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A STOCHASTIC CALCULUS MODEL OF CONTINUOUS TRADING: OPTIMAL PORTFOLIOS*

by

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Abstract

The problem of choosing a portfolio of securities so as to maximize the expected utility of wealth at a terminal planning horizon is solved via stochastic calculus and convex analysis. This problem is decomposed into two subproblems. With security prices modeled as semimartingales and trading strategies modeled as predictable processes, the set of terminal wealths is identified as a subspace in a space of integrable random variables. The first subproblem is to find the terminal wealth that maximizes expected utility. Convex analysis is used to derive necessary and sufficient conditions for optimality and an existence result. The second subproblem of finding the admissible trading strategy that generates the optimal terminal wealth is a martingale representation problem. The primary advantage of this approach is that explicit formulas can readily be derived for the optimal terminal wealth and the corresponding expected utility, as is shown for the case of an exponential utility function and a risky security modeled as geometric Brownian motion.

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1. Introduction

The optimal portfolio literature can be viewed as being in two parts according to the approach that is taken. The first is mean-variance analysis, as developed notably by Markowitz (1952, 1959) and Tobin (1958, 1965). Each security is modeled by two parameters: the mean and variance of its rate of return. With the additional specification of the correlations between securities, each particular portfolio or combination of securities is then naturally characterized by the mean and variance of its overall rate of return. The rational investor therefore focuses on the subset of portfolios comprising the "efficient frontier," that is, the collection of (undominated) portfolios which achieve the maximum mean for a given variance. The investor's final choice depends on his preferences toward risk in the trade-off between the mean and variance along the efficient frontier.

Mean-variance analysis is widely utilized because the approach both captures some important considerations in the optimal portfolio problem and is straightforward, if not easy, to implement. Furthermore, mean-variance analysis leads to a number of important consequences such as the Separation or Mutual Fund Theorem and the capital asset pricing model (Sharpe, 1964). However, mean-variance analysis has two significant limitations. While the simple model of security prices is a good one for many purposes, one would like to be able to consider alternative, perhaps more complex, models. In addition, the optimal portfolio yielded by mean-variance analysis is not necessarily optimal for the investor in terms of his underlying utility and preferences.

The second main part of the optimal portfolio literature deals with these limitations by directly addressing the problem of choosing the portfolio so as to maximize the expected utility of the outcome in question. The security prices are modeled as Markov processes, and so this problem is solved with dynamic programming and stochastic control theory. Massin (1968) and Samuelson (1969) analyzed discrete-time models, while Merton (1969, 1971) analyzed a diffusion process model.

While the expected utility approach satisfactorily addresses the limitations of mean-variance analysis, a price is paid due to the increase of . computational difficulties. The dynamic programming methodology reduces the maximum expected utility problem to a partial differential equation that usually remains intractable if not virtually impossible to solve. Explicit solutions can apparently be obtained only for the simplest cases. For example, Merton's (1971) simplest case involves a formidable partial differential equation; Fleming and Rishel (1975, pp. 160-161) present this same case as one of their few examples.

The purpose of this paper is to present an alternative method for solving the expected utility problem. This method, which involves stochastic calculus and convex analysis, seems to be computationally more efficient than dynamic programming. Partial differential equations can be entirely avoided, and, as seen by example, explicit solutions can readily be obtained. While the full range of cases and examples remains to be investigated, it appears that the method presented here has the potential for making optimal portfolio problems as easy to solve with the expected utility approach as they are with mean-variance analysis.

The basic problem studied here is to choose the trading strategy so as to maximize expected utility of the investor's wealth at a specified time horizon. No funds are added to or withdrawn from the portfolio during this time interval. The problem thus resembles Merton's (1969, 1971), except that he allows funds to be withdrawn. Although his problem is more general by allowing this consumption, the problem studied here is more general in terms of the allowable stochastic process models of the security prices. Indeed, the prices need not be Markov processes, but they must be semimartingales. The models of security prices and trading strategies used here are taken from Harrison and Pliska (1981).

The idea of using stochastic calculus and convex analysis to solve these kinds of problems was introduced by Pliska (1982). Briefly, the essence of this idea is to decompose the problem into three stages. First, you use martingale theory to characterize the set of attainable terminal wealths (a "terminal wealth," or contingent claim, is a random variable, and the attainable ones are "generated" by admissible trading strategies). Second, you use convex analysis to find the attainable wealth that maximizes expected utility. Finally, you use martingale theory again to determine the trading strategy which generates the optimal attainable wealth. This methodology was presented by Pliska (1982) in the context of a discrete time stochastic control problem where the underlying probability space was finite. While the concepts are the same, the mathematical machinery here is considerably more involved in order to accommodate the infinite probability spaces associated with continuous time models.

After a detailed formulation of the problem, Section 3 presents the main results: necessary and sufficient conditions for an attainable wealth to be optimal. Section 4 then focuses on the special case of complete markets. This means, roughly speaking, that all terminal wealths are attainable. The sufficient condition for optimality simplifies, and this leads to a result on the existence of an optimal attainable wealth as well as an algorithm for computing it. Sections 5 and 6 show how to use these results by studying an example involving an exponential utility function and geometric Brownian motion. The paper concludes with some remarks in Section 7.

2. Formulation and Preliminaries

The model of continuous trading to be used is almost the same as that in Harrison and Pliska (1981). Let (Ω, F, P) be a probability space, let $T < \infty$ be a fixed time horizon, and let $\mathbb{F} = \{F_t;\ 0 \le t \le T\}$ be a filtration satisfying les conditions habituelles with F_0 containing only Ω and the null sets of P and with $F_T = F$.

Let $S = \{S_t;\ 0 \le t = T\}$ be a vector valued stochastic process whose components $S^0, S^1, \ldots, S^K$ are adapted, right continuous with left limits, and strictly positive. Moreover, it is assumed that S^0 is a semimartingale with $S_0^0 = 1$. Here S_t^k represents the time t value of the kth security, so S is called the (undiscounted) price process. Upon defining $\beta = 1/S^0$, one defines the discounted price process $Z = (Z^1, \ldots, Z^K)$ by setting $Z^k = \beta S^k$ for $k = 1, \ldots, K$.

Let $\mathbb{P}$ be the set of probability measures Q on (Ω, F) that are equivalent to P and such that Z is a (vector) martingale under Q. It is assumed that $\mathbb{P}$ is nonempty, so S and Z are actually semimartingales under P. An arbitrary element $Q \in \mathbb{P}$ is selected and called the reference measure. Let E_Q denote the corresponding expectation operator. As explained in Harrison and Pliska (1981), the assumption that $\mathbb{P}$ is nonempty is made to rule out arbitrage opportunities that would permit investors to make unreasonable profits without any risk.

Let L(Z) denote the set of all vector valued, predictable processes $H = (H^1, \ldots, H^K) = \{H_t;\ 0 \le t \le T\}$ that are integrable with respect to the semimartingale Z (see Jacod (1979, p. 52) for details about L(Z)). An admissible trading strategy is any vector valued, predictable stochastic process $\phi = (\phi^0, \phi^1, \ldots, \phi^K) = \{\phi_t;\ 0 \le t \le T\}$ such that

(i) $(\phi^1, \ldots, \phi^K) \in L(Z)$

(ii) $\hat{V}(\phi) = \hat{V}_0(\phi) + \hat{G}(\phi)$, where $\hat{G}(\phi) = \int \phi dZ = \sum_{k=1}^{K} \int \phi^k dZ^k$ and

$\hat{V}(\phi) = \beta\phi S = \sum_{k=0}^{K} \phi^k Z^k$, and

(iii) $\hat{V}(\phi)$ is a martingale under Q.

Let Φ denote the set of all such admissible trading strategies. Here ϕ_t^k represents the number of shares or units of security k held by the investor at time t, $\hat{V}(\phi)$, the discounted value process, represents the discounted value of the portfolio, and $\hat{G}(\phi)$, the discounted gains process, represents the discounted net profit or loss due to the transactions by the investor. Thus (ii) says all changes in the value of the portfolio are due to the investment rather than due to infusion or withdrawal of funds. Condition (iii) serves to rule out certain foolish strategies that throw away money. Note that condition (iii) is the only one that might depend on the choice of the reference measure.

Harrison and Pliska (1981) specified a fourth condition, namely, that $\hat{V}(\phi) \ge 0$ for all $\phi \in \Phi$. This was done in order to rule out short sales that might leave the investor indebted to his broker. This condition is omitted here in order to facilitate the mathematical development. Little is lost from the modeling standpoint, since by assigning a sufficiently large negative utility to bankruptcy one achieves essentially the same effect as the constraint $\hat{V}(\phi) \ge 0$.

Let $\underline{\underline{X}}$ denote the set of all random variables X that are integrable under Q. Each $X \in \underline{\underline{X}}$ will be called a terminal wealth, although in the financial literature it is customary to call X a contingent claim. Such a terminal wealth is said to be <u>attainable</u> if there exists an admissible trading strategy $\phi \in \Phi$ such that $\hat{V}_T(\phi) = X$, in which case ϕ is said to <u>generate</u> X. Denoting

$$\tilde{\underline{\underline{C}}} = \{X \in \underline{\underline{X}} : X = V_0 + \hat{G}_T(\phi), \text{ some } \phi \in \Phi\},$$

one sees that $\tilde{\underline{\underline{C}}}$ is the set of all attainable terminal wealths (henceforth, <u>attainable wealths</u>).

Note that $\tilde{\underline{\underline{C}}}$ depends on the initial wealth V_0, the value of which is implicit in the definition of attainable wealths. In general, a terminal wealth $X \in \underline{\underline{X}}$ that is attainable for one initial wealth may not be attainable for others. Note also that the terminal wealths are discounted. Although it may seem more natural from the modeling standpoint to focus on undiscounted attainable wealths, discounting will facilitate the analysis.

The investor's preferences over $\underline{\underline{X}}$ are modeled with a state dependent utility function. Let $\tilde{u}$ be a real-valued function on $\mathbb{R} \times \Omega$, and set

$$\tilde{U}(X) = \int \tilde{u}(X(\omega),\omega)dP(\omega),$$

where P is the original probability measure. With additional assumptions, detailed below, about $\tilde{u}$, $\tilde{U}$ is an integral functional that is well-defined for each $X \in \underline{\underline{X}}$.

The investor's decision problem is now easy to state: for a given initial wealth V_0, choose a trading strategy $\phi \in \Phi$ so as to maximize $\tilde{U}(V_0 + \hat{G}_T(\phi))$, the expected utility of terminal wealth. Rather than using dynamic programming to solve this problem, two subproblems will be solved. The first is to find an <u>optimal</u> attainable wealth, that is, some $X_0 \in \tilde{\underline{\underline{C}}}$ such that $\tilde{U}(X_0) \geq \tilde{U}(X)$ for all $X \in \tilde{\underline{\underline{C}}}$. The second is to find the <u>optimal</u> trading strategy, that is, the $\phi \in \Phi$ that generates this optimal attainable wealth.

Before showing how to solve this first subproblem it is convenient to carry out a transformation. Let

$$\underline{\underline{C}} = \{X \in \underline{\underline{X}} : X = \hat{G}_T(\phi), \text{ some } \phi \in \Phi\},$$

so $\underline{\underline{C}} = \underline{\underline{\tilde{C}}} - V_0$ is the set of attainable wealths in the case where the initial wealth $V_0 = V$. Let $F = dP/dQ$ denote the Radon-Nikodym derivative, define the function

$$u(x,\omega) = \tilde{u}(x + V_0, \omega)F(\omega),$$

and define the new integral functional U by

$$U(X) = \int u(X(\omega),\omega)dQ(\omega).$$

Since $U(X) = \tilde{U}(V_0 + X)$ for all $X \in \underline{\underline{C}}$, it is apparent that the first subproblem can be restated as:

(1) Find $X_0 \in \underline{\underline{C}}$ such that $U(X_0) \geq U(X)$, all $X \in \underline{\underline{C}}$.

Thus without loss of generality the initial wealth V_0 will be taken to be zero and expected values will be calculated with the reference measure Q.

In order to ensure that U is a well-defined and suitably behaved integral functional, it is necessary to make the following

(2) Assumption.

(i) $x \to u(x,\omega)$ is concave and strictly increasing for each ω.

(ii) $-u$ is a normal, proper integrand

(iii) There exists at least one $X \in \underline{\underline{X}}$ such that $U(X) > -\infty$.

The theory of normal integrands with possibly infinite values was introduced and developed by Rockafellar in a series of papers including (1968), (1971), and (1976). For an understanding of (2ii) the reader should consult Rockafellar (1976, p. 173). Here it suffices to say that this assumption is technical in nature, but it is satisfied in most cases of practical interest. For example, if u is finite and independent of ω and if (2i) holds, then so does (2ii). In any event, (2i) and (2ii) together imply that U is a well-defined concave integral functional on $\underline{\underline{X}}$.

3. Main Results

The primary results here are necessary and sufficient conditions for $X_0 \in \underline{\underline{C}}$ to be a solution of subproblem (1). The methodology to be employed is convex analysis, as described in, for example, Rockafellar (1974). This section will also examine the matter of solving the second subproblem, finding the trading strategy $\phi \in \Phi$ that generates X_0.

It is clear from the nature of stochastic integrals that $\underline{\underline{C}}$ is a subspace of $\underline{\underline{X}}$. Viewing $\underline{\underline{X}}$ as an L^1 space of functions on the measure space (Ω, F, Q), let $\underline{\underline{Y}}$ denote its dual space, that is, the space of bounded random variables. The linear functionals on X are thus of the form $X \to E_Q[XY]$ for some $Y \in \underline{\underline{Y}}$. Let $\underline{\underline{C}}^{\perp}$ denote the orthogonal subspace of $\underline{\underline{C}}$, that is,

$$\underline{\underline{C}}^{\perp} = \{Y \in \underline{\underline{Y}} : E_Q[XY] = 0, \text{ all } X \in \underline{\underline{C}}\}.$$

For a brief digression, there is an interesting interpretation of $\underline{\underline{C}}^{\perp}$ that emerges from the theory of stochastic calculus. Let $\underline{\underline{H}}$ denote the set of stochastic integrals generated by the admissible trading strategies, that is,

$$\underline{\underline{H}} = \{\hat{G}(\phi) : \phi \in \Phi\}$$

Recalling the definition of $\underline{\underline{C}}$, it is apparent one can also write

$$\underline{\underline{H}} = \{\text{all martingales } M: M_t = E_Q[X|F_t], \text{ some } X \in \underline{\underline{C}}\}.$$

Moreover, $\underline{\underline{H}}$ is what is called a <u>stable subspace</u>, that is, it is closed in the space of L^1 martingales, it is closed under stopping (if $M \in \underline{\underline{H}}$ and T is a stopping time, then M stopped at time T is in $\underline{\underline{H}}$), and if $M \in \underline{\underline{H}}$ and $A \in F_0$

then $1_A M \in \underline{\underline{H}}$ (see Jacod (1979, Ch. IV) or Elliot (1982, Ch. 9) for the details). Now let $\underline{\underline{H}}^\perp$ be the weakly orthogonal subspace, that is,

$$\underline{\underline{H}}^\perp = \{\text{all martingales } N: N_t = E_Q[Y|\mathcal{F}_t], \text{ some } Y \in \underline{\underline{C}}^\perp\}.$$

By a standard argument (e.g., Meyer (1976, Ch. II)) $\underline{\underline{H}}^\perp$ is also stable, and if M and N belong to $\underline{\underline{H}}$ and $\underline{\underline{H}}^\perp$ respectively, then M and N are strongly orthogonal in the sense that their product is a martingale that is null at zero. Conversely, starting with $\underline{\underline{H}}$ one can define $\underline{\underline{H}}^\perp$ as the set of all bounded martingales N that are strongly orthogonal to all $M \in \underline{\underline{H}}$. Then

$$\underline{\underline{C}}^\perp = \{Y \in Y : Y = N_T, \text{ some } N \in \underline{\underline{H}}^\perp\}.$$

This connection between $\underline{\underline{C}}^\perp$ and $\underline{\underline{H}}^\perp$ will not be utilized further here, but it provides an intriguing interpretation of the dual variables for subproblem (1).

Denote

$$U^*(Y) = \inf_{X \in \underline{\underline{X}}} \{E_Q[X Y] - U(X)\}$$

for the concave conjugate functional of U and

$$u^*(y,\omega) = \inf_{x \in \mathbb{R}} \; xy - u(x,\omega)\}$$

for the concave conjugate functional of u. By Rockafellar (1975, Thm. 3C) and Assumption (2iii), U* is an integral functional given by

(3) $$U^*(Y) = \int u^*(Y(\omega),\omega)dQ(\omega).$$

The first main result is a sufficient condition for optimality.

(4) Theorem. Suppose $X_0 \in \underline{\underline{C}}$ and $Y_0 \in \underline{\underline{C}}^\perp$ satisfy

(5) $$u^*(Y_0(\omega),\omega) = X_0(\omega)Y_0(\omega) - u(X_0(\omega),\omega) \quad \text{a.s.}$$

Then

(6) $U(X) \le U(X_0) = -U^*(Y_0) \le -U^*(Y)$, all $X \in \underline{\underline{C}}$, $Y \in \underline{\underline{C}}^{\perp}$.

Proof. This proof uses standard arguments. Condition (5) and equation (3) imply

$$U^*(Y_0) = E_Q[X_0 Y_0] - U(X_0).$$

But $X_0 \in \underline{\underline{C}}$ and $Y_0 \in \underline{\underline{C}}^{\perp}$ so $E_Q[X_0 Y_0] = 0$ and

(7) $$U^*(Y_0) = -U(X_0),$$

which is one part of (6). Meanwhile, by the definition of the concave conjugate,

(8) $$U^*(Y_0) \le E_Q[X Y_0] - U(X), \quad \text{all } X \in \underline{\underline{X}},$$

so for any $X \in \underline{\underline{C}}$ one has $E_Q[X Y_0] = 0$ and $U^*(Y_0) \le -U(X)$. Combining this with (7) gives

$$U(X) \le U(X_0), \quad \text{all } X \in \underline{\underline{C}},$$

which is the second part of (6). Finally, taking arbitrary $Y \in \underline{\underline{C}}^{\perp}$, the definition of U^* implies $U^*(Y) \le E_Q[X_0 Y] - U(X_0)$, so as before

$$U^*(Y) \le -U(X_0) = U^*(Y_0),$$

and the third part of (6) is verified.

(9) Corollary. If $X_0 \in \underline{\underline{C}}$ and $Y_0 \in \underline{\underline{C}}^{\perp}$ satisfy (5), then $Y_0 \ge 0$.

Proof. Set $A = \{\omega \in \Omega : Y_0(\omega) < 0\}$ and consider $X = X_0 + 1_A$. Clearly $X \in \underline{\underline{X}}$, and by Assumption (2i) $U(X) \ge U(X_0)$. Now (7) and (8) imply

$$-U(X_0) \le E_Q[X Y_0] - U(X),$$

so $E_Q[X Y_0] \ge 0$. But $E_Q[X Y_0] = E_Q[X_0 Y_0] + E_Q[1_A Y_0] = E_Q[1_A Y_0]$, so $E_Q[1_A Y_0] \ge 0$. However, $1_A Y_0 \le 0$ by the definition of A, so one concludes $1_A Y_0 = 0$ a.s. and $Q(A) = P(A) = 0$.

Remark. If u is such that $X_1 \geq X_2$ and $X_1 \neq X_2$ imply $U(X_1) > U(X_2)$, then the idea of this proof can be used to show $Y_0 > 0$ a.s.

In order to assert that condition (5) is necessary for optimality, one apparently needs to make an additional assumption about the integral functional U. One such assumption is stated as a hypothesis in the following

(10) Theorem. Suppose U is finite and continuous everywhere on $\underline{\underline{X}}$. If $X_0 \in \underline{\underline{C}}$ satisfies $U(X_0) \geq U(X)$ for all $X \in \underline{\underline{C}}$, then there exists some $Y_0 \in \underline{\underline{C}}^{\perp}$ such that

$$u^*(Y_0(\omega),\omega) = X_0(\omega)Y_0(\omega) - u(X_0(\omega),\omega) \quad \text{a.s.}$$

Proof. Following Rockafellar (1966), one can set his function g = U and choose his function f on $\underline{\underline{X}}$ to be $f(X) = 0$ for $X \in \underline{\underline{C}}$ and $f(X) = 0$ for $X \notin \underline{\underline{C}}$ and thereby conclude from his Theorem 1 that there exists some $Y_0 \in \underline{\underline{C}}^{\perp}$ satisfying

$$U^*(Y_0) = E_Q[X_0 Y_0] - U(X_0).$$

This implies the desired equality (5) by Rockafellar (1976, Cor. 3E).

Remark. By Rockafellar (1976, Thm. 3L), U is finite and continuous on $\underline{\underline{X}}$ under Assumption (2) if there exists some positive scalar $a < \infty$ such that the partial derivative $\frac{\partial u(x,\omega)}{\partial x} \leq a$ for all x almost surely.

The key to using the preceding results to solve subproblem (1) is to successfully characterize the subspace $\underline{\underline{C}}$ of attainable wealths. If one has what is called a complete model, then as will be shown in the next section $\underline{\underline{C}}$ is easy to characterize and one can readily use Theorem (4) to solve for the optimal attainable wealth. On the other hand, if the model is not complete then $\underline{\underline{C}}$ does not offer such a convenient characterization. The analysis of this latter case remains as a subject for future research.

Before turning to the case of complete models, some remarks should be made about the final step in solving the optimal portfolio problem, viz., determining the admissible trading strategy that generates the optimal attainable wealth. Given $X_0 \in \underline{\underline{C}}$ one begins by computing the (discounted value process) martingale $\hat{V}$ via $\hat{V}_t = E_Q[X_0|F_t]$. It then remains to solve the martingale representation problem of finding the predictable process ϕ satisfying $d\hat{V} = \phi dZ$. This kind of problem, discussed in Jacod (1979, Ch. XI), generally requires <u>ad hoc</u> methods and the ingenious use of martingale theory. For example, Harrison and Pliska (1980, Sec. 5.3) solved a particular case with Ito's Formula, while in Section 6 below another particular case is solved by expressing both $\hat{V}$ and Z as stochastic integrals with respect to the same processes.

4. The Case of Complete Markets

Notice that $E_Q[X] = 0$ for all attainable wealths $X \in \underline{\underline{C}}$. This is because each discounted value process $\hat{V}(\phi)$ is a martingale under Q and is null at zero. In general the converse is not true, but if the model is complete, then

(11) $\underline{\underline{C}} = \{X \in X : E_Q[X] = 0\}$.

The model is said to be <u>complete</u> if for each $X \in \underline{\underline{X}}$ there exist V_0 and $\phi \in \Phi$ such that $V_0 + \hat{G}_T(\phi) = X$. This important concept was discussed at length by Harrison and Pliska (1981, 1983). They showed the model is complete if and only if the set of martingale measures $\mathbb{P}$ is a singleton. This, in turn, is true if and only if every martingale M can be represented as $M = M_0 + \hat{G}(\phi)$ for some $\phi \in \Phi$. Thus if $E_Q[X] = 0$, then taking the martingale M defined by $M_t = E_Q[X|F_t]$ it is apparent that $M = \hat{G}(\phi)$ for some $\phi \in \Phi$. Since $\hat{G}_T(\phi) = X$, this verifies (11).

If the model is complete, then several elements in the convex analysis set-up are simplified, as suggested by the following

(12) Proposition. If $X_0 \in \underline{\underline{C}}$ and $Y_0 \in \underline{\underline{C}}^{\perp}$ satisfy (5), then Y_0 is actually a positive constant in the case of a complete model.

Proof. Set $\hat{k} = E_Q[Y_0]$ and $Z = Y_0 - \hat{k}1$. Thus $E_Q[Z] = 0$, so $Z \in \underline{\underline{C}}$. But $Y_0 \in \underline{\underline{C}}^{\perp}$ and $\hat{k}1 \in \underline{\underline{C}}^{\perp}$, so $Z \in \underline{\underline{C}}^{\perp}$. Hence $Z = 0$, that is, $Y_0 = \hat{k}1$. Corollary (9) says $\hat{k} \geq 0$. If $\hat{k} = 0$, then sufficiency condition (5) is

$$u^*(0,\omega) = -u(X_0(\omega),\omega) = \inf_{x \in \mathbb{R}}\{-u(x,\omega)\} \quad \text{a.s.}$$

But this contradicts the assumption that $x \to u(x,\omega)$ is strictly increasing.

Knowing that Y_0 is a positive scalar greatly facilitates subproblem (1). The computational procedure is presented as the constructive proof of the following theorem, which provides a sufficient condition for the existence of an optimal attainable wealth.

(13) Theorem. Suppose the model is complete. Then there exists a solution X_0 to subproblem (1) if the following conditions all hold.

(i) the function $x \to u(x,\omega)$ is continuously differentiable a.s.

(ii) $\lim_{x \to \infty} \dfrac{\partial u(x,\omega)}{\partial x} = 0$ a.s.

(iii) $\lim_{x \to -\infty} \dfrac{\partial u(x,\omega)}{\partial x} = \infty$ a.s.

(iv) there exist finite scalars β and ε such that

$$\beta \geq \frac{\partial u(0,\omega)}{\partial x} \geq \varepsilon > 0, \text{ a.s.}$$

Proof. In view of Proposition (12) and the fact that Y_0 solves the dual problem in (6), the idea here is to find a finite, positive scalar $\hat{k}$, say, that maximizes $U^*(k1)$ over $k > 0$.

By (i) - (iii) and elementary calculus one sees that for each $k > 0$ and almost every ω there exists a number denoted $X(k,\omega)$ minimizing the convex function $x \to kx - u(x,\omega)$. In other words,

$$(14) \qquad u^*(k,\omega) = kX(k,\omega) - u(X(k,\omega),\omega), \; k > 0, \quad \text{a.s.}$$

By equation (3) one has

$$(15) \qquad U^*(k1) = E_Q[u^*(k,\omega)],$$

so the immediate objective is to find the value $\hat{k}$ that maximizes this concave expression. The issue is to show that $\hat{k}$ is strictly positive and finite. To do this, observe by the calculus used to compute $X(k,\omega)$ that

$$\frac{\partial u^*(k,\omega)}{\partial k} = X(k,\omega),$$

so

$$(16) \qquad \frac{dU^*(k1)}{dk} = E_Q\left[\frac{\partial u^*(k,\omega)}{\partial k}\right] = E_Q[X(k,\omega)], \quad k > 0.$$

Thus if k is very small, say $k < \varepsilon/2$, then by condition (13 iv) one has $X(k,\omega) > 0$ almost surely, in which case $\frac{dU^*(k1)}{dk} = E_Q[X(k,\omega)] > 0$. On the other hand, for very large k, say $k > 2\beta$, condition (13 iv) and the definition of $X(k,\omega)$ imply $X(k,\omega) < 0$ a.s., so by (16) one has $\frac{dU^*(k1)}{dk} < 0$.

Hence there exists a finite, strictly positive scalar $\hat{k}$ maximizing $U^*(k1)$ over $k > 0$. By the remark at the beginning of this proof it follows that $Y_0 = \hat{k}1$. Taking $X_0(\omega) = X(\hat{k},\omega)$ it is apparent from (14) that X_0 and Y_0 satisfy sufficient condition (5). Hence X_0 is a solution of subproblem (1) by Theorem (4).

Remark. The conditions (13i) - (13iv) are more than sufficient to guarantee existence, but the matter of making them less restrictive was not pursued in order to facilitate the exposition. In any event, the computational procedure can be summarized as follows. First compute $u^*(k,\omega)$ and the function $X(k,\omega)$ giving the minimizing values in the definition of u^*. Second, compute $U^*(k1)$ by (15). Third, compute the maximizing value $\hat{k}$. Fourth, substitute $\hat{k}$ in (15) to obtain the maximum expected utility of terminal wealth. Finally, take $X_0 = X(\hat{k},\cdot)$ for the optimal attainable wealth. This procedure will be illustrated in the following section.

5. Example: Exponential Utility

This section will illustrate how to carry out the procedure suggested in the proof of Theorem (13) for solving subproblem (1) in the case of a complete model.

Consider a utility function of the form

$$\tilde{u}(x,\omega) = a - \frac{b}{c}\exp(-ce^{rT}x),$$

where of the four scalar parameters a, b, c, and r all but a must be strictly positive. Here r can be interpreted as the discount rate, so if $e^{rT}x$ is the wealth at time T then x is the value of that wealth discounted to time zero. Taking initial wealth $V_0 = 0$, this gives

$$u(x,\omega) = F(\omega)[a - \frac{b}{c}\exp(-ce^{rT}x)]$$

for the transformed utility function, where F is the Radon-Nikodym derivative. It is easy to check that Assumption (2) and Conditions (13i) - (13iii) all hold. Condition (13iv) may not hold because it depends on F, but as will be seen below it is not critical. The utility function u fails to satisfy the condition in the remark following Theorem (10), so relationship (5) may not be necessary for optimality.

Using elementary calculus and the definition of the concave conjugate functional one computes

$$u^*(k,\omega) = \begin{cases} \frac{ke^{-rT}}{c} \log\left(\frac{be^{rT}F(\omega)}{k}\right) - aF(\omega) + \frac{ke^{-rT}}{c}, & k > 0, \\ -aF(\omega), & k = 0, \\ -\infty & k < \infty. \end{cases}$$

During this computation one notes the minimizing values of x for k > 0 are given by the function

$$(17) \qquad X(k,\omega) = \frac{e^{-rT}}{c} \log\left(\frac{be^{rT}F(\omega)}{k}\right).$$

Using equation (15) one immediately obtains

$$(18) \qquad U^*(kl) = \frac{ke^{-rT}}{c} \left\{\log b + E_Q[\log F] - \log k + rT + 1\right\} -a, \; k > 0.$$

Differentiating (18) with respect to k, one computes the maximizing value to be

$$\hat{k} = b\,e^{rT} \exp(E_Q[\log F]).$$

Substituting this into (18) gives the expected utility under the optimal attainable wealth, namely

$$(19) \qquad U(X_0) = a - \frac{b}{c} \exp(E_Q[\log F]).$$

Finally, substituting $\hat{k}$ into (17) gives the optimal attainable wealth

$$(20) \qquad X_0 = \frac{e^{-rT}}{c} \left\{\log F - E_Q[\log F]\right\}.$$

It is important to remark that not only have explicit formulas been obtained for these latter two quantities, but they depend on the underlying stochastic process model of the security prices only via the Radon-Nikodym derivative F. This will be the situation with any choice of utility function. Once the general formula for X_0 has been obtained for the utility function of interest, one can readily go forth and analyze a variety of security models (provided they all are complete). For each such model, the first step is to determine the reference measure Q and the Radon-Nikodym derivative F. Then one substitutes these into the formula at hand for X_0. Finally, one determines the trading strategy that generates X_0 by solving the martingale representation problem. This three step process will be illustrated in the next section.

6. Example: Geometric Brownian Motion

A single risky security S (the superscript 1 is omitted for ease of notation) is modeled as geometric Brownian motion in conjunction with a bond satisfying $S_t^0 = \exp(rt)$ and the exponential utility function of the preceding section. After solving this optimal portfolio problem with the methods of this paper, a comparison will be made with the dynamic programming approach offered by Merton (1971).

Following Harrison and Pliska (1980, Sec. 5), the discounted return process Y for S satisfies $Y_t = \sigma W_t + \mu t$, where W is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with $W_0 = 0$. The filtration is the one generated by W, and σ and μ are real constants. It follows that the discounted price process Z satisfies $Z_t = Z_0 \exp(Y_t - \frac{1}{2}\sigma^2 t)$, in which case $S_t = \exp(rt)Z_t$. In other words, S has return process $R_t = Y_t + rt$, a Brownian motion with variance σ^2 and drift $\mu + r$, and S satisfies the stochastic differential equation

$$\frac{dS}{S} = (\mu + r)dt + \sigma dW.$$

In order to apply the ideas of this paper, the first step is to specify the martingale measure Q. According to Harrison and Pliska (1980, Sec. 5.1) this is given by

$$dQ = M_T dP,$$

where M is the martingale (under P) given by

$$M_t = \exp\{-\mu W_t/\sigma - \mu^2 t/(2\sigma^2)\}$$

The martingale measure Q is unique, so the model is complete (see Harrison and Pliska (1983) and Harrison and Pliska (1980, Sec. 5.2)). Note the Radon-Nikodym derivative F = dP/dQ is given by

$$F = \exp\{\mu W_T/\sigma + \mu^2 T/(2\sigma^2)\}.$$

Anticipating a computation necessary for (19) and (20), the next step is to compute $E_Q[\log F]$. Substituting from above and using the fact that W_T is normally distributed with mean 0 and variance T,

$$E_Q[\log F] = E_Q[\mu W_T/\sigma + \mu^2 T/(2\sigma^2)]$$

$$= E[(\mu W_T/\sigma + \mu^2 T/(2\sigma^2))M_T]$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (\mu w/\sigma + \mu^2 T/(2\sigma^2))\exp\{-\mu w/\sigma - \mu^2 T/(2\sigma^2)\}\exp\{-w^2/(2T)\}dw$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (\mu w/\sigma + \mu^2 T/(2\sigma^2))\exp\{-(w + \mu T/\sigma)^2/(2T)\}dw.$$

Recognizing in this last integral the density function for a normal random variable with mean $-\mu T/\sigma$ and variance T, one easily concludes that

$$E_Q[\log F] = (\mu/\sigma)(-\mu T/\sigma) + \mu^2 T/(2\sigma^2)$$

$$= -\mu^2 T/(2\sigma^2).$$

Upon substituting this expression into (19) and (20) one immediately obtains

$$U(X_0) = a - \frac{b}{c}\exp\{-\mu^2 T/(2\sigma^2)\}$$

for the maximum expected utility and

$$X_0 = \frac{e^{-rT}}{c}\{\mu W_T/\sigma + \mu^2 T/(2\sigma^2) + \mu^2 T/(2\sigma^2)\}$$

$$= \frac{e^{-rT}}{c}\{\mu W_T/\sigma + \mu^2 T/\sigma^2\}$$

for the optimal attainable wealth.

It remains to determine ϕ, the optimal trading strategy specifying the number of shares in the risky asset. This is done by first deriving the discounted value process $\hat{V}$, which is given by $\hat{V}_t = E_Q[X_0|F_t]$. Proceeding as in the computation above of $E_Q[\log F]$, one computes

$$E_Q[W_T|F_t] = W_t + E_Q[W_T - W_t|F_t]$$

$$= W_t + E[W_{T-t}M_{T-t}]$$

$$= W_t + \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} w \exp\{-(w + \mu(T-t)/\sigma)^2/(2(T-t))\}dw$$

$$= W_t - \mu(T-t)/\sigma.$$

Hence

$$\hat{V}_t = \frac{e^{-rT}}{c}\{\mu W_t/\sigma - \mu^2(T-t)/\sigma^2 + \mu^2 T/\sigma^2\}$$

$$= \frac{e^{-rT}}{c}\{\mu W_t/\sigma + \mu^2 t/\sigma^2\}.$$

For the final step, using $d\hat{V} = \phi dZ$ and $dZ = ZdY = Z\sigma dW + Z\mu dt$ one sees that the optimal trading strategy ϕ must satisfy

$$\frac{\mu e^{-rT}}{c\sigma^2}\{\sigma dW + \mu dt\} = \phi Z\{\sigma dW + \mu dt\}.$$

Hence it is apparent that

(21) $$\phi = \frac{\mu e^{-rT}}{c\sigma^2 Z} .$$

Since $Z_t = e^{-rT} S_t$ and ϕS is the dollar value invested in the risky asset, this says it is optimal to invest $\mu e^{-r(T-t)}/(c\sigma^2)$ dollars in the risky asset at time t, holding the balance of one's wealth in the bond. In other words, the fraction $\theta_t = \phi_t S_t / V_t = \phi_t S_t / (e^{rt} \hat{V}_t)$ of one's wealth invested in the risky asset should be

$$\theta_t = \frac{1}{\sigma W_t + \mu t} .$$

It is instructive to compare how these results could have been derived with the dynamic programming approach outlined by Merton (1971). With $J(v,t)$ denoting the maximum expected terminal utility given current wealth is v at time t, one first must compute J by solving the partial differential equation

(22) $$0 = J_t + rv\, J_v - \mu^2 J_v^2 / (2\sigma^2 J_{vv})$$

subject to the boundary condition $J(v,T) = a - (b/c)\exp\{-cv\}$. This is a specialization of Merton's equation (28) (although the sign of one of his terms seems to be in error); the subscripts on J denote partial derivatives. With the solution J, one then computes the optimal fraction θ of one's wealth to invest in the risky asset by using Merton's equation (27), which is

(23) $$\theta_t = - \frac{\mu J_v(V_t, t)}{\sigma^2 V_t J_{vv}(V_t, t)} .$$

It is apparent that the key step in this procedure is to successfully solve the partial differential equation. This is generally a formidable undertaking and would even be difficult for the special case at hand if it were not for the fact that knowledge of the solution obtained with convex optimization allows one to make an educated guess about the solution J. Since the optimal amount to invest in the risky asset is independent of the current wealth v, at time t one should invest v in the bond and then act as if the wealth is zero and the time horizon is T-t. The investment of v dollars in the bond becomes exp(r(T-t))v at time T, so if the random variable X denotes the terminal discounted wealth from the remaining investments, then one has

$$J(v,t) = E[a - \frac{b}{c}\exp\{-c\, e^{r(T-t)}(v + X)\}].$$

By the same kind of calculations used above to derive $E_Q[\log F]$ and so forth, this means the conjectured solution is given by

$$J(v,t) = a - \frac{b}{c}\exp\{-c\, e^{r(T-t)}v\}\exp\{-\mu^2(T-t)/(2\sigma^2)\}.$$

It is straightforward and left for the reader to verify that this expression for J indeed satisfies the partial differential equation (22). Using equation (23) one computes the optimal fraction of one's wealth to hold in the risky asset to be given by

$$\theta_t = \frac{\mu e^{-r(T-t)}}{c\sigma^2 V_t}.$$

Since $\phi_t V_t = \phi_t S_t = \phi_t e^{rt} Z_t$ is the dollar value invested in the risky asset, this coincides with the result (21) obtained earlier by convex optimization.

7. Concluding Remarks

It is hoped that the methods and ideas in this paper will be used to analyze a variety of utility functions and stochastic process models of security prices, deriving formulas analogous to (19) and (20) as well as solutions analogous to (21). It would be especially desirable to see future research that extends the results of Section 3 to security price models that are not complete. The usefulness of this paper will eventually be measured by the extent to which its ideas are applied to a variety of interesting optimal portfolio problems.

In addition to using optimal portfolio theory to speculate on the open market, one could speculate about whether the basic idea of this paper, decomposing a control problem into two subproblems that are connected (in this case) by the set of attainable wealths, can be applied to other kinds of stochastic control problems. The concepts of reachability and controllability are well known for classical linear deterministic systems (cf. Kalman, Falb, and Arbib (1969, p. 32), Kailath (1980, pp. 84-90)). Since reachability is analogous to what is called attainability here, perhaps the development of these system theory concepts for stochastic models would lead to the successful application of this paper's decomposition idea to a variety of situations.

REFERENCES

R. J. Elliott, Stochastic Calculus and Applications, Berlin-Heidelberg-New York, Springer-Verlag, 1982.

W. H. Fleming and R. W. Rishel, Deterministic and Stochastic Optimal Control, Berlin-Heidelberg-New York: Springer-Verlag, 1975.

J. M. Harrison and S. R. Pliska, Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stoch. Proc. Appl. 11 (1981), 215-260.

J. M. Harrison and S. R. Pliska, A Stochastic Calculus Model of Continuous Trading: Complete Markets, Stoch. Proc. Appl. 15(1983), 313-316.

J. Jacod, Calcul Stochastique et Problèmes de Martingales, Lecture Notes in Mathematics 714, Berlin-Heidelberg-New York, Springer-Verlag, 1979.

T. Kailath, Linear Systems, Englewood Cliffs, N.J.: Prentice-Hall, 1980.

R. E. Kalman, P. L. Falb, and M. A. Arbib, Topics in Mathematical System Theory, New York: McGraw-Hill, 1969.

H. M. Markowitz, Portfolio Selection, J. of Finance Vol. VII, No. 1 March 1952, p. 89.

H. M. Markowitz, Portfolio Selection, New York: John Wiley & Sons, 1959.

J. Massin, Optimal Multi-period Portfolio Policies, J. Business 41 (1968), 215-229.

R. C. Merton, Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case, Rev. Econ. Statist. LI (1969), 247-257.

R. C. Merton, Optimal Consumption and Portfolio Rules in a Continuous Time Model, J. of Econ. Th. 3 (1971), 373-413.

P.-A. Meyer, Un cours sur les integrales stochastiques, Seminaire de Probabilité X, Lecture Notes in Mathematics 511, Berlin-Heidelberg-New York: Springer-Verlag, 1976, 245-400.

S. R. Pliska, A Discrete Time Stochastic Decision Model, Advances in Filtering and Optimal Stochastic Control, edited by W. H. Fleming and L. G. Gorostiza, Lecture Notes in Control and Information Sciences 42, Berlin-Heidelberg-New York: Springer-Verlag, 1982, 290-304.

R. T. Rockafellar, Extension of Fenchel's Duality Theorem for Convex Functions, Duke Mathematical Journal 33 (1966), 81-89.

R. T. Rockafellar, Integrals which are Convex Functionals, II, Pacific J. Math 39 (1971), 439-469.

R. T. Rockafellar, Conjugate Duality and Optimization, Regional Conference Series in Applied Mathematics 16, Philadelphia: Society for Industrial and Applied Mathematics, 1974.

R. T. Rockafellar, Integral Functionals, Normal Integrands, and Measurable Selections, Nonlinear Operators and the Calculus of Variations, edited by J. P. Gossez et al., Berlin-Heidelberg-New York: Springer-Verlag, 1976.

P. A. Samuelson, Lifetime Portfolio Selection by Dynamic Stochastic Programming, Rev. Econ. Statist. LI (1969), 239-246.

W. F. Sharpe, Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk, The J. of Finance, Sept. 1964, 425-552.

J. Tobin, Liquidity Preference as Behavior Toward Risk, Review of Economic Studies Vol. 25, 68-85, 1958.

J. Tobin, The Theory of Portfolio Selection, The Theory of Interest Rates, F. H. Hahn and F. P. R. Brechling, editors, London: MacMillan Co., 1965.