamma_D} = 0$ and any $\boldsymbol{\psi} \in V(\Omega)$.

Here $\tilde{\Theta}_\varepsilon \in C^{0,1}(\bar{\Omega})$ is a family of bounded functions in $H^1(\Omega)$, whose trace $\tilde{\Theta}_\varepsilon \underset{\varepsilon\downarrow 0}{\rightarrow} \Theta_D$ in $H^{\frac12}(\Gamma_D)$, $f_\varepsilon \in C^{0,1}(\mathbb{R})$ with $f_\varepsilon \underset{\varepsilon\downarrow 0}{\rightarrow} f$ uniformly on compact sets, χ_ε is defined by

$$\chi_\varepsilon(\tau) = \begin{cases} 1 & \text{for } \tau \geqslant 2\varepsilon \\ \tau/\varepsilon - 1 & \text{for } \varepsilon \leqslant \tau \leqslant 2\varepsilon \\ 0 & \text{for } \tau \leqslant \varepsilon \end{cases} \tag{29}$$

and g_ε, for each fixed $X \in \Gamma_N$, is the Yosida regularization of G_X, given in terms of its resolvent by

$$g_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon)\,, \qquad \text{where } J_\varepsilon = (I + \varepsilon G_X)^{-1}\,. \tag{30}$$

We shall prove in section 4 the existence of a solution $(\Theta_\varepsilon, \boldsymbol{w}_\varepsilon)$ to Problem (BS)$_\varepsilon$, together with the following estimates independently of $\varepsilon > 0$:

$$\|\Theta_\varepsilon\|_{H^1(\Omega)} \leqslant C\,, \quad -M \leqslant \Theta_\varepsilon(X) \leqslant M\,, \quad \text{for all } X \in \bar{\Omega}\,, \tag{31}$$

(32) $\|\Theta_\varepsilon\|_{C^{0,\alpha}(K)} \leqslant C(K)$, for any compact set $K \subset \Omega$, $(0 < \alpha < 1)$,

(33) $\|g_\varepsilon(\Theta_\varepsilon)\|_{L^q(\Gamma_N)} \leqslant C$,

(34) $\|\boldsymbol{w}_\varepsilon\|_{V(\Omega)} \leqslant C$.

Therefore from these estimates and well known compactness results one can select a subsequence, still labelled by ε, such that, for $\varepsilon \to 0$,

(35) $\Theta_\varepsilon \to \Theta$ in $H^1(\Omega)$-weak, in $L^p(\Gamma_N)$-strong for any $p < \infty$, uniformly in any compact subset $K \subset \Omega$ and pointwise a.e. in Γ_N;

(36) $g_\varepsilon(\Theta_\varepsilon) \rightharpoonup g$ in $L^q(\Gamma_N)$-weak;

(37) $\chi_\varepsilon(\Theta_\varepsilon) \rightharpoonup \chi$ in $L^\infty(\Omega)$-weak*, (since $0 \leqslant \chi_\varepsilon \leqslant 1$);

(38) $\boldsymbol{w}_\varepsilon \to \boldsymbol{w}$ in $V(\Omega)$-weak and $[L^4(\Omega)]^n$-strong $(n = 2, 3)$

for functions $\Theta \in H^1(\Omega) \cap C^{0,\alpha}(\Omega)$, $|\Theta| \leqslant M$ in $\bar{\Omega}$, $0 \leqslant \chi \leqslant 1$, $g \in L^q(\Gamma_N)$ and $\boldsymbol{w} \in V(\Omega)$.

Hence $f_\varepsilon(\Theta_\varepsilon) \to f(\Theta)$ in $L^p(\Omega)$ for all $p < \infty$ and we can let $\varepsilon \to 0$ in (27) obtaining (18).

Let $\boldsymbol{\psi}$ be any smooth solenoidal vector field such that

$$\operatorname{supp} \boldsymbol{\psi} \equiv K \subset \{\Theta < 0\} \equiv \{X \in \Omega \colon \Theta(X) < 0\}.$$

Note that $\{\Theta < 0\}$ is an open subset, by the continuity of Θ, and also one has $\max\limits_{X \in K} \Theta(X) < 0$. From (35) and (29) one deduces $\chi_\varepsilon(\Theta_\varepsilon) = 0$ on K for all ε small enough. Then, recalling that $\mu(\Theta_\varepsilon) \to \mu(\Theta)$ and $\boldsymbol{B}(\Theta_\varepsilon) \to \boldsymbol{B}(\Theta)$ uniformly on K, one can pass to the limit in (28), with such a fixed $\boldsymbol{\psi}$, obtaing (19). Since this holds for any smooth solenoidal vector field with compact support in $\{\Theta < 0\}$, (19) remains valid for all $\boldsymbol{\psi} \in V(\{\Theta < 0\})$ by density.

The above argument also shows that $\chi = 0$ a.e. in $\{\Theta < 0\}$ and so $\chi \leqslant 1 - \chi_{\{\Theta<0\}}$.

Consider now an arbitrary compact subset $K \subset \{\Theta > 0\}$. We also have $\min\limits_{X \in K} \Theta(X) > 0$, and from the uniform convergence we easily deduce that $\chi_\varepsilon(\Theta_\varepsilon(X)) = 1$ for all $X \in K$ and all ε, $0 < \varepsilon \leqslant \varepsilon_0(K)$. Therefore we deduce $\chi = 1$ a.e. in $\{\Theta > 0\}$ and $\chi_{\{\Theta>0\}} \leqslant \chi$. Choosing $\boldsymbol{\psi} = \boldsymbol{w}_\varepsilon$ in (28), since the first term is non negative by (21) and the second one is zero, we obtain for all $\varepsilon \leqslant \varepsilon_0(K)$

$$\frac{1}{\varepsilon}\int_K |\boldsymbol{w}_\varepsilon|^2 \leqslant \frac{1}{\varepsilon}\int_\Omega \chi_\varepsilon(\Theta_\varepsilon)|\boldsymbol{w}_\varepsilon|^2 \leqslant \max_{|\tau| \leqslant M} |\boldsymbol{B}(\tau)| \int_\Omega |\boldsymbol{w}_\varepsilon| \leqslant C$$

where C is independent of ε by (34). Therefore it follows that $\boldsymbol{w} = 0$ a.e. in K, and consequently also in $\{\Theta > 0\}$ by the arbitraryness of K.

To see that $(\Theta, g, \chi, \boldsymbol{w})$ is a weak solution, it remains to shows (15), i.e., $g(X) \in G(X, \Theta(X))$ a.e. $X \in \Gamma_N$.

We recall from the elementary properties of maximal monotone operators (see [B], Ch. II, § 4, for instance) that $g_\varepsilon(\Theta_\varepsilon) \in G_X(J_\varepsilon(\Theta_\varepsilon))$, together with (36) and $J_\varepsilon(\Theta_\varepsilon) \to \Theta$ in $L^2(\Gamma_N)$-strong (this follows first pontwise a.e. from (35) and (24) and, afterwards, in $L^2(\Gamma_N)$ by Lebesgue's theorem and assumption (26) which implies $|J_\varepsilon(X, \tau)| \leqslant |\tau| + \varepsilon |g_\varepsilon(X, 0)| \leqslant |\tau| + \varepsilon |g^0(X, 0)|$, for $|\tau| \leqslant M$), imply $g \in G_X(\Theta)$, that is (15). Note that in this step we have identified G_X with the maximal monotone operator $\partial \tilde{j}$ in $L^2(\Gamma_N)$, where $\tilde{j}$ is the convex functional associated with the integrand $j(X, \Theta) = \int_0^\Theta g^0(X, \tau)\, d\tau$. □

3. – An auxiliary nonlinear problem.

In the last section we have reduced our study to the approximating problem which is based in a fixed point argument and is postponed to the next section.

First we show the following auxiliary problem is well posed.

PROBLEM (A). – Find $\Theta \in C^0(\bar{\Omega}) \cap H^1(\Omega)$, such that $\Theta = \tilde{\Theta}$ on Γ_D and

$$\int_\Omega \{\nabla\Theta - f(\Theta)\boldsymbol{u} - h(\Theta)\boldsymbol{e}_z\}\cdot\nabla\zeta + \int_{\Gamma_N} g(\Theta)\zeta = 0\,, \qquad \forall \zeta \in H^1(\Omega)\colon \zeta|_{\Gamma_D} = 0\,, \tag{39}$$

with the following assumptions on the data: $\boldsymbol{u} = \boldsymbol{u}(X)$ is a fixed vector field in the following subspace $\mathfrak{L}^p_\sigma(\Omega)$, $p > n$,

$$\mathfrak{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in [L^p(\Omega)]^n\colon \int_\Omega \boldsymbol{v}\cdot\nabla\zeta = 0,\ \forall \zeta \in H^1(\Omega)\}\,; \tag{40}$$

$$\tilde{\Theta} \in C^{0,1}(\Gamma\colon)\,, \qquad \|\tilde{\Theta}\|_{L^\infty(\Omega)} \leqslant M\,; \tag{41}$$

(42) $f\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions;

(43) $g = g(X, \tau)\colon \Gamma_N \times \mathbb{R} \to \mathbb{R}$, is monotone increasing and continuous in τ for each fixed X, and satisfying conditions (25′) and (26) for some $q > n - 1$.

In this section all the results are valid for any dimension $n \geqslant 2$.

THEOREM 2. – Under assumptions (41)-(43), for each $\boldsymbol{u} \in L^p_\sigma(\Omega)$, $(p > n)$ there exists a unique solution to Problem (A), which verifies $\Theta \in C^{0,\alpha}(\bar{\Omega}) \cap H^1(\Omega)$, for some $0 < \alpha < 1$, and

$$\|\Theta\|_{L^\infty(\Omega)} \leqslant M\,. \tag{44}$$

PROOF. – i) *Existence.* We follow the argument of Proposition 1 of [R] (see also [CR]) and we begin by observing that (41) and the sign condition (25′) for g

imply the « a priori » estimate (44) for any solution to (A). In fact, letting $\zeta = (\Theta - M)^+$ in (39) we deduce $\Theta \leqslant M$ from

$$\int_\Omega |\nabla(\Theta - M)^+|^2 = \int_{\{\Theta > M\}} [\boldsymbol{u} f(\Theta) \cdot \nabla\Theta + h(\Theta)\, \partial_z \Theta] - \int_{\Gamma_N \cap \{\Theta > M\}} g(\Theta)(\Theta - M) \leqslant$$
$$\leqslant \int_\Omega \boldsymbol{u} \cdot \nabla F_M(\Theta) + \int_\Omega \partial_z H_M(\Theta) = \int_{\Gamma_D} H_M(\tilde{\Theta}) n_z = 0\,,$$

where $F_M(\tau) = \int_M^\tau f(t)\, dt$ and $H_M(\tau) = \int_M^\tau h(t)\, dt$ for $\tau \geqslant M$ and $F_M(\tau) = H_M(\tau) = 0$ for $\tau \leqslant M$ (recall $|\tilde{\Theta}| \leqslant M$ and (40)). Analogously one concludes $\Theta \geqslant -M$ with $\zeta = (\Theta + M)^-$.

Now denote f^M, g^M and h^M the truncated functions of f, g and h, respectively, i.e., $g^M(X, \tau) = g\big(X, \min(M, \max(-M, \tau))\big)$ and for

$$\tau \in B_R = \{\tau \in C^0(\bar{\Omega}) \colon \|\tau\|_{C^0(\bar{\Omega})} \leqslant R\}\,, \qquad (R \geqslant M > 0)\,,$$

define $\sigma = S(\tau)$ as the unique solution of the following mixed linear problem: find $\sigma \in H^1(\Omega)$, such that, $\sigma|_{\Gamma_D} = \tilde{\Theta}$ and

$$(45) \qquad \int_\Omega \nabla\sigma \cdot \nabla\zeta = \int_\Omega [f^M(\tau)\boldsymbol{u} + h^M(\tau)\boldsymbol{e}_z] \cdot \nabla\zeta - \int_{\Gamma_N} g^M(\tau)\zeta\,, \qquad \forall \zeta \in H^1(\Omega) \colon \zeta|_{\Gamma_D} = 0\,.$$

Using a Stampacchia's estimate [S], one concludes there exists $C > 0$ and $0 < \alpha < 1$, independent of τ, such that the solution verifies

$$(46) \qquad \|\sigma\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant C\big(\max_{|t| \leqslant M} |f(t)|\, \|\boldsymbol{u}\|_{L^p(\Omega)} + \max_{|t| \leqslant M} |h(t)| + \|\hat{g}_M\|_{L^q(\Gamma_N)} + \|\tilde{\Theta}\|_{C^{0,1}(\bar{\Gamma}_D)}\big) \equiv K$$

where we have denoted $\hat{g}_M(X) = \max\big(|g(X, M)|, |g(X, -M)|\big)$ (recall (43), (26)).

Then $S(B_K) \subset B_K$ and S is a continuous and compact mapping of B_K into itself. By the Schauder fixed point theorem there exists a function $\Theta = S(\Theta)$, which, of course, is a solution to (A), because the « a priori » estimate (44) equally holds with f^M, g^M and h^M.

ii) *Uniqueness*. Suppose Θ and $\hat{\Theta}$ are two solutions of (A), denote $\tau = \Theta - \hat{\Theta}$ and take in (39) $\zeta = \tau/(|\tau| + \delta)$ with $\delta > 0$. Recalling the monotonicity assumption on g (43), we obtain

$$\int_\Omega \nabla\tau \cdot \nabla\zeta \leqslant \int_\Omega \{[f(\Theta) - f(\hat{\Theta})]\boldsymbol{u} + [h(\Theta) - h(\hat{\Theta})]\boldsymbol{e}_z\} \cdot \nabla\zeta \leqslant \int_\Omega (L_f|\boldsymbol{u}| + L_h)|\tau||\nabla\zeta|\,,$$

where L_f and L_h denote the Lipschitz constants of f and h, respectively.

Since $\nabla\zeta = \delta\,\nabla\tau(|\tau| + \delta)^{-2}$, we deduce

$$\int_\Omega \frac{|\nabla\tau|^2}{(|\tau| + \delta)^2} \leqslant \int_\Omega (L_f|\boldsymbol{u}| + L_h)\frac{|\nabla\tau|}{(|\tau| + \delta)},$$

and, using Cauchy-Schwarz together with $|\nabla\tau| = |\nabla|\tau\|$, we have

$$\int_\Omega \left|\nabla \log\left(\frac{|\tau| + \delta}{\delta}\right)\right|^2 = \int_\Omega \frac{|\nabla\tau|^2}{(|\tau| + \delta)^2} \leqslant \int_\Omega (L_f|\boldsymbol{u}| + L_h)^2 = C\,.$$

Finally, by Poincaré's inequality, it follows

$$\int_\Omega |\log(|\Theta - \hat{\Theta}|/\delta + 1)|^2 \leqslant C' \qquad \text{(independent of } \delta)$$

and letting $\delta \to 0$ one must have $\Theta = \hat{\Theta}$, concluding the proof. □

REMARK 1. – In the existence part of the proof, the Lipschitz continuity of f and h can be relaxed to continuity. Also the monotonicity of g in Θ is not necessary if we replace the assumption (26) by (see (46))

$$(26') \qquad \forall R > 0,\ \exists \hat{g}_R \in L^q(\Gamma_N)\,, \quad q > n - 1\colon |g(X, \tau)| \leqslant \hat{g}_R(X) \quad \text{for } |\tau| \leqslant R\,.$$

PROPOSITION 1. – Under assumptions (41)-(43) we have the following continuous dependence result for Problem (A): if $\boldsymbol{u}_\eta \to \boldsymbol{u}_0$ in $[L^p(\Omega)]^n$-weak (resp. strong), for any $p > n$, then the corresponding solutions $\Theta_\eta \to \Theta_0$ in $H^1(\Omega)$-weak (resp. strong) and in $C^{0,\beta}(\bar{\Omega})$ for some fixed $0 < \beta < 1$.

PROOF. – For any $\boldsymbol{u}_0 \in \mathfrak{L}^p_\sigma(\Omega)$ take an arbitrary sequence

$$(47) \qquad \boldsymbol{u}_\eta \underset{\eta \to 0}{\rightharpoonup} \boldsymbol{u}_0 \quad \text{in } [L^p(\Omega)]^n\text{-weak, with } \boldsymbol{u}_\eta \in \mathfrak{L}^p_\sigma(\Omega)\,,$$

and denote by Θ_η the corresponding solutions of Problema (A), which by Theorem 2 verify the property (44). Then, from Stampacchia's estimate (46) we have

$$\|\Theta_\eta\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant K' \qquad \text{(independent of } \eta)$$

and, testing for each η (39) with $\zeta = \Theta_\eta - \tilde{\Theta}$, also

$$\|\Theta_\eta\|_{H^1(\Omega)} \leqslant C \qquad \text{(independent of } \eta)\,.$$

Hence one can select a subsequence, still denoted by $\eta \to 0$, such that

$$\Theta_\eta \to \Theta \quad \text{in } H^1(\Omega)\text{-weak and in } C^{0,\beta}(\bar{\Omega})$$

for any $0 \leq \beta < \alpha$. Passing to the limit in $(39)_\eta$, by uniqueness, one concludes $\Theta = \Theta_0 = \Theta(\boldsymbol{u}_0)$ and the convergence of the whole sequence.

If the convergence in (47) is strong, to prove the strong convergence in $H^1(\Omega)$, take $\zeta = \Theta_\eta - \Theta_0$ in (39) and use the uniform convergence. One has $f(\Theta_\eta) \to f(\Theta_0)$, and $h(\Theta_\eta) \to h(\Theta_0)$ uniformly in $\bar{\Omega}$, $g(X, \Theta_\eta(X)) \to g(X, \Theta_0(X))$ for all $X \in \Gamma_N$ and, by (43), (26) with Lebesgue's theorem, also in $L^q(\Gamma_N)$-strong. Therefore, using the trace and imbedding theorems, one has for $\tau_\eta = \Theta_\eta - \Theta_0$:

$$\int_\Omega |\nabla\tau_\eta|^2 = \int_\Omega \{[f(\Theta_\eta) - f(\Theta_0)]\boldsymbol{u}_\eta + f(\Theta_0)[\boldsymbol{u}_\eta - \boldsymbol{u}_0] + [h(\Theta_\eta) - h(\Theta_0)]\boldsymbol{e}_z\}\cdot\nabla\tau_\eta - \\ - \int_{\Gamma_N}[g(\Theta_\eta) - g(\Theta_0)]\tau_\eta \leq \Big(\int_\Omega |\nabla\tau_\eta|^2\Big)^{\frac{1}{2}} R_\eta\,,$$

with $R_\eta \to 0$ as $\eta \to 0$. Then, it follows

$$\int_\Omega |\nabla(\Theta_\eta - \Theta_0)|^2 \leq R_\eta^2\,,$$

which concludes the proof of the proposition. □

4. – Existence for the approximating problem.

In this section we rely again on Schauder fixed point Theorem in $\mathfrak{L}^4 \equiv [L^4(\Omega)]^n$, $n = 2, 3$, to solve the approximating problem. We recall, by well known inequalities (see [DL], [L] or [T]) there exists a positive constant K_0 (depending only on Ω and $n = 2, 3$) such that,

$$\|\boldsymbol{v}\|_{L^4} \leq \sqrt{2}\,\|\boldsymbol{v}\|_{V(\Omega)} \leq K_0\|D\boldsymbol{v}\|_{L^2}\,, \qquad \forall \boldsymbol{v} \in V(\Omega)\,. \tag{48}$$

Let $R > 0$ and introduce the following convex, closed bounded subset of $\mathfrak{L}^4$ (recall (40)),

$$\mathcal{B}_R = \{\boldsymbol{v} \in \mathfrak{L}^4_\sigma(\Omega)\colon \|\boldsymbol{v}\|_{L^4} \leq R\}\,. \tag{49}$$

Using the results of the preceding section we shall construct below a nonlinear mapping $\mathcal{T}\colon \mathcal{B}_R \to \mathcal{B}_R$, for an appropriate R, which will be shown to be continuous and compact.

THEOREM 3. – Assume $f_\varepsilon \in C^{0,1}(\mathbb{R})$, χ_ε and g_ε given by (29) and (30), respectively, and $\tilde{\Theta}_\varepsilon \in C^{0,1}(\bar{\Gamma}_D)$, $|\tilde{\Theta}_\varepsilon| \leq M$ in Ω. Then for each $\varepsilon > 0$, there exists a solution $(\Theta_\varepsilon, \boldsymbol{w}_\varepsilon) \in H^1(\Omega) \times V(\Omega)$ of Problem $(\mathrm{BS})_\varepsilon$, such that $\Theta_\varepsilon \in C^{0,\alpha}(\bar{\Omega})$, for some $0 < \alpha < 1$, and $|\Theta_\varepsilon| \leq M$ in Ω.

PROOF. – i) *Definition of* $\mathcal{T}$. For any $\boldsymbol{u} \in \mathfrak{L}^4_\sigma(\Omega)$ define $\tau_\varepsilon = \tau_\varepsilon(\boldsymbol{u})$ as the unique solution of the Problem (A) with $f = f_\varepsilon$, $h = bf_\varepsilon + \lambda b\chi_\varepsilon$, $g = g_\varepsilon$ and $\tilde{\Theta} = \tilde{\Theta}_\varepsilon$, which is in $C^{0,\alpha}(\bar{\Omega}) \cap H^1(\Omega)$ with $|\tau_\varepsilon| \leqslant M$ by the Theorem 2. Note that g_ε given by (30) satisfy the assumption (43), because (26) and $|g_\varepsilon(X, \tau)| \leqslant |g^0(X, \tau)|$, for any $X \in \Gamma_N$ and all $\tau \in [-M, M] \subset D(G_X)$, and the assumption (25) implies the sign condition (25′) for every g_ε. In fact, we see that $g_\varepsilon(M) = 1/\varepsilon(M - J_\varepsilon(M)) \geqslant 0$ since if $\sigma = J_\varepsilon(M)$ one has $M \in (I + \varepsilon G_X)(\sigma)$ and $G_X(\tau) \subset [0, \infty[$ for all $\tau \geqslant M$ implies $M \geqslant \sigma = J_\varepsilon(M)$. Analogously we see that $g_\varepsilon(-M) \leqslant 0$.

Now we define $\boldsymbol{v} = \mathcal{T}\boldsymbol{u}$ as the unique function $\boldsymbol{v} \in V(\Omega)$ which satisfies the linear problem:

$$(50)\qquad \int_\Omega 2\mu(\tau_\varepsilon(\boldsymbol{u}))\, D\boldsymbol{v} : D\boldsymbol{\psi} - \int_\Omega \boldsymbol{v}[(\boldsymbol{u} + b\boldsymbol{e}_z)\cdot\nabla\boldsymbol{\psi}] + \frac{1}{\varepsilon}\int_\Omega \chi_\varepsilon(\tau_\varepsilon(\boldsymbol{u}))\,\boldsymbol{v}\cdot\boldsymbol{\psi} = \int_\Omega \boldsymbol{B}(\tau_\varepsilon(\boldsymbol{u}))\cdot\boldsymbol{\psi}$$

for any $\boldsymbol{\psi} \in V(\Omega)$.

ii) $\mathcal{T}(\mathcal{B}_K) \subset \mathcal{B}_K$. Since, for fixed $\boldsymbol{u}$, the left member of (50) defines a coercive continuous bilinear form on $V(\Omega)$ (recall (21) and (48)) and the right side a continuous linear functional (recall $|\tau_\varepsilon| \leqslant M$), the existence and uniqueness of $\boldsymbol{v}$ follows by the Lax-Milgram theorem.

Taking $\boldsymbol{\psi} = \boldsymbol{v}$ in (50), using (48) and setting $\beta = \max_{|\tau| \leqslant M} |\boldsymbol{B}(\tau)|$, one has

$$(51)\qquad 4\mu_0 K_0^{-2}\|\boldsymbol{v}\|^2_{V(\Omega)} \leqslant \int_\Omega 2\mu(\tau_\varepsilon(\boldsymbol{u}))\, D\boldsymbol{v} : D\boldsymbol{v} \leqslant \beta|\Omega|^{\frac{3}{4}}\|\boldsymbol{v}\|_{L^4} \leqslant \sqrt{2}\,\beta|\Omega|^{\frac{3}{4}}\|\boldsymbol{v}\|_{V(\Omega)}$$

because, in the left side, the second term vanishes and the third one is nonnegative. Therefore one has the estimate

$$(52)\qquad \|\boldsymbol{v}\|_{L^4} \leqslant \beta|\Omega|^{\frac{3}{4}} K_0^2/2\mu_0 \equiv K\,,$$

which shows that if $\boldsymbol{u} \in \mathcal{B}_K$ also $\boldsymbol{v} = \mathcal{T}\boldsymbol{u} \in \mathcal{B}_K$.

iii) *Continuity and compactness of* $\mathcal{T}$. Take in $\mathfrak{L}^4_\sigma(\Omega)$ any sequence $\boldsymbol{u}_\eta \to \boldsymbol{u}_0$ in $\mathfrak{L}^4$-weak. From Proposition 1 one knows that $\tau_{\varepsilon\eta} = \tau_\varepsilon(\boldsymbol{u}_\eta) \to \tau_\varepsilon(\boldsymbol{u}_0) = \tau_{\varepsilon 0}$ uniformly in $\bar{\Omega}$ and consequently also $\mu(\tau_{\varepsilon\eta}) \to \mu(\tau_{\varepsilon 0})$, $\chi_\varepsilon(\tau_{\varepsilon\eta}) \to \chi_\varepsilon(\tau_{\varepsilon 0})$ and $\boldsymbol{B}(\tau_{\varepsilon\eta}) \to \boldsymbol{B}(\tau_{\varepsilon 0})$ in the topology of $C^0(\bar{\Omega})$. From (51), one has for $\boldsymbol{v}_\eta = \mathcal{T}\boldsymbol{u}_\eta$

$$\|\boldsymbol{v}_\eta\|_{V(\Omega)} \leqslant \sqrt{2}\,\beta|\Omega|^{\frac{3}{4}} K_0^2/2\mu_0\,,$$

and we can choose a subsequence, still labelled by η, and a function $\boldsymbol{v} \in V(\Omega)$ such that

$$\boldsymbol{v}_\eta \to \boldsymbol{v} \quad \text{in } V(\Omega)\text{-weak and } \mathfrak{L}^4\text{-strong}\,.$$

Then we can pass to the limit in $(50)_\eta$, since

$$\int_\Omega 2\mu(\tau_{\varepsilon\eta})\, D\boldsymbol{v}_\eta : D\boldsymbol{\psi} \to \int_\Omega 2\mu(\tau_{\varepsilon 0})\, D\boldsymbol{v} : D\boldsymbol{\psi}\,,$$

and

$$\int_\Omega \boldsymbol{v}_\eta(\boldsymbol{u}_\eta \cdot \nabla\boldsymbol{\psi}) \to \int_\Omega \boldsymbol{v}(\boldsymbol{u}_0 \cdot \nabla\boldsymbol{\psi})\,, \qquad \forall \boldsymbol{\psi} \in V(\Omega)$$

because $\boldsymbol{v}_\eta(\nabla\boldsymbol{\psi})^T \to \boldsymbol{v}(\nabla\psi)^T$ in $\mathfrak{L}^{\frac{4}{3}}$-strong.

By uniqueness, one has $\boldsymbol{v} = \boldsymbol{v}_0 = \mathcal{T}\boldsymbol{u}_0$, and the whole sequence $\boldsymbol{v}_\eta$ converges in $\mathfrak{L}^4$-strong to $\boldsymbol{v}_0$.

Then, the fixed point $\boldsymbol{w}_\varepsilon = \mathcal{T}\boldsymbol{w}_\varepsilon$ and $\Theta_\varepsilon = \tau_\varepsilon(\boldsymbol{w}_\varepsilon)$ form a solution to the approximating problem $(\mathrm{BS})_\varepsilon$. □

PROPOSITION 2. – Under the assumptions of the Theorem 1, all the solutions $(\Theta_\varepsilon, \boldsymbol{w}_\varepsilon)$ of the approximating problem $(\mathrm{BS})_\varepsilon$ verify the estimates (31)-(34) independently of ε.

PROOF. – As in the proof of Theorem 2, the « a priori » estimate $\|\Theta_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$ follows directly from (27). Then (33) is a direct consequence of (26), and taking $\boldsymbol{\psi} = \boldsymbol{w}_\varepsilon$ in (28) the estimate (34) follows as in (51), with $C = \sqrt{2}\,\beta|\Omega|^{\frac{1}{3}}K_0^2/2\mu_0$.

Using this information, the $H^1(\Omega)$-estimate for Θ_ε is now a standard consequence of the formulation (27) with $\zeta = \Theta_\varepsilon - \tilde{\Theta}_\varepsilon$, since $\tilde{\Theta}_\varepsilon$ have been chosen bounded in $H^1(\Omega)$.

Finally (32) is the local De Giorgi-Nash-Moser estimate (see [GT]) for the equation $-\Delta\Theta_\varepsilon = \nabla\cdot\boldsymbol{F}_\varepsilon$ in $\mathfrak{D}'(\Omega)$, where $\boldsymbol{F}_\varepsilon = f_\varepsilon(\Theta_\varepsilon)[\boldsymbol{w}_\varepsilon + b\boldsymbol{e}_z] + \lambda b\chi_\varepsilon(\Theta_\varepsilon)\boldsymbol{e}_z$ belongs to a bounded set of $\mathfrak{L}^4$. □

5. – A remark on the two-phase Rayleigh-Bénard problem.

It is a trivial remark to say that all the preceeding results, in particular the Theorem 1, still hold for $b = 0$ that is for the Boussinesq-Stefan problem without extraction, which have been considered already in [CDK] and [CD]. We shall obtain this problem with Dirichlet thermal boundary condition on Γ_N, the so-called two-phase Rayleigh-Bénard problem in [CB], by letting $b \to 0$ with an appropriate class of Neumann boundary conditions on that part of the boundary.

Consider a monotone increasing function η

$$\eta \in C^0(\mathbb{R}) \quad \text{and} \quad \eta(t)\,t \geqslant |t|^\sigma\,, \qquad t \in \mathbb{R},\ \sigma \geqslant 1\,, \tag{53}$$

for instance one can choose $\eta(t) = t|t|^{\sigma-2}$, and let β and Θ_N be two bounded functions on Γ_N:

(54) $$\Theta_N \in H^{\frac{1}{2}}(\Gamma_N)\,, \qquad \|\Theta_N\|_{L^\infty(\Gamma_N)} \leqslant M\,;$$

(55) $$k/b \geqslant \beta(X) \geqslant 1/kb > 0\,, \qquad \text{for some constant } k \geqslant 1\,.$$

Defining the nonlinear cooling on Γ_N by

(56) $$g(X, \tau) = \beta(X)\eta(\tau - \Theta_N(X))\,, \qquad \text{for } \tau \in \mathbb{R} \text{ and } X \in \Gamma_N\,,$$

one easily verifies conditions (23)-(26), by taking (53)-(55) into account.

Now we recall the definition of weak solution for the Rayleigh-Bénard problem introduced in [CBK]:

DEFINITION. – We say that $(\Theta, \boldsymbol{v})$ is a weak solution of problem (2)-(7) with $b = 0$ and Dirichlet thermal boundary condition, if

(57) $$\Theta \in C^0(\Omega) \cap H^1(\Omega)\,, \quad \Theta = \Theta_D \text{ on } \Gamma_D \text{ and } \Theta = \Theta_N \text{ on } \Gamma_N;$$

(58) $$\boldsymbol{v} \in V(\Omega)\,, \quad \boldsymbol{v} = 0 \quad \text{a.e. in } \{\Theta > 0\}\,;$$

(49) $$\int_\Omega [\nabla\Theta - f(\Theta)\boldsymbol{v}]\cdot\nabla\zeta = 0\,, \quad \forall \zeta \in H^1(\Omega)\colon \zeta|_{\Gamma_D \cup \Gamma_N} = 0\,;$$

(60) $$\int_{\{\Theta<0\}} [2\mu(\Theta)\, D\boldsymbol{v} \colon D\boldsymbol{\psi} - \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\boldsymbol{\psi})] = \int_{\{\Theta<0\}} \boldsymbol{B}(\Theta)\cdot\boldsymbol{\psi}\,, \quad \forall \boldsymbol{\psi} \in V(\{\Theta < 0\})\,.$$

The next theorem represents a continuous transition from one problem to another and gives an existence result for weak solutions to (57)-(60).

THEOREM 4. – Under the assumptions (20)-(22) and (53)-(56), if $b \to 0$ one can find functions $(\Theta_b, \boldsymbol{w}_b)$, solutions of (14)-(19) and a subsequence $b \to 0$, such that, for any compact set $K \subset \Omega$

(61) $$\Theta_b \to \Theta \text{ in } H^1(\Omega)\text{-strong and in } C^{0,\beta}(K)\,, \quad 0 < \beta < 1\,;$$

(62) $$\boldsymbol{w}_b \to \boldsymbol{v} \text{ in } V(\Omega)\text{-weak and } \mathfrak{L}^4\text{-strong}\,,$$

where $(\Theta, \boldsymbol{v})$ is a weak solution of the Rayleigh-Bénard problem, such that $|\Theta| \leqslant M$ in Ω.

PROOF. – Since the assumptions (53)-(56) preserve, in particular, the sign condition (25′), from the Theorem 1 one knows there exists Θ_b such that (with M and C independent of b)

(63) $$\|\Theta_b\|_{L^\infty(\Omega)} \leqslant M \quad \text{and} \quad \|\boldsymbol{w}_b\|_{V(\Omega)} \leqslant C\,, \quad \text{for any } b > 0\,.$$

Now, consider the unique solution $\zeta \in H^1(\Omega)$ to the problem

$$\Delta\xi = 0 \text{ in } \Omega\,, \quad \xi = \Theta_D \text{ on } \Gamma_D \text{ and } \xi = \Theta_N \text{ on } \Gamma_N\,,$$

and let $\zeta = \Theta_b - \xi$ in (18). We have, using (53) and (55),

$$(64)\qquad \frac{1}{2}\int_\Omega |\nabla(\Theta_b - \xi)|^2 + \frac{1}{kb}\int_{\Gamma_N} |\Theta_b - \Theta_N|^\sigma \leqslant C' \quad \text{(independent of } 0 < b \leqslant 1)\,,$$

for any Θ_b verifying (63). Then it follows

$$\|\Theta_b\|_{H^1(\Omega)} + \|\Theta_b\|_{C^{0,\alpha}(K)} \leqslant C''\,, \quad \text{for } 0 < b \leqslant 1\,,$$

for any compact $K \subset \Omega$ and some fixed $0 < \alpha < 1$. Therefore we can select a subsequence $b \to 0$ such that (for $0 < \beta < \alpha$)

$$(65)\qquad \Theta_b \to \Theta \quad \text{in } H^1(\Omega)\text{-weak, in } C^{0,\beta}(K) \text{ and a.e. on } \Gamma_N\,,$$

$$(66)\qquad \boldsymbol{w}_b \to \boldsymbol{v} \quad \text{in } V(\Omega)\text{-weak and } [L^4(\Omega)]^n\text{-strong } (n = 2, 3)\,,$$

from (64) one concludes that Θ verifies (57), since

$$\int_{\Gamma_N} |\Theta - \Theta_N|^\sigma \leqslant \liminf_{b\to 0} \int_{\Gamma_N} |\Theta_b - \Theta_N|^\sigma = 0\,,$$

and also (59) since one can easily let $b \to 0$ in $(18)_b$ for any fixed $\zeta \in H^1(\Omega)$: $\zeta = 0$ on $\Gamma_D \cup \Gamma_N$.

On the other hand, since $\Theta_b \to \Theta$ uniformly in any compact subset of Ω, if $K \subset \{\Theta > 0\}$ is an arbitrary compact set one must have also $K \subset \{\Theta_b > 0\}$ for b small enough and from $\boldsymbol{w}_b = 0$ a.e. in $\{\Theta_b > 0\}$ one deduces $\boldsymbol{v} = 0$ a.e. first in K and therefore also a.e. in $\{\Theta > 0\}$, that is $\boldsymbol{v}$ satisfies (58). Analogously, if $\boldsymbol{\psi}$ is any smooth solenoidal vector field such that $\operatorname{supp} \boldsymbol{\psi} \subset \{\Theta < 0\}$ one also has $\operatorname{supp} \boldsymbol{\psi} \subset \{\Theta_b < 0\}$ for b small enough and we can easily pass to the limit in $(19)_b$ for such a $\boldsymbol{\psi}$, concluding (60) by density.

It remains to prove the strong convergence in $H^1(\Omega)$. Set $\zeta = \Theta_b - \Theta$ in $(18)_b$ to get

$$\int_\Omega |\nabla\Theta_b|^2 + \frac{1}{kb}\int_{\Gamma_N} |\Theta_b - \Theta_N|^\sigma \leqslant \int_\Omega \nabla\Theta_b \cdot \nabla\Theta + R_b$$

with (using (63), (65) and (66))

$$R_b \equiv \int_\Omega \{f(\Theta_b)\boldsymbol{w}_b + b\boldsymbol{e}_z[f(\Theta_b) + \lambda\chi_b]\} \cdot \nabla(\Theta_b - \Theta) \to 0 \quad \text{as } b \to 0\,.$$

Hence one concludes $\nabla\Theta_b \to \nabla\Theta$ strongly in $\mathfrak{L}^2$, from

$$\int_\Omega |\nabla\Theta|^2 \leqslant \liminf_{b\to 0} \int_\Omega |\nabla\Theta_b|^2 \leqslant \limsup_{b\to 0} \int_\Omega |\nabla\Theta_b|^2 \leqslant \int_\Omega |\nabla\Theta|^2 . \quad \square$$

REFERENCES

[B] H. BRÉZIS, *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert,* Math. Studies, **5** North Holland (1983).

[CDK] J. R. CANNON - E. DIBENEDETTO - G. H. KNIGHTLY, *The Steady State Stefan Problem with Convection,* Arch. Rational Mech. Anal., **73** (1980), pp. 79-97.

[CD] J. R. CANNON - E. DIBENEDETTO, *The Steady State Stefan Problem with Convection, with Mixed Temperature and non-linear Heat Flux Boundary Conditions,* in « Free Boundary Problems », Ed. by E. MAGENES, Istit. Naz. Alta Matematica, Roma, 1980, Vol. I, pp. 231-265.

[CB] C. J. CHANG - R. A. BROWN, *Natural Convection in Steady Solidification: Finite Element Analysis of a Two-Phase Rayleigh-Bénard Problem,* J. Comput. Phys., **53** (1984), pp. 1-27.

[CR] M. CHIPOT - J. F. RODRIGUES, *On the Steady-State Continuous Casting Stefan Problem with Nonlinear Cooling,* Quart. Appl. Math., **40** (1983), pp. 476-491.

[DL] G. DUVAUT - J. L. LIONS, *Les inéquations en Méchanique et en Physique,* Dunod, Paris. 1972.

[F] M. C. FLEMINGS, *Solidification Processing,* McGraw-Hill, New York, 1974.

[GT] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order,* Springer, New York, 1977.

[L] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow* Gordon and Breach, New York, 1969.

[R] J. F. RODRIGUES, *Aspects of the Variational Approach to a Continuous Casting Problem,* Research Notes Math, Pitman, **120** (1985), pp. 72-83.

[S] G. STAMPACCHIA, *Problemi al contorno ellittici con dati discontinui dotati di soluzioni holderiane,* Ann. Mat. Pura Appl., **51** (1960), pp. 1-32.

[T] R. TEMAN, *Navier-Stokes Equations,* North-Holland Amsterdan, 1977.