# A still topical contribution of Gabrio Piola to Continuum Mechanics: the creation of peri-dynamics, non-local and higher gradient continuum mechanics



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**Institutions:** Sapienza University of Rome

**Published on:** 01 Jan 2014

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**Abstract**

Gabrio Piola's scientific papers are in some aspects still topical in the matematical-physics literature. Actually, even if some authors [10] dedicated many efforts to the aim of unveiling the true value of Gabrio Piola as a scientist, some deep parts of his scientific achievements remain not yet sufficiently illustrated. We start our considerations by discussing some of the phenomena which influence the storage and transmission of knowledge, being inspired by the work of Lucio Russo ([106]). Subsequently, our aim is to prove that non-local and higher gradient continuum mechanics is rigorously formulated already in Piola's works and then we try to explain the reasons of the unfortunate circumstance which caused the erasure of the memory of this aspect of his contribution. Finally some relevant differential relationships obtained in Piola [Piola, 1848] are carefully studied, as they are still nowadays too often ignored in the continuum mechanics literature while indeed they can be considered as the starting point of Levi-Civita's theory of Connection for Riemannian manifolds.

## 1 A premise about linguistic, ideological and cultural barriers impeding the transmission of knowledge

It is evident to many authors and it is very often recognized in the scientific literature that linguistic barriers may play a negative role in the transmission and advancement of science. We recall here, for instance, that Peano [96] in 1903, being aware of the serious consequences which a Babel effect can have on the effective collaboration among scientists, tried to push the scientific community towards the use of Latin or of an especially constructed *lingua franca* in scientific literature, i.e. the so-called latino sine flexione. Actually in Russo [105], [106] the author clearly analyses the consequences of the existence of those linguistic ideological and cultural barriers which did not permit the Latin speaking scholars to understand the depth of Hellenistic science: the beginning of the economical and social processes leading to the Middle Ages.

### 1.1 Gabrio Piola as a protagonist of Italian Risorgimento (Resurgence)

It is surprising that some important contributions to mechanics of a well-known scientist remained unnoticed and have been neglected for a so long a time. Actually, after a careful observation of distinct traces and by gathering hints and evidences, one can dare to propose a well-founded conjecture: Gabrio Piola has been a leading cultural and scientific protagonist of Italian Risorgimento (Resurgence).

The main evidence of this statement has been found, e.g., in his eulogy (that is translated in this volume) in memoriam of his *Maestro* Vincenzo Brunacci. This eulogy was written in 1818 (three years after the famous Rimini Proclamation by Giocchino Murat that, to give an idea of its content, started with *Italians! The hour has come to engage in your highest destiny* and which is generally considered as the beginning of the Italian Resurgence). In it (completely translated in this volume) there is a continuous reference to the Italian Nation which, in that time, could pursue some serious legal difficulties for the author of such a eulogy, leading eventually to the loss of his personal freedom. The eulogy starts with the words

" It is extremely painful for us to announce in this document the death of a truly great man, who, as during his life, was a glory for Italy, "

and ends with the words

"May these last achievements of such an inventive and ingenious Geometer be delivered up to a capable and educated scholar, who could enlighten them as they deserve, for the advancement of SCIENCES, for the glory of the AUTHOR and for the prestige of ITALY".

In the body of the eulogy one can find the following statements:

- " It seemed as if the Spirit of Italy who was in great sufferance because in that time the most brilliant star of all mathematical sciences, the illustrious Lagrangia, had left the Nation, that Spirit wanted to have the rise of another star, which being born on the banks of the river Arno, was bound to become the successor of the first one. This consideration is presenting itself even more spontaneous by when we will remark that Brunacci was the first admirer in Italy of the luminous Lagrangian doctrines, the scientist who diffused and supported them, the scientist who in his studies was always a very creative innovator in their applications. His first Maestri were two famous Italians, Father Canovai and the great geometer Pietro Paoli "

We remark that here Piola refers to Lagrange by his true and original Italian name, Lagrangia, that he refers to Italy as a unique cultural entity, that he deemed to exist the "Spirit of Italy", that he refers to as Italians two scientists who were Professors in Pisa (outside the Kingdom of Lombardy–Venetia, where Piola lived and worked) .

- "It is not licit for me neglecting to indicate another subject in which -with honored efforts- our professor distinguished himself. The Journal of Physical Chemistry of Pavia was illustrated in many of his pages by his erudite pen; I will content myself to indicate here three Memoirs where he examines the doctrine of capillary attraction of Monsieur Laplace, comparing it with that of Pessuti and where, with his usual frankness which is originated by his being persuaded of how well-founded was his case, he proves with his firm reasonings, **whatever it is said by the French geometers**, some propositions which are of great praise for the mentioned Italian geometer."

For the purposes of this paper, we note that Piola remarked that Brunacci gave in these memoirs of the Journal of Physical Chemistry, the role of the champion of Italian science to the Italian Pesutti as counterposed to the french geometer M. Laplace.

Brunacci greatly influenced Piola's scientific formation and rigorously cultivated his ingenious spirit, as Piola himself recognized in many places of his works. Piola was initiated by Brunacci to Mathematical Analysis but was immediately attracted -since his first original creations- to Mathematical Physics, which he based on the Principle of Virtual Velocities (as Lagrange called what has been later called the Principle of Virtual Work). Actually the aim of the whole scientific activity of Gabrio Piola has been to demonstrate that such a Principle can be considered the basis of the Postulation of every Mechanical Theory, see e.g. the papers [11], [12], [13], [14], [15], [16], [33], [34], [75], [76] for the inclusion of the dissipative effects. Indeed he developed -by using the Lagrangian Postulation- modern continuum mechanics, being -to our knowledge- the first author who introduced the dual in power of the gradient of velocity in the referential description of a continuous body. The coefficients of what will be recognized to be a distribution in the modern sense (as defined by Schwartz) were to be identified later, after the revolutionary theories introduced by Ricci and Levi-Civita, as a double tensor, the **Piola stress tensor**. Some of the results presented in Piola's works (e.g. those concerning continua the strain energy of which depends on higher gradients of the strain measures) can be regarded even nowadays as among the most advanced available in the literature.

## 1.2 Piola's works did not receive their due attention because they are written in Italian

It is clear that the strongest limiting factor to the full recognition of Piola's contribution to Continuum Mechanics must be found in his ideological choice: the use of the Italian language. Moreover he used a very elegant and erudite style which can be understood and appreciated only by a few specialists and he did not care if his works would be translated into other languages (as later was decided by Levi-Civita who -instead- cared to have some of his works translated into English and who wrote directly some others in French (see the works by Ricci-Curbastro and Levi-Civita [64], [65], [66], [104]).

We are persuaded that, in a historical period when all scientists of a given Nation were using their own language in higher studies, when in every University the official spoken language was the National one and where all textbooks, essays and scientific Memoirs were written in the mother language of the authors, Piola could not accept to admit the inferiority of his own mother language and decided to use it for publishing his works. A well-founded conjecture about this linguistic choice can be advanced: although Piola was surely fluent in French (he edited in Italian some works by Cauchy and cites long French excerpts by Poisson) he decided (*per la gloria dell'Italia*) for the glory of Italy to use his mother language, in a historical climate in which the Italian Nation was not yet the united and independent and therefore was not able to self-determine its destiny. This was a patriotic choice which was repaid by a nearly complete neglect of his contribution to mechanical science, exacerbated by the fact that Italian authors seem to have underestimated his contributions (for a detailed analysis of this point see [10]).

From a general point of view, the linguistic barriers often play a very puzzling role in the diffusion of ideas and theories. As discussed in [105], [106] the diffusion of Hellenistic science actually was slowed by the great barrier represented by the ignorance of the language used, but not stopped. The information slowly flowed from East to West, and althought it needed some centuries, and in the end, maybe translated into a Latin difficult to understand, still keeping Greek nomenclature and terminology, this science managed to pollinate the Italian and European Renaissance; however, the linguistic transfer corresponded to a nearly complete loss of the knowledge about the identity of the scientists who had first formulated the ideas at the basis of the scientific revolution. Even the true period of the appearance of the scientific method was finally postponed for more than a millennium. It has not to be considered astonishing, then, that the contribution of Piola while is still permeating the modern Continuum Mechanic literature, is however generally misunderstood, also by those who know better his contributions. Indeed linguistic barriers are very often insurmountable and prejudices, once they are rooted in the mind of scientists, are not easily removed.

## 2 Piola's impact on mechanical sciences

One should not believe that Piola's contribution to the mechanical sciences is completely ignored. Actually his contribution to the formulation of balance equations of force in Lagrangian description is universally recognized (this first re-discovery of Piola will be the object of another investigation). In this context the spirit of Piola's works can be recognized in many modern contributions (see e.g. [100]). One can undoubtedly say that the greatest part of his novel contributions to Mechanics, although having imparted a great momentum to and substantial influence on the work of many prominent mechanicians, is in fact generally ignored. Although the last statement may seem at first sight exaggerated, the aim of the present paper is exactly to prove it while presenting the evidence of a circumstance which may seem surprising: some parts the works of Gabrio Piola represent a topical contribution as late as the year 2013.

Those who have appreciated the works of Russo [105], [106] will not be at all shocked by such a statement, as there is evidence that many scientific contributions remained unsurpassed for centuries, if not millennia. Therefore one thesis that we want to put forward in this paper is that the contribution of Gabrio Piola should not be studied with the attitude of the historian of science but rather with the mathematical rigor needed to understand a contemporary textbook or a research paper. On the other hand, the authors question the concept of *historical method* especially when applied to history of science and history of mathematics. We claim that there is not any peculiar historical method to be distinguished from the generic scientific method which has to be applied to describe any other kind of phenomena, although the subject of the investigation is as complex as those involved in the transmission, storage and advancement of scientific knowledge. A fortiori, however, imagine that one could determine precisely in what constitutes such a historical method: then it MUST include the capability of the historian to understand, master and reconstruct rigorously the mathematical theories which he has decided to study from the historical point of view. In other words: a historian of a particular branch of mathematics has to master completely the theory whose historical development he wants to describe. It is rather impossible, for instance, that somebody who does not know the theory of integration could recognize that (see [106]) Archimedes actually used rigorous arguments leading to the proof of the existence of the integral of a quadratic function. Moreover, together with the linguistic barriers (one has to know doric Greek to understand Archimedes and XIXth century Italian to understand Gabrio Piola), there are also notational difficulties: one should not naively believe that HIS OWN notations are advanced and modern while the notations found in the sources are clumsy and primitive. Actually notations are a matter of arbitrary choice and from this point of view - remember that mathematics is based on axiomatic definition of abstract concepts to which the mathematician assigns a meaning by means of axioms and definitions - all notations are equally acceptable. Very often historians of mathematics [1] decide that a theory is much more modern than it actually is, simply because they do not find the modern symbols or the modern nomenclature in old textbooks. For instance, if one does not find in a textbook the symbol $\Delta$ , this does not mean that the integral was not known to the author of that textbook. It could simply mean that the technology of printing at the age of that textbook required the use of another symbol or of another symbolic method. Indeed some formulas by Lagrange or Piola seem at first sight to the authors of the present paper to resemble, for their complicated length, lines of commands for LaTeX. Actually the historian has to READ carefully the textbooks which he wants to assess and interpret: when these books are books whose content is a mathematical theory, reading them implies reading all the fundamental definitions, lemmas and properties, which are needed to follow its logical development.

In the authors' opinion, in Truesdell and Toupin [140] the contribution of Piola to mechanical sciences is accounted for only partially while in Truesdell [139] it is simply overlooked. It has to be remarked that authors [10] dedicated many efforts to the aim of unveiling the true value of Gabrio Piola as a scientist; however, some deep parts of his scientific results remain not yet sufficiently illustrated. Our aim is

- to prove that non-local and higher gradients continuum mechanics was conceived already in Piola's works starting from a clever use of the Principle of Virtual Work

- to explain the unfortunate circumstances which caused the erasure from memory of this aspect of Piola's contribution, although his pupils respected so greatly his scientific standing that they managed to dedicate an important square to his name in Milan (close to the Politecnico), while a statue celebrating him was erected in the Brera Palace, also in Milan.

Finally some differential relationships obtained in [Piola, 1848] are carefully discussed, as they are still nowadays too often used without proper attribution in the continuum mechanics literature and can be considered the starting point of the Levi-Civita theory of Connection in Riemannian manifolds. The main source for the present paper is the work which is translated nearly word by word in this volume

[1] See for instance Russo (2003) page 53 and ff. for what concerns the difficulties found by historians who did not know trigonometry in recognizing that Hellenist science had formulated it but with different fundamental variables and notations

**Piola, G., Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione, Modena, Tipi del R.D. Camera, (1845-1846)**

but the authors have also consulted other works by Piola [Piola, 1825], [Piola, 1833], [Piola, 1836], [Piola, 1856]. In all the above-cited papers by Piola the kinematical descriptor used is simply the placement field defined on the reference configuration: in these works there is no trace of more generalized models of the type introduced by the Cosserats [21]. However the spirit of Piola's variational formulation (see, e.g., [6], [7], [25], [52], [53], [60], [74], [137], [138]) and his methods for introducing generalized stress tensors can be found in the papers by Green and Rivlin ([54], [55] [56] and [57]) and also many modern works authored for instance by Neff and his co-workers, [87], [90], [94] and of by Forest and his co-workers [49], [50].

# 3 Non-Local Continuum Theories in Piola's works

In the work by Piola [Piola, 1848] the homogenized theory which is deduced by means of the identification of powers in the discrete micro-model and in the continuous macro-model can be called (in the language used by Eringen [46], [48]) a non-local theory. Also some Italian authors (see e.g. [99]), who contributed to the field with important papers, seem not to give explicit recognition that they were reformulating (and extending) the results already found by Piola.

In this volume we translate in English the Piola's works which are most relevant in the present context and in this section we translate into modern symbols the formulas which the reader may find in such a translation in their original form. Moreover, we will recall in a less suggestive, but more direct and modern language the statements made by Piola. It is our opinion that some of Piola's arguments can compete in depth and generality, even nowadays, with those which can be found in some of the most advanced modern presentations. Postponing the analysis of Piola's homogeneization process to a subsequent investigation, we limit ourselves here to describe the continuum model which he deduces from the Principle of Virtual Velocities for a discrete mechanical system constituted by a finite set of molecules, which he considers to be (or, because of his controversy with Poisson, he must accept as) the most fundamental Principle in his Postulation process.

In Piola [Piola, 1848] (Capo I, pag. 8) the reference configuration of the considered deformable body is introduced by labelling each material particle with the three Cartesian coordinates $(a, b, c)$. It is suggestive to remark that the same notation is used in Hellinger [59], see e.g., pag.605. We will denote by the symbol $X$ the position occupied by each of the considered material particles in the reference configuration. The placement of the body is then described by the set of three scalar functions (Capo I, pag.8 and then pages 11-14)

$$x\,(a, b, c)\,, y\,(a, b, c)\,, z\,(a, b, c)$$

which, by using a compact notation, we will denote with the symbol $\chi$ mapping any point in the reference configuration into its position in the actual one.

## 3.1 Piola's non-local internal interactions

In Capo VI, on page 149 Piola [Piola, 1848] introduces:

"the quantity $\rho$ (equations (3),(5), (6)) has the value given by the equation

$$\begin{aligned}\rho^2 &= [x\,(a+f, b+g, c+k) - x\,(a, b, c)]^2 \\ &+ [y\,(a+f, b+g, c+k) - y\,(a, b, c)]^2 \\ &+ [z\,(a+f, b+g, c+k) - z\,(a, b, c)]^2 .\text{"}\end{aligned} \tag{8}$$

So by denoting with the symbol $\bar{X}$ the particle labelled by Piola with the coordinates $(a+f,\ b+g,\ c+k)$ we have, in modern notation, that

$$\rho^2(X, \bar{X}) = \left\|\chi(\bar{X}) - \chi(X)\right\|^2 . \tag{8bis}$$

In Capo VI on page 150 we read the following expression for the internal work, relative to a virtual displacement $\delta\chi$, followed by a very clear remark:

"

$$\Delta\, da\, \Delta\, db\, \Delta\, dc\, \Delta\, df\, \Delta\, dg\, \Delta\, dk \cdot \frac{1}{2} K \delta\rho \tag{10}$$

[...] In it the integration limits for the variables $f, g, k$ will depend on the surfaces which bound the body in the antecedent configuration, and also on the position of the molecule $m$, which is kept constant, that is by the variables $a, b, c$ which after the first three will also vary in the same domain."

Here the scalar quantity $K$ is introduced as the *intensity* of the force (see the page 147 of the translatation of [Piola, 1848] ) exerted by the particle $\bar{X}$ on the particle $X$ and the 1/2 is present as the action reaction principle holds. The quantity $K$ is assumed to depend on $\bar{X}, X$ and $\rho$ and manifestly it is measured in $\left[N\,(m)^{-6}\right]$ (SI Units). In the number 72 starting on page 150 of [Piola, 1848], Piola discusses the physical meaning of this scalar quantity and consequently establishes some restrictions on the constitutive equations which have to be assigned to it. Indeed he refrains from any effort to obtain for it an expression in terms of microscopic quantities and limits himself to require its objectivity by assuming its dependence on $\rho$, an assumption which will have in the sequel some important consequences. Moreover he argues that if one wants to deal with continua more general than fluids (for a discussion of this point one can have a look on the recent paper [5]) then it may depend (in a symmetric way) also on the Lagrangian coordinates of both $\bar{X}$ and $X$ : therefore

$$K(\bar{X}, X, \rho) = K(X, \bar{X}, \rho).$$

On Pages 151 and 152 [Piola, 1848] we then read some statements which cannot be rendered clearer:

"As a consequence of what was were said up to now we can, by adding up the two integrals (1), (10), and by replacing the obtained sum in the first two parts of the general equation (1) num°.16., formulate the equation which includes the whole molecular mechanics. Before doing so we will remark that it is convenient to introduce the following definition

$$\Lambda = \frac{1}{4}\frac{K}{\rho} \tag{11}$$

by means of which it will be possible to introduce the quantity $\Lambda\delta\rho^2$ instead of the quantity $\frac{1}{2}K\delta\rho$ in the sextuple integral (10); and that inside this sextuple integral it is suitable to isolate the part relative to the triple integral relative to the variables $f, g, k$, placing it under the same sign of triple integral with respect to the variables $a, b, c$ which includes the first part of the equation: which is manifestly allowed. In this way the aforementioned general equation becomes

$$\Delta\, da\, \Delta\, db\, \Delta\, dc \cdot \left\{ \left( X - \frac{d^2x}{dt^2} \right) \delta x + \left( Y - \frac{d^2y}{dt^2} \right) \delta y + \left( Z - \frac{d^2z}{dt^2} \right) \delta z \right.$$

$$\left. + \Delta\, df\, \Delta\, dg\, \Delta\, dk \cdot \Lambda\delta\rho^2 \right\} + \Omega = 0 \tag{12}$$

where it is intended that (as mentioned at the beginning of the num°.71.) it is included in the $\Omega$ the whole part which may be introduced because of the forces applied to surfaces, lines or well-determined points and also because of particular conditions which may oblige some points to belong to some given curve or surface. "

Piola is aware of the technical difficulty which he could be obliged to confront in order to calculate the first variation of a square root: as he knows that these difficulties have no physical counterparts, instead of $K$ he introduces another constitutive quantity $\Lambda$ which is the dual in work of the variation $\delta\rho^2$.

**Remark 1.** ***Boundedness and attenuation assumptions on*** $K$ ***and*** $\Lambda$. *Note that Piola explicitly assumes the summability of the function* $\Lambda\delta\rho^2 = \frac{1}{4}\frac{K}{\rho}\delta\rho^2 = \frac{1}{2}K\delta\rho$ *and the boundedness of the function* $K$. *As a consequence when* $\rho$ *is increasing then* $\Lambda$ *decreases.*

**Remark 2.** ***Objectivity of Virtual Work.*** *Note that* $\delta\rho^2$ *and* $\Lambda(X, \bar{X}, \rho)$ *are invariant (see [127]) under any change of observer and as Piola had repeadedly remarked, see e.g. Capo IV, num.48, page 86-87, the expression for virtual work has to verify this condition. Remark also that, as the work is a scalar, in this point Piola's reasoning is rendered difficult by his ignorance of Levi-Civita's tensor calculus. In another formalism the previous formula can be written as follows*

$$\Delta_{\mathcal{B}}[(b_m(X) - a(X))\,\delta\chi(X) + \left(\Delta_{\mathcal{B}}\,\Lambda(X, \bar{X}, \rho)\delta\rho^2\mu(\bar{X})d\bar{X}\right)]\mu(X)dX + \delta W(\partial\mathcal{B}) = 0 \tag{12bis}$$

*where* $\mathcal{B}$ *is the considered body,* $\partial\mathcal{B}$ *its boundary,* $\mu$ *is the volume mass density,* $b_m(X)$ *is the (volumic) mass specific externally applied density of force,* $a(X)$ *the acceleration of material point* $X$, *and* $\delta W(\partial\mathcal{B})$ *the work expended on the virtual displacement by actions on the boundary* $\partial\mathcal{B}$ *and eventually the first variations of the equations expressing the applied constraints on that boundary times the corresponding Lagrange multipliers.*

In Eringen [48], [46], [47], the non-local continuum mechanics is founded on a Postulation based on Principles of balance of mass, linear and angular momenta, energy and entropy. However in [47] a chapter on variational principles is presented. One can easily recognize by comparing, for example, the presentation in [47] with (12bis) that in the works by Piola the functional

$$\left(\Delta_{\mathcal{B}}\,\Lambda(X, \bar{X}, \rho)\delta\rho^2\mu(\bar{X})d\bar{X}\right) \tag{N1}$$

is assumed to satisfy a slightly generalized version of what in [47] pag. 34 is called the

*Smooth Neighborhood Hypothesis*

which reads (in Eringen's work the symbol $V$ is used with the same meaning as our symbol $\mathcal{B}$, $X'$ instead of $\bar{X}$, $x$ instead of $\chi$, $t'$ denotes a time instant, the symbol $()_{,K_i}$ denotes the partial derivatives with respect to $K_i - th$ coordinate of $X$, and is assumed the convention of sums over repeated indices) as follows:

"Suppose that in a region $V_0 \subset V$, appropriate to each material body, the independent variables admit Taylor series expansions in $X' - X$ in $V_0$ [...] terminating with gradients of order $P, Q$, etc.,

$$x(X',t') = x(t') + \left(X'_{K_1} - X_{K_1}\right) x_{,K_1}(t') \\ + ... + \frac{1}{P!}\left(X'_{K_1} - X_{K_1}\right) .... \left(X'_{K_P} - X_{K_P}\right) x_{,K_1...K_P}(t'),$$

and [...]. If the response functionals are sufficiently smooth so that they can be approximated by the functionals in the field of real functions

$$x(t'), x_{,K_1}(t'), ...., x_{,K_1...K_P}(t'), \quad [...] \tag{3.1.6}$$

we say that the material at $X$ [...] satisfies a *smooth neighborhood hypothesis. Materials of this type, for* $P > 1, Q > 1$ are called *nonsimple materials of gradient type."*

Actually Piola is not truncating the series and keeps calculating the integrals on the whole body $\mathcal{B}$. Although no explicit mention can be found in the text of Piola, because of the arguments presented in remark 1, it is clear that he uses a weaker form of the *Attenuating Neighborhood Hypotheses* stated on page 34 of [47]. To be persuaded of this statement the reader will need to proceed to the next section. To conclude this section we need to remark that in very recent times, as a karstic river, the ideas of Piola are back on the stage of Continuum Mechanics.

The idea of an internal interaction which does not fall in the framework of Cauchy continuum mechanics is again attracting the attention of many researchers. Following Piola's original ideas modern peridynamics[2] assumes that the force applied on a material particle of a continuum actually depends on the deformation state of a whole neighbourhood of the particle.

## 3.2 An explicit calculation of the Strong Form of the Variational Principle (12bis).

A more detailed discussion about the eventual novelties contained in the formulation of peridynamics when compared with e.g. Eringen's non-local continuum mechanics is postponed to further investigations. In this section we limit ourselves to compute explicitly the Euler-Lagrange equation corresponding to the Variational Principle (12bis). To this end we need to treat algebraically the expression

$$\Delta_{\mathcal{B}}\left(\Delta_{\mathcal{B}}\Lambda(X,\bar{X},\rho)\delta\rho^2\mu(\bar{X})d\bar{X}\right)\mu(X)dX \tag{N2}$$

by calculating explicitly

$$\delta\rho^2 = \delta\left(\sum_{i=1}^{3}\left(\chi_i(\bar{X}) - \chi_i(X)\right)\left(\chi_i(\bar{X}) - \chi_i(X)\right)\right)$$

With simple calculations we obtain that (Einstein convention is applied from now on)

$$\delta\rho^2 = \left(2\left(\chi^i(\bar{X}) - \chi^i(X)\right)\left(\delta\chi_i(\bar{X}) - \delta\chi_i(X)\right)\right)$$

which once placed in (N2) produces

$$\Delta_{\mathcal{B}}\Delta_{\mathcal{B}}\left(2\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X}) - \chi^i(X)\right)\right)\left(\delta\chi_i(\bar{X}) - \delta\chi_i(X)\right)d\bar{X}dX = \\ = \frac{1}{2}\left(\Delta_{\mathcal{B}} f^i(\bar{X})\delta\chi_i(\bar{X})d\bar{X} + \Delta_{\mathcal{B}} f^i(X)\delta\chi_i(X)dX,\right)$$

where we have introduced the internal interaction force (recall that Piola, and we agree with his considerations as presented in his num.72 on pages 150-151, assumes that $\Lambda(X,\bar{X},\rho) = \Lambda(\bar{X},X,\rho)$) by means of the definition

$$f^i(\bar{X}) := \Delta_{\mathcal{B}}\left(4\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X}) - \chi^i(X)\right)\right)dX$$

[2] We remark that (luckily!) the habit of inventing new names (alhough sometimes the related concepts are not so novel) is not lost in modern science (see [106] for a discussion of the importance of this attitude in science) and that the tradition of using Greek roots for inventing new names is still alive.

By a standard localization argument one easily gets that (12bis) implies

$$a^i(X) = b^i_m(X) + f^i(X) \tag{N3}$$

which (see also Appendices) is exactly the starting point of modern peridynamics.

Many non-local continuum theories were formulated since the first formulation by Piola: we cite here for instance [46], [47], [48], [123]. Remarkable also are the following more modern papers [26], [27], [31], [35], [38], [39], [63], [111], [121], [133], [134], [136]. The non-local interaction described by the integral operators introduced in the present subsections are not to be considered exclusively as interactions of a mechanical nature: indeed recently a model of biologically driven tissue growth has been introduced (see e.g. [3], [72], [73]) where such a non-local operator is conceived to model the biological stimulus to growth.

## 3.3 Piola's higher gradient continua

The state of deformation of a continuum in the neighbourhood of one of its material points can be approximated by means of the Green deformation measure and of all its derivatives with respect to Lagrangian referential coordinates. Piola never considers the particular case of linearized deformation measures (which is physically rather unnatural): his spirit has been recovered in many modern works, among which we cite [122], [128]. Indeed in Capo VI, on page 152, Piola develops in Taylor series $\delta\rho^2$ (also by using his regularity assumptions about the function $\Lambda(X, \bar{X}, \rho)$ and the definition (11)) and replaces the obtained development in (N1).

In a more modern notation (see in this volume the word by word translation) starting from

$$\chi_i(\bar{X}) - \chi_i(X) = \sum_{N=1}^{\infty} \frac{1}{N!}\left(\frac{\partial^N \chi_i(X)}{\partial X_{i_1} \ldots \partial X_{i_N}}(\bar{X}_{i_1} - X_{i_1})\ldots(\bar{X}_{i_N} - X_{i_N})\right)$$

Piola gets an expression for the Taylor expansion with respect to the variable $\bar{X}$ of center $X$ for the function,

$$\rho^2(\bar{X}, X) = \left(\chi^i(\bar{X}) - \chi^i(X)\right)\left(\chi_i(\bar{X}) - \chi_i(X)\right)$$

He estimates and explicitly writes first, second and third derivatives of $\rho^2$ with respect to the variable $\bar{X}$. This is what we will do in the sequel, repeating his algebraic procedure with the only difference consisting in the use of Levi-Civita tensor notation.

We start with the first derivative

$$\frac{1}{2}\frac{\partial\rho^2(\bar{X}, X)}{\partial\bar{X}_\alpha} = \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\alpha} \tag{N4}$$

We remark that when $\bar{X} = X$ this derivative vanishes. Therefore the first tem of Taylor series for $\rho^2$ vanishes. We now proceed by calculating the second and third order derivatives :

$$\frac{1}{2}\frac{\partial^2\rho^2(\bar{X}, X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta} = \frac{\partial\chi^i(\bar{X})}{\partial\bar{X}_\beta}\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\alpha} + \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial^2\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta} =$$
$$=: C_{\alpha\beta}(\bar{X}) + \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial^2\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta};$$

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X}, X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} = \frac{\partial C_{\alpha\beta}(\bar{X})}{\partial\bar{X}_\gamma} + \frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\gamma}\frac{\partial^2\chi^i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta} + \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} \tag{N5}$$

The quantities of this last equation are exactly those described in [Piola, 1848] on page 157 concerning the quantities appearing in formulas (14) on page 153.

We now introduce the result (formula (N12)) found in Appendices (in order to remain closer to Piola's presentation we refrain here from using Levi-Civita alternating symbol)

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right)$$

so that by replacing in (N5) we get

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X}, X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} + \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right) + \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} \tag{N6}$$

so that when $\bar{X} = X$ we get that the third order derivatives of $\rho^2$ can be expressed in terms of the first derivatives of $C_{\gamma\beta}$. Now we go back to read in Capo VI sect. 73 page 152-153:

“ 73. What remains to be done in order to deduce useful consequences from the equation (12) is simply a calculation process. Once recalled the equation (8), it is seen, transforming into series the functions in the brackets, so that one has

$$\begin{aligned}\rho^2 = &\left(f\frac{dx}{da} + g\frac{dx}{db} + k\frac{dx}{dc} + \frac{f^{\ 2}}{2}\frac{d^2x}{da^2} + ec.\right)^2\\ + &\left(f\frac{dy}{da} + g\frac{dy}{db} + k\frac{dy}{dc} + \frac{f^{\ 2}}{2}\frac{d^2y}{da^2} + ec.\right)^2\\ + &\left(f\frac{dz}{da} + g\frac{dz}{db} + k\frac{dz}{dc} + \frac{f^{\ 2}}{2}\frac{d^2z}{da^2} + ec.\right)^2;\end{aligned}$$

and by calculating the squares and gathering the terms which have equal coefficients:

$$\begin{aligned}\rho^2 &= f^{\ 2}t_1 + g^{\ 2}t_2 + k^{\ 2}t_3 + 2fgt_4 + 2fkt_5 + 2gkt_6\\ &+ f^{\ 3}T_1 + 2f^{\ 2}gT_2 + 2f^{\ 2}kT_3 + f\ g^2T_4 + ec.\end{aligned} \tag{13}$$

in which expression the quantities $t_1, t_2, t_3, t_4, t_5, t_6$ represent the six trimonials which are alreay familiar to us, as we have adopted such denominations since the equations (6) in the num°.34.; and the quantities $T_1, T_2, T_3, T_4, ec.$ where the index goes to infinity, represent trinomials of the same nature in which derivatives of higher and higher order appear. ”

Then the presentation of Piola continues with the study of the algebraic structure of the trinomial constituting the quantities $T_1, T_2, T_3$, as shown by the formulas appearing in Capo VI, n.73 on pages 153-160. The reader will painfully recognize that these huge component-wise formulas actually have the same structure which becomes easily evident in formula N6 and in all formulas deduced, with Levi-Civita Tensor Calculus, in the Appendices .

What Piola manages to recognize (also with a courageous conjecture, see Appendices) is that in the expression of Virtual Work all the quantities which undergo infinitesimal variation (which are naturally to be chosen as measures of deformation) are indeed either components of the deformation measure $C$ or components of one of its gradients.

Indeed in the sect. 74 page 156 one reads:

“74. A new proposition, to which the reader should pay much attention, is that all the trinomials $T_1, T_2, T_3$,etc. where the index goes to infinity, which appear in the previous equation (17), can be expressed by means of the only first six $t_1, t_2, t_3,\ t_4, t_5, t_6$, and of their derivatives with respect to the variables $a, b, c$ of all orders. I started to suspect this analytical truth because of the necessary correspondence which must hold between the results which are obtained with the way considered in this Capo and those results obtained with the way considered in the Capo III and IV. ”

This statement is true and its importance is perfectly clear to Piola: for a discussion of the mathematical rigor of his proof the reader is referred to the discussion in one of the Appendices. In order to transform the integral expression (N1)

$$(\Delta_{\mathcal{B}}\,\Lambda(X, \bar{X}, \rho)\delta\rho^2(X, \bar{X})\mu(\bar{X})d\bar{X})$$

Piola remarks that (pages 155-156):

“When using the equation (13) to deduce the value of the variation $\delta\rho^2$ , it is clear that the characteristic $\delta$ will need to be applied only to the trinomials we have discussed up to now, so that we will have:

$$\begin{aligned}\delta\rho^2 &= f^{\ 2}\delta t_1 + g^{\ 2}\delta t_2 + k^{\ 2}\delta t_3 + 2fg\ \delta t_4 + 2fk\ \delta t_5 + 2gk\ \delta t_6\\ &+ f^{\ 3}\delta T_1 + 2f^{\ 2}g\delta T_2 + 2f^{\ 2}k\delta T_3 + f\ g^2\delta T_4 + ec.\end{aligned} \tag{16}$$

Indeed the coefficients $f^{\ 2}, g^{\ 2}, k^{\ 2}, 2fg$, etc. are always of the same form as the functions giving the variables $x, y, z$ in terms of the variables $a, b, c$, and therefore cannot be affected by that operation whose aim is simply to change the form of these functions. Vice versa, by multiplying the previous equation (16) times $\Lambda$ and then integrating with respect to the variables $f, g, k$ in order to deduce from such calculation the value to be given to the forth term under the triple integral of the equation (12), such an operation is affecting only the quantities $\Lambda f^{\ 2}, \Lambda g^{\ 2}$, etc. and the variations $\delta t_1, \delta t_2, \delta t_3 .... \delta T_1, \delta T_2, ec.$ cannot be affected by it, as the trinomials $t_1, t_2, t_3 .... T_1, T_2, ec.$ (one has to consider carefully which is their origin) do not contain the variables $f, g, k$ : therefore such variations result to be constant factors, times which are to be multiplied the integrals to be calculated in the subsequent terms of the series. ”

Using a modern notation we have that

$$\rho^2(\bar{X}, X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left.\frac{\partial^N \rho^2(\bar{X}, X)}{\partial \bar{X}_{i_1} .... \partial \bar{X}_{i_N}}\right|_{X=\bar{X}} (\bar{X}_{i_1} - X_{i_1}) .... (\bar{X}_{i_N} - X_{i_N})$$

and therefore that

$$\delta\rho^2(\bar{X},X)=\sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta\left.\frac{\partial^N\rho^2(\bar{X},X)}{\partial\bar{X}_{i_1}....\partial\bar{X}_{i_N}}\right|_{X=\bar{X}}\right)(\bar{X}_{i_1}-X_{i_1})....(\bar{X}_{i_N}-X_{i_N}).$$

As a consequence

$$\Delta_{\mathcal{B}}\,\Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})d\bar{X}=$$

$$=\sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta\left.\frac{\partial^N\rho^2(\bar{X},X)}{\partial\bar{X}^{i_1}....\partial\bar{X}^{i_N}}\right|_{X=\bar{X}}\right)\left(\Delta_{\mathcal{B}}\,\Lambda(X,\bar{X},\rho)\left((\bar{X}^{i_1}-X^{i_1})....(\bar{X}^{i_N}-X^{i_N})\right)\mu(\bar{X})d\bar{X}\right)$$

If we introduce the tensors

$$T_{\cdot}^{i_1...i_N}(X):=\left(\Delta_{\mathcal{B}}\,\Lambda(X,\bar{X},\rho)\left((\bar{X}^{i_1}-X^{i_1})....(\bar{X}^{i_N}-X^{i_N})\right)\mu(\bar{X})d\bar{X}\right)$$

we get, also by recalling formula (N18) from appendices,

$$\Delta_{\mathcal{B}}\,\Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})d\bar{X}=\sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta L_{i_1....i_n}\left(C(X),..,\nabla^{n-2}C(X)\right)\right)T_{\cdot}^{i_1...i_N}(X)$$

Piola then states that

"After these considerations it is manifest the truth of the equation:

$$\Delta\,df\,\Delta\,dg\,\Delta\,dk\cdot\Lambda\delta\rho^2= \tag{17}$$

$$\begin{aligned}&(1)\ \ \delta t_1+(2)\ \ \delta t_2+(3)\ \ \delta t_3+(4)\ \ \delta t_4+(5)\ \ \delta t_5+(6)\ \ \delta t_6\\&+(7)\ \ \delta T_1+(8)\ \ \delta T_2+(9)\ \ \delta T_3+(10)\ \ \delta T_4+ec.\end{aligned}$$

where the coefficients (1), (2), etc. indicated by means of numbers in between brackets, must be regarded to be each a function of the variables $a,b,c$ as obtained after having performed the said definite integrals. "

In order to establish the correct identification between Piola's notation and the more modern notation which we have introduced the reader may simply consider the following table ($i=1,2,....n,....$)

$$T_{\cdot}^{i_1...i_N}\leftrightarrows(1),(2),ec.\quad\delta L_{i_1....i_n}\left(C,..,\nabla^{n-2}C\right)\leftrightarrows\delta T_i\ \ .$$

After having accepted Piola's assumptions the identity (12bis) becomes

$$\Delta_{\mathcal{B}}\left(\left(b_m(X)-a(X)\right)\delta\chi(X)+\sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta L_{i_1....i_n}\left(C(X),..,\nabla^{n-2}C(X)\right)\right)T_{\cdot}^{i_1...i_N}(X)\right)\mu(X)dX$$
$$+\,\delta W(\partial\mathcal{B})=0$$

By a simple re-arrangement and by introducing a suitable notation the last formula becomes

$$\Delta_{\mathcal{B}}\left(\left(b_m(X)-a(X)\right)\delta\chi(X)+\sum_{N=1}^{\infty}\left\langle\nabla^N\delta C(X)|S_{\cdot}(X)\right\rangle\right)\mu(X)dX+\delta W(\partial\mathcal{B})=0 \tag{12tris}$$

where $S$ is a $N-th$ order contravariant totally symmetric tensor[3] and the symbol $\langle|\rangle$ denotes the total saturation (inner product) of a pair of totally symmetric contravariant and covariant tensors.

Indeed on pages 159-160 of [Piola, 1848] we read:

"75. Once the proposition of the previous num. has been admitted, it is manifest that the equation (17) can assume the following other form

$$\Delta\,df\,\Delta\,dg\,\Delta\,dk\cdot\Lambda\delta\rho^2= \tag{18}$$

[3]The constitutive equations for such tensors must verify the condition of frame invariance. When these tensors are defined in terms of a deformation energy (that is when the Principle of Virtual Work is obtained as the first variation of a Least Action Principle) the objectivity becomes a restriction on such an energy. The generalization of the results in Steigmann (2003) to the N-the gradient continua still needs to be found.

$$(\alpha)\ \delta t_1 + (\beta)\ \delta t_2 + (\gamma)\ \delta t_3 + .... + (\epsilon)\ \frac{\delta dt_1}{da} + (\zeta)\ \frac{\delta dt_1}{db} + (\eta)\ \frac{\delta dt_1}{dc}$$
$$+ (\vartheta)\ \frac{\delta dt_2}{da} + .... + (\lambda)\ \frac{\delta d^2 t_1}{da^2} + (\mu)\ \frac{\delta d^2 t_1}{dadb} + .... + (\xi)\ \frac{\delta d^2 t_2}{da^2} + (o)\ \frac{\delta d^2 t_2}{dadb} + ec.$$

in which the coefficients $(\alpha), (\beta)\ ....\ (\epsilon)\ ....\ (\lambda)\ ....ec.$ are suitable quantities given in terms of the coefficients $(1), (2)\ ....\ (7), (8)\ ....$ of the equation (17): they depend on the quantities $t_1, t_2....t_6$, and on all order derivatives of these trinomials with respect to the variables $a, b, c$ . Then the variations $\delta t_1$, $\delta t_2$....(with the index varying up to infinity) and the variations of all their derivatives of all orders $\frac{\delta dt_1}{da}, \frac{\delta dt_1}{db}, ec.$ appear in the (18) only linearly ".

# 4 Weak and Strong Evolution Equations for Piola Continua

To our knowledge a formulation of the Principle of Virtual Work for $N-th$ gradient Piola Continua equivalent to (12tris) is found in the literature only in [32], but the authors were unaware of the previous work of Piola. The reader is referred to the aforementioned paper for the detailed presentation of the needed Postulation process and the subsequent procedure of integration by parts needed for transforming the weak formulation of evolution equations given by (12tris) into a strong formulation in which suitable bulk equations and the corresponding boundary conditions are considered.

We shortly comment here about the relative role of Weak and Strong formulations, framing it in a historical perspective. Since at least the pioneering works by Lagrange the Postulation process for Mechanical Theories was based on the Least Action Principle or on the Principle of Virtual Work. One can call Variational both these Principles as the Stationarity Condition for Least Action requires that for all admissible variations of motion the first variation of Action must vanish, statement which, as already recognized by Lagrange him-self, implies a form of the Principle of Virtual Work. However in order to compute the motions relative to given initial data the initiators of Physical Theories needed to integrate by parts the Stationarity Condition which they had to handle. In this way they derived some PDEs with some boundary conditions which sometimes were solved by using analytical or semi-analytical methods. From the mathematical point of view this procedure is applicable when the searched solution have a stronger regularity than the one strictly needed to formulate the basic variational principle. It is a rather ironic circumstance that nowadays very often those mathematicians who want to prove well-posedness theorems for PDEs (which originally were obtained by means of an integration by part procedure) start their reasonings by applying in the reverse direction the same integration by parts process: indeed very often the originating variational principle of all PDEs is forgotten. Some examples of mathematical results which exploit in an efficient way the power of variational methods are those presented for instance in Neff [89], [91], [92].

Actually, even if one refuses to accept the idea of basing all physical theories on variational principles, he is indeed obliged, in order to find the correct mathematical frame for his models, to prove the validity of a weak form appliable to his painfully formulated balance laws. In reality (see [29]) his model will not be acceptable until he has been able to reformulate it in a weak form. It seems that the process which occurred in mathematical geography, described in Russo [105]-[106], occurs very frequently in science. While the reader is referred to the cited works for all details, we recall here the crucial point of Russo's argument, as needed for our considerations. Ptolemy presented in his Almagest a useful tool for astronomical calculations: actually his *Handy Tables* tabulate all the data needed to compute the positions of the Sun, Moon and planets, the rising and setting of the stars, and eclipses of the Sun and Moon. The main calculation tools in Ptolemy's treatise are the deferents and epicycles, which were introduced by Apollonius of Perga and Hipparchus of Rhodes in the framework of astronomic theories much more advanced than the one formulated by Ptolemy (if Russo's conjecture is true). Unfortunately Ptolomy misunderstood the most ancient (and much deeper) theories and badly re-organized the knowledges, observations and theories presented in the treatise by Hipparchus (treatise which has been lost): indeed Ptolemy being a practical scientist gives a too high importance to the calculation tools by blurring in a list of logical incongruences the rigorous and deep (and eliocentric!) theories formulated by Hipparchus four centuries before him. Actually in Ptolemy's vision the calculation tools become the fundamental ingredients of the mathematical model which he presents. This seems to have occurred also in Continuum Mechanics: the Euler-Lagrange equations, obtained by means of a process of integration by parts, were originally written, starting from a variational principle, to supply a calculation tool to applied scientists. They soon became (for simplifying) the bulk of the theories and often the originating variational principles were forgotten (or despised as too mathematical). For a period balance equations were (with some difficulties which are discussed e.g. in [29]) postulated on physical grounds.

When the need of proving rigorous existence and uniqueness theorems met the need of developing suitable numerical methods, and when the many failures of the finite difference schemes became evident, the variational principles were re-discovered *starting from the balance equations.* The variational principles represent at first the starting point of mechanical theories and were used to get, by means of algebraic manipulation, some tools for performing practical calculations: i.e., the associated Euler-Lagrange equations or (using another name) balance equations. However, with a strange exchange of roles, if their basic role is forgotten and balance equations are regarded as the basic principles from which one has to start the formulation of the theories, then variational principles need to be recovered as a computational tool.

One question needs to be answered: why in the modern paper [32] a strong formulation was searched of the evolution equation for $N-th$ gradient continua? The answer is: beacause of the need of finding for those theories the most suitable boundary conditions ! This point is discussed also in [Piola, 1848] as remarked already in [5]. Piola[Piola, 1848], on pages 160-161, claims that:

"Now it is a fundamental principle of the calculus of variations (and we used it also in this Memoir in the num.° 36 and elsewhere) that series as the previous one, where the variations of some quantities and the variations of their derivatives with respect to the fundamental variables $a, b, c$ appear linearly can be always be transformed into one expression which contains the first quantities without any sign of derivation, with the addition of other terms which are exact derivatives with respect to one of the three simple independent variables. As a consequence of such a principle, the expression which follows to the equation (18) can be given by

$$\Delta\, df\, \Delta\, dg\, \Delta\, dk \cdot \Lambda \delta \rho^2 = \tag{19}$$

$$A\delta t_1 + B\ \delta t_2 + C\ \delta t_3 + D\delta t_4 + E\delta t_5 + F\ \delta t_6$$
$$+ \frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc}.$$

The values of the six coefficients $A, B, C, D, E, F$ are series constructed with the coefficients $(\alpha), (\beta), (\gamma) \ldots. (\epsilon), (\zeta) \ldots. (\lambda), ec.$ of the equation (18) which appear linearly, with alternating signs and affected by derivations of higher and higher order when we move ahead in the terms of said series: the quantities $\Delta, \Theta, \Upsilon$ are series of the same form of the terms which are transformed, in which the coefficients of the variations have a composition similar to the one which we have described for the six coefficients $A, B, C, D, E, F$. Once -instead of the quantity under the integral sign in the left hand side of the equation (12)- one introduces the quantity which is on the right hand side of the equation (19), it is clear to everybody that an integration is possible for each of the last three addends appearing in it and that as a consequence these terms only give quantities which supply boundary conditions. What remains under the triple integral is the only sestinomial which is absolutely similar to the sestinomial already used in the equation (10) num.° 35. for rigid systems. Therefore after having remarked the aforementioned similarity the analytical procedure to be used here will result perfectly equal to the one used in the num.° 35, procedure which led to the equations (26), (29) in the num.° 38 and it will become possible the demonstration of the extension of the said equations to every kind of bodies which do not respect the constraint of rigidity, as it was mentioned at the end of the num.° 38. It will also be visible the coincidence of the obtained results with those which are expressed in the equations (23) of the num.°50. which hold for every kind of systems and which were shown in the Capo IV by means of those intermediate coordinates $p, q, r$, whose consideration, when using the approach used in this Capo, will not be needed."

The novel content in [32] consists in the determination of

- the exact structure of the tensorial quantity whose components are called $A, B, C, D, E, F$ by [Piola, 1848]
- the exact structure of the boundary conditions resulting when applying Gauss' theorem to the divergence field called by [Piola, 1848]

$$\frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc}$$

  on a suitable class of contact surfaces.

The considerations sketched about the history of celestial mechanics should persuade the reader that it is not too unlikely that some ideas by Piola needed 167 years for being further developed (even if the fact that the authors did not manage to find any intermediate reference does not mean that such a reference does not exist, maybe in a language even less understandable than Italian).

Earlier papers (nowadays considered fundamental) by Mindlin [79], [80], [81], [107], [108] had developed a more complete study of Piola Continua, at least up to those whose deformation energy depends on the Third Gradient, completely characterizing the nature of contact actions in these cases, or for continua having a kinematics richer than that considered by Piola, including microdeformations and micro-rotations. Many important applications can be conceived for higher gradient materials, as for instance those involving the phenomena described for instance in [1], [27], [30], [51], [71], [72], [103], [112], [113], [114], [115], [116], [117], [118], [119], [141], [142], [143], [144].

# 5 One- and Two-dimensional Continua and Micro-Macro identification procedure as introduced by Piola (1845-6)

On page 19 Piola justifies the introduction of one-dimensional or two-dimensional bodies as follows:

"11. Sometimes mathematicians are used to consider the matter configured not in a volume with three dimensions but [configured] in a line or in a surface: in these cases we have the so called linear or surface systems. Indeed [these systems] are nothing else than abstractions and it is just for this reason that the Geometer should pay the major attention to three dimensional systems. Nevertheless, it is useful to consider [these systems] because the several analyses for the three kind of systems provide feedbacks that make clear [such analyses], and moreover [such analyses] are useful for physical applications, eventhough always in an approximate way, because the bodies, rigorously speaking, being never deprived in Nature of one or two dimensions. Although for both linear and surface systems we need special considerations in order to represent the distribution of the molecules, and [in order] to form the idea of the density and the measure of the mass, yet [the idea and the measure] are at all similar to the above referred for three dimensional systems: thus, I will expound them shortly. "

On page 39 num. 24 and on page 46 num. 29 of [Piola, 1848] is studied the structure of the Principle of Virtual Work in the case in which one or two dimensions of the considered body can be neglected in the description of its motion. Piola uses these parts to prepare the reader for the micro-macro identification process for three-dimensional bodies which he will study later in full detail. This identification process

- starts from a discrete system of material particles which are placed in a reference configuration at the nodes of a suitably introduced mesh,
- proceeds with the introduction of a suitable placement field $\chi$ having all the needed regularity properties
- assumes that the values of $\chi$ at the aforementioned nodes can be considered an approximation of the displacements of the discrete system of material particles
- and is based on the identification of Virtual Work expressions in the discrete and continuous models.

While the detailed description of aforementioned identification (see [1], [4], [58]) process is postponed to further studies, we want here to remark that non-local and higher gradient theories for beams and shells are already implicitly formulated in [Piola, 1848], although the main subject there is the study of three-dimensional bodies.

The authors have found interesting connections in this context with many of the subsequent works and the most suggestive are those concerning the theory of shells and plates; namely, [42], [43], [44], [45], [70], [97], where interesting phenomena involving phase transition are considered, or the papers by Neff [87], [88], [89], [90], [92], [93]. Moreover the methods started by Piola are used also when describing bidimensional surfaces carrying material properties as for instance in [62], [77], [78], [98], [124], [125], [131], [132].

Also interesting analogies for what concerns the connections between discrete and continuous models can be found with papers dealing with one-dimensional continua and their stability as for instance [67] and [68], where are studied the dynamics of beams or chains of beams, [69], where the non-linear equations for inextensible cables deduced by Piola are applied to very interesting special motions, [126] where the case of prestressed networks is considered, [129]and [130], where the spirit of Piola's contribution is adapted to the context of spatial rods and the nonlinear theory for spatial lattices. Concerning the micro-macro identification procedure in the recent literature one can find many continuators of Piola's works. Notable are the works [8], [9], [17] in which Piola continua are obtained by means of homogenization procedures starting from lattice beam microstructures. It is possible to cite also some studies which consider visco-elastic continuum theories with damage (see [18], [19], [20], [23], [24]) for microscopically granular or discrete systems as for instance [82], [83], [84], [85], [101], [102] or other studies of phenomena involving multiscale coupling (see e.g. [86]).

# 6 A Conclusion: Piola as precursor of the Italian School of Differential Geometry

The most important contribution of Gabrio Piola to mechanical sciences is the universally recognized Piola transformation, which allows for the transformation of some equations in a conservative form from Lagrangian to Eulerian description. The differential geometric content of this contribution does not need to be discussed, as it has been treated in many works and textbooks: we simply refer to [40] and to the references there cited for a detailed discussion of this point and more considerations about the relationship between continuum mechanics and differential geometry (see also [41]).

In the present paper we have shown that there are other major contributions to mechanics by Gabrio Piola which have been underestimated: we also have tried a first analysis of the reasons for which this circumstance occurred. In this concluding section we want to remark that also those results by Piola which we have described in the present paper have a strong connection with differential geometry (in this context see also [109], [110]). The readers is referred to the discussion about historical method which was developped in the Introduction: knowledge of the basic ideas of differential geometry is required to follow the considerations which we present here. The criticism usually given to the kind of reconstructions which we want to present is usually based on the following statement: the historian wanted to read something which could not be written in such an early stage of knowledge.

We dismiss a priori this criticism on the basis of the following statements

- The inaugural lecture by Riemann dates to 1854 therefore Piola's results are surely antecedent but very close in time.
- Riemann is considered one of the founders of Riemannian geometry even if he did not write any formula using the indicial notation developed by Ricci and Levi-Civita
- Riemannian tensor is named after Riemann even if there is no formal definition of the concept of tensor in Riemann's works.

In his inaugural lecture Riemann discusses one of his main contributions to geometry: i.e. the condition for which a Riemannian manifold is flat. This study (indirectly influenced by Gauss) started a flow of investigations in which the Italian School has played a dominant role. We recall here e.g. Ricci's Lemma and Identities, the concept of Levi-Civita parallel transport and the Levi-Civita Theorem about parallel transports compatible with a Riemannian structure. Also referring to the last Appendix for substantiating our statement we claim that it was indeed Continuum Mechanics which originated Differential Geometry and that the Italian School in differential geometry may have been originated in the works of Piola. Indeed in the Appendices we have proven that Piola has obtained (component-wise, exactly in the same form in which Riemann obtained all his results) the equation (N14),

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right).$$

This equation is equivalent (see [135] vol.II page 184 ) to the Riemannian condition of flatness.

# 7 Appendices

## 7.1 Peridynamics: A new/old model for deformable bodies

The celebrated and fundamental textbook by Lagrange [61] is, with few and biased exceptions, generally regarded as a milestone in Mechanical Sciences and unanimously as novel in its content and style of presentation. Indeed Lagrange himself, differently from what was done by his epigones, puts his work in the correct perspective, by giving the due credit to all his predecessors. Indeed the Mécanique Analytique starts with an interesting historical introduction, which can be considered the initiation of the modern history of mechanics. Unfortunately also this aspect of the Lagrangian lesson is not very often followed in modern science.

A very new Continuum Mechanical Theory has been recently announced and developed: Peridynamics. Actually the ideas underlying Peridynamics are very interesting and most likely they deserve the full attention of experts in continuum, fracture and damage mechanics. Indeed starting from a balance law of the form (N3) for instance in [36], [37] and [120] (but many other similar treatments are available in the literature) one finds a formulation of Continuum Mechanics which relaxes the standard one transmitted by the apologists of the Cauchy format and seems suitable (see the few comments below) to describe many and interesting phenomena e.g. in crack formation and growth.

However even those scientists whose mother language is Italian actually seem unaware of the contribution due to Gabrio Piola in this field: this loss of memory and this lack of credit to the major sources of our knowledge, even in those cases in which their value is still topical, is very dangerous, as proven in detail by the analysis developed in Russo [105], [106]. Unfortunately this tendency towards a mindless modernism seems to become more and more aggravated.

In [120] the analysis started by Piola is continued, seemingly as if the author, Silling, were one of his closer pupils: arguments are very similar and also a variational formulation of the presented theories is found and discussed. In [63] and in [122] it is stated in the abstracts that:

"The peridynamic model is a framework for continuum mechanics based on the idea that pairs of particles exert forces on each other across a finite distance. The equation of motion in the peridynamic model is an integro- differential equation. In this paper, a notion of a peridynamic stress tensor derived from nonlocal interactions is defined."

"The peridynamic model of solid mechanics is a nonlocal theory containing a length scale. It is based on direct interactions between points in a continuum separated from each other by a finite distance. The maximum interaction distance provides a length scale for the material model. This paper addresses the question of whether the peridynamic model for an elastic material reproduces the classical local model as this length scale goes to zero. We show that if the motion, constitutive model, and any nonhomogeneities are sufficiently smooth, then the peridynamic stress tensor converges in this limit to a Piola-Kirchhoff stress tensor that is a function only of the local deformation gradient tensor, as in the classical theory. This limiting Piola-Kirchhoff stress tensor field is differentiable, and its divergence represents the force density due to internal forces."

The reader is invited to compare these statements with those which can be found in the translated Piola's works.

It is very interesting to see how fruitful can be the ideas formulated 167 years ago by Piola. It is enough to read the abstract of [2]

"The paper presents an overview of peridynamics, a continuum theory that employs a nonlocal model of force interaction. Specifically, the stress/strain relationship of classical elasticity is replaced by an integral operator that sums internal forces separated by a finite distance. This integral operator is not a function of the deformation gradient, allowing for a more general notion of deformation than in classical elasticity that is well aligned with the kinematic assumptions of molecular dynamics. Peridynamics' effectiveness has been demonstrated in several applications, including fracture and failure of composites, nanofiber networks, and polycrystal fracture. These suggest that peridynamics is a viable multiscale material model for length scales ranging from molecular dynamics to those of classical elasticity."

Or also the abstract of the paper by Parks et al. [95].

"Peridynamics (PD) is a continuum theory that employs a nonlocal model to describe material properties. In this context, nonlocal means that continuum points separated by a finite distance may exert force upon each other. A meshless method results when PD is discretized with material behavior approximated as a collection of interacting particles. This paper describes how PD can be implemented within a molecular dynamics (MD) framework, and provides details of an efficient implementation. This adds a computational mechanics capability to an MD code enabling simulations at mesoscopic or even macroscopic length and time scales "

It is remarkable how strictly related are non-local continuum theories with the discrete theories of particles bound to the nodes of a lattice. How deep was the insight of Piola can be understood by looking at the literature about the subject which includes for instance [2], [35], [36], [37], [38], [39], [63], [111], [120], [121], [122].

## 7.2 On an expression for $\nabla F$ deduced in Piola (1845-6) on pages 158-159

In this appendix we deduce, by means of the Levi-Civita tensor calculus, the expression for the second gradient of placement that is needed to transform eqn. (N4) into eqn. (N5) and that is obtained by [Piola, 1848]. The original calculations are rather lengthy and cumbersome: it is however the opinion of the authors that Piola had caught their tensorial or at least their algebraic structure. Indeed the notation he used made rather easy the identification of the tensorial objects involved. We start from the following identification between modern and Piola's notation

$$F^i_\alpha \leftrightarrows \begin{pmatrix} \frac{dx}{da} & \frac{dy}{da} & \frac{dz}{da} \\ \frac{dx}{db} & \frac{dy}{db} & \frac{dz}{db} \\ \frac{dx}{dc} & \frac{dy}{dc} & \frac{dz}{dc} \end{pmatrix} \quad \det F \leftrightarrows H \quad (\det F)\left(F^{-1}\right)^\beta{}_j \leftrightarrows \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \tag{N7}$$

so that we can state that equation (28) on page 26 of [Piola, 1848] is equivalent to the following one

$$(\det F)\, F^{-1} F = (\det F)\, I$$

where $I$ is the identity matrix. Moreover the equation (6) on page 57 is equivalent to the following one

$$\begin{pmatrix} t_1 & t_4 & t_5 \\ t_4 & t_2 & t_6 \\ t_5 & t_6 & t_3 \end{pmatrix} \leftrightarrows C = F^T F \quad C_{\alpha\beta} = F^i_\alpha F_{i\beta} \tag{N8}$$

The equation on pages 158-159 in [Piola, 1848] is written in tensorial form as follows:

$$\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} := D_{\alpha\beta\eta}\left(F^{-1}\right)^{i\eta}, \quad F_{i\eta}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} = \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\eta} = D_{\alpha\beta\eta} \tag{N9}$$

Now by recalling that

$$\frac{\partial F_{i\beta}}{\partial X^\gamma} = \frac{\partial F_{i\gamma}}{\partial X^\beta}$$

we have the symmetry of $D$ with respect to the first two indeces,

$$D_{\alpha\beta\eta} = D_{\beta\alpha\eta} \tag{N10}$$

and, because of such expression, we can perform the following simple calculations (usual symmetrization, $A_{(ab)} = A_{ab} + A_{ba}$, and skew-symmetrization, $A_{[ab]} = A_{ab} - A_{ba}$, conventional symbols are used)

$$\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} = \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma}$$

$$2\frac{\partial C_{\alpha[\beta}}{\partial X^{\gamma]}} = \frac{\partial C_{\alpha\beta}}{\partial X^{\gamma}} - \frac{\partial C_{\alpha\gamma}}{\partial X^{\beta}} =$$

$$= \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma} - \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\gamma} - F^i_\alpha \frac{\partial F_{i\gamma}}{\partial X^\beta} =$$

$$= \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} - \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\gamma} = D_{\alpha\gamma\beta} - D_{\alpha\beta\gamma} = 2D_{\alpha[\gamma\beta]}$$

$$\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} = \frac{\partial \left(F^i_\alpha F_{i\beta}\right)}{\partial X^\gamma} = \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma} =$$

$$= D_{\gamma\alpha\beta} + D_{\gamma\beta\alpha} = 2D_{\gamma(\alpha\beta)}$$

By decomposing $D$ into its skew and symmetric parts (with respect to the second and third index, see also (N10)) one gets

$$D_{\gamma\alpha\beta} = D_{\gamma(\alpha\beta)} + D_{\gamma[\alpha\beta]} = \frac{1}{2}\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma[\beta}}{\partial X^{\alpha]}} = \frac{1}{2}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right) \tag{N11}$$

The third order tensor $D_{\gamma\alpha\beta}$ which we have introduced allows us to reproduce in the compact form (N9) the formula which occupies nearly two pages of Piola's work. Moreover we have obtained the formula N11 with an easy calculation process which is much less involved than the one first conceived by Piola. From (N11) we have

$$F_{i\beta}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\gamma} = \frac{1}{2}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right) \tag{N12}$$

which is equivalent to

$$\frac{\partial^2 \chi^j}{\partial X^\alpha \partial X^\gamma} = \frac{1}{2}\left(F^{-1}\right)^{j\beta}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right) \tag{N13}$$

To compare the two formalisms let us state the identification of the left-hand side of one line, i.e. of the 11th one divided by $2H$ of the formula appearing on page 158 in [Piola, 1848], i.e.,

$$\frac{\partial^2 \chi^2}{\partial X^1 \partial X^2} \leftrightarrows \frac{d^2 y}{dadb}.$$

Thus, from (N11) with $\alpha = 1$, $j = \gamma = 2$, by recalling the symmetry of the tensor $C$ and the identifications (N7) and (N8),

$$\frac{\partial^2 \chi^2}{\partial X^1 \partial X^2} = \frac{1}{2}\left(F^{-1}\right)^{2\beta}\left(\frac{\partial C_{1\beta}}{\partial X^2} + \frac{\partial C_{2\beta}}{\partial X^1} - \frac{\partial C_{21}}{\partial X^\beta}\right) =$$

$$= \frac{1}{2}\left(F^{-1}\right)^{21}\left(\frac{\partial C_{11}}{\partial X^2} + \frac{\partial C_{21}}{\partial X^1} - \frac{\partial C_{21}}{\partial X^1}\right) +$$

$$+ \frac{1}{2}\left(F^{-1}\right)^{22}\left(\frac{\partial C_{12}}{\partial X^2} + \frac{\partial C_{22}}{\partial X^1} - \frac{\partial C_{21}}{\partial X^2}\right) +$$

$$+ \frac{1}{2}\left(F^{-1}\right)^{23}\left(\frac{\partial C_{13}}{\partial X^2} + \frac{\partial C_{23}}{\partial X^1} - \frac{\partial C_{21}}{\partial X^3}\right) =$$

$$= \frac{1}{2}\left(F^{-1}\right)^{21}\frac{\partial C_{11}}{\partial X^2} + \frac{1}{2}\left(F^{-1}\right)^{22}\frac{\partial C_{22}}{\partial X^1} +$$

$$+ \frac{1}{2}\left(F^{-1}\right)^{23}\left(\frac{\partial C_{13}}{\partial X^2} + \frac{\partial C_{23}}{\partial X^1} - \frac{\partial C_{21}}{\partial X^3}\right) \leftrightarrows$$

$$\leftrightarrows \frac{l_2}{2H}\frac{dt_1}{db} + \frac{m_2}{2H}\frac{dt_2}{da} + \frac{n_2}{2H}\left(\frac{dt_5}{db} + \frac{dt_6}{da} - \frac{dt_4}{dc}\right)$$

which is, multiplying by $2H$ both members, the 11th equality on page 158 in [Piola, 1848]

$$2H\,\frac{d^2 y}{dadb} = l_2\frac{dt_1}{db} + m_2\frac{dt_2}{da} + n_2\left(\frac{dt_5}{db} + \frac{dt_6}{da} - \frac{dt_4}{dc}\right)$$

Piola continued the calculations by considering the third order derivatives. However the obtained expressions are too long for being reproduced in printed form. So he states:

"The trinomials with third order derivatives are of three kinds: there are those constituted by derivatives of first and third order, and one can count 30 of them: there are those constituted by derivatives of second and third order, and they are 60 in number: and there are those which contain only third order derivatives and they are 55 in number. I am not writing them, as everybody who is given the needed patience can easily calculate them by himself, as it can be also done for those trinomials containing derivatives of higher order."

As we can use Levi-Civita tensor calculus it is easier for us to find the needed patience, at least for calculating the trinomials constituted by derivatives of first and third order. Indeed from

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right) \tag{N14}$$

by differentiating the (N14) we get

$$\frac{\partial}{\partial X^\eta}\left(F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta}\right) = \frac{\partial}{\partial X^\eta}\left(\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right)\right)$$

and rearranging the terms,

$$F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha \partial X^\beta \partial X^\eta} = -\frac{\partial^2\chi_i}{\partial X^\gamma \partial X^\eta}\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta} + \frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta \partial X^\beta} + \frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta \partial X^\alpha} - \frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta \partial X^\gamma}\right). \tag{N15}$$

By replacing the following equality due to (N9)

$$\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(F^{-1}\right)^{i\delta}\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta} + \frac{\partial C_{\beta\delta}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right) \tag{N16}$$

in the identity (N15) one gets

$$\begin{aligned} F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha \partial X^\beta \partial X^\eta} = &-\frac{1}{2}\left(F^{-1}\right)_i^{\nu}\left(\frac{\partial C_{\gamma\nu}}{\partial X^\eta} + \frac{\partial C_{\eta\nu}}{\partial X^\gamma} - \frac{\partial C_{\eta\gamma}}{\partial X^\nu}\right)\frac{1}{2}\left(F^{-1}\right)^{i\delta}\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta} + \frac{\partial C_{\beta\delta}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right) \\ &+\frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta \partial X^\beta} + \frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta \partial X^\alpha} - \frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta \partial X^\gamma}\right) \end{aligned}$$

which can easily be rewritten in the form

$$\begin{aligned} F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha \partial X^\beta \partial X^\eta} = &-\frac{1}{4}\left(C^{-1}\right)^{\nu\delta}\left(\frac{\partial C_{\gamma\nu}}{\partial X^\eta} + \frac{\partial C_{\eta\nu}}{\partial X^\gamma} - \frac{\partial C_{\eta\gamma}}{\partial X^\nu}\right)\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta} + \frac{\partial C_{\beta\delta}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right) \\ &+\frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta \partial X^\beta} + \frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta \partial X^\alpha} - \frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta \partial X^\gamma}\right) \end{aligned}$$

which has the structure sought after by Piola. With easy calculation, from the last equation we get

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X},X)}{\partial \bar{X}_\alpha \partial \bar{X}_\beta \partial \bar{X}_\gamma} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} + \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right) + \left(\chi_i(\bar{X}) - \chi_i(X)\right)\frac{\partial^3\chi^i(\bar{X})}{\partial \bar{X}_\alpha \partial \bar{X}_\beta \partial \bar{X}_\gamma} \tag{N17}$$

In the following appendix an induction argument will be presented which allows us to prove the conjecture put forward by Piola at the beginning of his sect. 74, pag.156.

### 7.3 After these calculations *I abandoned myself to the analogy*

We are not so enthusiastic about the work of Piola to the extent that we cannot see clearly the limits of his mathematical proofs. Indeed the important property which he discusses in the num.74 is obtained by means of a proof by analogy which is not considered acceptable nowadays. Although there are examples of mathematical induction which are very ancient (see the discussion in [106] and references therein) only after Boole and Dedekind it became a universally known and (nearly universally) accepted method. Actually Piola states here that, because of objectivity, the expression of Virtual Work must depend only on deformation measure $C_{\gamma\beta}$ and its derivatives. However, as we have already pointed out, his proof is based, for higher derivatives, on an argument which the majority of contemporary mathematicians would consider no more than a (maybe well-grounded) conjecture. Indeed at the beginning of page 157 of [Piola, 1848] one reads

**after these calculations I abandoned myself to the analogy: and this will be sooner or later unavoidable, because our series is infinite and it will be impossible to check all its terms.**

We reproduce here an inductive argument which indeed follows the original spirit of Piola. Let us start by proving that:

**Lemma 1.** ***Representation of placement higher order derivatives.*** *For every $n$ there exist a family of (polynomial) functions*

$M_{\gamma\alpha_1...\alpha_n}$ *of the tensor variables* $C, \nabla C, ..\nabla^{n-1}C$ *such that*

$$\left(\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma}\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}}\right) = M_{\gamma\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right) \tag{N18}$$

As we have proven such a lemma for $n = 2$ that is,

$$F_{i\gamma}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right). \tag{N19}$$

In order to prove (N18) for every $n$ it is sufficient to prove that if it is valid for all $N \leq n$ then it is valid also for $N = n + 1$. Let us start by remarking that (N18) implies that

$$\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}} = \left(F^{-1}\right)^{i\eta} M_{\eta\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right) \tag{N20}$$

Let us then differentiate (N18) assumed valid for $N = n$ to get

$$\frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma}\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}}\right) = \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(M_{\gamma\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\right)$$

which implies

$$\begin{aligned}\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma}\frac{\partial^{n+1} \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}\partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(M_{\gamma\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\right) \\ &- \frac{\partial^2 \chi_i(\bar{X})}{\partial \bar{X}^\gamma \partial \bar{X}^{\alpha_{n+1}}}\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}}\end{aligned}$$

Now by replacing equation (N20) two times (for $n = 2$ and for $N = n$) we get

$$\begin{aligned}\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma}\frac{\partial^{n+1} \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}\partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(M_{\gamma\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\right) \\ -\left(\left(F^{-1}\right)_i^{\eta} M_{\eta\alpha_{n+1}}\left(C, .., \nabla^{n-1}C\right)\right)&\left(\left(F^{-1}\right)^{i\beta} M_{\beta\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\right) = \\ &= M_{\gamma\alpha_1...\alpha_n\alpha_{n+1}}\left(C, .., \nabla^{n-1}C\right),\end{aligned}$$

where we have introduced the definition

$$\begin{aligned}M_{\gamma\alpha_1...\alpha_n\alpha_{n+1}}\left(C, .., \nabla^{n-1}C\right) &:= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(M_{\gamma\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\right) \\ &- \left(C^{-1}\right)^{\beta\eta} M_{\eta\alpha_{n+1}}\left(C, .., \nabla^{n-1}C\right) M_{\beta\alpha_1...\alpha_n}\left(C, .., \nabla^{n-1}C\right)\end{aligned}$$

The proof by induction of the lemma is thus complete. To prove that also the generic $n - th$ order derivative of $\rho^2$ can be expressed, when $\bar{X} = X$ , in terms of the $(n-2) - th$ order derivatives of $C_{\gamma\beta}$ we can use again a simple recursion argument based on the previous lemma. Indeed the following other lemma is true:

**Lemma 2.** ***Representation of the derivatives of the distance function*** $\rho$. *For every* $n$ *there exist a family of (polynomial) functions* $L_{\alpha_1....\alpha_n}$ *of the variables* $C, .., \nabla^{n-2}C$ *such that*

$$\frac{\partial^n \rho^2(\bar{X}, X)}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}} = L_{\alpha_1....\alpha_n}\left(C, .., \nabla^{n-2}C\right) + \left(\left(\chi_i(\bar{X}) - \chi_i(X)\right)\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}}\right) \tag{N21}$$

To prove the Lemma we assume by inductive hypothesis that it is true for $N = n$ and prove that it is true for $N = n + 1$. As we have proven formula (N17), that is the previous lemma for $n = 3$, then the Lemma follows by the Mathematical Induction Principle. Therefore by differentiating equation (N21) one gets

$$\begin{aligned}\frac{\partial^{n+1} \rho^2(\bar{X}, X)}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}}\left(L_{\alpha_1....\alpha_n}\left(C, .., \nabla^{n-2}C\right)\right) + \left(\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^{\alpha_{n+1}}}\frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}}\right) \\ &+ \left(\sum_{i=1}^{3}\left(\chi_i(\bar{X}) - \chi_i(X)\right)\frac{\partial^{n+1} \chi_i(\bar{X})}{\partial \bar{X}^{\alpha_1}...\partial \bar{X}^{\alpha_n}\partial \bar{X}^{\alpha_{n+1}}}\right)\end{aligned}$$

which by replacing equation (N18) becomes

$$\frac{\partial^{n+1}\rho^2(\bar{X},X)}{\partial\bar{X}^{\alpha_1}...\partial\bar{X}^{\alpha_{n+1}}} = \frac{\partial}{\partial\bar{X}^{\alpha_{n+1}}}\left(L_{\alpha_1....\alpha_n}\left(C,..,\nabla^{n-2}C\right)\right) + M_{\alpha_{n+1}\alpha_1...\alpha_n}\left(C,..,\nabla^{n-1}C\right)$$
$$+\left(\sum_{i=1}^{3}\left(\chi_i(\bar{X})-\chi_i(X)\right)\frac{\partial^{n+1}\chi_i(\bar{X})}{\partial\bar{X}^{\alpha_1}...\partial\bar{X}^{\alpha_n}\partial\bar{X}^{\alpha_{n+1}}}\right)$$

which proves the Lemma once one has introduced the following recursive definition

$$L_{\alpha_1....\alpha_n\alpha_{n+1}}\left(C,..,\nabla^{n-2}C\right) := \frac{\partial}{\partial\bar{X}^{\alpha_{n+1}}}\left(L_{\alpha_1....\alpha_n}\left(C,..,\nabla^{n-2}C\right)\right) + M_{\alpha_{n+1}\alpha_1...\alpha_n}\left(C,..,\nabla^{n-1}C\right).$$

## 7.4 An Italian Mathematical Genealogy

In [105] it is discussed a most likely loss of knowledge occurred at the end of the Punic wars. More generally all the processes of erasure and removal of previosly well-established scientific knowledge are related to the simultaneous occurrence of two circumstances: the loss of continuity in the chain between Maestro and student in the academic institutions and the loss of the awareness -in the whole society- of the strict connection which exists between science (in all its most abstract expressions, including mathematics) and technology (see also [106]).

The final result of the simultaneous occurrence of these two circumstances is that the societies in which they occur do not invest resources in the storage and transmission of theoretical knowledge and that, as a consequence, it is broken the contact between maestro and pupil, established when a living scientist teaches to his students the content of the most difficult and important textbooks. As a final result, in those societies, at first the theoretical knowledge, and subsequently after a more or less long time period, also the technological capabilities are lost. We want to underline in this appendix that there is a direct genealogy starting from Gabrio Piola and leading to the founders of absolute tensor calculus. The Italian school of the XIX Century was started under the momentum impressed by the Napoleonic reforms of the political organization of the Italian Nation: in this context the reader should see his Eulogy where Piola, talking about the textbook of Mathematical Analysis written by his Maestro Vincenzo Brunacci writes

"was also pushed by the advice of that Sovereign who was an investigator of the stars who, being in Milan, wrote to persuade him to start this oeuvre in the year 1800, believing that he was the only one among the Italians who was capable to complete it successfully.".

Gabrio Piola never accepted a university chair: however his pupil Francesco Brioschi was the founder of the Politecnico di Milano. Brioschi mentored Enrico Betti and Eugenio Beltrami. Ulisse Dini was pupil of Enrico Betti, being also his successor as the chair of Mathematical Analysis and Geometry at the Università di Pisa. Gregorio Ricci Curbastro was pupil of Ulisse Dini, Eugenio Beltrami and Enrico Betti. Tullio Levi-Civita was pupil of Gregorio Ricci Curbastro. The strength of the Italian School of Mathematical Physics, Mathematical Analysis and Differential Geometry has been weakened by two processes, one which it shares with all other National Schools and in general with all groups of scientists, the second one which is more peculiar to the Italian Nation.

1. It happens very often that some theories need to be rediscovered and reformulated several times in different circles before becoming a universally recognized part of knowledge. For instance, the basic ideas of functional analysis and its founding concept of functional (which goes back to the calculus of variations and that can be defined with the sentence *a function whose argument is a function*) were already treated in the papers by Erik Ivar Fredholm and in Hadamard's 1910 textbook and had previously been introduced in 1887 by the Italian mathematician and physicist Vito Volterra. The theory of nonlinear functionals was continued by students of Hadamard, in particular Fréchet and Lévy. Hadamard also founded the modern school of linear functional analysis, further developed by Riesz and the group of Polish mathematicians around Stefan Banach. However, Heisenberg and Dirac did need to rediscover many parts of a theory already known and they developed such a theory until the moment at which von Neumann could recognize that actually Quantum Mechanics had been formulated in terms of what he called Hilbert Spaces. Simple laziness or the difficulty of understanding the formalism introduced by other authors, lack of time or of economical means. All of these may lead some very brilliant scientists to ignore results obtained by other scientists, which are nevertheless relevant for their work. Many of the mathematicians listed in the previous genealogy rediscovered many times the results which their predecessors had already obtained because of the aforementioned first process. Such a process could be called removal and/or ignorance of the results which appear not to be relevant. This first removal process is indeed observed very often in the history of science applied to the most various groups, independently of their nationality[4], and the case of the rediscovery of functional analysis is a striking example of its occurrence.

2. Napoleon favoured the birth of an Italian mathematical school, and among many other actions he pushed Vincenzo Brunacci to write the first Italian textbook in Mathematical Analysis. However he could not enforce in the Italian School the habit, always

[4] The authors are indebted to Prof. Mario Pulvirenti for having attracted their attention to this first process and also for having recalled to them the example concerning functional analysis.

followed by the French School, which leads all French Scientists to recognize, to develop and to glorify the contributions of their compatriots. Instead the Italian scientists always preferred to follow the tradition of their predecessors, i.e. the scientists of Greek language who developped the Hellenistic science (see [106]). Hellenic tradition is based on the intentional removal and contempt of the contribution due to the compatriots and on the continuous preference for the approval of foreign scientists. The described process leads the members of a national group to consider the other national groups always stronger, more qualified and more productive, while actively acting to impeach the cultural, political and academic growth of the compatriot scientists.

The momentum given to the Italian School by Napoleon eventually lead to the birth of Tensor Calculus, but was exhausted by the typical Italian negative attitude towards compatriots, which was exemplified by the removal of Levi-Civita, due to Mussolini's racial persecutions, from his chair in Rome, that was immediately occupied by Signorini. Finally, however, it has to be recognized at least that

i) the strict relationship between differential geometry and continuum mechanics has been discovered and developed by the Italian School started by Piola and culminating in Levi-Civita

ii) the great advancement of Riemannian geometry produced by the recognition of the unicity of the parallel transport compatible with a Riemannian metric (the so-called Levi-Civita Theorem) has its deep roots already in Piola's works (recall the well-known concept of Piola's Transformation)

iii) Ulisse Dini's Theorem clarifies mathematically the concept of constraint intensively used in the works of Piola. Indeed the crucial concept of independent constraints (defined as those having non-singular Jacobian) was clearified by Dini several decades after Piola had proven its importance in continuum mechanics.

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