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A stochastic ergodic theorem for superadditive processes

M. A. AKCOGLU AND L. SUCHESTON

Department of Mathematics, University of Toronto, Toronto, Ontario M5S1A1, *Canada*; *Department of Mathematics, Ohio State Univeristy, Columbus, Ohio* 43210, *USA*

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Abstract. An elementary proof is given of Krengel's stochastic ergodic theorem in the setting of multiparameter superadditive processes.

1. *Introduction*

The stochastic ergodic theorem due to U. Krengel [**4**] asserts that if T is a linear contraction of the L_1 space of a probability space and if $f \in L_1$ then the Cesaro averages

$$n^{-1}(f + Tf + \cdots + T^{n-1}f)$$

converge in probability. Our purpose in this paper is to give an extension of this result to multiparameter superadditive processes (theorem (3.12)). That the existence of 'exact dominants' [**1**] implies the one-parameter extension to the superadditive case was shown by Fong [**3**]. The multiparameter additive case was also proved recently by Krengel [**5**]. Our proof is by a truncation argument already used by us earlier [**1**], depending only on certain properties of L_1 as a Banach lattice. The extension to Banach lattices is however less elementary, and we hope to present it in a separate article.

Note that the definition of superadditivity used in this article is the one given by Smythe [**6**].

An apparently new result of some independent interest is theorem 2.4, which gives the decomposition of the space into a 'positive' and a 'null' part for an arbitrary family of positive contractions on L_1.

2. *Basic Results*

Consider the real L_1 space of a measure space $(X, \mathscr{F}, \mu)$. Functional relations below are often to be understood modulo sets of measure zero. We will assume that μ is σ-finite. This is no loss of generality, as we are going to deal only with countable classes of functions. The characteristic function of a set $F \in \mathscr{F}$ is χ_F. The cone of non-negative functions in L_1 is denoted by L_1^+. By a contraction on L_1 we mean a linear operator $T: L_1 \to L_1$ with $\|T\| \le 1$. An operator $T: L_1 \to L_1$ is called positive if $TL_1^+ \subset L_1$.

Definition (2.1). A sequence f_n in L_1 is said to *converge stochastically* to $f \in L_1$ if

$$\lim_n \mu(A \cap \{x \mid |f_n(x) - f(x)| > \alpha\}) = 0$$

whenever $\alpha > 0$ and whenever $A \in \mathscr{F}$ has finite measure. If μ is a finite measure then this is, of course, just convergence in measure.

Although this is the usual definition of stochastic convergence (c.f. [**5**]), we will use a different formulation given in the following lemma.

LEMMA (2.2). *Let f_n be a sequence in L_1 and let $f \in L_1$. Then the following are equivalent*:

(i) *f_n converges to f stochastically*;

(ii) $\lim_n \| |f_n - f| \Lambda \phi \| = 0$ *whenever* $\phi \in L_1^+$.

Proof. Let $g_n = |f_n - f|$. Assume (i) and let $\phi \in L_1^+$, $\varepsilon > 0$ be given. Choose $A \in \mathscr{F}$ with $\mu(A) < \infty$ such that

$$\|\chi_{A^c}\phi\| < \varepsilon.$$

Choose $\delta > 0$ such that $\|\chi_E \phi\| < \varepsilon$ whenever $\mu(E) < \delta$. Choose n_0 such that $\mu(A \cap B_n) < \delta$ whenever $n \geq n_0$, where

$$B_n = \{x \mid g_n(x) > \varepsilon(\mu(A))^{-1}\}.$$

Then

$$\|g_n \Lambda \phi\| \leq \|\chi_{A^c}\phi\| + \|\chi_{A \cap B_n}\phi\| + \|\chi_{A \cap B_n^c} g_n\| \leq 3\varepsilon \qquad \text{whenever } n \geq n_0.$$

Conversely, assume (ii) and let $A \in \mathscr{F}$, with $\mu(A) < \infty$, and $\alpha > 0$ be given. If

$$B_n = \{x \mid g_n(x) > \alpha\} \quad \text{and} \quad \phi = \alpha \chi_A$$

then

$$\alpha\mu(A \cap B_n) \leq \|g_n \Lambda \phi\|$$

which shows that

$$\lim_n \mu(A \cap B_n) = 0. \qquad \square$$

If f_n is a sequence in L_1^+ dominated by a fixed $\phi \in L_1^+$, then f_n has a subsequence that converges weakly in L_1. If, in addition,

$$\limsup_n \|f_n\| > 0,$$

then this sequence can be chosen to have a non-zero weak limit. If f_n and ϕ_k are two arbitrary sequences in L_1^+, then f_n has a subsequence f_{n_i} such that

$$\text{w-}\lim_i f_{n_i} \Lambda \phi_k = g_k$$

exists for each k, where w-lim denotes the weak limit in L_1. We omit the easy and familiar proofs of these facts.

THEOREM (2.3). *Let f_n be a sequence in L_1^+ such that*

$$\sup_n \|f_n\| = M < \infty.$$

Then either f_n converges stochastically to zero, or there is a non-zero $g \in L_1^+$ such that $Tg = g$, whenever $T: L_1 \to L_1$ is a positive contraction such that

$$\lim_n \|f_n - Tf_n\| = 0.$$

Furthermore, if the support of each f_n is contained in a set $E \in \mathcal{F}$, then the support of g is also contained in E.

Proof. Assume that f_n does not converge stochastically to zero. Let $\phi \in L_1^+$ be such that

$$\limsup_n \|f_n \Lambda \phi\| > 0.$$

We will assume, without loss of generality, that $\phi > 0$ a.e. on X. By passing to a subsequence, if necessary, we may assume that

$$\text{w-}\lim_n f_n \Lambda(k\phi) = g_k \quad \text{and} \quad \lim_n \|f_n - f_n \Lambda(k\phi)\| = \alpha_k$$

exist for each integer $k \geq 1$, and that $g_1 \neq 0$. Then $\lim_k g_k = g$ exists a.e. and in L_1-norm, as g_k is a non-decreasing bounded sequence in L_1^+. Hence g is a non-zero function in L_1^+. It is also clear that if each f_n has support contained in E then the support of g is also contained in E. We will now show that if $T: L_1 \to L_1$ is a positive contraction and if

$$\lim_n \|f_n - Tf_n\| = 0,$$

then $Tg = g$. Let

$$r_n^k = f_n \Lambda(k\phi), \qquad u_n^k = (Tf_n)\Lambda(k\phi)$$

and

$$f_n = r_n^k + s_n^k, \qquad Tf_n = u_n^k + v_n^k.$$

Since $\|f_n - Tf_n\| \to 0$, the uniform integrability of $(r_n^k - u_n^k)$ implies that of $(s_n^k - v_n^k)$, hence we have

$$\|r_n^k - u_n^k\| \to 0, \quad \|s_n^k - v_n^k\| \to 0 \qquad \text{as } n \to \infty,$$

for each fixed k. Note that

$$\alpha_k = \lim_n \|s_n^k\| = \lim_n \|v_n^k\|$$

is non-increasing in k and

$$\alpha = \lim_k \alpha_k \geq 0$$

exists. Given $\varepsilon > 0$, find k such that

$$\alpha_k - \alpha < \varepsilon \quad \text{and} \quad \|g - g_k\| < \varepsilon.$$

Then find $l > k$ such that

$$T(k\phi) \leq l\phi + h$$

for some $h \in L_1^+$ with $\|h\| < \varepsilon$. Since $r_n^k \leq k\phi$, then

$$Tr_n^k \leq l\phi + h.$$

Now

$$Tf_n = Tr_n^k + Ts_n^k = u_n^l + v_n^l$$

shows that we can write

$$u_n^l = (Tr_n^k - h_n) + w_n \quad \text{and} \quad v_n^l = (Ts_n^k - w_n) + h_n$$

where

$$0 \leq h_n = Tr_n^k - (u_n^l \Lambda Tr_n^k) \leq Tr_n^k - (l\phi \Lambda Tr_n^k) \leq h$$

and

$$0 \leq w_n = Ts_n^k \Lambda [u_n^l - (u_n^l \Lambda Tr_n^k] \leq Ts_n^k.$$

Hence

$$\|v_n^l\| = \|Ts_n^k\| - \|w_n\| + \|h_n\| \to \alpha_l \qquad \text{as } n \to \infty,$$

which implies that

$$\limsup_n \|w_n\| = \limsup_n (\|Ts_n^k\| + \|h_n\|) - \alpha_l$$

$$\leq \limsup_n (\|s_n^k\| + \|h_n\|) - \alpha_l$$

$$\leq \alpha_k + \varepsilon - \alpha_l < 2\varepsilon.$$

Therefore

$$\limsup_n \|u_n^l - Tr_n^k\| \leq \limsup_n (\|h_n\| + \|w_n\|) \leq 3\varepsilon$$

which shows that $\|g_l - Tg_k\| < 3\varepsilon$ so that $\|g - Tg\| < 5\varepsilon$. □

THEOREM (2.4). *Let $\{T_i\}$, $i \in I$, be an arbitrary family of positive L_1 contractions. Then there exists a unique (modulo sets of measure zero) partition of X into two sets P and N such that*

(i) *there is a $g \in L_1^+$ such that $T_i g = g$ for all $i \in I$ and such that P is the support of g: $P = \{x \mid g(x) > 0\}$;*

(ii) *if f_n is a bounded sequence in L_1^+ such that*

$$\lim_n \|f_n - T_i f_n\| = 0 \qquad \text{for each } i \in I$$

then $\chi_N f_n$ converges stochastically to zero.

Proof. We will obtain P as the largest set that supports an invariant function. Let $\mathcal{G}$ be the collection of all sets $G \in \mathcal{F}$ that can be obtained as

$$G = \{x \mid g(x) > 0\},$$

where $g \in L_1^+$ and $T_i g = g$ for all $i \in I$. Let ν be any finite measure on $(X, \mathcal{F})$, equivalent to μ, and let

$$\alpha = \sup\{\nu(G) \mid G \in \mathcal{G}\}.$$

Let G_n be a sequence in $\mathcal{G}$ such that $\nu(G_n) \to \alpha$. For each n, choose $g_n \in L_1^+$ such that

$$G_n = \{x \mid g_n(x) > 0\} \quad \text{and} \quad T_i g_n = g_n \qquad \text{for all } i \in I.$$

We may also assume that $\|g_n\| = 1$. Then

$$g = \sum_{n=1}^{\infty} (1/2^n) g_n$$

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is another function in L_1^+ such that $T_i g = g$ for all $i \in I$ and such that $P = \{x \mid g(x) > 0\} \in \mathscr{G}$ satisfies $\nu(P) = \alpha$. Let $N = X - P$. Then (i) is satisfied.

To see that (ii) is also satisfied, let f_n be a bounded sequence L_1^+ such that

$$\lim_n \|f_n - T_i f_n\| = 0 \qquad \text{for all } i \in I.$$

Let $h_n = \chi_N f_n$ and let $T'_i : L_1 \to L_1$ be defined as

$$T'_i f = \chi_N T_i f, \qquad f \in L_1,$$

which is a positive contraction for each $i \in I$. We claim that

$$\|h_n - T'_i h_n\| \to 0 \qquad \text{as } n \to \infty.$$

First observe that

$$\chi_N T_i \chi_P f = T'_i \chi_P f = 0 \qquad \text{for any } f \in L_1,$$

which follows easily from the fact that

$$\|\chi_P f - (\chi_P f \wedge kg)\| \to 0 \qquad \text{as } k \to \infty$$

and that

$$T_i(\chi_P f \wedge kg) \leq kg.$$

Hence

$$\|h_n - T'_i h_n\| = \|h_n - T'_i h_n - T'_i \chi_P f_n\| = \|\chi_N f_n - \chi_N T_i f_n\| \leq \|f_n - T_i f_n\| \to 0$$

as $n \to \infty$. Therefore, if h_n does not converge to zero stochastically, then by the previous theorem there is a non-zero $h \in L_1^+$ such that $T'_i h = h$ for all $i \in I$. Since each h_n has support in N, the support of h is also in N. Now, if $\chi_P T_i h = r_i$, then

$$T_i h = \chi_N T_i h + \chi_P T_i h = T'_i h + r_i = h + r_i$$

implies that $r_i = 0$, since T_i is a contraction. Hence $T_i h = h$ for all $i \in I$. This a contradiction because now the support of $g + h$ also belongs to $\mathscr{G}$ and its ν-measure is strictly larger than the ν-measure of P.

To see the uniqueness, note that if $\phi \in L_1^+$ is a function such that $T_i \phi = \phi$ for all $i \in I$, then the support of ϕ is contained in any set P such that $N = X - P$ satisfies (ii). In fact, in this case the constant sequence $f_n = \phi$ satisfies the hypothesis in (ii) and consequently $\chi_N f_n = \chi_N \phi$ must converge to zero stochastically. □

Definition (2.5). The sets P and N obtained in theorem 2.4 will be called, respectively, the *positive* and the *null* parts of the family $\{T_i\}$.

3. *Superadditive Processes*

Let V be the set consisting of K-dimensional vectors $v = (v_1, \ldots, v_K)$ with non-negative integer coordinates, and let $e = (1, \ldots, 1)$ be the vector with all coordinates equal to 1. If $u, v \in V$, then $[u, v)$ denotes the set

$$\{w \mid w \in V, \quad u_k \leq w_k < v_k, \quad k = 1, \ldots, K\}$$

and $\pi(v)$ denotes the number of points in $[0, v)$. If $u, v \in V$, then uv will be defined as the vector with coordinates

$$(uv)_k = u_k v_k.$$

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Finally, $\lim_v$ denotes the limit as $v_k \to \infty$ for each k, independently of each other.

For each $v \in V$, let T_v be a positive L_1 contraction and assume that $\{T_v\}$ is a semi-group:

$$T_u T_v = T_{u+v} \qquad \text{for all } u, v \in V.$$

If, in addition,

$$\int T_v f \, d\mu = \int f \, d\mu \qquad \text{for all } v \in V, \quad \text{for all } f \in L_1,$$

then $\{T_v\}$ will be called a *Markovian* semi-group. For $u, v \in V$, let

$$S_u^v = \sum_{w \in [0,u)} T_{wv}$$

and

$$A_u^v = [\pi(u)]^{-1} S_u^v, \qquad \text{if } \pi(u) > 0$$
$$A_u^v = 0 \qquad \text{if } \pi(u) = 0.$$

If $v = e$ then these are, respectively, the ordinary sums and the averages of the semi-group over $[0, u)$, and will be denoted as S_u and A_u. Observe that $A_v A_u^v = A_{uv}$.

Let P and N be the positive and null parts of the family $\{T_v\}$, as obtained in § 2. Let $L_1(P)$ be the set of L_1 functions with supports contained in P. Note that T_v maps $L_1(P)$ into $L_1(P)$ and the restriction of $\{T_v\}$ to $L_1(P)$ is a Markovian semi-group. We need the mean ergodic theorem in the following K-dimensional form.

THEOREM (3.1). *If $f \in L_1(P)$ then $\lim_v A_v f$ exists in the L_1-norm.*

This is a direct consequence of the corresponding 1-dimensional form, which, in turn, follows from the Kakutani-Yosida mean ergodic theorem.

Finally note that if $f \in L_1$, then $\lim_u \chi_N A_u f = 0$ stochastically, in the sense that

$$\lim_u \| |\chi_N A_u f| \wedge \phi \| = 0 \qquad \text{whenever } \phi \in L_1^+.$$

This follows from theorem (2.4), since

$$\lim_u \|T_v A_u f - A_u f\| = 0 \qquad \text{for all } v \in V.$$

Definition (3.2). A partition of a set $E \subset V$ is called an *admissible* partition if it is the common refinement of finitely many partitions of the form

$$(E \cap \{w \mid w_k < n\}, E \cap \{w \mid w_k > n\}),$$

where k and n are integers and $1 \le k \le K$.

Definition (3.3). A function $F: V \to L_1$ is called a *superadditive process* (with respect to $\{T_v\}$) if

$$F_v \ge \sum_{i=1}^{n} T_{u^i} F_{v^i - u^i}$$

whenever $\{[u^i, v^i)\}_{i=1}^n$ is an admissible partition of $[0, v)$. The function $f: V \to L_1$ defined as

$$f_v = [\pi(v)]^{-1} F_v \qquad \text{if } \pi(v) > 0$$
$$f_v = 0 \qquad \text{if } \pi(v) = 0$$

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is called the average of F. If $\sup_v \|f_v\| < \infty$, then F is called a *bounded* superadditive process. If $F_v \in L_1^+$ for each $v \in V$, then F is called *positive*.

If both F and $-F$ are superadditive then F is called *additive*. Note that F is additive if and only if $F_v = S_v h$, where $h = F_e \in L_1$. An additive process is always bounded, and if $h \in L_1^+$ then $S_v h$ is also positive. Also, the sum of two superadditive processes is again a superadditive process.

LEMMA (3.4). *Any superadditive process is the difference of two positive superadditive processes. If the original process is bounded then both of these positive processes are also bounded.*

Proof. Since $\{[u, u+e)\}_{u\in[0,v)}$ is an admissible partition of $[0, v)$, we have that

$$F_v \geq S_v F_e.$$

Hence $F_v - S_v F_e$ is a positive superadditive process. Then both

$$F'_v = (F_v - S_v F_e) + S_v F_e^- \quad \text{and} \quad F''_v = S_v F_e^+$$

are positive superadditive processes and

$$F = F' - F''.$$

The boundedness is obvious. □

LEMMA (3.5). *Let f be the average of a positive bounded superadditive process with respect to a Markovian semi-group $\{T_v\}$ and let $\gamma = \sup_v \|f_v\|$. Let $0 < \delta < \gamma$, and u, $v \in V$ be such that*

$$\|f_v\| \geq \gamma - \delta \quad \textit{and} \quad \lambda = \pi(v)[\pi(v+u)]^{-1} \geq 1 - (\delta/\gamma).$$

Then the L_1-distance between any two of the functions f_v, f_{v+u}, $T_u f_v$ is less than

$$\frac{4\delta\gamma}{\gamma - \delta}.$$

Proof. The set $[0, v+u)$ has in particular the following two admissible partitions: one containing $[0, v)$ as an atom and the other containing $[u, v+u)$ as an atom. Hence

$$0 \leq F_v \leq F_{v+u} \quad \text{and} \quad 0 \leq T_u F_v \leq F_{v+u},$$

because of the positivity of F. Therefore,

$$0 \leq \lambda f_v \leq f_{v+u} \quad \text{and} \quad 0 \leq \lambda T_u f_v \leq f_{v+u}.$$

But

$$\|\lambda f_v\| = \|\lambda T_u f_v\| \geq \lambda(\gamma - \delta) \geq \gamma - 2\delta \quad \text{and} \quad \|f_{v+u}\| \leq \gamma.$$

Hence $\|\lambda f_v - \lambda T_u f_v\| \leq 4\delta$ i.e.

$$\|f_v - T_u f_v\| \leq 4\delta/(1 - (\delta/\gamma)).$$

Also $\|f_{v+u} - \lambda f_v\| \leq 2\delta$ implies that

$$\|f_{v+u} - f_v\| \leq 2\delta + \|f_v - \lambda f_v\| \leq 2\delta + (\delta/\gamma)(\gamma - \delta) \leq 3\delta.$$

Similarly,

$$\|f_{v+n} - T_u f_v\| \leq \|f_{v+u} - \lambda T_u f_v\| + \|T_u f_v - \lambda T_u f_v\| \leq 2\delta + (\delta/\gamma)(\gamma - \delta) \leq 3\delta. \quad □$$

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THEOREM (3.6). *Let f be the average of a positive bounded superadditive process with respect to a Markovian semi-group $\{T_v\}$. Then, for each $\varepsilon > 0$ there is a $z \in V$ such that*

$$\limsup_v \|A_v f_z - f_v\| < \varepsilon.$$

Proof. Let $\gamma = \sup_v \|f_v\| = 1$, and choose δ, $0 < \delta < 1$, such that $4\delta/(1-\delta) < \varepsilon/8$. Let $z \in V$ be such that $\pi(z) > 0$ and $\|f_z\| > 1-\delta$. Finally choose $u^0 \in V$ such that

$$\pi(uz)[\pi(uz+w)]^{-1} > 1-\delta \qquad \text{whenever } u \geq u^0, \quad w \in [0, z).$$

We claim that

$$\|A_v f_z - f_v\| < \varepsilon \qquad \text{whenever } v \geq u^0 z.$$

First note that any $v \geq u^0 z$ can be expressed as $v = uz + w$, where $u \geq u^0$ and $w \in [0, z)$. But

$$\|A_{uz+w} h - A_{uz} h\| \leq 2\delta \|h\| \qquad \text{for any } h \in L_1,$$

by a simple computation. Hence

$$\|A_{uz} f_z - A_{uz+w} f_z\| \leq 2\delta\gamma < \varepsilon/4.$$

To complete the proof it is now enough to show that

$$\|A_{uz} f_z - f_{uz+w}\| < 3\varepsilon/4$$

whenever $u \geq u^0$ and $w \in [0, z)$. Since $\{[rz, (r+e)z)\}_{r \in [0,u)}$ is an admissible partition of $[0, uz)$, we have that

$$0 \leq S_u^z F_z \leq F_{uz} \quad \text{or} \quad 0 \leq A_u^z f_z \leq f_{uz},$$

noticing that $\pi(uz) = \pi(u)\pi(z)$. Hence $1-\delta < \|f_z\| = \|A_u^z f_z\| \leq \|f_{uz}\| \leq 1$ implies that

$$\|A_u^z f_z - f_{uz}\| < \delta < \varepsilon/8.$$

Since $\|f_{uz}\| > 1-\delta$, we now apply lemma 3.5 to obtain

$$\|T_w f_{uz} - f_{uz}\| < \varepsilon/8.$$

Hence, together with

$$\|T_w A_u^z f_z - T_w f_{uz}\| < \varepsilon/8,$$

this gives

$$\|T_w A_u^z f_z - f_{uz}\| < \varepsilon/4.$$

Taking the average over $w \in [0, z)$, we then have

$$\|A_z A_u^z f_z - f_{uz}\| = \|A_{uz} f_z - f_{uz}\| < \varepsilon/4.$$

Then, again by lemma 3.5, we have that $\|f_{uz} - f_{uz+w}\| < \varepsilon/8$ and consequently,

$$\|A_{uz} f_z - f_{uz+w}\| < 3\varepsilon/8 < 3\varepsilon/4. \qquad \square$$

LEMMA (3.7). *Let F be a positive superadditive process with respect to $\{T_v\}$ and assume that P is the positive part of $\{T_v\}$. Then $\chi_P F$ is also a superadditive process.*

Proof. Let $\{[u^i, v^i)\}_{i=1}^n$ be an admissible partition of $[0, v)$. Then

$$\chi_P F_v \geq \chi_P \sum_{i=1}^n T_{u^i} F_{v^i - u^i} \geq \sum_{i=1}^n \chi_P T_{u^i} \chi_P F_{v^i - u^i} = \sum_1^n T_{u^i} \chi_P F_{v^i - u^i}.$$

Here the first inequality follows from the superadditivity of F, the second inequality from the positivity of F (and T_{u^i}) and the equality from the fact that $\chi_N T_u \chi_P h = 0$ for all $h \in L_1$ and $u \in V$. □

Remark. Call a set $A \in F$ an *absorbing* set for $\{T_v\}$, if

$$\chi_{X-A} T_v \chi_A h = 0 \qquad \text{for all } h \in L_1, \quad v \in V.$$

It is clear that the above proof shows that $\chi_A F$ is superadditive whenever A is absorbing and F is positive and superadditive.

Results close to the following were obtained by Derriennic and Krengel ([**2**, theorem 5.2]).

LEMMA (3.8). *Let f be the average of a positive and bounded superadditive process with respect to $\{T_v\}$ and let P be the positive part of $\{T_v\}$. Then* $\lim_v \chi_P f_v$ *exists in the L_1-norm.*

Proof. Let $f' = \chi_P f$ and let T'_v be the restriction of T_v to $L_1(P)$. Then $\{T'_v\}$ is a Markovian semi-group and f' is the average of a positive bounded process, superadditive with respect to this semi-group. Then theorem (3.6) shows that for each $\varepsilon > 0$ there is a $z \in V$ such that

$$\limsup_v \|A'_v f'_z - f'_v\| < \varepsilon,$$

where $A'_v f'_z$ are the averages with respect to T'_v, equal to $A_v f'_z$. Now $\lim_v A_v f'_z$ exists in the L_1 norm, for each $z \in V$, by the mean ergodic theorem (3.1), since $f'_z \in L_1(P)$. Hence $\lim_v f'_v$ must also exist in the L_1-norm. □

LEMMA (3.9). *Let f be the average of a positive superadditive process F with respect to $\{T_v\}$ and let N be the null part of $\{T_v\}$. Then* $\lim_v \chi_N f_v = 0$ *stochastically if at least one of the following conditions is satisfied*:

(3.10) *$\{T_v\}$ is a Markovian semi-group and F is a bounded process*;

(3.11) *F is dominated by an additive process*; *i.e. there is an $h \in L_1$ such that $F_v \le S_v h$ for all $v \in V$.*

Proof. If (3.10) is satisfied then for each $\varepsilon > 0$ there is a $z \in V$ such that

$$\limsup_v \|A_v f_z - f_v\| < \varepsilon.$$

But $\lim_v \chi_N A_v f_z = 0$ stochastically, i.e.

$$\lim_v \|(\chi_N A_v f_z) \Lambda \phi\| = 0 \quad \text{for each } \phi \in L_1^+.$$

Hence we obtain

$$\limsup_v \|(\chi_N f_v) \Lambda \phi\| < \varepsilon \qquad \text{for all } \phi \in L_1^+.$$

Therefore $\lim_v \chi_N f_v = 0$ stochastically. The proof under the assumption (3.11) is clear, since

$$0 \le \chi_N f_v \le \chi_N A_v h.$$ □

The positivity assumption in lemmas (3.8) and (3.9) can be removed easily. First, since any bounded superadditive process is the difference of two positive and

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bounded superadditive processes, it is clear that lemma (3.8) and a part of lemma (3.9), under the hypothesis (3.10), extend directly. To see that lemma (3.9) holds under the assumption (3.11), assume that $F_v \leq S_v h$ with $h \in L_1$. Then

$$0 \leq F_v - S_v F_e \leq S_v(h - F_e)$$

and lemma (3.9) applies to $F_v - S_v F_e$. But it also applies to $S_v F_e^+$ and $S_v F_e^-$ and consequently to $S_v F_e$. Collecting these results, we have the following theorem.

THEOREM (3.12). *Let f be the average of a superadditive process F with respect to a semi-group* $\{T_v\}$, $v \in V$, *of positive* L_1 *contractions. Let P be the positive part of* $\{T_v\}$. *Assume that at least one of the following two conditions is satisfied*:

(3.10) $\{T_v\}$ *is a Markovian semi-group and F is a bounded process*;

(3.11) *there is an* $h \in L_1$ *such that* $F_v \leq S_v h$ *for all* $v \in V$.

Then $\lim_v f_v = \bar{f}$ *exists stochastically*; *i.e. there is an* $\bar{f} \in L_1$ *such that*

$$\lim_v \| |f_v - \bar{f}| \wedge \phi \| = 0 \qquad \text{for all } \phi \in L_1^+.$$

Furthermore, $\lim_v \chi_P f_v = \bar{f}$ *in the* L_1*-norm.*

There is an obvious extension of this result to additive processes with respect to semi-groups $\{\tilde{T}_v\}$ of general, not necessarily positive, L_1 contractions, assuming that $\{\tilde{T}_v\}$ is dominated by a semi-group $\{T_v\}$ of positive L_1 contractions, in the sense that

$$|\tilde{T}_v f| \leq T_v |f| \qquad \text{for all } f \in L_1, \quad v \in V.$$

(Note that this is the case if the linear moduli of the generators of $\{\tilde{T}_v\}$ commute.)

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