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A Steady State Potential Flow Model for Semiconductors (*).

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Summary. – *We present a three-dimensional steady state irrotational flow model for semiconductors which is based on the hydrodynamic equations. We prove existence and local uniqueness of smooth solutions under a smallness assumptions on the data. This assumption implies subsonic flow of electrons in the semiconductors device.*

1. – Introduction.

The hydrodynamic model describing the electron flow in a unipolar semiconductor is given by [1, 2]:

$$\left.\begin{aligned} &(1.1) \qquad \rho_t + \operatorname{div}(\rho u) = 0 \\ &(1.2) \qquad u_t + (u \cdot \nabla) u + \frac{1}{\rho}\nabla p - \nabla \phi = -\frac{u}{\tau} \\ &(1.3) \qquad \Delta \phi = \rho - C(x) \end{aligned}\right\} \quad x = (x_1, x_2, x_3) \in \Omega, \ t > 0,$$

where ρ, u, ϕ denote the electron density, electron velocity and the electrostatic potential, respectively. The constant $\tau > 0$ is the velocity relaxation time, $\Omega \subseteq \mathbb{R}^3$ the bounded domain occupied by the semiconductor, $C = C(x)$ the prescribed density of positive background ions (doping profile) and

$$J = \rho u \tag{1.4}$$

is the (negative) electron current density. Since in this paper we only consider isentropic flow, the energy equation of the hydrodynamic model is replaced by the pressure-density relation

$$p = p(\rho). \tag{1.5}$$

Also, we only consider the steady state case and, additionally, we make the assump-

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tion of irrotational flow:

$$\operatorname{curl} u = 0. \tag{1.6a}$$

Introducing the velocity potential ψ by

$$u = -\nabla\psi \tag{1.6b}$$

and using the identity

$$(u\cdot\nabla)u = \frac{1}{2}\nabla(|u|^2) - u\times \operatorname{curl} u, \tag{1.7}$$

the equation (1.2) reduces to

$$\nabla\left(\frac{1}{2}|\nabla\psi|^2 + h(\rho) - \phi\right) = \frac{\nabla\psi}{\tau}, \qquad h'(\rho) := \frac{1}{\rho}p'(\rho). \tag{1.8}$$

The system (1.1), (1.2), (1.3) then can be written as:

$$\left.\begin{aligned} &\frac{1}{2}|\nabla\psi|^2 + h(\rho) = \phi + \frac{\psi}{\tau} && (1.9)\\ &\operatorname{div}(\rho\nabla\psi) = 0 && (1.10)\\ &\Delta\phi = \rho - C(x) && (1.11)\end{aligned}\right\} \quad x\in\Omega.$$

The electron current is now given by

$$J = -\rho\nabla\psi. \tag{1.12}$$

The existence of irrotational subsonic steady state flows in the gas-dynamics case (i.e. equations (1.9) with $\tau=\infty$, $\phi\equiv 0$ and (1.10)) is well-known (see. e.g. [9]). In this paper we shall employ a different analytical approach, which makes explicit use of the Poisson equation (1.11) and allows us to incorporate boundary conditions appropriate for the semiconductor case.

Also, we remark that (we shall see later on) the assumption $\tau\equiv$ const is crucial for the existence of irrotational steady states.

The boundary $\partial\Omega$ is assumed to split into N disjoint, closed and connected «contact» segments $\Gamma_1,\dots,\Gamma_N$ and «insulating» segments, whose union we denote by Γ_{ins}.

We impose the following boundary conditions:

$$\rho/\Gamma_i = \rho_D/\Gamma_i, \qquad i=1,\dots,N, \tag{1.13}$$

$$(\nabla\psi\times\nu)/\Gamma_i = 0, \qquad i=1,\dots,N, \tag{1.14}$$

$$\nabla\rho\cdot\nu/\Gamma_{\text{ins}} = 0, \tag{1.15}$$

$$\nabla\psi\cdot\nu/\Gamma_{\text{ins}} = 0, \tag{1.16}$$

where ν denotes the outward unit normal to $\partial\Omega$. Note that (1.16) specifies that the normal component of the velocity vanishes along Γ_{ins}, which implies no current flow through Γ_{ins} and (1.14) implies that ψ is constant on each contact segment Γ_i. For a current driven device the values of these constants are determined by imposing the currents I_i, $i = 1, \dots, N$, flowing out of the contacts:

$$-\int_{\Gamma_i} \rho \nabla \psi \cdot \nu \, ds = I_i, \qquad i = 1, \dots, N. \tag{1.17}$$

Because of the conservation equation (1.10) the data I_i, $i = 1, \dots, N$ must satisfy

$$\sum_{i=1}^{N} I_i = 0. \tag{1.18}$$

We shall demonstrate now that the boundary conditions (1.14), (1.17) lead to Dirichlet data for the velocity potential ψ. Therefore we introduce the functions χ_i as solutions of

$$\left.\begin{aligned} &\operatorname{div}(\rho \nabla \chi_i) = 0, \; x \in \Omega && (1.19)\\ &\chi_i / \Gamma_j = \delta_{ij}, \; j = 1, \dots, N && (1.20)\\ &\nabla \chi_i \cdot \nu / \Gamma_{ins} = 0 && (1.21)\end{aligned}\right\} \quad i = 1, \dots, N,$$

where δ_{ij} denotes the Kronecker symbol. Then we define the influence matrix $D = (D_{ij})_{\substack{i = 1, \dots, N \\ j = 1, \dots, N}}$ by

$$D_{ij} = \int_{\Gamma_j} \rho \nabla \chi_i \cdot \nu \, ds. \tag{1.22}$$

Since we can decompose (with as yet unknown constants ψ_i):

$$\psi = \sum_{j=1}^{N} \psi_j \chi_j \tag{1.23}$$

we obtain from (1.17)

$$-\sum_{j=1}^{N} D_{ji} \psi_j = I_i. \tag{1.24}$$

It is easily shown that D is a symmetric nonnegative matrix of rank $N - 1$. Because of the condition (1.18), the system (1.24) can be inverted if we prescribe one of the values ψ_i (e.g. $\psi_1 = 0$). We thus obtain

$$\psi / \Gamma_i = \psi_i, \tag{1.25}$$

where ψ_i depends on $I_j, j = 1, \dots, N$, and by the definition of χ_i, on ρ. More information on the influence matrices and their use in vector field decomposition can be found in [8].

By analogy to the drift-diffusion model for semiconductors [3, 4], the velocity potential ψ can be regarded as a quasi-Fermi level for electrons. Indeed, when all the ψ_i's are zero, then there is no current flowing and, thus, the device is in thermal equilibrium. In this situations, we obtain the following equation for the electrostatic potential ϕ from (1.9), (1.10), (1.11):

$$\phi = h(\rho), \qquad \Delta\phi = \rho - C(x) \tag{1.26}$$

which agrees with the drift-diffusion equilibrium, if $h(\rho) = K \ln \rho$, $K > 0$, i.e. if $p = K\rho$ holds (linear pressure-density relationship).

For a «voltage driven device» the values ψ_i, $i = 1, \ldots, N$ (applied poentials) are prescribed (with, say, $\psi_1 = 0$) and the outflow currents I_i can be computed a posteriorily from (1.24), (1.18).

In Section 2, we present an existence and (local) uniqueness result under the assumption $\Gamma_{\text{ins}} = \{\ \}$. The reason for this is that the proof heavily relies on regularity results for elliptic equations in $W^{2,q}(\Omega)$, which in full generality can only be obtained for the Dirichlet problem (see [5]). It is possible to generalize the existence and uniqueness result to the mixed Newmann-Dirichlet boundary value problem under additional very stringent regularity and geometry assumptions on the boundary segments.

The electron flow in the semiconductor is called subsonic if

$$|u| = |\nabla\psi| < \sqrt{p'(\rho)}, \qquad x \in \Omega \tag{1.27}$$

or, equivalently,

$$|J|^2 \leqslant \rho^2 p'(\rho), \qquad x \in \Omega, \tag{1.28}$$

holds. The quantity $\sqrt{p'(\rho)}$ is called electron sound speed [6]. Obviously, shocks may occur if the flow is (partly) supersonic, and, thus, the main assumption for the existence and uniqueness result is a restriction on the magnitude of the boundary datum for ψ, which will imply a fully subsonic flow.

The subsonic one-dimensional steady state case was analysed in [7], where a condition for fully subsonic flow, which is verifyable in terms of the data, was given. The present paper extends this result to the three-dimensional case, however, the smallness assumption cannot be verified explicitely anymore.

2. – Existence of a smooth solution.

We apply the Laplace-operator to (1.9), use (1.10), (1.11) and obtain:

$$\Delta\left(\frac{1}{2}|\nabla\psi|^2 + h(\rho)\right) + \frac{1}{\tau}\frac{\nabla\psi}{\rho}\cdot\nabla\rho - \rho = -C(x), \qquad x \in \Omega. \tag{2.1}$$

In order to eliminate the third derivatives of ψ in (2.1) we calculate

$$\frac{1}{2}\Delta(|\nabla\psi|^2) = Q(\psi) + \sum_{i=1}^{3} \psi_{x_i}(\Delta\psi)_{x_i}, \tag{2.2}$$

where $Q(\psi)$ is given by

(2.3) $$Q(\psi) = (\psi_{x_1x_1})^2 + (\psi_{x_2x_2})^2 + (\psi_{x_3x_3})^2 + 2(\psi_{x_1x_2})^2 + 2(\psi_{x_1x_3})^2 + 2(\psi_{x_2x_3})^2 .$$

By sustituting $\Delta\psi - (\nabla\rho/\rho)\nabla\psi$ (which is the equation (1.10)) into (2.2) we obtain

(2.4) $$\frac{1}{2}\Delta(|\nabla\psi|^2) = Q(\psi) + \frac{1}{\rho^2}(\nabla\psi\cdot\nabla\rho)^2 - \frac{1}{\rho}\sum_{i,j=1}^{3}\psi_{x_i}\psi_{x_j}\rho_{x_ix_j} - \frac{1}{\rho}\sum_{i,j=1}^{3}\psi_{x_i}\rho_{x_j}\psi_{x_ix_j}$$

and (2.1), (1.10) can be written as coupled system:

(2.5) $$\Delta h(\rho) - \frac{1}{\rho}\sum_{i,j=1}^{3}\psi_{x_i}\psi_{x_j}\rho_{x_ix_j} + \frac{1}{\rho^2}(\nabla\psi\cdot\nabla\rho)^2 + \frac{1}{\tau}\frac{\nabla\psi}{\rho}\nabla\rho - \frac{1}{\rho}\sum_{i,j=1}^{3}\psi_{x_i}\rho_{x_j}\psi_{x_ix_j} - \rho = - Q(\psi) - C(x), \qquad x\in\Omega,$$

(2.6) $$\Delta\psi + \frac{\nabla\rho}{\rho}\nabla\psi = 0, \qquad x\in\Omega.$$

As mentioned in the introduction we assume that Γ_{ins} is empty and pose the Dirichlet boundary conditions:

(2.7) $$\rho/\partial\Omega = \rho_D/\partial\Omega, \qquad \psi/\partial\Omega = \psi_D/\partial\Omega .$$

Assume now that (ψ,ρ), $\rho \geq \underline{\rho} > 0$, is a strong solution of the system (2.5), (2.6), (2.7). Then the electrostatic potential ϕ can be obtained from the relation (1.9). Going from (2.5), (2.6) back to (2.1), we easily conclude that the Poisson equation (1.11) is satisfied. Also, a sufficiently regular solution of (1.9), (1.10), (1.11), solves (2.5), (2.6). In this sense the two problems are equivalent.

At first we prove that, for given ψ, ellipticity of (2.5) is equivalent to the condition that the flow is subsonic.

LEMMA 2.1. – *Assume that $\rho(x) > 0$, $p'(\rho(x)) > 0$ hold for some $x\in\Omega$. Then the equation* (2.5) *is elliptic at x if and only if*

(2.8) $$|\nabla\psi(x)| < \sqrt{p'(\rho(x))}$$

holds.

PROOF. – We write the principal part of (2.5) as $L(\rho,\nabla\psi) = \sum_{i,j=1}^{3} a_{ij}(\rho,\nabla\psi)\rho_{x_ix_j}$ and compute the eigenvalues of the matrix $A = (a_{ij})$. We obtain $\lambda_1 = \lambda_2 = h'(\rho) > 0$, $\lambda_3 = h'(\rho) - (1/\rho)|\nabla\psi|^2$. The result follows since $h'(\rho) = (1/\rho)p'(\rho)$. ∎

Lemma 2.1 indicates that it is essential for the existence proof to control $\|\nabla\psi\|_{L^\infty(\Omega)}$ and to bound ρ from below.

We make the following assumptions:

(A.1) $$p\in C^3([0,\infty)\to[0,\infty)), \qquad p'(\rho) > 0 \ \forall\rho > 0,$$

(A.2) $$C \in L^{\infty}(\Omega), \qquad 0 < \underline{C} \leq C(x) \leq \overline{C} \ \forall x \in \Omega,$$

(A.3) $$\psi_D \in C^{2,\delta}(\overline{\Omega}) \quad \text{for some } \delta,\ 0<\delta<1;\ \rho_D \in W^{2,\infty}(\Omega),\ 0 < \underline{\rho} \leq \rho_D(x) \leq \overline{\rho}\ \forall x \in \partial\Omega$$

(A.4) Ω is a bounded convex $C^{2,\delta}$-domain in $\boldsymbol{R}^3$.

The existence result is stated in:

THEOREM 2.2. – *Let the assumptions* (A.1)-(A.4) *hold. Then there exists* $\varepsilon > 0$ *such that the problem* (1.9), (1.10), (1.11), (2.7) *has a solution* $(\psi, \rho, \phi) \in C^{2,\delta}(\overline{\Omega}) \times W^{2,q}(\Omega) \times C^{4,\delta}(\overline{\Omega})$, $1 \leq q < \infty$, *which satisfies* $\rho \geq \mathrm{Min}(\underline{\rho}, \underline{C}) > 0$ *if*

(2.9) $$\|\psi_D\|_{C^{2,\delta}(\overline{\Omega})} < \varepsilon$$

holds.

PROOF. – We shall use Schauder's fixed point Theorem. Therefore we set up the map $T: \sigma \to \rho$ defined as follows:

(*A*) Solve

(2.10*a*) $$\Delta\psi + \frac{\nabla\sigma}{\sigma}\nabla\psi = 0 \quad \text{in } \Omega$$

(2.10*b*) $$\psi/\partial\Omega = \psi_D/\partial\Omega$$

for $\psi = \psi[\sigma]$.

(*B*) Set

(2.11) $$\rho := g(v), \qquad g := h^{-1}$$

where v solves

(2.12*a*) $$\Delta v - \frac{g'(h(\sigma))}{\sigma}\sum_{i,j=1}^{3}\psi_{x_i}\psi_{x_j}v_{x_ix_j} - \frac{g''(h(\sigma))}{\sigma g'(h(\sigma))}\sum_{i,j=1}^{3}\psi_{x_i}\psi_{x_j}\sigma_{x_i}v_{x_j} +$$
$$+ g'(h(\sigma))\left(\frac{\nabla\psi\cdot\nabla\sigma}{\sigma^2}\nabla\psi + \frac{\nabla\psi}{\tau\sigma} - \frac{1}{\sigma}\sum_{i=1}^{3}\psi_{x_i}\nabla\psi_{x_i}\right)\nabla v - g(v) = -Q(\psi) - C(x), \qquad x \in \Omega$$

(2.12*b*) $$v/\partial\Omega = h(\rho_D)/\partial\Omega.$$

It is an easy exercise to show that every (sufficiently regular positive) fixed point ρ^* of T corresponds to a solution (ρ^*, ψ^*) of (2.5), (2.6), (2.7) where ψ^* is determined by solving (2.6) with $\rho = \rho^*$ subject to the boundary condition $\Psi^*/\partial\Omega = \psi_D/\partial\Omega$.

For the analysis of the semilinear equation (2.12) we use the following

LEMMA 2.2. – *Assume that the following conditions hold*:

(i) Ω *is a* $C^{1,1}$*-domain,*

(ii) $a_{ij} \in C^{0,\delta}(\overline{\Omega})$ *for some* $0 < \delta \leq 1$, $a_{ij} = a_{ji}$, $1 \leq i, j \leq 3$,

(iii) $\exists \underline{a} > 0$: $\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \geq \underline{a}|\xi|^2$, $\forall x \in \Omega$, $\forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

(iv) $\boldsymbol{b} \in L^\infty(\Omega)$,

(v) $g \in C^1(\mathbb{R})$, $g' \geq 0$,

(vi) $f \in L^\infty(\Omega)$,

(vii) $v_D \in W^{2,\infty}(\Omega)$, $v_1 \leq v_D(x) \leq v_2$ *for all* $x \in \partial\Omega$.

Additionally, assume that there are $\underline{v}, \overline{v} \in \mathbb{R}$ *such that*

$$g(\underline{v}) \leq -f(x), \qquad g(\overline{v}) \geq -d(x) \text{ for all } x \in \Omega \text{ with } d \in L^\infty(\Omega) \tag{2.13}$$

holds. Then the problem

$$\sum_{i,j=1}^{3} a_{ij}(x) v_{x_i x_j} + \boldsymbol{b}(x)\cdot\nabla v - g(v) = f(x), \qquad x \in \Omega, \tag{2.14a}$$

$$v/\partial\Omega = v_D/\partial\Omega, \tag{2.14b}$$

has a unique solution which satisfies $v \in W^{2,q}(\Omega)$, $1 < q < \infty$, *and*

$$\min(v_1, \underline{v}) \leq v(x) \leq \max(v_2, \overline{v}) + D_1\|f - d\|_{L^\infty(\Omega)}, \qquad x \in \Omega, \tag{2.15a}$$

$$\|v\|_{W^{2,q}(\Omega)} \leq D_2[\|v_D\|_{W^{2,q}(\Omega)} + \|f\|_{L^\infty(\Omega)} + |g(\|v\|_{L^\infty(\Omega)})|], \qquad 1 < q < \infty, \tag{2.15b}$$

where D_1 *is independent of* a_{ij}, $\boldsymbol{b}$ *and* D_2 *is bounded when* $\|\boldsymbol{b}\|_{L^\infty(\Omega)}$, $\max_{1\leq i,j\leq 3}\|a_{ij}\|_{C^{0,\delta}(\overline{\Omega})}$, $\underline{a}$ *vary in compact subsets of* $[0, \infty)$, $(0, \infty)$ *and* $(0, \infty)$ *respectively.*

PROOF. – The existence of a solution of (2.14) is standard. The estimate (2.15*a*) follows from the classical maximum principle and (2.15*b*), from the $W^{2,q}$-estimate associated with Theorem 9.15 in [5]. ■

We define the set $A_{\rho_1,\rho_2,\kappa} := \{\rho \in C^{1,\delta}(\overline{\Omega})/\rho_1 \leq \rho(x) \leq \rho_2 \text{ for all } x \in \Omega, \|\rho\|_{C^{1,\delta}(\overline{\Omega})} \leq \kappa\}$ for positive numbers $0 < \rho_1 \leq \rho_2 \leq \kappa$. We shall prove that the map T is a compact self map of $A_{\rho_1,\rho_2,\kappa}$ when the parameters ρ_1, ρ_2, κ are appropriately chosen.

Now let $\sigma \in A_{\rho_1,\rho_2,\kappa}$. Since $\|\nabla\sigma/\sigma\|_{C^{0,\delta}(\overline{\Omega})} \leq \text{const}\,\kappa^2/\rho_1^2$ holds, the standard Hölder-estimate for elliptic equations [5, Theorem 6.6] applied to (2.10*a*), (2.10*b*) gives, using (2.9):

$$\|\psi\|_{C^{2,\delta}(\overline{\Omega})} \leq K_1(\rho_1, \kappa)\|\psi_D\|_{C^{2,\delta}(\overline{\Omega})} \leq K_1(\rho_1, \kappa)\,\varepsilon \tag{2.16}$$

(from now on we denote by K_i functions which are bounded when their arguments vary in compacts subsets of $(0, \infty)$).

We shall define the quantities ρ_i, ρ_2, κ by applying Lemma 2.2 to (2.12) (with the obvious identifications of the coefficients and the right hand side).

The equation (2.12a) is uniformly elliptic in Ω (with $\underline{a} = 1/2$) if

$$1 - \frac{g'(h(\sigma))}{\sigma} |\nabla\psi|^2 \geqslant \frac{1}{2} \qquad \forall x \in \Omega \tag{2.17}$$

or equivalently

$$|\nabla\psi| \leqslant \sqrt{\frac{1}{2} p'(\sigma)} \qquad \forall x \in \Omega \tag{2.18}$$

hold. We now choose $\varepsilon = \varepsilon(\rho_1, \rho_2, \kappa)$ such that

$$K_1(\rho_1, \kappa)\,\varepsilon \leqslant \sqrt{\frac{1}{2} \min_{\rho_1 \leqslant \alpha \leqslant \rho_2} p'(\alpha)} \tag{2.19}$$

holds. Then, by (2.16) we conclude (2.18).

We now estimate the coefficients of (2.12a):

$$\left\| \frac{g'(h(\sigma))}{\sigma} \psi_{x_i} \psi_{x_j} \right\|_{C^{0,\delta}(\bar{\Omega})} \leqslant K_2(\rho_1, \rho_2, \kappa) \|\nabla\psi\|^2_{C^{0,\delta}(\bar{\Omega})} \leqslant K_3(\rho_1, \rho_2, \kappa)\,\varepsilon^2, \tag{2.20a}$$

$$\left\| \frac{g''(h(\sigma))}{\sigma g'(h(\sigma))} \psi_{x_i} \psi_{x_j} \sigma_{x_i} \right\|_{C^{0,\delta}(\bar{\Omega})} \leqslant K_3(\rho_1, \rho_2, \kappa)\,\varepsilon^2, \tag{2.20b}$$

$$\left\| g'(h(\sigma)) \left(\frac{\nabla\psi \cdot \nabla\sigma}{\sigma^2} \nabla\psi + \frac{\nabla\psi}{\tau\sigma} - \frac{1}{\sigma} \sum_{i=1}^{3} \psi_{x_i} \nabla\psi_i \right) \right\|_{L^\infty(\Omega)} \leqslant \tag{2.21}$$

$$\leqslant K_4(\rho_1, \rho_2, \kappa)(\|\nabla\psi\|^2_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)} \|\psi\|_{C^{2,\delta}(\bar{\Omega})}) \leqslant K_5(\rho_1, \rho_2, \kappa)\,\varepsilon,$$

$$\|Q(\psi)\|_{L^\infty(\Omega)} \leqslant K_6 \|\psi\|^2_{C^{2,\delta}(\bar{\Omega})} \leqslant K_7(\rho_1, \kappa)\,\varepsilon^2. \tag{2.22}$$

We set $\underline{v} := h(\underline{C})$ and obtain

$$g(\underline{v}) = \underline{C} \leqslant C(x) + Q(\psi)$$

since $Q(\psi) \geqslant 0$. The left inequality (2.15a) gives $v(x) \geqslant \min(h(\underline{C}), h(\underline{\rho}))$ and we obtain

$$\rho(x) \geqslant \rho_1, \tag{2.23}$$

where we set

$$\rho_1 := \min(\underline{C}, \underline{\rho}). \tag{2.24}$$

With $\bar{v} := h(\bar{C})$ we compute

$$g(\bar{v}) = \bar{C} \geqslant C(x)$$

and the right inequality (2.15a) gives

$$v(x) \leqslant \operatorname{Max}(h(\bar{C}), h(\bar{\rho})) + D_1 \|Q(\psi)\|_{L^\infty(\Omega)} \leqslant \operatorname{Max}(h(\bar{C}), h(\bar{\rho})) + K_8(\kappa)\,\varepsilon^2$$

(from now on the dependence of K_i on ρ_1 is suppressed since ρ_1 was already defined in (2.24)). We now restrain $\varepsilon = \varepsilon(\rho_2, \kappa)$ such that

$$K_8(\kappa)\,\varepsilon^2 \leqslant a \tag{2.25}$$

holds where $a > 0$ is chosen so small that $\mathrm{Max}(h(\overline{C}), h(\overline{\rho})) + a$ is in the domain of g. Then we obtain

$$\rho(x) \leqslant \rho_2 \tag{2.26}$$

with

$$\rho_2 := g(\max\,(h(\overline{C}), h(\overline{\rho})) + a)\,. \tag{2.27}$$

Thus, we are left with fixing κ. Therefore we use (2.15*b*) with $q > 3/(1-\delta)$ to estimate

$$\|v\|_{W^{2,q}(\Omega)} \leqslant K_9(\|v_D\|_{W^{2,q}(\Omega)} + \overline{C} + K_{10}(\kappa)\,\varepsilon^2 + \rho_2)\,, \tag{2.28}$$

where K_9 only depends on a product of the form $K_{11}(\kappa)\,\varepsilon$ (also, the dependence of K_i on ρ_2 is suppressed from now on). We now choose $\varepsilon = \varepsilon(\kappa)$ such that $K_{11}(\kappa)\,\varepsilon \leqslant 1$, $K_{10}(\kappa)\,\varepsilon^2 \leqslant 1$. Then we obtain from (2.28)

$$\|v\|_{W^{2,q}(\Omega)} \leqslant A(\|v_D\|_{W^{2,q}(\Omega)} + \overline{C} + 1 + \rho_2)\,, \tag{2.29}$$

with A being independent of ε and κ.

Because of the compact imbedding $W^{2,q}(\Omega) \to C^{1,\delta}(\overline{\Omega})$ for $q > 3/(1-\delta)$, we have

$$\|\rho\|_{C^{1,\delta}(\overline{\Omega})} \leqslant \kappa\,, \tag{2.30}$$

where we set

$$\kappa = \rho_2 + (\max_{h(\rho_1) \leqslant \beta \leqslant h(\rho_2)} (|g'(\beta)| + |g''(\beta)|))\,K_{12}A(\|h(\rho_D)\|_{W^{2,q}(\Omega)} + \overline{C} + 1 + \rho_2)\,, \tag{2.31}$$

where K_{12} is the bound of the imbedding.

We thus proved that the operator T is a self map of $A_{\rho_1,\rho_2,\kappa}$ if ρ_1, ρ_2, κ are chosen as in (2.24), (2.27) and (2.31) resp., and if ε (cf. (2.9)) is sufficiently small. Then, $T(A_{\rho_1,\rho_2,\kappa})$ is precompact in $C^{1,\delta}(\overline{\Omega})$ because, as mentioned above, the imbedding $W^{2,q}(\Omega) \to C^{1,\delta}(\overline{\Omega})$ is compact. The continuity of T, regarded as a map of a subset of $C^{1,\delta}(\overline{\Omega})$ into $C^{1,\delta}(\overline{\Omega})$ can be proved by standard arguments based on $W^{2,q}$-estimates for solutions of linear elliptic equation and are omitted here.

Therefore, we conclude the existence of a fixed point of T from the Schauder Theorem and the proof of Theorem 2.1 is completed. ∎

The local uniqueness result is stated in:

THEOREM 2.2. – *Let κ_*, ρ_* be arbitrary positive constants. Then, there exists* $\varepsilon = \varepsilon(\kappa*, \rho*) > 0$ *such that the solution* (ψ, ρ, ϕ) *of* (1.9), (1.10), (1.11), (2.7) *is unique in* *the set of functions*

$$C(\bar{\Omega}) \times \{\rho \in C^{1,\delta}(\bar{\Omega}),\ \|\rho\|_{C^{1,\delta}(\bar{\Omega})} \leq \kappa*,\ \rho \geq \rho*\} \times C(\bar{\Omega})$$

if

$$\|\psi_D\|_{C^{2,\delta}(\bar{\Omega})} \leq \varepsilon \tag{2.32}$$

holds.

PROOF. – Let (ψ_1, ρ_1, ϕ_1), (ψ_2, ρ_2, ϕ_2) be two solutions in the considered class. We set $\alpha = \psi_1 - \psi_2$, $\beta = \rho_1 - \rho_2$, $\gamma = \phi_1 - \phi_2$. Then, by subtracting the equations, we obtain the following boundary value problem:

$$\frac{1}{2}\nabla(\psi_1 + \psi_2)\cdot\nabla\alpha + h'(\xi)\beta = \gamma + \frac{\alpha}{\tau}, \qquad \beta|_{\partial\Omega} = 0, \tag{2.33}$$

$$\operatorname{div}(\rho_1\nabla\alpha + \beta\nabla\psi_2) = 0, \qquad \alpha|_{\partial\Omega} = 0, \tag{2.34}$$

$$\Delta\gamma = \beta, \qquad \gamma|_{\partial\Omega} = \gamma_D|_{\partial\Omega}, \qquad \gamma_D = \frac{1}{2}\nabla(\psi_1 + \psi_2)\cdot\nabla\alpha, \tag{2.35}$$

where ξ is between ρ_1 and ρ_2.

We multiply (2.23) by β, (2.34) by α (2.35) by $\gamma - \gamma_D$ and integrate

$$\int_\Omega \rho_1 |\nabla\alpha|^2 dx = -\int_\Omega \beta\nabla\psi_2\cdot\nabla\alpha\, dx, \tag{2.36}$$

$$\frac{1}{2}\int_\Omega \beta\nabla(\psi_1 + \psi_2)\cdot\nabla\alpha\, dx + \int_\Omega h'(\xi)\beta^2 dx = \int_\Omega \beta\gamma\, dx + \frac{1}{\tau}\int_\Omega \alpha\beta\, dx, \tag{2.37}$$

$$\int_\Omega \beta\gamma\, dx = \frac{1}{2}\int_\Omega \beta\nabla(\psi_1 + \psi_2)\cdot\nabla\alpha\, dx - \int_\Omega |\nabla\gamma|^2 dx + \int_\Omega \nabla\gamma\cdot\nabla\gamma_D\, dx. \tag{2.38}$$

Subtracting (2.38) from (2.37) gives

$$\int_\Omega h'(\xi)\beta^2 dx + \int_\Omega |\nabla\gamma|^2 dx = \frac{1}{\tau}\int_\Omega \alpha\beta\, dx + \int_\Omega \nabla\gamma\cdot\nabla\gamma_D\, dx. \tag{2.39}$$

We estimate (2.36):

$$\int_\Omega |\nabla\alpha|^2 \mathrm{d}x \leq M\|\nabla\psi_2\|_{L^\infty(\Omega)}\|\beta\|_{L^2(\Omega)}\|\nabla\alpha\|_{L^2(\Omega)} \tag{2.40}$$

and thus

$$\|\nabla\alpha\|_{L^2(\Omega)} \leq M\varepsilon\|\beta\|_{L^2(\Omega)} \tag{2.41}$$

follows. Here and in the sequel we denote by M not necessarily equal constants, which only depend on the data and on κ_*, ρ_*.

By using the Poincaré inequality, we can therefore estimate the first term on the right hand side of (2.39)

$$\frac{1}{\tau}\int_{\Omega} \alpha\beta\, dx \leqslant M\varepsilon\|\beta\|^2_{L^2(\Omega)}. \tag{2.42}$$

We obtain for the second term

$$\int_{\Omega} \nabla\gamma\cdot\nabla\gamma_D\, dx \leqslant \|\nabla\gamma\|_{L^2(\Omega)}\|\nabla\gamma_D\|_{L^2(\Omega)} \tag{2.43}$$

and from the definition (2.35) of γ_D

$$\|\nabla\gamma_D\|_{L^2(\Omega)} \leqslant M\|\psi_1+\psi_2\|_{W^{2,q}(\Omega)}\|\alpha\|_{H^2(\Omega)} \leqslant M\varepsilon\|\alpha\|_{H^2(\Omega)}. \tag{2.44}$$

We carry out the divergence in (2.34)

$$\rho_1\Delta\alpha + \nabla\rho_1\cdot\nabla\alpha + \beta\Delta\psi_2 + \nabla\beta\cdot\nabla\psi_2 = 0$$

and estimate

$$\|\Delta\alpha\|_{L^2(\Omega)} \leqslant M\|\nabla\alpha\|_{L^2(\Omega)} + M\varepsilon\|\beta\|_{L^2(\Omega)} + M\varepsilon\|\nabla\beta\|_{L^2(\Omega)}.$$

We thus obtain using (2.41)

$$\|\alpha\|_{H^2(\Omega)} \leqslant M\varepsilon(\|\beta\|_{L^2(\Omega)} + \|\nabla\beta\|_{L^2(\Omega)}). \tag{2.45}$$

We consider (2.33) with $h'(\xi)\beta$ replaced by $h(\rho_1)-h(\rho_2)$ and apply the gradient:

$$\nabla\gamma_D + h'(\rho_1)\nabla\beta + h''(\eta)\beta\nabla\rho_2 = \nabla\gamma + \frac{1}{\tau}\nabla\alpha, \tag{2.46}$$

where η is between ρ_1 and ρ_2. Taking the L^2-norm and using that $h'(\rho_1)\geqslant\lambda>0$ gives

$$\lambda\|\nabla\beta\|_{L^2(\Omega)} \leqslant M\|\beta\|_{L^2(\Omega)} + \|\nabla\gamma\|_{L^2(\Omega)} + M\|\nabla\alpha\|_{L^2(\Omega)} + \|\nabla\gamma_D\|_{L^2(\Omega)}. \tag{2.47}$$

We use (2.44), (2.45), (2.41) and derive

$$\lambda\|\nabla\beta\|_{L^2(\Omega)} \leqslant M\|\beta\|_{L^2(\Omega)} + M\varepsilon^2\|\nabla\beta\|_{L^2(\Omega)} + \|\nabla\gamma\|_{L^2(\Omega)}. \tag{2.48}$$

Then, for ε sufficiently small, we have

$$\|\nabla\beta\|_{L^2(\Omega)} \leqslant M\|\beta\|_{L^2(\Omega)} + \|\nabla\gamma\|_{L^2(\Omega)} \tag{2.49}$$

and from (2.45)

$$\|\alpha\|_{H^2(\Omega)} \leqslant M\varepsilon(\|\beta\|_{L^2(\Omega)} + \|\nabla\gamma\|_{L^2(\Omega)}). \tag{2.50}$$

We use (2.50) to estimate (2.44) and consequently (2.43):

$$\int_{\Omega} \nabla\gamma\cdot\nabla\gamma_D\, dx \leqslant M\varepsilon^2(\|\beta\|^2_{L^2(\Omega)} + \|\nabla\gamma\|^2_{L^2(\Omega)}). \tag{2.51}$$

Finally, we estimate (2.39) using (2.42), (2.51):

$$\lambda\|\beta\|^2_{L^2(\Omega)} + \|\nabla\gamma\|^2_{L^2(\Omega)} \leqslant M\varepsilon\|\beta\|^2_{L^2(\Omega)} + M\varepsilon^2\|\nabla\gamma\|^2_{L^2(\Omega)} \tag{2.52}$$

and conclude $\beta = \gamma = \alpha = 0$ for ε sufficiently small. ∎

From a practical standpoint the most severe restriction of the presented model is the assumption that the velocity relaxation time τ is constant. Note that, for non constant τ, the equation (1.8) does not admit solutions if

$$\operatorname{rot}\left(\frac{\nabla\psi}{\tau(x)}\right) = -\frac{1}{\tau^2(x)}\nabla\tau(x) \times \nabla\psi \not\equiv 0\,. \tag{2.53}$$

Thus, nonconstant relaxation times, in particular current and/or density dependent models, are generally a source of vorticity.

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