A Stochastic Dynamic Programming Approach for Pricing Options on Stock-Index Futures

by

Tymur Kirillov

Master of Science in Management

Submitted in partial fulfillment of the requirements for the degree of

Master of Science in Management.

Faculty of Business, Brock University
St. Catharines, Ontario

This thesis is dedicated to my mother and father.

Abstract

The aim of this thesis is to price options on equity index futures with an application to standard options on S&P 500 futures traded on the Chicago Mercantile Exchange. Our methodology is based on stochastic dynamic programming, which can accommodate European as well as American options. The model accommodates dividends from the underlying asset. It also captures the optimal exercise strategy and the fair value of the option. This approach is an alternative to available numerical pricing methods such as binomial trees, finite differences, and ad-hoc numerical approximation techniques. Our numerical and empirical investigations demonstrate convergence, robustness, and efficiency. We use this methodology to value exchange-listed options. The European option premiums thus obtained are compared to Black's closed-form formula. They are accurate to four digits. The American option premiums also have a similar level of accuracy compared to premiums obtained using finite differences and binomial trees with a large number of time steps. The proposed model accounts for deterministic, seasonally varying dividend yield. In pricing futures options, we discover that what matters is the sum of the dividend yields over the life of the futures contract and not their distribution.

Contents

1. Introduction

In this paper, we present an alternative methodology for pricing American futures options with an application to options on S&P 500 stock-index futures. The proposed method, which is based on stochastic dynamic programming, can be used to price options on futures, and it accommodates constant as well as seasonally varying dividend payouts from the spot asset. Along with the existing spectrum of available methodologies, this alternative could help traders in price discovery, especially for over the counter (OTC) illiquid contracts. The proposed alternative method respects the true dynamics of the underlying asset. It is also competitive with other option valuation methods like finite differences and binomial trees. In addition, our method can be extended to accommodate high-dimensional pricing problems. The following background information and literature review revisit some futures options basics, and previously available pricing methodologies.

An option with a futures contract as an underlying is called an option on a futures contract or a *futures option*. The holder of a call/put futures option has the right, but not the obligation, to assume a long/short position in the underlying futures contract upon exercise and to claim an amount equal to the difference between the ongoing futures price and the strike price of the option, if positive. The option expires worthless otherwise. Depending on the style of the option, exercise can occur before or strictly at the maturity of the option. Exchange-traded futures options are by and large of American style, which may not have the same maturity as the underlying futures contract.

In the U.S., trading in options on futures contracts can be traced back to 1982 when the Commodity Futures Trading Commission (CFTC) allowed experimental trading in options with futures contracts as the underlying. Although initial option trading was limited to only one type of underlying futures contract on each exchange, this limitation was lifted and permanent trading was authorized in 1987. Trading in options on futures contracts has since proliferated to include a wide variety of futures contracts with underlying spot assets such as stock indices, Eurodollars, Treasury instruments, currencies, metals and agricultural products, which have been extensively used for

position hedging and speculative purposes (Chance and Brooks, 2007). In this paper, we value futures options with an application to options on S&P 500 index futures, which are actively traded on the largest futures exchange in the world—the Chicago Mercantile Exchange (CME). Index-futures and option contracts on the S&P500 were introduced at the CME in 1982 and 1983 respectively. Today, the S&P 500 futures contract accounts for over 90 percent of all U.S. stock index futures trading, and over the years, it, along with the corresponding option contract has experienced remarkable growth in trading volume. Thus, the trading volume for the S&P 500 futures and option contracts traded at the pit as well as on the electronic platform consisted of 9,346,637 and 10,338,677 contracts respectively in 2009 (CME Volume Report). Currently, at any given time, there are 11 standard S&P 500 futures option contracts listed on the CME. These 11 contracts consist of 8 quarterly cycle contracts (March, June, September and December), which expire in the same month as the underlying futures; and 3 serial-month contracts (for months other than those in the March-December cycle), which expire into the nearest quarterly underlying-futures contract. This means that the difference between the expiration date of the option and the underlying futures contract could be as great as ninety days.

The popularity of futures options has escalated over the years. There are some advantages to trading futures options over options on the underlying spot asset itself. Two major advantages are higher liquidity and lower trading costs. Trading futures options is often easier than trading spot options because futures prices are quoted on futures exchanges and are readily available. Furthermore, the delivery of the underlying is also much easier in the case of futures options as it is cheaper and more convenient to deliver a futures contract rather than a physical asset, upon exercise. One of the main reasons for the popularity of index-futures options, which was recently pointed out by the CME, would be the margining advantage. For instance, a financial manager may employ a strategy that includes a position in a short index-futures option and a long index-futures position for the purpose of cash equitization. In this case, during the market rallies the margin due from the futures exchange can easily offset the margin on the short option position through the same margin account. Now, if the short option is on the spot then the manager would have to withdraw the margin due from the exchange on the futures

position and deposit it with the option clearing corporation (OCC). In this example, the convenience is obvious. In addition, for some traders, it is important to maintain market exposure even after maturity of the option and the serial-month futures option contracts offer this type of convenience, as these futures options do not expire simultaneously with the underlying futures.

The European futures options can be priced using Black's (1976) closed-form solution, which can also accommodate constant, continuous, interim payouts from the underlying spot. On the other hand, American futures options demand special attention. Like American options on the spot, pricing American futures options is challenging. The challenge arises from the early-exercise feature embedded in the option contract. The early-exercise feature adds value to the option in the form of an early-exercise premium, which varies in magnitude depending on the moneyness of the option (Ramaswamy and Sundaresan, 1985; Whaley, 1986). There is no analytic result for valuing the benefit of the early-exercise feature; however, there are approximation techniques that make the attempt.

Over the years, various methodologies have been proposed to price American futures options, including numerical methods and analytic approximations. Explicit and implicit finite differences, binomial trees as well as ad-hoc methodologies are representatives of numerical-approximation methods, while the quadratic-approximation (Whaley, 1986) and compound-option approaches (Shastri and Tandon, 1986) are representatives of analytic approximation methods. The method proposed in this paper is numerical in nature.

Ramaswamy and Sundaresan (1985) as well as Brenner et al. (1989), in their option valuation procedures, assume that the dividends on the stock index are paid at a constant proportional rate. Harvey and Whaley (1992) find that such an assumption could lead to pricing errors when valuing options on indices. However, given the special relationship between spot and futures prices, it is conceivable that this problem might appear in index-futures option valuation as well. Consequently, we pre-emptively extend the constant-dividend yield assumption and adjust our methodology to accommodate deterministic, seasonally varying dividend yield.

The rest of this thesis is organized as follows. Section 2 considers some of the pertinent existing literature in detail. Section 3 introduces assumptions and describes the model. Section 4 presents the stochastic dynamic programming framework. Section 5 reports the results of numerical investigation, and Section 6 presents the results of an empirical investigation.

2. Literature Review

Black (1976) provides a thorough characterization of forward and futures contracts and, under Black-Scholes (1973) assumptions introduces, a framework for pricing European options on futures contracts. If used to price American options, Black's (1976) option-pricing model would misprice the premiums because it does not capture the early-exercise premium inherent in the American-style options. As shown in the literature, and unlike call options on the spot, call options on futures have a positive probability of early exercise, independent of payouts from the spot. Consequently—and again unlike call options on the spot, where early-exercise is solely related to interim payments—the ability to capture early-exercise premiums is much more relevant for options on futures.

In general, most studies on American futures options primarily focus on one or more of the following issues: capturing and valuing the early-exercise possibility, estimating its significance, examining comparative statics, and evaluating the performance of the proposed pricing model with an empirical investigation or simulation. In the literature, early-exercise premiums are evaluated against Black's model and performance of the proposed model is usually evaluated by comparing the simulation results to option market-prices and/or results produced by other pricing methodologies such as finite differences, binomial trees, and others. Some also attempt to point out the differences between how spot and futures option prices respond to changes in model parameters.

Brenner et al. (1985) use finite differences to price American options on the spot and corresponding futures with and without interim payments and they examine the difference between the two. They find that, under some scenarios, it might be optimal to

exercise a futures option but not an otherwise identical option on the spot. Brenner et al. report that, for options based on assets with no interim payments, call options on futures have a higher value than call options on the spot. Conversely, put options on futures have a lower value than corresponding puts on the spot. They find that the difference in price is more pronounced for puts than calls. They also observe that, when interim payments from the spot asset are introduced, the observed differences in option price decrease with an increase in the magnitude of the payments. As for the early-exercise boundary, Brenner et al. find that it is a non-decreasing function of volatility and time to maturity and a non-increasing function of the interest rate. They also observe that increases in the magnitude of interim payments decrease the probability of early-exercise for futures calls and increase such a probability for futures puts. The intuition behind this observation is as follows: For the calls, the higher the payout from the spot asset the less likely it is that the underlying futures price would reach the early-exercise boundary and trigger an early exercise. For the puts, a higher payout from the spot would propel the price toward the exercise boundary, thus enhancing the probability of an early-exercise.

Brenner et al. (1989) examine stock-index options and stock-index futures options. They find that the greater the difference between the interest rate and the dividend yield, the more prominent is the difference between their prices. In fact, this observation can be inferred on a theoretical level by noting that the futures price and the underlying spot price are identical if the dividend yield is equal to the risk-free interest rate. Brenner et al. find that Black's (1976) value is adequate for near-term, out-of-the-money American call and put futures options. They show that the early-exercise premium associated with either an in-the-money option or an option with long maturity contributes significantly to the overall option value and must not be neglected.

Ramaswamy and Sundaresan (1985) derive rational pricing restrictions and use finite differences to value American options on stock-index futures. They examine and compare the response in the early-exercise frontier to changes in the risk-free rate for both options on the spot and options on the futures. They also examine the magnitude of option mispricing due to the constant risk-free interest-rate assumption by introducing a mean-reverting square-root diffusion process instead. They find that Black's formula

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works best for at-or in-the-money options. Moreover, they find that the early-exercise frontier for spot and futures options is affected in a different way by changes in the level of the risk-free interest rate under a constant risk-free interest-rate assumption. They conclude that the optimal exercise frontier is a decreasing function of the risk-free interest rate for call options on futures and an increasing function for call options on the spot. They report, that under a constant risk-free interest-rate assumption, the early-exercise frontier is an increasing function of time to maturity for both types of options. They discover that the constant risk-free interest-rate assumption creates a mispricing error that varies between 7% and -5%, depending on the scenario, when compared to prices simulated under a stochastic interest-rate assumption. Ramaswamy and Sundaresan conclude that the price differences are due to the location of the current interest rate with respect to the long-run mean.

Shastri and Tandon (1986a) conduct a two-step analysis of American options on futures. As a first step in their analysis, Shastri and Tandon adapt the Geske and Johnson (1984) compound-option approach for pricing American spot options to American futures options valuation and evaluate the significance of the early-exercise premium under various scenarios. Their findings are consistent with those in the existing literature. For instance, they observe that, for out-of-the-money options, both the European and American options values are almost identical. They observe a divergence in prices for at- or in-the-money options that increases with time to maturity. They also find that Black's formula works best in conjunction with low volatility and risk-free interest-rate levels irrespective of the moneyness of the option. This result is consistent with theory since the early-exercise feature has a relatively low value under low risk-free interest rate-levels. In the second step of their analysis, Shastri and Tandon evaluate the performance of their American futures option pricing model and Black's model by comparing their results to option market-prices on the S&P 500 and the West German Mark futures, traded on the CME. They find that the predictive ability of Black's model is comparable to their American option pricing model. In their subsequent study Shastri and Tandon (1986b) conduct an empirical test of their adapted Geske and Johnson model. Both historical and implied volatilities are used. They discover that the market premiums substantially deviate from the prices predicted by the model. Shastri and Tandon show that the

mispricing is related to the moneyness and time to maturity of the option. They conclude that abnormal profits can be earned by exploiting such mispricing but that transaction costs would be too high to sustain the strategy.

Whaley (1986) evaluates American futures options using an adaptation of the Barone-Adesi and Whaley quadratic-approximation technique for pricing American spot options. Whaley finds that the early-exercise premium contributes meaningfully to the overall option premium. Whaley also performs a comparative empirical investigation against market prices on S&P 500 futures options. He determines the moneyness and maturity biases. In particular, he observes that out-of-the-money calls are underpriced and in-the-money calls are overpriced relative to the model. Out-of-the-money puts are overpriced and in-the-money puts are underpriced relative to the model. He finds that the maturity bias is identical for both types of options—short-term options are underpriced and long-term options are overpriced; however, the bias is more severe for the puts. Like Shastri and Tandon (1986), Whaley (1986) confirms the possibility of abnormal profits due to mispricing by employing a riskless hedging strategy. He also notes that due to transaction costs, the strategy cannot be sustained by a retail investor.

Using Whaley's quadratic approximation model, Cakici et al. (1993) evaluate options on T-Note and T-Bond futures contracts. They find that the prices obtained using Black (1976) and quadratic approximation models are identical. They show that the market overprices in-the-money calls relative to both models; however, no mispricing is detected for out or at-the-money calls. Statistically significant mispricing is observed only for short-term in-the-money options. Systematic put-mispricing tendencies are identical to those found by Whaley (1986). These results however must be interpreted with caution for, as Overdahl (1988) points out, Whaley's model systematically underestimates the critical futures price for calls and overestimates it for puts. He observes that the bias thus created varies across maturities, and that its direction is consistent to those found by Whaley and Cakici.

Kim (1994) builds on the Kim (1990) result and proposes an analytic approximation to value American futures options. Kim identifies the optimal exercise boundary by using a two-stage regression and then arrives at the futures options values by

implementing numerical integration. He reports that his approach provides more accurate values for longer-maturity options in comparison to the quadratic-approximation approach.

As a rule, researchers and practitioners rely on numerical methods for quantifying the early-exercise premium of American options. The option prices obtained by using these methods are considered to be extremely accurate. The finite differences approach is looked at by Schwartz (1977), Brenner, Courtadon and Subrahmanyam (1985) as well as Ramaswamy and Sundaresan (1985) and the lattice approach is looked at by Parkinson (1977), Cox, Ross, and Rubinstein (1979). These are the two main frameworks for numerical methods. Within these two frameworks, the time to maturity of the option is fragmented into minute intervals and, using backward induction by applying the boundary condition at every decision point, a fair value of the option is obtained at inception. It is the methodology used to evaluate an option over those small intervals that sets the lattice and finite difference framework apart. In the lattice approach, a discrete, higher-order distribution is used to approximate the evolution of the stochastic process and the value of the option at every node is the discounted risk-neutral expected payoff at the end of each interval. The finite difference approach estimates continuous partial derivatives in the differential equation and solves for the option values at each interval, such that the differential equation is satisfied at each decision point.

3. The Model

3.1 Assumptions

The stock index level S_t is a Markov process modeled as a geometric Brownian motion. Thus S_t satisfies

$$dS_t = (r - \delta(t))S_t dt + \sigma S_t dZ_t, \quad \text{for } 0 \leq t \leq T,$$

where r is a constant riskless rate; $\delta(.)$ is a deterministic function of time and represents a continuous proportional dividend yield on the index; σ is the volatility of the index

returns, which is assumed to be constant; and Z_t is the standard Brownian process. Under these assumptions

$$S_T = S_t e^{\left(r - \frac{1}{(T-t)}\int_t^T \delta(w)dw - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z}$$

where $t \leq T$ and Z is a random drawing from $N(0,1)$.

It must be noted that additional assumptions such as complete markets, impossibility of arbitrage, continuous trading, no restriction on short selling, an equal borrowing and lending rate, as well as the absence of transaction costs and taxes are standard and assumed to hold throughout. We also extend the traditional assumption of a constant proportional dividend yield to a deterministic function of time. Table 1 presents the monthly dividend yields for the year 2009. In this table the dividends are extracted using the methodology in Cornell and French (1983). The dividends are measured by taking the difference between the daily-value-weighted returns on S&P 500 including dividends and value-weighted returns excluding dividends (available from CRSP), which are then converted to annualized continuously compounded yields. In Table 1, the variability is obvious and justifies our extension.

Table1   *2009 Monthly Dividend Yields For the S&P 500 Index*

January	*February*	*March*	*April*	*May*	*June*
1.07%	3.61%	3.51%	1.45%	2.78%	2.33%
July	*August*	*September*	*October*	*November*	*December*
1.03%	2.49%	2.04%	0.89%	2.92%	2.03%

When the underlying asset pays dividends at a constant proportional dividend rate, we have

$$F_t = S_t e^{(r-\delta)(T-t)},$$

where F_t is the price of a forward contract at t for delivery at T.

Now, assuming that the dividend yield is a deterministic function of time, then the relationship between the spot and the corresponding forward can be restated as follows:

$$F_t = S_t e^{\left(r - \frac{1}{(T-t)} \int_t^T \delta(w) dw\right)(T-t)}. \tag{1}$$

This formulation is inferred from the results presented in Duffie and Stanton (1992). This relationship holds even for the futures prices as long as the risk-free interest rate is non-stochastic (Cox, Ingersoll, and Ross, 1981). A stochastic dividend-yield assumption would also break down this relationship (Lioui, 2006). Since our assumptions about the risk-free interest rate and dividend rate are deterministic in nature, we henceforth hold this relationship to be true for the purposes of subsequent futures option pricing.

Clearly, the futures price depends on the dividend stream only through the following:

$$\int_t^T \delta(w) dw.$$

This is true no matter how the dividends are distributed over the life of the futures contract. We investigate this question for European as well as American futures option contracts.

3.2 European Case for Futures Options

Black (1976) was the first to tackle the problem of futures option pricing. Black proposes a framework to price a European option on futures under the following assumptions. The risk-free rate is a fixed constant; futures prices are log normally distributed; and markets are perfect, promote liquidity and support continuous trading. Black's model simply combines the Black-Scholes option-pricing model with the cost-of-carry futures pricing model. In particular, given the current futures price F_t, stock price S_t, and exercise price K, Black solves for the European call and put option as follows:

$$c = E^* \left[e^{-r(T-t)} \max\left(0, F_T - K\right) \right]$$
$$= e^{-r(T-t)} \left[F_t N(d_1) - KN(d_2) \right],$$

for a call, and

$$p = E^* \left[e^{-r(T-t)} \max\left(0, K - F_T\right) \right]$$
$$= e^{-r(T-t)} \left[KN\left(-d_2\right) - F_t N\left(-d_1\right) \right]$$

for a put respectively, where r is the constant risk-free rate, $N(.)$ is the cumulative univariate normal distribution,

$$d_1 = \left[\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 \left(T - t\right) \right] \Big/ \sigma\sqrt{T-t},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Black effectively replaces S_t with $e^{-r(T-t)}F_t$ in the Black and Scholes pricing formula. Thus if F_t in the above formula is replaced by $S_t e^{r(T-t)}$ then, under a constant proportional risk-free rate Black's formula would collapse into the Black-Scholes (1973) pricing formula for a European option on the spot. It must be noted that Black's formula also works for European options on futures with constant proportional payout from the spot. To see this, one needs to replace F_t in the above formula with $S_t e^{(r-\delta)(T-t)}$, where δ is the continuous payout from the spot. As a result, Black's formula would collapse into Merton's (1973) formula for European options on the spot with continuous payout. This has to be true if the no-arbitrage condition is to be satisfied.

To gain insight into European-option valuation, consider a European option on a futures contract with stock/index as the underlying. Let $\{S\}$ be a Markov stochastic process for the stock price, T_2 be the maturity of the futures contract, and $t \in [t_0, t_n = T_1]$ be any date between inception and maturity of the futures option $(T_1 \leq T_2)$. Let $F_t(s)$ be the futures price at t for index level $S_t = s$. Here, $F_t(.)$ is a function of s which is determined by the arbitrage-free relationship in (1).

It is important to note that it is quite reliable to use the mathematical relationship between the spot and the futures price due to the fact that the state variable, here the stock

price, is an investment-grade asset. Investment-grade assets generally lack a convenience yield, hence forcing associated futures prices to obediently adhere to arbitrage-free futures price bounds. On the other hand, futures prices on certain commodity assets generally fail to have lower bounds due to the existence of the convenience yield. The convenience yield precludes arbitrage, thus neutralizing the market forces responsible for arbitrage-free futures prices. This is precisely why, in the case of some commodities, the above relationship does not hold and the log-normal assumption imposed on futures prices breaks down.

The holder of the European futures option pays the premium at the inception of the option contract and exercises the option at maturity if it is in-the-money; otherwise, the option expires worthless. From the perspective of an investor at time t_m, define $s = S_{t_m}$ and $F(s) = F_{t_m}(S_{t_m})$. The exercise value of European futures option holder on date t_m for $m = n$ is given by

$$v_{t_m}^e\left(F(s)\right) = v_{t_m}\left(F(s)\right) = \begin{cases} \max\left(0, F(s) - K\right), & \text{for a call} \\ \max\left(0, K - F(s)\right), & \text{for a put} \end{cases},$$

where $s = S_{t_m}$ and $F(s) = F_{t_m}(S_{t_m})$.

The holding value at date t_0 is given by

$$v_{t_0}^h\left(F(s)\right) = E\left[e^{-r(T_1 - t_0)} v_{t_n}\left(F_{t_n}(S_{t_n})\right) \mid S_{t_0} = s\right], \tag{2}$$

where $F(s) = F_{t_0}(S_{t_0})$. The holding value at t_n is given by

$$v_{t_n}^h\left(F(s)\right) = 0,$$

where $F(s) = F_{t_n}(S_{t_n})$. The expectation in (2) is computed under risk-neutral probability measure. Consequently, the option premium paid by the option holder at inception is the discounted expected payoff of the option at maturity under risk-neutral probabilities. It is

clear that the payoff is determined by the position of the stock at maturity, as intermediate price levels do not matter. On the other hand the picture is quite different when the option is American and can be exercised prior to maturity.

3.3 American Case for Futures Options

Consider an identical option, as above, except that it is American. As before, the option holder pays the premium at inception. Only this time, he may choose to exercise the option and obtain exercise proceeds on any decision date t_m, for $m = 0,...,n$. The exercise value is given by

$$v_{t_m}^e\left(F(s)\right) = \begin{cases} \max\left(0, F(s) - K\right), & \text{for a call} \\ \max\left(0, K - F(s)\right), & \text{for a put} \end{cases}.$$

The holding value of the option at t_m is

$$v_{t_m}^h\left(F(s)\right) = E\left[\rho_{t_m} v_{t_{m+1}}\left(F_{t_{m+1}}\left(S_{t_{m+1}}\right)\right) \mid S_{t_m} = s\right], \quad \text{for } m = 0,...,n-1, \qquad (3)$$

where $\rho_{t_m} = e^{-r(t_{m+1} - t_m)}$. A rational option holder would formulate his optimal strategy as follows. Throughout the life of the option, the option holder evaluates the benefit of immediate exercise compared to the benefit of holding the option for at least until the subsequent decision instant. It is optimal to exercise the option only if the exercise value exceeds the holding value of the option. As a result, any excess in exercise proceeds over the holding value would trigger an immediate exercise of the option. Otherwise, the holder would postpone the exercise until the optimality condition is met, or else let the option expire worthless.

The overall value function is given by

$$v_{t_m}\left(F(s)\right) = \begin{cases} \max\left(v_{t_m}^e\left(F(s)\right), v_{t_m}^h\left(F(s)\right)\right), & \text{for } m = 0,...,n-1 \\ v_{t_m}^e\left(F(s)\right), & \text{for } m = n \end{cases}. \qquad (4)$$

And the optimal exercise region is defined as follows:

$$\left\{ \left(t_m, s \right) \text{ such that } v_{t_m}^e \left(F(s) \right) > v_{t_m}^h \left(F(s) \right) \right\}.$$

There are two possible scenarios:

1. When $T_1 = T_2$, the price of the futures contract converges to the spot price of the underlying, resulting in

$$v_{t_n}^e \left(F(s) \right) = \max \left(0, s - K \right).$$

This is a typical situation for standard CME options on S&P 500 futures contracts expiring in the quarterly March-December cycle. As noted earlier, these option contracts expire in the same month as the underlying futures.

2. When $T_1 < T_2$, the exercise-value function at t_n is given by

$$v_{t_n}^e \left(F(s) \right) = \max \left(0, F(s) - K \right).$$

This situation is true in the case of the standard CME serial month options on the S&P 500 futures. These option contracts expire into the nearest futures contract, which in turn, expire in one of the March-December quarterly months.

From the perspective of an investor at maturity, s is known and $v_{t_n} \left(F(s) \right)$ can be easily computed for all s. However, v_{t_m} is usually unknown for $m = 0, ..., n-1$. We approximate the overall value function using a piecewise linear interpolation over a finite grid. Our methodology in approximating v_{t_m} is identical to Ben-Ameur et al. (2004), with adaptations to fit the case of futures contracts as the underlying asset. The approximation details are discussed in Section 4 and are built on the assumptions discussed in section 3.1.

3.4 Treatment of Dividends

Consistent with our assumptions the dividends throughout the life of the futures contract are treated as follows. A desired collection of annualized, continuously compounded dividend yields δ_j, for $j = 1, ..., N$, along with the corresponding incidence

points t_j, for $j = 1, \ldots, N$, are superimposed on a time line. The number (N), the magnitude and the frequency $\left(\Delta t = t_{j+1} - t_j \right)$ of the selected dividend yields are pre-specified and fixed to fit a particular scenario. The timeline either spans the period from inception to maturity of the option in the case of $t_N = T_1 = T_2$ or from inception of the option to maturity of the futures contract in the case of $T_1 < T_2 = t_N$. Thus by construction, δ_t is a piecewise linear function of time and, for any point in time t, the corresponding dividend yield is given by

$$\delta_t = \gamma_j + \lambda_j t, \text{ for } t \in \left[t_j, t_{j+1} \right],$$

where γ_j and λ_j are the intercept and slope of the piecewise linear segment defined on $\left[t_j, t_{j+1} \right]$ and can be obtained as follows:

$$\gamma_j = \frac{\delta_j \times t_{j+1} - \delta_{j+1} \times t_j}{t_{j+1} - t_j}, \tag{5}$$

$$\lambda_j = \frac{\delta_{j+1} - \delta_j}{t_{j+1} - t_j}. \tag{6}$$

At this point, it is trivial to compute the annualized accumulated divided yield between any two points on the time line. Thus, the accumulated annualized dividend yield on $[t, u]$ can be evaluated in closed form as follows:

For $t_j \leq t < u \leq t_{j+1}$, we have

$$\tfrac{1}{u-t} \int_t^u \delta(w) \, dw = \gamma_j + \tfrac{1}{2} \lambda_j (u+t), \tag{7}$$

and for $t_j < t < t_{j+1} < u \leq t_{j+2}$, we have

$$\tfrac{1}{u-t} \int_t^u \delta(w) \, dw = \tfrac{1}{u-t} \left[\left(\gamma_j \left(t_{j+1} - t \right) + \tfrac{1}{2} \lambda_j \left(t_{j+1}^2 - t^2 \right) \right) + \left(\gamma_{j+1} \left(u - t_{j+1} \right) + \tfrac{1}{2} \lambda_{j+1} \left(u^2 - t_{j+1}^2 \right) \right) \right]. \tag{8}$$

These computations are employed at various stages while solving the DP equation.

4. Dynamic Programming Framework

Except for particular cases, American options cannot be priced in closed form. Here, we use a dynamic program for pricing American futures options, which is as follows.

Let $G=\{a_0=0,a_1...,a_p,a_{p+1}=+\infty\}$ be a grid of points representing the stock index. Assume the availability of a piecewise linear approximation $\hat{v}_{t_{m+1}}(.)$ for the overall value function $v_{t_{m+1}}(.)$, seen as a function of the stock index $s=S_{t_{m+1}}$ through the futures price $F_{t_{m+1}}$ at time t_{m+1}. This assumption is not a strong one since we do know the true value function $v_{t_n}(.)$ at the maturity of the option T_1. The approximation $\hat{v}_{t_{m+1}}(.)$ can be expressed as follows:

$$\hat{v}_{t_{m+1}}(F(s))=\sum_{i=0}^{p}\left(\alpha_i^{m+1}+\beta_i^{m+1}F_{t_{m+1}}(s)\right)\mathrm{I}\left(a_i<s\le a_{i+1}\right), \qquad (9)$$

where $s=S_{t_{m+1}}$. The relationship between $S_{t_{m+1}}$ and $F_{t_{m+1}}$ comes from the cost-of-carry relationship:

$$F_{t_{m+1}}=S_{t_{m+1}}e^{\left(r-\eta_{t_{m+1}}\right)\left(T_2-t_{m+1}\right)}.$$

Here, T_2 is the maturity of the futures contract $(T_2\ge T_1)$ and

$$\eta_{t_{m+1}}=\frac{1}{T_2-t_{m+1}}\int_{t_{m+1}}^{T_2}\delta(w)\,dw,$$

where $\delta(.)$ is a deterministic function of time for the continuous proportional dividend rate on the index.

The local coefficients α_i^{m+1} and β_i^{m+1}, for $i=0,...,p$, are

$$\beta_i^{m+1} = \frac{\hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_{i+1}\right)\right) - \hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_i\right)\right)}{F_{t_{m+1}}\left(a_{i+1}\right) - F_{t_{m+1}}\left(a_i\right)}$$

$$= \frac{\hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_{i+1}\right)\right) - \hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_i\right)\right)}{\left(a_{i+1} - a_i\right)e^{\left(r - \eta_{m+1}\right)\left(T_2 - t_{m+1}\right)}}$$

and

$$\alpha_i^{m+1} = \frac{\hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_i\right)\right)F_{t_{m+1}}\left(a_{i+1}\right) - \hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(a_{i+1}\right)\right)F_{t_{m+1}}\left(a_i\right)}{F_{t_{m+1}}\left(a_{i+1}\right) - F_{t_{m+1}}\left(a_i\right)}.$$

Here, β_i^{m+1} is the slope on $\left[a_i, a_{i+1}\right]$, and is analogous to the delta of the option, as approximated by finite differences.

The slope and intercept at $i = p$ are

$$\alpha_p^{m+1} = \alpha_{p-1}^{m+1} \text{ and } \beta_p^{m+1} = \beta_{p-1}^{m+1}.$$

The no-arbitrage pricing gives the holding-value function $v_{t_m}^h(.)$ at t_m as an average under the risk-neutral probability measure of the overall value function $v_{t_{m+1}}(.)$, discounted back from time t_{m+1} to time t_m as follows:

$$v_{t_m}^h\left(F(s)\right) = E\left[\rho_{t_m} v_{t_{m+1}}\left(F_{t_{m+1}}\left(S_{t_{m+1}}\right)\right) \mid S_{t_m} = s\right], \text{ for all } s,$$

where

$$\rho_{t_m} = e^{-r\left(t_{m+1} - t_m\right)}.$$

The holding value of the option, like the overall value of the option, cannot be obtained in closed form. The idea is to approximate it over the finite grid G:

$$\tilde{v}_{t_m}^h\left(F\left(a_k\right)\right) = E\left[\rho_{t_m} \hat{v}_{t_{m+1}}\left(F_{t_{m+1}}\left(S_{t_{m+1}}\right)\right) \mid S_{t_m} = a_k\right], \text{ for all } a_k \in (G) \qquad (10)$$

$$= \rho_{t_m} \sum_{i=0}^{p}\left(\alpha_i^{m+1} A_{ki}^m + \beta_i^{m+1} B_{ki}^m\right),$$

where

$$A_{ki}^m = E\left[I\left(a_i < S_{t_{m+1}} \le a_{i+1}\right) | S_{t_m} = a_k \right]$$

and

$$B_{ki}^m = E\left[F_{t_{m+1}}\left(S_{t_{m+1}}\right) I\left(a_i < S_{t_{m+1}} \le a_{i+1}\right) | S_{t_m} = a_k \right].$$

The coefficients A_{ki}^m and B_{ki}^m can be interpreted as transition parameters. They characterize the dynamics of the stock index. Our numerical procedure can be implemented efficiently as long as the transition parameters are derived in closed form. Under the geometric Brownian motion hypothesis, we have A_{ki}^m and B_{ki}^m in closed-form as follows:

$$A_{ki}^m = E[I(\frac{a_i}{a_k} < e^{(r-\theta_{t_m}-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}Z} \le \frac{a_{i+1}}{a_k})]$$

$$= \begin{cases} N(c_{k,1}), & \text{for } i = 0 \\ N(c_{k,i+1}) - N(c_{k,i}), & \text{for } 1 \le i \le p\text{-}1 \\ 1 - N(c_{k,p}), & \text{for } i = p \end{cases}$$

and

$$B_{ki}^m = E[a_k e^{(r-\theta_{t_m}-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}Z} I(\frac{a_i}{a_k} < e^{(r-\theta_{t_m}-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}Z} \le \frac{a_{i+1}}{a_k})]$$

$$= \begin{cases} a_k N(c_{k,1}-\sigma\sqrt{\Delta t_m})e^{(r-\theta_{t_m})\Delta t}, & \text{for } i = 0 \\ a_k [N(c_{k,i+1}-\sigma\sqrt{\Delta t_m}) - N(c_{k,i}-\sigma\sqrt{\Delta t_m})]e^{(r-\theta_{t_m})\Delta t}, & \text{for } 1 \le i \le p-1, \\ a_k [1-N(c_{k,p}-\sigma\sqrt{\Delta t_m})]e^{(r-\theta_{t_m})\Delta t}, & \text{for } i = p \end{cases}$$

where

$$\theta_{t_m} = \frac{1}{t_{m+1}-t_m}\int_{t_m}^{t_{m+1}} \delta(w)\,dw.$$

The futures option pricing DP algorithm runs as follows:

1. Set $m = n-1$. Compute $\tilde{v}_{t_n}\left(F\left(a_k\right)\right) = v_{t_n}^e\left(F\left(a_k\right)\right)$, for all $a_k \in G$, as in $\left(4\right)$.

2. Interpolate $\tilde{v}_{t_n}\left(.\right)$ defined on G to $\hat{v}_{t_n}\left(.\right)$, which is defined on the overall state space, as in $\left(9\right)$.

3. For $m = n-1,...,0$, do;

   a) Compute $\tilde{v}_{t_m}^h\left(.\right)$ on G, as in $\left(10\right)$;

   b) Compute $\tilde{v}_{t_m}\left(.\right) = \max\left(\tilde{v}_{t_m}^h\left(.\right), v_{t_m}^e\left(.\right)\right)$ on G, as in $\left(4\right)$;

   c) Interpolate $\tilde{v}_{t_{m+1}}\left(.\right)$ defined on G to $\hat{v}_{t_m}\left(.\right)$ defined on the overall state space, using $\left(9\right)$.

At time t_0, we obtain the value function $\hat{v}_{t_0}\left(.\right)$ defined on the overall state space, and the optimal exercise strategy defined over the time period $\left[t_0, T_1\right]$. The latter is as follows: Exercise at the first observation date t_m and stock index level $s = S_{t_m}$, where

$$v_{t_m}^e\left(F\left(s\right)\right) > \hat{v}_{t_m}^h\left(F\left(s\right)\right).$$

The DP procedure does respect the true dynamics of the underlying asset through the transition parameters A_{ki}^m and B_{ki}^m. This is the major advantage when compared to competing methodologies.

5. Numerical Investigation

The following numerical investigation assesses the degree of comparability of our results with those available in the existing literature. In particular, we compare the American futures option prices obtained using the DP approach to those obtained using other methodologies such as the finite differences, the binomial trees, the analytic approximation approach (Whaley, 1986) and others. The European futures option prices obtained using the DP approach are also compared to those obtained using Black's formula for the purposes of observing convergence.

In Tables 2–6, presented below, the call option is on the stock/index futures contract with exercise price $K = 100$. The inception and maturity for the option and the futures contracts are identical $(T_1 = T_2)$. The decision dates are equally spaced and their number is fixed to the number of days left until the maturity of the option contract. Specific sets of time to maturity, volatility, risk-free interest rate, and dividend yield are used for comparison purposes. In most tables, the initial futures prices vary from deeply out-of-the-money to deeply in-the-money within each scenario. The values reported under the DP approach are obtained by setting the grid size to 2000 $(p = 2000)$ points, unless otherwise stated. For each specific scenario, each table presents the American option values obtained using the DP approach and other alternative methodologies, as well as the corresponding European option values. The early-exercise premium, which is measured against Black's values and the computation (CPU) time in seconds are also reported. Our code lines are executed using a 2.13 GHz Windows PC.

In Table 2, the underlying futures price ranges between 80 and 120, and the remaining time to maturity is either 3 or 6 months. The volatility parameter is either 20% or 40%, and the risk-free interest rate is either 8% or 12%. In the case of European option prices, convergence to Black's prices is evident as grid size increases from 400 to 1600 points. American option prices converge from above as the grid size p increases. In Table 2, our American option prices are compared to those reported in Chamberlain and Chiu (1990). They report the prices obtained using three different methods, namely binomial trees, explicit and implicit finite differences. Our results are comparable to those obtained using the binomial-tree method. The results obtained using the explicit finite differences are close to those obtained using the DP and the binomial approaches. However, the results obtained using the implicit finite differences approach show divergence from both. It is evident from Table 2 and consistent with theory that the deeper the option is in-the-money, the higher is the early-exercise premium; and, the higher the volatility and time to maturity, the higher is the fair value of the option.

Table 2

	F	European				American				Chamberlain and Chiu (1990)		
		DP			Black's	DP						
		Grid 400	*Grid 800*	*Grid 1600*	*Formula*	*Grid 400*	*Grid 800*	*Grid 1600*	*ExPrem*	*Binomial*	*ExpFD*	*ImpFD*
Scenario 1	80	0.0409	0.0398	0.0393	0.0391	0.0413	0.0399	0.0394	0.0003	0.04	0.05	0
T=0.25	90	0.6997	0.6986	0.6984	0.6983	0.7009	0.6999	0.6996	0.0013	0.7	0.77	0.03
r=0.08	100	3.9103	3.9092	3.9089	3.9088	3.9247	3.9236	3.9233	0.0145	3.93	4.18	1.07
σ=0.20	110	10.7387	10.7375	10.7371	10.7370	10.8190	10.8178	10.8175	0.0805	10.82	11.21	10
δ=0	120	19.7513	19.7491	19.7486	19.7484	20.0279	20.0268	20.0265	0.2781	20.03	20.32	20
Scenario 2	80	1.1657	1.1633	1.1627	1.1624	1.1677	1.1652	1.1646	0.0022	1.17	1.37	0
T=0.25	90	3.5210	3.5188	3.5182	3.5180	3.5298	3.5277	3.5271	0.0091	3.53	3.92	0.03
r=0.08	100	7.8107	7.8086	7.8080	7.8078	7.8396	7.8374	7.8369	0.0290	7.85	8.47	1.13
σ=0.40	110	14.0119	14.0097	14.0092	14.0090	14.0856	14.0834	14.0829	0.0738	14.09	15.25	10
δ=0	120	21.7117	21.7095	21.7089	21.7088	21.8696	21.8674	21.8668	0.1581	21.86	22.21	20
Scenario 3	80	1.1540	1.1517	1.1511	1.1509	1.1571	1.1548	1.1542	0.0033	1.16	1.36	0
T=0.25	90	3.4859	3.4837	3.4832	3.4830	3.5001	3.4980	3.4975	0.0144	3.5	3.89	0.03
r=0.12	100	7.7330	7.7309	7.7303	7.7301	7.7794	7.7772	7.7767	0.0465	7.79	8.41	1.13
σ=0.40	110	13.8725	13.8703	13.8698	13.8696	13.9912	13.9891	13.9885	0.1189	14	15.16	10
δ=0	120	21.4958	21.4935	21.4929	21.4927	21.7509	21.7486	21.7481	0.2553	21.74	22.04	20
Avg. CPU		8	33	133		12	49	210				
Scenario 4	80	0.2994	0.2976	0.2972	0.297	0.3003	0.2985	0.298	0.001	0.29	0.4	0
T=0.5	90	1.7048	1.7034	1.7031	1.703	1.7132	1.7118	1.7115	0.0085	1.7	1.96	0.03
r=0.08	100	5.4182	5.4167	5.4163	5.4162	5.4641	5.4626	5.4622	0.046	5.47	5.91	1.07
σ=0.20	110	11.7347	11.733	11.7326	11.7324	11.8998	11.8982	11.8977	0.1653	11.9	12.9	10
δ=0	120	19.9106	19.9086	19.9081	19.908	20.356	20.3542	20.3538	0.4458	20.36	20.51	20
Scenario 5	80	0.2954	0.2923	0.2914	0.2911	0.2969	0.2936	0.2928	0.0016	0.29	0.39	0
T=0.5	90	1.6728	1.6701	1.6695	1.6692	1.6864	1.6837	1.6831	0.0139	1.68	1.92	0.03
r=0.12	100	5.3130	5.3099	5.3092	5.3089	5.3873	5.3843	5.3835	0.0746	5.39	5.81	1.07
σ=0.20	110	11.5048	11.5013	11.5004	11.5001	11.7729	11.7693	11.7685	0.2683	11.77	12.68	10
δ=0	120	19.5195	19.5152	19.5141	19.5137	20.2447	20.2413	20.2404	0.7266	20.24	20.29	20
Avg. CPU		13	56	253		18	73	323				

Table 3 reports prices for American futures options under similar scenarios as in Table 2. The objective is to compare the prices obtained using the analytic approximation (Whaley, 1986) to prices obtained using the DP approach. The grid size is fixed at 2000 points with futures prices varying between 80 and 120. The volatility is either 15% or 30% and time to maturity is either 3 or 6 months. Table 3 also reports the early-exercise premium and computation time in seconds. According to Table 3 the prices obtained using the DP approach behave in a similar manner to those in Table 2. In particular, as the moneyness of the option increases, the early-exercise premium consistently increases under each scenario. Tendencies such as the increase in the fair price of the option due to increase in volatility and increase in the exercise premium due to increase in the risk-free interest rate are identical to those found in Table 2.

Table 3		Whaley(1986)		Black's	DP		
	F	American	ExPrem	European	American	ExPrem	CPU(sec)
Scenario 1	80	0.0029	0.0002	0.0027	0.0028	0.0001	307
T=0.25	90	0.2547	0.0018	0.2529	0.2533	0.0004	300
r=0.08	100	2.9458	0.0137	2.9321	2.9429	0.0108	307
σ=0.15	110	10.2627	0.0875	10.1752	10.2715	0.0963	306
δ=0	120	20	0.3761	19.6239	20	0.3761	307
Scenario 2	80	0.003	0.0003	0.0027	0.0028	0.0001	303
T=0.25	90	0.2533	0.0029	0.2504	0.2510	0.0006	301
r=0.12	100	2.9257	0.0227	2.9030	2.9204	0.0174	300
σ=0.15	110	10.2205	0.1465	10.0740	10.2300	0.1561	297
δ=0	120	20	0.5714	19.4286	20	0.5714	304
Scenario 3	80	0.3986	0.0030	0.3956	0.3962	0.0006	305
T=0.25	90	1.9913	0.0096	1.9817	1.9862	0.0045	302
r=0.08	100	5.8878	0.0277	5.8601	5.8819	0.0217	308
σ=0.30	110	12.3237	0.0710	12.2527	12.3253	0.0725	312
δ=0	120	20.647	0.1694	20.4776	20.6639	0.1863	294
Scenario 4	80	0.0603	0.0020	0.0583	0.0587	0.0004	462
T=0.5	90	0.8256	0.0106	0.8150	0.8186	0.0036	453
r=0.08	100	4.1099	0.0463	4.0636	4.0984	0.0348	469
σ=0.15	110	10.8584	0.1752	10.6832	10.8621	0.1790	461
δ=0	120	20.0018	0.5913	19.4105	20.0075	0.5970	451

Table 4 presents the option prices obtained using implicit finite differences, Kim's (1994) approach and the DP approach. The futures prices vary between 90 and 110, the interest rate is either 8% or 12% and the volatility is either 20% or 30%. Time to maturity is either 6 months or 3 years. The prices obtained using the DP approach are identical to

those obtained using the finite differences approach and Kim's approximation technique. The values obtained by DP correspond with Kim's results. In particular, the early-exercise premium tends to increase with the moneyness and volatility of the underlying. Prolonging the time to maturity increases the early-exercise premium.

Table 4		Kim (1994)			Black's		DP	
	F	ImpFD	American	ExPrem	European	American	ExPrem	CPU(sec)
Scenario 1	90	1.7100	1.7100	0.0070	1.703	1.7116	0.0086	460
T=0.5	95	3.2400	3.2400	0.0183	3.2217	3.2430	0.0213	465
r=0.08	100	5.4600	5.4600	0.0438	5.4162	5.4626	0.0464	471
σ=0.20	105	8.3700	8.3700	0.0899	8.2801	8.3716	0.0915	474
δ=0	110	11.9000	11.9000	0.1676	11.7324	11.8985	0.1661	471
Scenario 2	90	3.8600	3.8600	0.0266	3.8334	3.8565	0.0231	475
T=0.5	95	5.7800	5.7800	0.0398	5.7402	5.7816	0.0414	477
r=0.08	100	8.1800	8.1800	0.0642	8.1158	8.1854	0.0696	465
σ=0.30	105	11.0500	11.0500	0.1112	10.9388	11.0501	0.1113	483
δ=0	110	14.3400	14.3400	0.1725	14.1675	14.3371	0.1696	546
Scenario 3	90	7.2500	7.2500	0.4406	6.8094	7.2521	0.4427	2366
T=3	95	9.3300	9.3300	0.6359	8.6941	9.3309	0.6368	2269
r=0.08	100	11.7000	11.7000	0.8831	10.8169	11.7054	0.8885	1981
σ=0.20	105	14.3700	14.3700	1.2074	13.1626	14.3701	1.2075	2180
δ=0	110	17.3200	17.3200	1.6066	15.7134	17.3170	1.6036	2150
Scenario 4	90	12.5200	12.5200	0.8280	11.692	12.5189	0.8269	2094
T=3	95	14.8900	14.8900	1.0600	13.83	14.8896	1.0596	2285
r=0.08	100	17.4600	17.4600	1.3351	16.1249	17.4588	1.3339	2130
σ=0.30	105	20.2200	20.2200	1.6542	18.5658	20.2188	1.6530	2171
δ=0	110	23.1600	23.1600	2.0183	21.1417	23.1619	2.0202	2103
Scenario 5	90	1.6800	1.6800	0.0108	1.6692	1.6830	0.0138	463
T=0.5	95	3.1900	3.1900	0.0321	3.1579	3.1919	0.0340	477
r=0.12	100	5.3800	5.3800	0.0711	5.3089	5.3834	0.0745	486
σ=0.20	105	8.2600	8.2600	0.1439	8.1161	8.2635	0.1474	469
δ=0	110	11.7700	11.7700	0.2699	11.5001	11.7684	0.2683	468
Scenario 6	90	3.7900	3.7900	0.0325	3.7575	3.7943	0.0368	472
T=0.5	95	5.6900	5.6900	0.0635	5.6265	5.6927	0.0662	482
r=0.12	100	8.0700	8.0700	0.1149	7.9551	8.0668	0.1117	464
σ=0.30	105	10.9000	10.9000	0.1778	10.7222	10.9011	0.1789	478
δ=0	110	14.1600	14.1600	0.2730	13.887	14.1604	0.2734	465

In Table 5, only the futures price varies within each scenario. The DP futures option prices are compared to prices obtained using the implicit finite differences. The interest rate and the dividend yield are fixed at 10% and 5% respectively. Although in their paper Ramaswamy and Sundaresan report that volatility of 25% was used to compute the option values, their European option values are close to Blacks's values at the volatility level of

15%. Hence, we use 15% volatility to compute our values. The option values computed using DP are comparable to those reported by Ramaswamy and Sundaresan.

Table 5		Ramaswamy and Sundaresan			DP	Black's	DP	
	F	American	European	ExPrem	American	European	ExPrem	CPU(sec)
	80.9924	0.0054	0.0054	0	0.0050	0.0049	0.0001	296
	86.0544	0.0577	0.0577	0	0.0577	0.0575	0.0002	302
Scenario 1	91.1165	0.3542	0.3535	0.0007	0.3627	0.3619	0.0008	298
T=0.25	96.1785	1.3577	1.3535	0.0042	1.3854	1.3808	0.0046	302
r=0.10	101.2405	3.5692	3.551	0.0182	3.6095	3.5899	0.0196	299
σ=0.15	106.3025	7.0703	7.0107	0.0596	7.1062	7.0438	0.0624	302
δ=0.05	111.3646	11.4961	11.3411	0.155	11.5222	11.363	0.1592	300
	116.4266	16.4266	16.087	0.3396	16.4465	16.1022	0.3443	299
	121.4886	21.4886	20.9772	0.5114	21.5094	20.9902	0.5192	309
	81.9971	0.1073	0.107	0.0003	0.1101	0.1096	0.0005	443
	87.1219	0.422	0.4202	0.0018	0.4336	0.4316	0.0021	444
Scenario 2	92.2468	1.2341	1.2265	0.0076	1.2618	1.2535	0.0082	451
T=0.5	97.3716	2.8403	2.8419	*-0.0016*	2.8848	2.8580	0.0268	447
r=0.10	102.4964	5.4089	5.3396	0.0693	5.4627	5.3907	0.072	448
σ=0.15	107.6212	8.9023	8.7408	0.1615	8.9563	8.7901	0.1662	448
δ=0.05	112.746	13.1367	12.8054	0.3313	13.1859	12.8473	0.3386	450
	117.8709	17.9011	17.2874	0.6137	17.9456	17.3214	0.6242	451
	122.9957	22.9957	21.9931	1.0026	23.0378	22.0213	1.0165	447
	83.0143	0.3683	0.3659	0.0024	0.3799	0.3771	0.0028	593
	88.2027	0.9953	0.9865	0.0088	1.021	1.0114	0.0096	593
Scenario 3	93.3911	2.2075	2.1808	0.0267	2.2506	2.2224	0.0282	619
T=0.75	98.5795	4.176	4.1076	0.0684	4.2343	4.1631	0.0712	619
r=0.10	103.7679	6.9619	6.8089	0.153	7.0292	6.8712	0.158	618
σ=0.15	108.9563	10.5127	10.2063	0.3064	10.5825	10.268	0.3145	610
δ=0.05	114.1447	14.7079	14.1475	0.5604	14.7763	14.2038	0.5725	611
	119.3331	19.4167	18.4649	0.9518	19.4826	18.5141	0.9685	614
	124.5215	24.5215	23.0169	1.5046	24.5854	23.0593	1.5261	615

6. Empirical Investigation

6.1 Data

In our empirical investigation, we implement the DP approach to value four standard CME S&P 500 futures options, which traded on the CME in 2009. The results are compared to the corresponding closing prices quoted on the CME at the end of the first trading day. The underlying for the selected options is the S&P 500 December 2009 futures contract. The inception dates for all four options span the period between June and August 2009. Options 1 and 4 have roughly 4 months remaining to maturity. Options 2 and 3 have roughly 6 months as the remaining time to maturity. Options 1, 2 and 3 are quarterly options and expire in the same month as the underlying futures contract ($T_1 = T_2$). Option 4 is a serial-month option and expires in November 2009, which is prior to the expiry of the underlying futures ($T_1 < T_2$). This enables us to test the model under cases 1 and 2, which were discussed in Section 3.3. Options 1 and 3 are out-of-the-money. Options 2 and 4 are at-the-money and in-the-money, respectively. The daily closing prices of the selected options and the corresponding underlying futures contracts along with the corresponding inception and maturity dates are provided by Datastream. Our state variable, which is the S&P 500 index level on the day of inception, is provided by CRSP. The risk-free interest rate inputs for each option are constructed from the discount yields on Treasury Bills, that are available from the Federal Reserve website. Hull (2009) states that, when the cost of carry and the convenience yield (dividend yield) are functions only of time, it can be shown that the volatility of the futures price is the same as the volatility of the underlying asset. Therefore for our volatility parameter we use historical volatility estimated using the value-weighted log-returns on the S&P 500 index, which are also available from CRSP. As a separate case, we also value the options using implied volatility measures, which are provided by Datastream. In our experiment, we employ a seasonally varying dividend yield, which varies month to month throughout the life of the underlying futures contract. The dividend yield is measured using the Cornell and French (1983) methodology, as discussed Section 3.1. The monthly annualized dividend yields employed herein are presented in Table1. For the sake of exposition, the dividend flow for options 2 and 3 with

an inception date in June 2009 is 2.33%, 1.03%, 2.49%, 2.04%, 0.89%, 2.92% and 2.03% for June, July, August, September, October, November and December, respectively.

6.2 Results

As we proceed with the empirical investigation we value each option under various volatility estimates. When it comes to volatility parameters for the purpose of option valuation, Hull (2009) recommends the daily historical volatility estimates based on the most recent 90 to 180 days, annualized by the square root of 252 days per year. Hull also recommends a rule of thumb for a volatility parameter. He suggests estimating historical volatility over the number of days to which this volatility would be applied, i.e., using the estimation window equal to the remaining time to maturity of the option. In our experiment we evaluate the selected options under historical volatility estimates based on 30, 60, 90 and 180 day estimation windows. We also evaluate the options under an implied volatility measure based on Black's formula. The historical volatility, as estimated following Hull, somewhat agrees with the implied volatility provided by CRSP for the estimation window of 60 and 90 days. When we enlarge the estimation window to 180 days, the historical volatility no longer agrees with the implied volatility measure. Indeed this would be the case since it is a well-known fact that the S&P500 index was extremely volatile at the end of 2008, and beginning of 2009. Our model is a constant volatility model and therefore cannot capture this change in the volatility structure. Table 7 presents the results of our empirical investigation.

In Table 7, the volatility parameters are reported in the first row, the DP option values in index points are reported in the second row, the European (Black's) values are reported in the third row and the early-exercise premiums in the fourth. Black's model does not require that the option contract and the underlying futures contract mature at the same time (Hull, 2009); therefore, we report Black's European option values for option 4 as well.

The sensitivity of the option values and the early-exercise premiums to varying volatility estimates is evident from the table. For options 2 and 3, the table reports the highest early-exercise premiums, which correspond to relatively higher volatility estimates. Table 8 further summarises the results of the empirical investigation.

For each option, Table 8 presents the closing price, maturity in days, closest prices obtained using the DP approach with historical and implied volatility, corresponding volatility estimation windows (EW) in days and the pricing errors as a percentage of the closing price. The pricing errors reported are as high as 4.04% and as low as 0.20% under the historical volatility measure.

Table 7

| | | Closing Price | Volatility Estimation Window (Days) | | | | |
			30	60	90	180	IMP VOL
Option 1	t=26/08/09						
	S=1028.12	28.2	0.1820	0.1900	0.2290	0.3162	0.2305
	F=1022.3		18.2338	19.8829	27.7726	46.4914	28.0830
	K=1085		18.2280	19.8767	27.7651	46.4807	28.0754
	r=0.0015		0.0058	0.0062	0.0075	0.0107	0.0076
	T_1=0.31						
Option 2	t=30/06/09						
	S=919.32	63.4	0.2278	0.2591	0.3366	0.4705	0.2634
	F=911.3		55.7222	63.5241	82.8091	115.9950	64.6054
	K=915		55.7007	63.4996	82.7771	115.9497	64.5805
	r=0.0035		0.0215	0.0245	0.0321	0.0453	0.0249
	T_1=0.47						
Option 3	t=24/06/09						
	S=900.94	63.8	0.2346	0.2747	0.3437	0.4864	0.2791
	F=893.8		53.4030	63.3858	80.5585	115.9107	64.4934
	K=905		53.3812	63.3599	80.5257	115.8631	64.4671
	r=0.0035		0.0219	0.0259	0.0328	0.0476	0.0263
	T_1=0.49						
Option 4	t=27/07/09						
	S=982.18						
	F=975.5	85.6	0.2262	0.2358	0.2851	0.4039	0.2668
	K=925		77.2369	79.1415	89.0609	113.5526	85.3536
	r=0.0027		77.2211	79.1252	89.0426	113.5291	85.3361
	T_1=0.32		0.0159	0.0163	0.0183	0.0235	0.0175
	T_2=0.40						

Pricing errors, as high as 1.90% and as low as -0.29%, are reported for the implied volatility measure. In this case, our results are almost perfect. For the best performance among historical volatility measures, the relative error is as high as 4.04% and as low as 0.20% of the closing prices. Now, if options 2 and 3 were priced with a 180-day volatility

estimate, the pricing errors would have been around 83% and 82% respectively. This result underscores the importance of the volatility measure used in the model.

Table 8				Historical Volatility		Implied Volatility	
	Price	Maturity	EW	Price	Error %	Price	Error %
Option 1	28.2	111	90	27.7726	-1.52%	28.0830	0.42%
Option 2	63.4	170	60	63.5241	0.20%	64.6054	1.90%
Option 3	63.8	176	60	63.3858	-0.65%	64.4934	1.09%
Option 4	85.6	143	90	89.0609	4.04%	85.3536	-0.29%

7. Conclusion

In this paper, we propose a methodology based on stochastic dynamic programming for valuing options on stock index futures. Our numerical investigation demonstrates convergence and robustness when used to value both the European as well as the American options on futures contracts. Our results are comparable to those available in the existing literature obtained using alternative numerical methodologies and approximation techniques. The early-exercise premium mimics the behaviour observed in the existing literature. In our empirical investigation, we use the DP approach to value the standard S&P 500 futures option contracts under various estimates of volatility and an assumption of a seasonally varying dividend yield. The selected futures options contracts are traded in a very liquid and efficient environment. We consider both the case where the option and the underlying futures contract mature in the same month and the case where the option expires prior to the underlying futures. The results of the empirical investigation are almost perfect and produce accurate futures options prices. Further avenues of research may include extensions to accommodate stochastic volatility, or modelling volatility as a deterministic function of time. As long as the transition parameters can be computed in closed form, these extensions would not interfere with the efficiency of the model. The accuracy of the technique can also be improved by extending linear interpolations to high-order polynomial approximations (Ben-Ameur et al., 2002). The dynamic program can be further extended to accommodate stochastic volatility and jump-diffusion processes. It can also be combined with Monte Carlo simulation to accommodate high-dimensional problems (Longstaff and Schwartz, 2001).

References

H. Ben-Ameur, M. Breton and P. Francois. A Dynamic Programming Approach to Price Installment Options." *European Journal of Operational Research* 169 (2006), 667-76.

H. Ben-Ameur, M. Breton and P. L'Écuyer. "A Dynamic Programming Procedure for Pricing American-style Asian Options." *Management Science* 48 (2002), 625-43.

F. Black. "The Pricing of Commodity Contracts." *Journal of Financial Economics* 3 (1976), 167-79.

F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (1973), 637-54.

M. Brenner, G.R. Courtadon and M. Subrahmanyam. "Options on the Spot and Options on the Futures." *Journal of Finance* 40 (1985), 1303-17.

M. Brenner, G.R. Courtadon and M. Subrahmanyam. "Options on Stock Indices and Options on Futures." *Journal of Banking and Finance* 13 (1989), 773-82.

N. Cakici, S. Chatterjee and A. Wolf. "Empirical Tests of Valuation Models for Options on T-Note and T-Bond Futures." *Journal of Futures Markets* 13 (1993), 1-13.

T.W. Chamberlain and R.W.T. Chiu. "The Valuation of American Calls on Futures Contracts: A Comparison of Methods." *Quarterly Journal of Business and Economics* 29 (1990), 3-25.

D.M. Chance and R. Brooks. *Derivatives and Risk Management*, Thomson (2007).

B. Cornell and K.R. French. "The Pricing of Stock Index Futures." *Journal of Futures Markets* 3 (1983), 1-14.

J.C. Cox, J.E. Ingersoll and S.A. Ross. "The Relation Between Forward Prices and Futures Prices." *Journal of Financial Economics* 9 (1981), 321-46.

J.C. Cox, S.A. Ross and M. Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7 (1979), 229-63.

D. Duffie and R. Stanton. "Pricing Continuously Resettled Contingent Claims." *Journal of Economic Dynamics and Control* 16 (1992), 561-73.

R. Geske and H.E. Johnson. "The American Put Option Valued Analytically." *Journal of Finance* 39 (1984), 1511-24.

C.R. Harvey and R.E. Whaley. "Dividends and S&P 100 Index Option Valuation." *Journal of Futures Markets* 12 (1992), 123-37.

J.C. Hull. *Options, Futures and Other Derivatives*, Pearson/Prentice Hall (2009).

I.J. Kim. "Analytic Approximation of the Optimal Exercise Boundaries for American Futures Options." *Journal of Futures Markets* 14 (1994), 1-24.

I.J. Kim. "The Analytic Valuation of American Options." *Review of Financial Studies* 3 (1990), 547-72.

A. Lioui. "Stochastic Dividend Yields and Derivatives Pricing in Complete Markets." Working Paper (2006), EDHEC Business School.

F.A. Longstaff and Schwartz. "Valuing American Options by Simulation: A Simple Least-Squares Approach." *The Review of Financial Studies* 14 (2001), 113-47.

R.C. Merton. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (1973), 141-83.

J.A. Overdahl. "The Early Exercise of Options on Treasury Bond Futures." *Journal of Financial and Quantitative Analysis* 23 (1988), 437-49.

M. Parkinson. "Option Pricing: The American Put." *Journal of Business* 50 (1977), 21-36.

K. Ramaswamy and S.M. Sundaresan. "The Valuation of Options on Futures Contracts." *Journal of Finance* 40 (1985), 1319-40.

E.S. Schwartz. "The Valuation of Warrants: Implementing a New Approach." *Journal of Financial Economics* 4 (1977), 79-93.

K. Shastri and K. Tandon. "Options on Futures Contracts: A Comparison of European and American Pricing Models." *Journal of Futures Markets* 6 (1986), 593-618.

K. Shastri and K. Tandon. "An Empirical Test of Valuation Model for American Options on Futures Contracts." *Journal of Financial and Quantitative Analysis* 21 (1986), 377-92.

R.E. Whaley. "Valuation of American Futures Options: Theory and Empirical Tests." *Journal of Finance* 41 (1986), 127-50.

Appendix I

Numerical Example

Let t_0 and t_1 be the inception and maturity of the option respectively. Consider a European option on stock-index futures, with $S_{t_0} = 100$, $K = 100$, $\sigma = 0.25$, $r = 0.08$, $\delta = 0$, and $T_1 = T_2 = 0.25$. Since the maturity of the option contract and the futures contract are the same, the futures price converges to the stock-index level at maturity. Given these parameters, let $G = \{a_0 = 0, a_1 = 89.70, a_2 = 95.92, a_3 = 101.23, a_4 = 106.83, a_5 = 114.24, a_6 = +\infty\}$ be a grid of points representing the possible stock-index levels at maturity. Compute the transition parameters A_{ki}^m and B_{ki}^m under the geometric Brownian motion as discussed in the paper from t_0 to t_1. The transition parameters A_{ki}^m and B_{ki}^m are reported in tables 9 and 10 respectively.

Table 9	$[a_0, a_1]$	$[a_1, a_2]$	$[a_2, a_3]$	$[a_3, a_4]$	$[a_4, a_5]$	$[a_5, a_6]$
a_1	0.46	0.21	0.14	0.10	0.06	0.03
a_2	0.26	0.20	0.17	0.15	0.13	0.10
a_3	0.14	0.16	0.16	0.17	0.18	0.19
a_4	0.07	0.10	0.13	0.16	0.21	0.33
a_5	0.02	0.05	0.08	0.12	0.20	0.54

Table 10	$[a_0, a_1]$	$[a_1, a_2]$	$[a_2, a_3]$	$[a_3, a_4]$	$[a_4, a_5]$	$[a_5, a_6]$
a_1	37.70	19.34	13.59	9.91	7.00	3.97
a_2	21.91	18.41	16.69	15.28	13.85	11.73
a_3	12.09	14.43	16.02	17.61	19.56	23.56
a_4	5.73	9.44	12.82	16.91	23.03	41.05
a_5	1.81	4.32	7.52	12.45	21.92	68.53

The value of the option at maturity on individual points of the grid is set to the exercise value $\tilde{v}_{t_1}(a_k) = v_{t_1}^e(a_k) = \max(0, a_k - K)$. The grid points with corresponding exercise values are presented in Table 11.

Table 11	a_0	a_1	a_2	a_3	a_4	a_5
$\tilde{v}_{t_1}$	0	0	0	1.23	6.83	14.24
a_k	0	89.70	95.92	101.23	106.83	114.24

Now, interpolate $\tilde{v}_{t_1}$ on G and join $\tilde{v}_{t_1}$ with linear pieces. The computations are shown for the interval $[a_2, a_3]$ and are as follows:

$$\beta_2^{t_1} = \frac{1.23 - 0}{101.23 - 95.92} = 0.23,$$

$$\alpha_2^{t_1} = \frac{0 \times 101.23 - 1.23 \times 95.92}{101.23 - 95.92} = -22.24.$$

The intercept and slope for all intervals are presented in Table 12.

Table 12	α_i^n	β_i^n
$[a_0, a_1]$	0	0
$[a_1, a_2]$	0	0
$[a_2, a_3]$	-22.24	0.23
$[a_3, a_4]$	-100.01	1.00
$[a_4, a_5]$	-99.96	1.00
$[a_5, a_6]$	-99.96	1.00

Once we interpolate piece-wise linearly, we have $\hat{v}_{t_1}$ defined on the overall state space. $\hat{v}_{t_1}$ is the approximation of the overall value of the option, v_{t_1}. Now, that we have $\hat{v}_{t_1}$, we set the program to t_0 and use the transition parameters A_{ki}^m and B_{ki}^m to compute the holding value on the grid points $a_1, \ldots, a_5$. We use equation 10 to make the necessary computations. For the purpose of exposition we show the computation for $\tilde{v}_{t_0}^h(a_1)$, which is as follows:

$$\tilde{v}_{t_0}^h(a_1) = e^{-0.08 \times 0.25}\big[(0 \times 0.46 + 0 \times 37.70) + (0 \times 0.21 + 0 \times 19.37) +$$
$$+ (-22.24 \times 0.14 + 0.23 \times 13.59) + (-100.01 \times 0.10 +$$
$$+ 1 \times 9.91) + (-99.96 \times 0.06 + 1 \times 7) + (-100 \times 0.03 +$$
$$+ 1 \times 3.97)\big] = 1.71$$

The holding values for other grid points are presented in Table 13.

Table 13

a_k	$\tilde{v}_{t_0}^h(a_k)$
a_1	1.71
a_2	3.93
a_3	6.81
a_4	10.73
a_5	16.96

Now, we can approximate the value of the European option. Since, $S_{t_0} \in [a_2, a_3]$, we interpolate on the segment as follows:

$$\beta_2^{t_0} = \frac{6.81 - 3.93}{e^{0.08 \times 0.25}(101.23 - 95.92)} = 0.53,$$

$$\alpha_2^{t_0} = \frac{2.20 \times 101.23 - 4.81 \times 95.92}{101.23 - 95.92} = -48.09.$$

The approximation of the option value is given by

$$\hat{v}_{t_0}\big(F_{t_0}(s)\big) = \alpha_2^{t_0} + \beta_2^{t_0} \times F_{t_0}(s).$$

So, we compute as follows:

$$-48.09 + 0.53 \times 100 \times e^{0.08 \times 0.25} = 5.98$$

This is the approximated value of the option with grid size of 5 points $(p = 5)$. The Black's formula gives us 5.99. The discrepancy is due to limited grid-size and rounding.

Appendix II

Code lines written in VBA - Excel 2007

```vba
Option Explicit
Sub NewProject()
Dim S As Double, r As Double, X As Double, sigma As Double, OM As Double, FM As
Double, dd As Integer
Dim a() As Double, p As Double, increment As Double, Z As Double, prob As Double,
Thold() As Double, Thold1() As Double
Dim TP() As Double, Bi() As Double, SettlementDay As Integer, rho As Double, v() As
Double, ve() As Double
Dim vh() As Double, alpha() As Double, wa() As Double, intercept() As Double, slope()
As Double
Dim opt As Double, d As Double, Div As Double, t0 As Date, t1 As Date, td As Double
Dim Ti(), m As Integer, N As Integer, i As Integer, j As Integer, k As Integer, l As Integer,
Optype As Integer
Dim DivFreq As Integer, DivYield() As Double, DateLine() As Double, CurrentDay As
Double
Dim DaysInYear As Integer, StepSize As Double, DivCurve() As Double
Dim c As Integer, Db As Double, Dbar As Double, BlacksValue As Double, Request As
Variant, RequestH() As Double, RequestE() As Double, RequestO() As Double
Dim FuturesPrices() As Double, NumIntervals As Double, NumDiv As Integer,
DivWindow As Double, Length As Integer, caseA As Integer, caseB As Integer
Dim caseC As Integer, div0_fm As Double, divOM_FM As Double
Worksheets("FuturesOption").Activate
S = Range("B1")
sigma = Range("B2")
OM = Range("B3")
r = Range("B4")
p = Range("B5")
X = Range("B6")
FM = Range("B7")
DivFreq = Range("B8")
N = Range("B9")
Optype = Range("B10")
Request = Range("B11")
SettlementDay = 0
DaysInYear = 360
CurrentDay = OM / DaysInYear              'Current Day =Maturity for loop
c = 0      """ captures OM on dateline
d = 0
Div = 0
BlacksValue = 0
DivWindow = DivFreq
caseA = 0
caseB = 0
caseC = 0
```

```
increment = 1 / (p + 1)
prob = increment
StepSize = (OM / N) / DaysInYear
rho = Exp(-r * StepSize)
If VarType(Request) <> vbString And Request > N Then
    MsgBox ("Request date is outside decision scope")
    Exit Sub
End If
If FM < OM Then
    MsgBox ("Option Maturity cannot exceed Futures Maturity")
    Exit Sub
End If
ReDim a(0 To p), TP(1 To p, 0 To p), Bi(1 To p, 0 To p)
ReDim ve(0 To p), v(0 To p), vh(0 To p), wa(1 To p), alpha(0 To p)
ReDim intercept(0 To p), slope(0 To p)
ReDim RequestH(1 To p), RequestE(1 To p), RequestO(1 To p)
ReDim FuturesPrices(1 To p) As Double
ReDim DivCurve(0 To N), Thold(0 To N)
''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''

If FM > OM Then                '''''''''''''''''''''''''''''''''' Construction of TimeLine
    NumIntervals = FM / DivWindow
Else
    NumIntervals = OM / DivWindow
End If
If FM > OM Then
    If NumIntervals <> Fix(NumIntervals) Then
        NumDiv = Fix(NumIntervals) + 2
        caseA = 1
    ElseIf NumIntervals = Fix(NumIntervals) Then
        NumDiv = Fix(NumIntervals) + 1
        caseB = 1
    End If
Else
    If NumIntervals <> Fix(NumIntervals) Then
        NumDiv = Fix(NumIntervals) + 2
        caseC = 1
    ElseIf NumIntervals = Fix(NumIntervals) Then
        NumDiv = NumIntervals + 1
        c = NumDiv - 1
    End If
End If
OM = OM / DaysInYear
FM = FM / DaysInYear
DivWindow = DivFreq / DaysInYear
ReDim DivYield(0 To NumDiv - 1) As Double '''numdiv-1 because starting point is zero
accounts for 1 div
```

```
ReDim DateLine(0 To NumDiv - 1) As Double
For i = 0 To NumDiv - 1
   DivYield(i) = Cells(i + 1, 10)
   DateLine(i) = i * DivWindow
Next i '''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''
'''''''''''''''''''''''''''''''''''''''''cut the divline at either OM or FM
If caseA = 1 Then    ''''caseA FM>OM & +2 div
   DivYield(NumDiv - 1) = inter(DivYield(NumDiv - 2), DivYield(NumDiv - 1),
DateLine(NumDiv - 2), DateLine(NumDiv - 1)) + _
      slp(DivYield(NumDiv - 2), DivYield(NumDiv - 1), DateLine(NumDiv - 2),
DateLine(NumDiv - 1)) * FM
   DateLine(NumDiv - 1) = FM
   For i = 1 To NumDiv - 1
      If DateLine(i) > OM Then Exit For
   Next i
   If (DateLine(i - 1) * DaysInYear = OM * DaysInYear) Then GoTo here
   DivYield(i - 1) = inter(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) + _
      slp(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) * OM
   DateLine(i - 1) = OM
   c = i - 1
ElseIf caseB = 1 Then ''''caseB FM>OM & +1 div
   For i = 1 To NumDiv - 1
       If DateLine(i) > OM Then Exit For
    Next i
    If (DateLine(i - 1) * DaysInYear = OM * DaysInYear) Then GoTo here
    DivYield(i - 1) = inter(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) + _
       slp(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) * OM
    DateLine(i - 1) = OM
    c = i - 1
ElseIf caseC = 1 Then ''''caseC OM=FM & +2 div
   DivYield(NumDiv - 1) = inter(DivYield(NumDiv - 2), DivYield(NumDiv - 1),
DateLine(NumDiv - 2), DateLine(NumDiv - 1)) + _
      slp(DivYield(NumDiv - 2), DivYield(NumDiv - 1), DateLine(NumDiv - 2),
DateLine(NumDiv - 1)) * OM
   DateLine(NumDiv - 1) = OM
   c = NumDiv - 1
End If
here: c = i - 1
'''''''''''''''''''''''''''''''''''''''''''''''''ready for integration
For i = c To 1 Step -1    '''''''''''''''''''''''''''''Div integration t0 to OM
   Div = Div + inter(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) *
(DateLine(i) - DateLine(i - 1)) + _
      0.5 * slp(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) * ((DateLine(i)) ^
2 - (DateLine(i - 1)) ^ 2)
Next i
Div = Div * (1 / OM)
```

```vb
For i = 1 To p                              '''Grid construction
    Z = Application.WorksheetFunction.NormSInv(prob)
    a(i) = S * grid(r, sigma, Div, OM, Z)
    prob = prob + increment
Next i
a(0) = 0
'a(0) = S * grid(r, sigma, Div, OM, -3.5)          '''Set a0 to -3.5 std dev
Div = 0
If FM > OM Then                             """"Integrating dividend OM to FM
    For i = NumDiv - 1 To c + 1 Step -1
        Div = Div + inter(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) *
(DateLine(i) - DateLine(i - 1)) + _
        0.5 * slp(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i)) * ((DateLine(i)) ^
2 - (DateLine(i - 1)) ^ 2)
    Next i
Div = Div * (1 / (FM - OM))
End If
t0 = Timer ""'Start Time
For i = 1 To p
    ve(i) = Application.WorksheetFunction.Max((a(i) * Exp((r - Div) * (FM - OM)) - X), 0)
'''Computing Exercise Values at maturity
    alpha(i) = ve(i)
Next i
For i = 0 To p - 1
    intercept(i) = inter(alpha(i), alpha(i + 1), a(i), a(i + 1))   '''Piecewise
    slope(i) = slp(alpha(i), alpha(i + 1), a(i) * Exp((r - Div) * (FM - OM)), a(i + 1) * Exp((r -
Div) * (FM - OM)))
Next i
intercept(p) = intercept(p - 1)
slope(p) = slope(p - 1)
'ReDim Thold1(0 To N)
Thold(N) = X
'Thold1(N) = X
ReDim Ti(0 To N)        'Requirement for inflation factor
For i = N - 1 To 0 Step -1
    Ti(i) = StepSize * (N - i) """"(OM / N)=stepsize
Next i
Ti(N) = 0
CurrentDay = CurrentDay – StepSize
""""""""""""""""""MAIN LOOP""""""""""""""""
For m = N - 1 To 0 Step -1
    If m = 0 Then
        CurrentDay = SettlementDay
    End If
    d = 0
    For i = 0 To NumDiv - 1
```

```
        If DateLine(i) > CurrentDay Then Exit For   "'Decision day position with respect to
div timeline
        Next i
        If ((CurrentDay + StepSize) <= DateLine(i)) Then      ""'Dividend Integration tm-
tm+1 Annualized
            d = Case1(DivYield(i - 1), DivYield(i), DateLine(i - 1), DateLine(i), CurrentDay,
(CurrentDay + StepSize))
        ElseIf ((CurrentDay + StepSize) > DateLine(i)) Then
            d = Case2(DivYield(i - 1), DivYield(i), DivYield(i + 1), DateLine(i - 1),
DateLine(i), DateLine(i + 1), CurrentDay, (CurrentDay + StepSize), StepSize)
        End If
  For k = 1 To p                               "'Computing transitional probability (Stock)
        For i = 0 To p
            If i = 0 Then
                TP(k, i) = c1(r, sigma, d, a(k), a(i + 1), OM / N)
            ElseIf i < p Then
                TP(k, i) = Trans(r, sigma, d, a(k), a(i + 1), a(i), OM / N)
            Else
            TP(k, i) = 1 - c1(r, sigma, d, a(k), a(i), OM / N)
            End If
        Next i
    Next k
  For k = 1 To p                               "'Computing trunkated means (Stock)
        For i = 0 To p
            If i = 0 Then
                Bi(k, i) = beta0(r, sigma, d, a(k), a(i + 1), OM / N)
            ElseIf i < p Then
                Bi(k, i) = beta(r, sigma, d, a(k), a(i + 1), a(i), OM / N)
            Else
                Bi(k, i) = betap(r, sigma, d, a(k), a(i), OM / N)
            End If
        Next i
    Next k
  d = d * StepSize 'disannualizing
  For k = 1 To p
        wa(k) = 0
        For i = 0 To p
        wa(k) = wa(k) + TP(k, i) * intercept(i) + slope(i) * Bi(k, i) * Exp((r - Div) * (Ti(m
+ 1) + (FM - OM))) "'Computing expected payoff at +1dd
        Next i
        vh(k) = wa(k) * rho                          "' Discounting expected payoff
        Div = (1 / (FM - CurrentDay)) * (Div * (FM - (CurrentDay + StepSize)) + d)
"'unwinding annualization tm+1-OM and annualizing tm-OM
        If Optype = 1 Then
        ve(k) = Application.WorksheetFunction.Max(a(k) * Exp((r - Div) * (Ti(m) + (FM -
OM))) - X, 0) "' Computing exercise values -1 decision date if American
```

```vba
        v(k) = Application.WorksheetFunction.Max(ve(k), vh(k))  '' Setting overall value
for at a decision point at specific grid point
        Else
        v(k) = vh(k)                              '' For European option overall value is
discounted payoff no exercise as -1
        End If
        If m = Request Then
            For i = 1 To p
                FuturesPrices(i) = a(i) * Exp((r - Div) * (Ti(m) + (FM - OM)))
            Next i
        End If
If (k < p) Then  """""""""""
        If m = N - 1 And OM = FM Then
            Div = (Div * (OM - CurrentDay) - d)
        Else
            Div = (Div * (FM - CurrentDay) - d) * (1 / (FM - CurrentDay - StepSize))
'''resetting the dividend  to tm+1-OM
        End If
    End If
Next k
If m = Request Then
        For k = 1 To p
            RequestH(k) = vh(k)
            RequestE(k) = ve(k)
            RequestO(k) = v(k)
        Next k
    End If
If Optype <> 0 Then
        For i = 1 To p  """"""""""""""""""""""""""""""""""""""""""""""
            If ve(i) > vh(i) Then Exit For
        Next i
        If i > p Then
            Thold(m) = 0
        Else
            Thold(m) = (-inter(vh(i - 1), vh(i), a(i - 1), a(i)) - X) / (slp(vh(i - 1), vh(i), a(i - 1) *
Exp((r - Div) * (Ti(m) + (FM - OM))), a(i) * Exp((r - Div) * (Ti(m) + (FM - OM)))) - 1)
        End If
    End If
For i = 1 To p
        alpha(i) = v(i)                              ''Setting values for each grid point for dd
    Next i
 For i = 0 To p - 1
        intercept(i) = inter(alpha(i), alpha(i + 1), a(i), a(i + 1))    '' Interpolation
        slope(i) = slp(alpha(i), alpha(i + 1), a(i) * Exp((r - Div) * (Ti(m) + (FM - OM))), a(i
+ 1) * Exp((r - Div) * (Ti(m) + (FM - OM))))
    Next i
```

```
    intercept(p) = intercept(p - 1)
    slope(p) = slope(p - 1)
    CurrentDay = CurrentDay - StepSize
Next m
'''''''''''''''''''''''''''''''''''''''''''''''''''''''''''

CurrentDay = CurrentDay + StepSize
For i = 0 To p
    If a(i) > S Then Exit For        '''Locating initial stock price
Next i
                        '''Interpolating for corresponding futures option price
opt = inter(alpha(i - 1), alpha(i), a(i - 1), a(i)) + slp(alpha(i - 1), alpha(i), a(i - 1) * Exp((r -
Div) * FM), a(i) * Exp((r - Div) * FM)) * S * Exp((r - Div) * FM)
t1 = Timer ''''''''''Stop Time
td = t1 - t0
''''''''''''''''''''''''''''''''''''''''''''''Printing to screen
Call Blacks(S, X, Div, r, sigma, OM, FM, BlacksValue)
Sheets.Add After:=Sheets(Sheets.Count)
Range("A1") = "S" '''''''''''''''''''''''''Checking the Inputs
Range("B1") = S
Range("A2") = "sigma"
Range("B2") = sigma
Range("A3") = "OM"
Range("B3") = OM
Range("A4") = "r"
Range("B4") = r
Range("A5") = "p"
Range("B5") = p
Range("A6") = "X"
Range("B6") = X
Range("A7") = "FM"
Range("B7") = FM
Range("A8") = "DivFreq"
Range("B8") = DivFreq
Range("A9") = "dd"
Range("B9") = N
Range("A10") = "Optype"
Range("B10") = Optype
Range("A11") = "Request"
Range("B11") = Request ''''''''''''''''''''''''''''''''''''''''''''
Range("D1") = "Option Value"
Range("E1") = opt
Range("D2") = "Blacks Value"
Range("E2") = BlacksValue
Range("D3") = "CPU"
Range("E3") = td
If (VarType(Request) <> vbString) Then
```

```
      Call GraphV(RequestH, RequestE, RequestO, FuturesPrices, p)
End If
If Optype <> 0 Then
  Call GraphT(Thold, N)
End If
End Sub
'"Grid Function"'
Function grid(r As Double, sigma As Double, Div As Double, T As Double, Z As Double)
As Double
    grid = Exp((r - Div - (sigma ^ 2) / 2) * T + sigma * Sqr(T) * Z)
End Function
'"Transition Probability Function"'
Function Trans(r As Double, sigma As Double, Div As Double, ak As Double, c As
Double, _
B As Double, T As Double) As Double
Dim q1 As Double
Dim q2 As Double
    q1 = (Log(B / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    q2 = (Log(c / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
   Trans = Application.WorksheetFunction.NormSDist(q2) _
        - Application.WorksheetFunction.NormSDist(q1)
End Function
Function c1(r As Double, sigma As Double, Div As Double, ak As Double, a As Double, T
As Double) As Double
    Dim q1 As Double
    q1 = (Log(a / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    c1 = Application.WorksheetFunction.NormSDist(q1)
End Function
'"Trunkated Mean Function"'
Function beta(r As Double, sigma As Double, Div As Double, ak As Double, B As Double,
_
a As Double, T As Double) As Double
    Dim q1 As Double
    Dim q2 As Double
    q1 = (Log(a / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    q2 = (Log(B / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    beta = ak * (Application.WorksheetFunction.NormSDist(q2 - sigma * Sqr(T)) _
        - Application.WorksheetFunction.NormSDist(q1 - sigma * Sqr(T))) * Exp((r - Div)
* T)
End Function
Function beta0(r As Double, sigma As Double, Div As Double, ak As Double, _
B As Double, T As Double) As Double
    Dim q1 As Double
q1 = (Log(B / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    beta0 = ak * (Application.WorksheetFunction.NormSDist(q1 - sigma * Sqr(T))) * Exp((r
- Div) * T)
```

```
End Function
Function betap(r As Double, sigma As Double, Div As Double, ak As Double, _
a As Double, T As Double) As Double
    Dim q1 As Double
q1 = (Log(a / ak) - (r - Div - (sigma ^ 2) / 2) * (T)) / (sigma * Sqr(T))
    betap = ak * (1 - (Application.WorksheetFunction.NormSDist(q1 - sigma * Sqr(T)))) *
Exp((r - Div) * T)
End Function
'"Intercept Function'"
Function inter(v0 As Double, v1 As Double, a0 As Double, a1 As Double) As Double
    inter = (v0 * a1 - v1 * a0) / (a1 - a0)
End Function
'"Slope Function'"
Function slp(v0 As Double, v1 As Double, a0 As Double, a1 As Double) As Double
    slp = (v1 - v0) / (a1 - a0)
End Function
'"Dividend Integration Fuctions'"
Function Case1(DivYield0 As Double, DivYield1 As Double, DateLine0 As Double, _
DateLine1 As Double, CurrentDay As Double, CurrentDayPlusStep As Double) As Double
    Case1 = (inter(DivYield0, DivYield1, DateLine0, DateLine1) + _
        0.5 * slp(DivYield0, DivYield1, DateLine0, DateLine1) * (CurrentDayPlusStep +
CurrentDay))
End Function
Function Case2(DivYield0 As Double, DivYield1 As Double, DivYield2 As Double, _
DateLine0 As Double, DateLine1 As Double, DateLine2 As Double, CurrentDay As
Double, CurrentDayPlusStep As Double, StepSize As Double) As Double
    Case2 = (inter(DivYield0, DivYield1, DateLine0, DateLine1) * (DateLine1 -
CurrentDay) + _
        0.5 * slp(DivYield0, DivYield1, DateLine0, DateLine1) * (DateLine1 ^ 2 -
CurrentDay ^ 2) + _
        inter(DivYield1, DivYield2, DateLine1, DateLine2) * (CurrentDayPlusStep -
DateLine1) + _
        0.5 * (slp(DivYield1, DivYield2, DateLine1, DateLine2)) * (CurrentDayPlusStep ^
2 - DateLine1 ^ 2)) * _
        (1 / StepSize)
End Function
```