6-1985

A Stochastic Model of Investment, Marginal q and the Market Value of the Firm

Andrew B. Abel
University of Pennsylvania

Recommended Citation

Abel, A. B. (1985). A Stochastic Model of Investment, Marginal q and the Market Value of the Firm. *International Economic Review, 26* (2), 305-322. http://dx.doi.org/10.2307/2526585

A Stochastic Model of Investment, Marginal q and the Market Value of the Firm

Abstract

This paper presents closed-form solutions for the investment and valuation of a competitive firm with a Cobb-Douglas production function and a constant elasticity adjustment cost function in the presence of stochastic prices for output and inputs. The value of the firm is a linear function of the capital stock. The optimal rate of investmentis an increasing function of the slope of the value function with respect to the capital stock (marginal q). A mean preserving spread of the distribution of future price increases investment. An increase in the scale of the random component of a price can increase, decrease or not affect the rate of investment depending on the sign of the covariance of this price with a weighted average of all prices.

Disciplines

Finance and Financial Management

NBER WORKING PAPER SERIES

A STOCHASTIC MODEL OF INVESTMENT, MARGINAL q AND THE MARKET VALUE OF THE FIRM

Andrew B. Abel

Working Paper No. 1484

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1984

The research reported here is part of the NBER's research program in Economic Fluctuations. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

NBER Working Paper #1484
October 1984

A Stochastic Model of Investment, Marginal q and the Market Value of the Firm

ABSTRACT

This paper presents closed-form solutions for the investment and valuation of a competitive firm with a Cobb-Douglas production function and a constant elasticity adjustment cost function in the presence of stochastic prices for output and inputs. The value of the firm is a linear function of the capital stock. The optimal rate of investment is an increasing function of the slope of the value function with respect to the capital stock (marginal q).

A mean preserving spread of the distribution of future price increases investment. An increase in the scale of the random component of a price can increase, decrease or not affect the rate of investment depending on the sign of the covariance of this price with a weighted average of all prices.

Andrew B. Abel
Harvard University
Littauer Center 111
Cambridge, MA 02138
(617) 495-1869

A Stochastic Model of Investment, Marginal q and the Market Value of the Firm

Andrew B. Abel[1]

1. Introduction

In this paper we develop a stochastic model of the production and investment behavior of a competitive firm and use this model to examine the effects of uncertainty on the optimal rate of investment. The framework for this analysis is a stochastic version of the q theory of investment. Following a line of argument presented by Keynes [1936], Tobin [1969] defined (average) q as the ratio of the market value of a firm to the replacement cost of its capital and then argued that investment is an increasing function of q. A more rigorous foundation for the q theory of investment is based on the adjustment cost literature developed by Eisner and Strotz [1963], Lucas [1967], Gould [1968], and Treadway [1969]. It has been shown by Mussa [1977], Abel [1979, 1982] and Yoshikawa [1980] that in the presence of convex adjustment costs, investment is an increasing function of the shadow price of installed capital (marginal q). More recently, Hayashi [1982] has shown that under certain linear homogeneity and price-taking assumptions, the shadow price of installed capital is equal to the market value of the firm divided by the replacement cost of its capital; that is, marginal q equals average q. In situations in which marginal q and average q are not equal, it is marginal q which is relevant for investment.

The literature cited above has developed the q theory in a deterministic framework with adjustment costs. Stochastic models of investment in the presence of adjustment costs have been developed by Lucas and Prescott [1971], Hartman [1972], Pindyck [1982], and Abel [1983]. Using a discrete-time stochastic model, Hartman showed that for a competitive firm with constant returns to scale, increased uncertainty about future output prices or factor prices leads to increased current investment. More recently, Pindyck [1982] and Abel [1983] have analyzed investment behavior in continuous time models in which the price of output evolves according to an Ito process, and Abel demonstrated that Hartman's results carry over to continuous time. This paper extends Abel [1983] by incorporating several variable factors of production, with stochastic prices, and analyzes the effects of increased uncertainty. By extending the model to include several stochastic prices, we are led to examine different types of increases in uncertainty. A payoff to this extension is that we find that different types of (mean-preserving) increases in uncertainty can have qualitatively different effects on the rate of investment.

In analyzing the effects of increased uncertainty about prices, we examine two types of increase in uncertainty: (1) a mean-preserving spread, and (2) an increase in scale. Although an increase in scale is a mean-preserving spread for a scalar random variable, we show that for a multivariate random variable, an increase in scale is not, in general, a mean-preserving spread. More importantly, we show that these two types of increase in uncertainty about prices have different effects on investment. As shown by Hartman [1972], a mean-preserving spread tends to increase investment; however, an increase in the scale of the random component of a single price will raise, lower, or not affect the rate of investment depending on whether the covariance of this price with a weighted average of all prices is positive, negative, or zero.

Section 2 develops the model of the competitive firm and discusses the stochastic processes for the output price and the factor prices. The strategy of the paper is to restrict the specification of technology enough (constant elasticity) so that we can obtain explicit solutions for investment, marginal q and the market value of the firm. We present these solutions and provide an economic interpretation for them in Section 3. In Section 4 we define and analyze the effects of two alternative types of increase in uncertainty. The effects of increased uncertainty on the required rate of return are discussed in Section 5. Concluding remarks are presented in Section 6.

2. The Model of the Firm

Consider a competitive firm with a neoclassical production function $F(X_{1t}, \ldots X_{nt}, K_t)$ where X_{it}, $i = 1, \ldots, n$, is the amount of the ith variable factor used at time t and K_t is the amount of capital used at time t. Let p_t denote the price of output at time t and let w_{it}, $i = 1, \ldots, n$, denote the price of the ith variable factor at time t. The firm can accumulate capital by undertaking gross investment I_t at a cost $w_{n+1,t}C(I_t)$, where $w_{n+1,t}$ is a multiplicative shock to the adjustment cost function. Following the adjustment cost literature, we assume that $C(I_t)$ is an increasing convex function ($C' > 0$, $C'' > 0$) and that $C(0) = 0$. The accumulation of capital is given by

$$dK_t = (I_t - \delta K_t)dt \tag{1}$$

where δ is the constant proportional rate of depreciation.

The price of output, the prices of the variable factors, and the multiplicative adjustment cost shock are generated by Ito processes. To economize on notation, we let $w_{0,t}$ denote the price of output p_t and specify the evolution of w_{it}, $i = 0,\ldots,n+1$ as

$$\frac{dw_{it}}{w_{it}} = \pi_i dt + \sigma_i dZ_i \qquad i = 0,1,2,\ldots, n+1 \tag{2}$$

where dZ_i are Wiener processes with mean zero and unit variance such that

$$E(dZ_i dZ_j) = \rho_{ij} dt \tag{3}$$

The correlation coefficients ρ_{ij} satisfy $-1 \leq \rho_{ij} \leq 1$ and $\rho_{ii} = 1$.

There are several properties of these stochastic processes for w_{it} which should be noted.[2] The expected growth rate of w_{it}, $\frac{1}{dt}E_t(\frac{dw_{it}}{w_{it}})$ is equal to π_i and the instantaneous variance of w_{it} is $w_{it}^2\sigma_i^2$. The instantaneous covariance of w_{it} and w_{jt} is $\rho_{ij}\sigma_i\sigma_j w_{it}w_{jt}$. Finally, note that conditional on w_{it}, the future value of w_i, say w_{is}, $s > t$, is log-normally distributed with mean $w_{it}e^{\pi_i(s-t)}$ and variance $[e^{\sigma_i^2(s-t)}-1]e^{2\pi_i(s-t)}w_{it}^2$.[3] Thus,

$$E_t(w_{is}) = w_{it}e^{\pi_i(s-t)} \qquad s \geq t \tag{4}$$

where $E_t(\ )$ denotes the expectation conditional on information at time t. Observe in (4) that the conditional expected value of w_{is}, $s \geq t$, is independent of the variance of the process generating w_i.

The value of a risk-neutral firm at time t is the maximized expected present value of net cash flow from time t onward. Assuming that the discount rate r is constant, the value of the firm can be expressed as a time-invariant function of w_{it}, $i = 0,\ldots, n+1$, and the capital stock K_t,

$$V(w_{0,t}, \ldots, w_{n+1,t}, K_t) = \tag{5}$$
$$\max E_t \int_t^\infty [w_{0s}F(X_{1s}, \ldots, X_{ns}, K_s) - \sum_{i=1}^n w_{is}X_{is} - w_{n+1,s}C(I_s)]e^{-r(s-t)}ds$$

where the maximization in (5) is over the decision variables I and $X_1, \ldots, X_n$ and is subject to the constraints in (1) and (2). Optimality requires that

$$(6)\quad rV\,dt = \max E_t\{[w_{0t}F(X_{1t}, \ldots X_{nt}, K_t) - \sum_{i=1}^{n} w_{it}X_{it} - w_{n+1,t}C(I_t)]dt + dV\}$$

Equation (6) has a simple economic interpretation. The term in square brackets on the right hand side of (6) is the net cash flow over a small interval dt of time and the term dV is the change in the value of the firm. Equation (6) simply states that the expected rate of return on the firm (net cash flow plus capital gain divided by the value of the firm) must be equal to the discount rate r.

To calculate dV we use Ito's Lemma to obtain

$$(7)\quad dV = \sum_{i=0}^{n+1} \frac{\partial V}{\partial w_{it}} dw_{it} + \frac{\partial V}{\partial K_t} dK_t + \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} \frac{\partial^2 V}{\partial w_{it}\partial w_{jt}}(dw_{it})(dw_{jt})$$
$$+ \sum_{i=0}^{n+1} \frac{\partial^2 V}{\partial w_{it}\partial K_t}(dw_{it})(dK_t) + \frac{1}{2}\frac{\partial^2 V}{\partial K_t\partial K_t}(dK_t)^2$$

The expected value of dV is easily calculated using (1) and (2) and the fact that $E_t(dZ_i) = 0 = dt^2 = E_t(dt)(dZ_i)$ to obtain

$$(8)\quad E_t(dV) =$$

$$\{\sum_{i=0}^{n+1} \frac{\partial V}{\partial w_{it}} \pi_i w_{it} + \frac{\partial V}{\partial K_t}(I_t - \delta K_t) + \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} \frac{\partial^2 V}{\partial w_{it}\partial w_{jt}} \rho_{ij}\sigma_i\sigma_j w_{it}w_{jt}\}dt$$

Substituting (8) into (6) and defining $V_i = \frac{\partial V}{\partial w_{it}}$, $V_K = \frac{\partial V}{\partial K_t}$ and $V_{ij} = \frac{\partial^2 V}{\partial w_{it}\partial w_{jt}}$, we obtain

$$(9) \quad rV = \max_{I_t, X_{1t}, \ldots, X_{nt}} \{ w_{0t} F(X_{1t}, \ldots, X_{nt}, K_t) - \sum_{i=1}^{n} w_{it} X_{it} - w_{n+1,t} C(I_t)$$

$$+ \sum_{i=0}^{n+1} V_i \pi_i w_{it} + (I_t - \delta K_t) V_K + \frac{1}{2} \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} V_{ij} \rho_{ij} \sigma_i \sigma_j w_{it} w_{jt} \}$$

The nonlinear partial differential equation in (9) is the Bellman equation. In general, the Bellman equation cannot be solved explicitly. The strategy in this paper is to restrict the specification of technology enough to obtain a closed form solution to the Bellman equation.

2.1 Constant Elasticity Technology

In order to make the Bellman equation easily solvable we assume that the production function is Cobb-Douglas and that the adjustment cost function $C(I_t)$ has a constant elasticity. Specifically

$$(10) \qquad F(X_1, \ldots, X_n, K) = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} K^{\varphi}$$

$$(10a) \qquad \text{where} \quad \alpha_i > 0 \quad i = 1,\ldots,n \quad \text{and} \quad \varphi = 1 - \sum_{i=1}^{n} \alpha_i > 0$$

and

$$(11) \qquad C(I_t) = I_t^{\beta} \qquad , \ \beta > 1.$$

Given this specification of technology we can now maximize the right hand side of (9) with respect to $X_{1t}, \ldots, X_{nt}$. Since X_{it}, $i = 1,\ldots, n$, affect only current output and current variable cost, they are chosen to maximize current cash flow. It is straightforward to show that with the Cobb-Douglas production function in (10),[4]

$$(12)\quad \max_{X_{it}, \ldots, X_{nt}} w_{0t}F(X_{1t}, \ldots, X_{nt}, K_t) - \sum_{i=1}^{n} w_{it}X_{it} = \gamma \prod_{j=0}^{n} w_{jt}^{\frac{-\alpha_j}{\phi}} K_t$$

$$(12a)\quad \text{where} \quad \alpha_0 \equiv -1, \qquad \gamma \equiv \phi[\prod_{j=1}^{n} \alpha_j^{\alpha_j}]^{\frac{1}{\phi}}$$

The optimal rate of investment is found by differentiating (9) with respect to I_t and setting the derivative equal to zero. Using the fact that $C'(I_t) = \beta I_t^{\beta-1}$, the optimal rate of investment satisfies $w_{n+1,t}\beta I_t^{\beta-1} = V_K$ from which it follows that

$$(13)\quad I_t = [V_K/(w_{n+1,t}\beta)]^{\frac{1}{\beta-1}}$$

and

$$(14)\quad -w_{n+1,t}C(I_t) + I_tV_K = (\beta-1)w_{n+1,t}C(I_t)$$

Substituting (12) and (14) into (9), letting $p_tF_{K_t}$ denote the marginal revenue product of capital, observing that $C(I_t) = I_t^{\beta}$, and using (13), we obtain

$$(15)\quad rV = p_tF_{K_t}K_t + (\beta-1)w_{n+1,t}C(I_t) - \delta K_tV_K + \sum_{i=0}^{n+1} V_i\pi_iw_{it} + \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} V_{ij}\rho_{ij}\sigma_i\sigma_jw_{it}w_{jt}$$

$$(15a)\quad \text{where} \quad p_tF_{K_t} = \gamma \prod_{j=0}^{n} w_{jt}^{\frac{-\alpha_j}{\phi}}$$

$$(15b)\quad C(I_t) = [V_K/(w_{n+1,t}\beta)]^{\frac{\beta}{\beta-1}}$$

The solution to the nonlinear partial differential equation in (15) is derived in Appendix A and is discussed in the next section.

3. Investment, q, and the Valuation of the Firm

In this section we present and analyze explicit solutions for the value of the firm, marginal q and the optimal rate of investment. As shown in the Appendix, the value of the firm can be written as

$$\text{(16a)}\qquad V(w_{0t}, \ldots, w_{n+1,t}, K_t) = \mu_1 p_t F_{K_t} K_t + \mu_2(\beta-1) w_{n+1,t} C(I_t)$$

or (equivalently)

$$\text{(16b)}\qquad V(w_{0t}, \ldots, w_{n+1,t}, K_t) = \mu_1 \gamma \prod_{j=0}^{n} w_{jt}^{\frac{-\alpha_j}{\varphi}} K_t + \mu_2(\beta-1)\left[\frac{\mu_1\gamma}{\beta} \prod_{j=0}^{n+1} w_j^{-\alpha_j/\varphi}\right]^{\frac{\beta}{\beta-1}}$$

where $\alpha_{n+1} \equiv \varphi/\beta$

where

$$\text{(16c)}\qquad \mu_1 = \left[r+\delta+\sum_{i=0}^{n} \frac{\alpha_i}{\varphi}(\pi_i - \tfrac{1}{2}\sigma_i^2) - \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\alpha_i}{\varphi}\frac{\alpha_j}{\varphi}\rho_{ij}\sigma_i\sigma_j\right]^{-1}$$

$$\text{(16d)}\qquad \mu_2 = \left[r + \sum_{i=0}^{n+1} \frac{\beta\alpha_i}{(\beta-1)\varphi}(\pi_i - \tfrac{1}{2}\sigma_i^2) - \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\left[\frac{\beta\alpha_i}{(\beta-1)\varphi}\right]\left[\frac{\beta\alpha_j}{(\beta-1)\varphi}\right]\rho_{ij}\sigma_i\sigma_j\right]^{-1}$$

Equations (16a) and (16b) are equivalent to each other; equations (16c) and (16d) give the values of the constants μ_1 and μ_2. Equation (16b) expresses the value of the firm in terms of the state variables $w_{0t}, \ldots, w_{n+1,t}$ and K_t. Equation (16a) expresses the value of the firm in terms of more easily interpretable economic variables. Examination of the equations in (16) leads to several results.

Result 1: The value of the firm at time t is a linearly homogeneous function of $w_{0,t}, w_{1,t}, \ldots, w_{n+1,t}$.

To derive this result observe that the sum of the exponents of w_{jt} in the first term in (16b) is $\sum_{j=0}^{n} \frac{-\alpha_j}{\varphi}$ and the sum of the exponents of w_{jt} in the second term in (16b) is $\frac{\beta}{\beta-1} \sum_{j=0}^{n+1} \frac{-\alpha_j}{\varphi}$. Recalling that $\alpha_0 = -1$, $\alpha_{n+1} = \varphi/\beta$ and $\varphi = 1 - \sum_{j=1}^{n} \alpha_j$, it is clear that each of the sums of coefficients is equal to one. Therefore, we obtain Result 1.

Result 2:[5] The value of the firm at time t is a linear function of K_t.

The slope of the value function with respect to K_t, i.e., V_K, is equal to $\mu_1 p_t F_{K_t} = \mu_1 \gamma \prod_{j=0}^{n} w_{jt}^{-\alpha_j/\varphi}$, which, as we will show, is equal to the expected present value of the marginal revenue products of capital. Since the firm is a price-taker and the production function is linearly homogeneous, the marginal revenue product of capital is independent of the level of the capital stock. Hence, the expected present value of marginal revenue products is independent of K_t and the slope of the value function is independent of K_t.

In order to show that V_K is equal to the expected present value of the marginal revenue products of capital, we first present the following lemma which permits easy calculation of the expected present value of the marginal products of capital.

Lemma 1 Suppose $G_t = G(w_{0t}, \ldots, w_{n+1,t}) = \prod_{i=0}^{n+1} w_{i,t}^{c_i}$, where c_i are known constants and $w_{i,t}$ evolve according to (2) and (3). Then the present value of G_s $s \geq t$ discounted at rate λ is

$$(17a) \qquad E_t \int_t^\infty G_s e^{-\lambda(s-t)} ds = \frac{G_t}{\lambda + \sum_{i=1}^{n+1} c_i(\pi_i - \frac{1}{2}\sigma_i^2) - \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} c_i c_j \rho_{ij}\sigma_i\sigma_j}$$

$$(17b) \qquad = \frac{G_t}{\lambda + \frac{1}{dt}[E_t(d \ln G_t) - \frac{1}{2}\operatorname{var}(d \ln G_t)]}$$

Proof. See Appendix B.

If we let the discount rate λ be $r+\delta$ and let G_t be the marginal revenue product of capital at time t, $\gamma \prod_{i=0}^{n} w_{it}^{-\alpha_i/\varphi}$, so that $c_i = -\alpha_i/\varphi$, $i=\varphi, \ldots, n$ and $c_{n+1} = 0$, then it follows immediately from (16c) and Lemma 1 that $\mu_1 p_t F_{K_t}$ is the expected present value of marginal revenue products accruing to capital from time t onward. The discount factor λ reflects both the rate of interest r as well as physical depreciation at rate δ. Thus $\mu_1 p_t F_{K_t}$ is the expected present value of marginal revenue products accruing to the undepreciated portion of a unit of capital which is in place at time t.

It is convenient to define q_t as the marginal valuation of capital divided by $w_{n+1,t}$ (the shock to the adjustment cost function). Therefore, from (13) we obtain

$$(18a) \qquad I_t = \beta^{\frac{1}{1-\beta}} q_t^{\frac{1}{\beta-1}}$$

$$(18b) \qquad \text{where} \quad q_t = V_{K,t}/w_{n+1,t}$$

Inspection of (18) leads to

Result 3: The optimal rate of investment is an increasing function of q_t with elasticity $\frac{1}{\beta-1}$ where β is the (constant) elasticity of $C(I)$ with respect to I. Also, q_t and I_t are homogeneous of degree zero in $w_{0,t}, \ldots, w_{n+1,t}$.

The relation between the valuation of the firm and the rate of investment can be interpreted with the use of Figure 1.

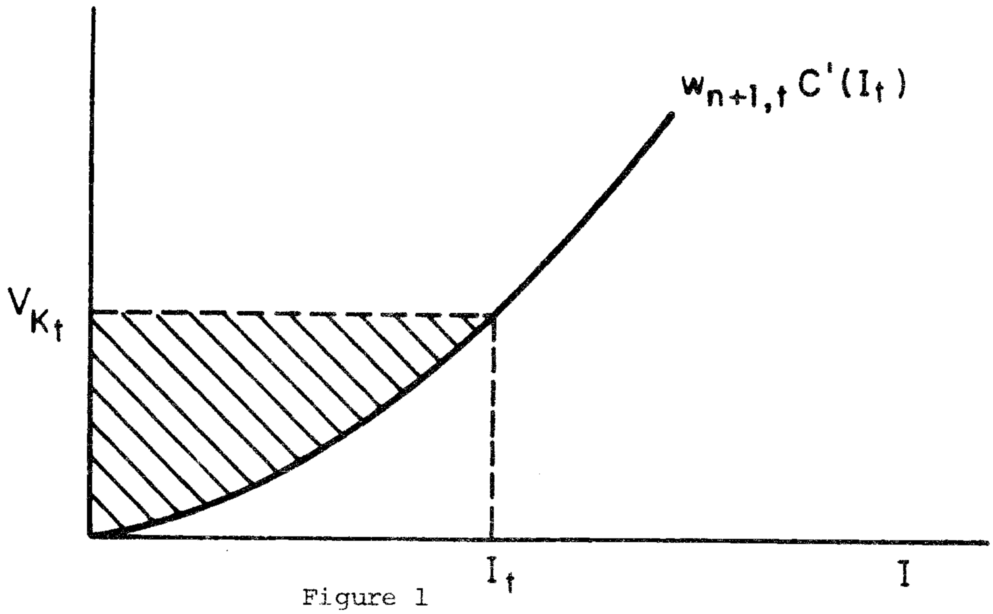

Figure 1

The optimal rate of investment is chosen to equate the marginal valuation of capital, V_{K_t}, with the marginal adjustment cost $w_{n+1,t}C'(I_t)$, as shown in Figure 1. Thus the optimal rate of investment is related to the slope (with respect to K_t) of the valuation of the firm. The constant term in the valuation equation is related to the shaded area in Figure 1. This shaded area is equal to $I_tV_{K_t} - w_{n+1,t}C(I_t)$, which is the expected present value of rentals accruing to infra-marginal units of investment at date t; it is the amount by which the valuation of current investment, $I_tV_{K_t}$, exceeds the cost of current investment $w_{n+1,t}C(I_t)$. According to (14) this present value of infra-marginal rents is equal to $(\beta-1)w_{n+1,t}C(I_t)$. Therefore, the constant term in the valuation equation (16a) is equal to the area of the shaded region in Figure 1 multiplied by μ_2. Since $(\beta-1)w_{n+1,t}C(I_t)$ is equal to $(\beta-1)[\frac{\mu_1\gamma}{\beta}\prod_{j=0}^{n+1} w_j^{-\alpha_j/\phi}]^{\frac{\beta}{\beta-1}}$, it follows from (16d) and Lemma 1 that the constant term in the valuation equation is equal to the expected present value of infra-marginal rents to current and future investment (To apply Lemma 1, let $\lambda = r$ and $c_i = \frac{-\beta\alpha_i}{(\beta-1)\phi}$ for $i=0,1, \ldots, n+1$).

To summarize, the value of the firm at time t is a linear function of K_t. The linear term in K_t represents the expected present value of marginal revenue products accruing to capital currently in place at time t. The constant term represents the expected present value of rents to infra-marginal units of current and future investment.

4. The Effects of Increasing Uncertainty

In this section we examine the effects of increased uncertainty on the optimal rate of investment and on the market value of the firm. In a discrete-time model, Hartman [1972] has shown that if w_{it}, $i=0,\ldots, n+1$, undergoes a mean preserving spread, then there is an increase in the rate of investment. In a continuous time model with a single variable factor of production, Abel [1983] has shown that Hartman's result continues to hold.

In this section we extend the results of Abel [1983] to a model with several (n+2) random variables. The extension is non-trivial as explained below. We consider two types of increases in uncertainty: (1) a mean preserving spread (MPS); and (2) an increase in the scale of one of the random variables (IS). In the case of a single random variable, an increase in scale is a mean preserving spread. However, with several random variables, an increase in the scale of one variable is a mean preserving spread if and only if that variable is uncorrelated with all other random variables; if the variable whose scale is increased has a nonzero covariance with any other random variable, then an increase in scale is <u>not</u> a mean preserving spread.

The effects of an MPS increase in uncertainty differ from the effects of an IS increase in uncertainty. We will show that, consistent with Hartman's findings, an MPS increase in uncertainty will increase investment. However, the effects on investment of an IS increase in the uncertainty associated with $w_{j,t}$ depends on the covariance of $\ln w_{j,t}$ with $\sum_{i=0}^{n} \frac{\alpha_i}{\varphi}(\ln (w_i/w_j))$. Depending on whether this covariance is positive, negative, or zero, an IS increase in uncertainty will increase, decrease or leave unchanged the rate of investment.

We will examine the effects on investment of increasing uncertainty holding constant the current values of $w_{i,t}$. Since investment is an increasing function of q_t, we can focus on the effects of uncertainty on q_t. For given values of $w_{i,t}$ $i = 0,\ldots, n$, the effects on q_t and investment can be determined simply by determining the effects on μ_1: the effects on q_t and investment are in the same direction as the effects on μ_1.

We will first compare optimal investment under certainty and under uncertainty. In all cases we will examine changes in uncertainty which leave $E_t(w_{is})$, $s \geq t$, unchanged. Observe from (4) that $E_t(w_{is})$ is independent of all σ_j and all ρ_{jk}. Therefore, the certainty case relevant for comparison to any uncertainty case is obtained simply by setting all σ_i equal to zero. From (16c) it follows that μ_1 (and hence q_t and I_t) is greater under uncertainty that under certainty if and only if

$$\sum_{i=0}^{n} \frac{\alpha_i}{\phi}\sigma_i^2 + \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\alpha_i}{\phi}\frac{\alpha_j}{\phi}\rho_{ij}\sigma_i\sigma_j > 0 \tag{19}$$

We can prove that (19) holds by using

Lemma 2. Suppose $x_i > 0$ for $i = 1, 2, \ldots, m$ and that $\sum_{i=0}^{m} x_i = -1$. Define

$$S(x_0, \ldots, x_m) \equiv \sum_{i=0}^{m} x_i\sigma_i^2 + \sum_{i=0}^{m}\sum_{j=0}^{m} x_i x_j \rho_{ij}\sigma_i\sigma_j \quad \text{where } \rho_{ij} = \text{cov}(dZ_i, dZ_j).$$

Then $S(x_0, \ldots, x_m) \geq 0$, with strict inequality unless $\text{Var}(\sigma_i dZ_i - \sigma_0 dZ_0) = 0$ for all i.

Proof. See Appendix C.

If we let $x_i = \frac{\alpha_i}{\phi}$ and $m = n$, then (19) follows immediately from Lemma 2 (provided that there is not perfect correlation among all dZ_i). Hence, as shown by Hartman [1972] and Abel [1983] the optimal rate of investment is higher under uncertainty than under certainty.

4.1 Mean Preserving Spread

We follow Hartman's extension to several random variables of the Rothschild-Stiglitz [1970] definition of a mean preserving spread. Specifically, if x is a random vector and if u is a random vector (with the same dimension as x) such that $E(u|x) = 0$, then the distribution of the random vector $y = x+u$ is a mean preserving spread of the distribution of x. Observe that the covariance matrix of y exceeds the covariance matrix of x by a nonnegative definite matrix (we allow some elements of u to be nonstochastic).

We now consider the effects of a mean preserving spread on the distribution of the Wiener process dZ_i. In particular, we add an uncorrelated process to the Ito process for $\frac{dw_{it}}{w_{it}}$ to obtain

$$\frac{dw_{i,t}}{w_{i,t}} = \pi_i dt + \sigma_i dZ_i + \sigma_i^* dZ_i^* \qquad i=0,1,\dots,n+1 \tag{20}$$

where $E_t(dZ_i)(dZ_j^*) = 0$ and $E_t(dZ_i^*)(dZ_j^*) = \rho_{ij}^* dt$. The expected growth rate of w_{it}, $\frac{1}{dt}E_t(\frac{dw_{it}}{w_{it}})$, is equal to π_i as before. However, the instantaneous variance of w_{it} is now $w_{it}^2(\sigma_i^2 + \sigma_i^{*2})$ and instantaneous covariance of w_{it} and w_{jt} is now $w_{it}w_{jt}(\rho_{ij}\sigma_i\sigma_j + \rho_{ij}^*\sigma_i^*\sigma_j^*)$. The effect of performing this MPS on dZ_i is to reduce μ_1^{-1} by $A^* = \frac{1}{2}\sum_{i=0}^{n}\frac{\alpha_i}{\varphi}\sigma_i^{*2} + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{\alpha_i}{\varphi}\frac{\alpha_j}{\varphi}\rho_{ij}^*\sigma_i^*\sigma_j^*$. It follows immediately from Lemma 2 that $A^* > 0$ and hence that a MPS increases in uncertainty leads to an increase in μ_1, q_t and investment.

4.2 Increase in Scale

Consider a scalar random variable Z with mean $\bar{Z}$. We will say that the scalar random variable y represents an increase in scale for the random variable Z, if $y-\bar{Z} = (1+b)(Z-\bar{Z})$ for some constant $b > 0$. Thus from (2) an IS increase in uncertainty of $\frac{dw_{it}}{w_{it}}$ corresponds to an increase in σ_i but has no effect on the distribution of dZ_{it}. Thus, in a multivariate context, an IS increase in the uncertainty of $\frac{dw_{it}}{w_{it}}$ has no effect on (dZ_{it}) (dZ_{jt}) and hence does not affect the correlation matrix of $(\frac{dw}{w})_t \equiv [\frac{dw_{ot}}{w_{ot}}, \dots, \frac{dw_{n+1,t}}{w_{n+1,t}}]$ which has ρ_{ij} as the $(i+1,j+1)$ element. The effect on the covariance matrix of $(\frac{dw}{w})_t$ is to multiply row $(i+1)$ and col $(i+1)$ by some constant greater than 1.

This effect on the covariance matrix is to be contrasted (see Lemma 3 below) with the effect of a MPS increase in uncertainty which adds a positive semi-definite matrix to the covariance matrix of $\frac{dw}{w}$.

We examine the effects of an IS increase in uncertainty by differentiating μ_1 with respect to σ_i holding constant all ρ_{ij} and σ_j, $j \neq i$. Differentiating (16c) with respect to σ_i we obtain

$$(21) \qquad \frac{\partial \mu_1}{\partial \sigma_i} = \mu_1^2 \frac{\alpha_i}{\phi \sigma_i} (\sigma_i^2 + \sum_{j=0}^{n} \frac{\alpha_j}{\phi} \rho_{ij} \sigma_i \sigma_j)$$

Recalling that $\sum_{j=0}^{n} \frac{\alpha_j}{\phi} = -1$, equation (21) may be rewritten as

$$(22) \qquad \frac{\partial \mu_1}{\partial \sigma_i} = \mu_1^2 \frac{\alpha_i}{\phi \sigma_i} (\sum_{j=0}^{n} \frac{\alpha_j}{\phi} (\rho_{ij} \sigma_i \sigma_j - \sigma_i^2))$$

Now observe that $\text{Cov}(\ln(w_j/w_i), \ln w_i) = \rho_{ij}\sigma_i\sigma_j - \sigma_i^2$ so that (22) can be expressed as

$$(23) \qquad \frac{\partial \mu_1}{\partial \sigma_i} = \frac{\alpha_i \mu_1^2}{\phi \sigma_i} \text{Cov}[\sum_{j=0}^{n} \frac{\alpha_j}{\phi} \ln(w_j/w_i), \ln w_i]$$

From equation (23), $\frac{\partial \mu_1}{\partial \sigma_i}$ is positive, negative, or zero depending on whether the covariance of $\sum_{j=0}^{n} \frac{\alpha_j}{\phi} \ln(w_j/w_i)$ and $\ln w_i$ is positive, negative, or zero. Thus an IS increase in uncertainty will increase, decrease or have no effect on the optimal rate of investment depending on whether $\text{Cov}[\sum_{j=0}^{n} \frac{\alpha_j}{\phi} \ln(w_j/w_i), \ln w_i]$ is positive, negative or zero. Observe from (21) that in the special case in which $\rho_{ij} = 0$, $i \neq j$, $\frac{\partial \mu_1}{\partial \sigma_i} = \mu_1^2 \sigma_i \frac{\alpha_i}{\phi}(1+\frac{\alpha_i}{\phi}) > 0$ so that an IS increase on uncertainty leads to an increase in the rate of investment.

At first glance it may appear inconsistent that the effect on investment of an MPS increase in uncertainty is unambiguously positive, whereas the effect on investment of an IS increase in uncertainty can be positive, negative, or zero. These two findings are reconciled by the fact that, in general, an IS increase in uncertainty is not an MPS increase in uncertainty. Only if dZ_i is uncorrelated with all dZ_j, $j \neq i$, is it the case that an IS increase in uncertainty of $\frac{dw_{it}}{w_{it}}$ is an MPS increase in uncertainty.

To show that an IS increase in uncertainty of dZ_i is not, in general, a MPS, we will use the following lemma:

Lemma 3. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of

$$A = \begin{bmatrix} a_1 & a_2 \cdots a_m \\ a_2 & 0 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ a_m & 0 \cdots 0 \end{bmatrix}$$

where $a_1 > 0$ and $m \geq 2$. Then $\lambda_1\lambda_2 = -\sum_{i=2}^{m} a_i^2 \leq 0$, $\lambda_1 + \lambda_2 = a_1 > 0$ and, if $m \geq 3$, $\lambda_3 = \cdots = \lambda_m = 0$.

Proof. See Appendix D.

Using the fact that all eigenvalues of a symmetric nonnegative definite matrix are non-negative we obtain the following

Corollary. The matrix A in Lemma 3 is nonnegative definite if and only if $a_2 = \cdots = a_m = 0$.

Using the corollary above we can now prove the following

Proposition. An IS increase in uncertainty of dZ_i is not an MPS increase in uncertainty unless dZ_i has zero correlation with all dZ_j, $j \neq i$.

Proof. Without loss of generality, we examine an IS increase in uncertainty of $\frac{dw_{0,t}}{w_{0,t}}$ which increases the covariance matrix from $\Sigma \equiv (\rho_{ij}\sigma_i\sigma_j)$ to $\Sigma + D$ where

$$D = \begin{bmatrix} b^2\sigma_0^2 & b\rho_{01}\sigma_0\sigma_1 & \cdots & b\rho_{0n}\sigma_0\sigma_n \\ b\rho_{01}\sigma_0\sigma_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b\rho_{0n}\sigma_0\sigma_n & 0 & \cdots & 0 \end{bmatrix}$$

From the Corollary to Lemma 3, D is nonnegative definite if and only if $\rho_{0i} = 0$ for $i = 1,\ldots,n$. Since an MPS increase in uncertainty causes the covariance matrix to increase by a nonnegative definite matrix, the IS increase in uncertainty cannot be a MPS if $\rho_{0i} \neq 0$ for any $i \geq 1$. On the other hand, if $\rho_{0i} = 0$, $i = 1,\ldots,n$, then the IS increase in uncertainty is equivalent to the following MPS: In (20) let $\sigma_0{}^* = b\sigma_0$ and let $\sigma_i{}^* = 0$, $i = 1,\ldots, n$. q.e.d.

In this section we have examined two different concepts of increasing uncertainty in a multivariate context: an MPS increase in uncertainty and an IS increase in uncertainty. We have shown that an MPS increase in uncertainty unambiguously raises the rate of investment whereas an IS increase in uncertainty will raise, lower or leave unchanged the rate of investment depending on whether a certain covariance is positive, negative, or zero.

As a final comment on the effects of uncertainty, it should be emphasized that it is uncertainty of relative prices which has an effect on investment. If all w_{it} are perfectly (positively) correlated and have the same proportional variance, then all relative prices w_{it}/w_{jt} are non-stochastic. In this case, the rate of investment under uncertainty is the same as under certainty.

5. The Required Rate of Return[6]

Up to this point our analysis of the firm's behavior has been conducted under the assumption of risk-neutrality. In particular, we have assumed that the required rate of return on the firm's equity, r, remains unchanged when the uncertainty of output price and factor prices is changed. It should be noted that risk-neutrality *per se* is not required for the invariance of r with respect to changes in uncertainty. More generally, in the traditional capital asset pricing model, the required rate of return on a firm is independent of the variance of its own prices (output prices and factor prices) if the rate of return on the firm is uncorrelated with the return on the market portfolio. In the context of more recent asset pricing models of Lucas (1978) and Breeden (1979), the required rate of return on a firm will be independent of the variances of prices if the rate of return on the firm is uncorrelated with the marginal utility of consumption. Thus, risk-neutrality *per se* is not required for the results in this paper to hold.

If we drop the assumption that the return on the firm is uncorrelated with the market portfolio (or with the marginal utility of consumption), then the required rate of return on the firm is an increasing function of the covariance of the firm's return with the return on the market portfolio. If the increase in price uncertainty causes this covariance to increase, then the required rate of return also increases which tends to decrease both q_t and investment. Alternatively, if the increase in price uncertainty leads to a decrease in the relevant covariance, then the required rate of return decreases so that q_t and investment each tend to increase.

It is clear that to reach any conclusions about the effect of uncertainty on the required rate of return we would have to impose some structure on the covariance of the rate of return on the firm and the rate of return on the market portfolio (or the marginal utility of consumption). The results in earlier sections can be used to calculate the random component of the rate of return on the firm. However, without developing a complete general equilibrium dynamic stochastic model, we have tremendous latitude in specifying a stochastic process for the rate of return on the market portfolio and thus could "derive" results which show the required rate of return increasing or decreasing in response to an increase in uncertainty.

The analysis of this paper is explicitly partial equilibrium in nature. We have argued above that to reach any conclusions about the effect of increased uncertainty on the required rate of return (without, in effect, being free to assume the conclusion by strategically specifying the stochastic process for the rate of return on the market portfolio) would require a general equilibrium model. Of course, in a general equilibrium framework, the analysis of uncertainty should focus not on the effects of price uncertainty but rather on the effects of uncertainty about preferences and technology. Such analysis is beyond the scope of this paper.

6. Concluding Remarks

We have analyzed the optimal production and investment behavior of a competitive firm facing random prices for output and factors of production. By restricting the production function to be Cobb-Douglas and the adjustment technology to have constant elasticity, we were able to obtain closed-form solutions for investment, marginal q and the market value of the firm. In particular, the market value of the firm is a linear function of the firm's capital stock; investment is an increasing function of the slope of this value function.

Using the closed-form solution for the optimal rate of investment, we examined the effects on investment of two alternative types of increase in uncertainty about the random vector of prices. The effect of a mean-preserving spread is to increase investment. However, the effect of an increase in the scale of the random component of a single price is to increase, decrease, or leave unchanged the rate of investment depending on whether the covariance of this price with a (geometric) weighted average of all prices is positive, negative, or zero.

Harvard University

and

National Bureau of Economic Research

Footnotes

1. I thank Ernst Berndt, Stanley Fischer, Robert McDonald, Peter Merrill and Robert Pindyck for helpful discussions on earlier drafts of this paper. I also thank the participants in workshops at Columbia University, Harvard University and M.I.T. and two anonymous referees for their helpful comments. Research support from the Department of Energy and the National Science Foundation is gratefully acknowledged.

2. For good discussions of stochastic calculus set in an economic context, the reader is referred to Brock, Chow [1981], Fischer [1975], and Merton [1971].

3. The solution to the stochastic differential equation in (2) is

$$(*) \qquad w_{is} = w_{it} \exp[(\pi_i - \tfrac{1}{2}\sigma_i^2)(s-t) + \sigma_i \int_t^s dZ_i]$$

(See, for example, Fischer [1975], equation (13A)). The solution may be rewritten as

$$(**) \qquad \ln w_{is} = \ln w_{it} + (\pi_i - \tfrac{1}{2}\sigma_i^2)(s-t) + \sigma_i \int_t^s dZ_i$$

from which it follows that $\ln w_{is}$ is normally distributed with mean $\ln w_{it} + (\pi_i - \frac{1}{2}\sigma_i^2)(s-t)$ and variance $\sigma_i^2(s-t)$. Using the facts that if $\ln x$ is normally distributed with mean μ and variance σ^2, then $E(x) = \exp[\mu + \frac{1}{2}\sigma^2]$ and $Var(x) = [\exp(\sigma^2) - 1] \cdot [\exp(2\mu + \sigma^2)]$, we find that $E_t(w_{is}) = w_{it}e^{\pi_i(s-t)}$ and $Var_t(w_{is}) = w_{it}^2 e^{2\pi_i(s-t)}[e^{\sigma_i^2(s-t)} - 1]$.

4. Choosing $X_1, \ldots, X_n$ to maximize $pF(X_1, \ldots, X_n, K) - \sum_{i=1}^{n} w_i X_i$ where $F(\ )$ is the Cobb-Douglas production function in (10) yields

$$\frac{w_i X_i}{\alpha_i} = pF \qquad i=1,\ldots,n \tag{4.1}$$

which reflects the fact that α_i is the (constant) share of variable factor i. Using (4.1) for X_i and X_j yields

$$X_j = \frac{w_i X_i}{\alpha_i} \frac{\alpha_j}{w_j} \tag{4.2}$$

Substituting (4.2) into the production function for j=1,...,n yields

$$F = \left(\frac{w_i X_i}{\alpha_i}\right)^{\sum_{j=1}^{n} \alpha_j} \left[\prod_{j=1}^{n} (\alpha_j/w_j)^{\alpha_j}\right] K^{\varphi} \tag{4.3}$$

Combining (4.1) and (4.3) and recalling that $\varphi = 1 - \sum_{j=1}^{n} \alpha_j$ yields

$$\left(\frac{w_i X_i}{\alpha_i}\right)^{\varphi} = p\left[\prod_{j=1}^{n} (\alpha_j/w_j)^{\alpha_j}\right] K^{\varphi} \tag{4.4}$$

so that

$$w_i X_i = \alpha_i p^{1/\varphi} \left[\prod_{j=1}^{n} (\alpha_j/w_j)^{\alpha_j}\right]^{1/\varphi} K \tag{4.5}$$

From (4.1) the maximized value of $pF - \sum_{i=1}^{n} w_i X_i$ is equal to φpF which using (4.1) and (4.5) is equal to

$$\varphi p^{1/\varphi} \left[\prod_{j=1}^{n} (\alpha_j/w_j)^{\alpha_j}\right]^{1/\varphi} K \tag{4.6}$$

Equation (4.6) is equivalent to equation (12) in the text.

5. Mussa [1974] showed that for a linearly homogeneous production function, the value of the firm under certainty is linear in K_t.

6. I thank an anonymous referee for suggesting that I consider the effects of uncertainty on the required rate of return.

References

Abel, Andrew B., Investment and the Value of Capital (New York: Garland Publishing Co., 1979).

_______________, "Dynamic Effects of Permanent and Temporary Tax Policies in a q Model of Investment," Journal of Monetary Economics, 9 (May 1982), 353-73.

_______________, "Optimal Investment Under Uncertainty, American Economic Review, 73,1 (March 1983), 228-233.

Breeden, Douglas, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," Journal of Financial Economics, (September 1979), 265-296.

Brock, William A., "Introduction to Stochastic Calculus: A User's Manual," mimeo, University of Chicago.

Chow, Gregory C., Econometric Analysis by Control Methods (New York: John Wiley and Sons, 1981).

Eisner and Strotz, "Determinants of Business Investment," in Commission on Money and Credit, Impacts of Monetary Policy (Englewood Cliffs, NJ: Prentice-Hall, 1963), pp. 59-337.

Fischer, Stanley, "The Demand for Index Bonds," Journal of Political Economy 83 (1975), pp. 509-534.

Gould, John P., "Adjustment Costs in the Theory of Investment of the Firm," Review of Economic Studies 35 (January 1968), pp. 47-55.

Hartman, Richard, "The Effects of Price and Cost Uncertainty on Investment," Journal of Economic Theory 5 (October 1972), 258-66.

Hayashi, Fumio, "Tobin's Marginal and Average q: A Neoclassical Interpretation," Econometrica, 50 (January 1982), 213-224.

Keynes, John Maynard, The General Theory of Employment, Interest and Money (London: Macmillan, 1936).

Lucas, Robert E., Jr., "Adjustment Costs and the Theory of Supply," Journal of Political Economy 75 (August 1967), pp. 321-34.

______________________, "Asset Prices in an Exchange Economy," Econometrica (November 1978), 1429-1445.

______________________ and Prescott, Edward C., "Investment Under Uncertainty," Econometrica, 39,5 (September 1971), 659-681.

Merton, Robert C., "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," Journal of Economic Theory 3 (December 1971), pp. 373-413.

Mussa, Michael, "Market Value and the Investment Decision in an Adjustment Cost Model of Firm Behavior," Department of Economics, University of Rochester Discussion Paper 74-15, July 1974.

______________, "External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment, Economica 47 (May 1977), 163-78.

Pindyck, Robert S., "Adjustment Costs, Demand Uncertainty, and the Behavior of the Firm," American Economic Review, 72 (June 1982), 415-27.

Tobin, James, "A General Equilibrium Approach to Monetary Theory," Journal of Money, Credit and Banking 1 (February 1969), 15-29.

Treadway, Arthur B., "On Rational Entrepreneurial Behavior and the Demand for Investment," Review of Economic Studies 36 (1969), 227-239.

Yoshikawa, Hiroshi, "On the "q" Theory of Investment," American Economic Review 70 (September 1980), 739-743.

Appendix A

We solve the Bellman equation in (15) using the method of undetermined coefficients. We hypothesize that the solution takes the form

(A1) $V(w_0, \ldots, w_{n+1}, K) = V^{(1)}(w_0, \ldots, w_n)K + V^{(2)}(w_0, \ldots, w_{n+1})$

(A1a) where $V^{(1)} = \mu_1 pF_K = \mu_1 \gamma \prod_{j=0}^{n} w_j^{-\alpha_j/\phi}$

(A1b) $$V^{(2)} = \mu_2(\beta-1)w_{n+1}C(I) = \mu_2(\beta-1)\left(\frac{\mu_1\gamma}{\beta}\right)^{\frac{\beta}{\beta-1}} \left[\prod_{j=0}^{n+1} w_j^{-\alpha_j/\phi}\right]^{\frac{\beta}{\beta-1}}$$

Letting $V_i^{(k)}$ denote $\frac{\partial V^{(k)}}{\partial w_i}$ and $V_{ij}^{(k)}$ denote $\frac{\partial^2 V^{(k)}}{\partial w_i \partial w_j}$, we can differentiate (A1a) and (A1b) to obtain

(A2) $$V_K = V^{(1)}$$

(A3) $$w_i V_i^{(1)} = \frac{-\alpha_i}{\phi} V^{(1)}$$

(A4) $$w_i^2 V_{ii}^{(1)} = \frac{\alpha_i}{\phi}\left(1 + \frac{\alpha_i}{\phi}\right) V^{(1)}$$

(A5) $$w_i w_j V_{ij}^{(1)} = \frac{\alpha_i \alpha_j}{\phi^2} V^{(1)} \qquad i \neq j$$

(A6) $$w_i V_i^{(2)} = \frac{\beta}{\beta-1} \frac{-\alpha_i}{\phi} V^{(2)}$$

(A7) $$w_i^2 V_{ii}^{(2)} = \frac{\beta}{\beta-1} \frac{\alpha_i}{\phi}\left(1 + \frac{\beta}{\beta-1} \frac{\alpha_i}{\phi}\right) V^{(2)}$$

(A8) $$w_i w_j V_{ij}^{(2)} = \left(\frac{\beta}{\beta-1}\right)^2 \frac{\alpha_i}{\phi} \frac{\alpha_j}{\phi} V^{(2)} \qquad i \neq j$$

Recognizing that $V_i = V_i^{(1)}K + V_i^{(2)}$ and that $V_{ij} = V_{ij}^{(1)}K + V_{ij}^{(2)}$, we substitute (A2)-(A8) into (15) to obtain

$$rV^{(1)}K + rV^{(2)} = \mu_1^{-1}V^{(1)}K + \mu_2^{-1}V^{(2)} - \delta V^{(1)}K + \sum_{i=0}^{n}(\frac{-\alpha_i}{\phi}\pi_i + \frac{1}{2}\frac{\alpha_i}{\phi}\sigma_i^2)V^{(1)}K$$

$$+ \sum_{i=0}^{n+1}\frac{\beta}{\beta-1}(\frac{-\alpha_i}{\phi}\pi_i + \frac{\alpha_i}{\phi}\frac{1}{2}\sigma_i^2)V^{(2)} + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{\alpha_i}{\phi}\frac{\alpha_j}{\phi}\rho_{ij}\sigma_i\sigma_j V^{(1)}K$$

$$+ \frac{1}{2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}(\frac{\beta}{\beta-1})^2\frac{\alpha_i}{\phi}\frac{\alpha_j}{\phi}\rho_{ij}\sigma_i\sigma_j V^{(2)} \tag{A9}$$

Equating the coefficients of $V^{(1)}K$ on both sides of (A9) yields the value of μ_1 shown in (16c) and equating the coefficients of $V^{(2)}$ on both sides of (A9) yields the value of μ_2 in (16d).

Appendix B

Proof of Lemma 1

Observe that $\ln G_s = \sum_{i=0}^{n+1} c_i \ln w_{is}$ so that $E_t(\ln G_s) = \sum_{i=0}^{n+1} c_i E_t(\ln w_{is})$ and $Var_t(\ln G_t) = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_i c_j \operatorname{cov}_t(\ln w_{is}, \ln w_{js})$. Observe that (see footnote 3) $E_t(\ln w_{is}) = \ln w_{it} + (\pi_i - \frac{1}{2}\sigma_i^2)(s-t)$ and $\operatorname{cov}_t(\ln w_{is}, \ln w_{js}) = \rho_{ij}\sigma_i\sigma_j(s-t)$ which yields

(B1) $$E_t(\ln G_s) = \sum_{i=0}^{n+1} c_i[\ln w_{it} + (\pi_i - \frac{1}{2}\sigma_i^2)(s-t)]$$

(B2) $$Var_t(\ln G_s) = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_i c_j \rho_{ij}\sigma_i\sigma_j(s-t)$$

Since $\ln w_{is}$ is (conditionally) normally distributed, so is $\ln G_s$. Therefore

(B3) $$E_t(G_s) = \exp[E_t(\ln G_s) + \frac{1}{2} Var_t(\ln G_s)]$$

Substituting (B1) and (B2) into (B3) yields

(B4) $$E_t(G_s) = G_t \exp\{[\sum_{i=0}^{n+1} c_i(\pi_i - \frac{1}{2}\sigma_i^2) + \frac{1}{2}\sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_i c_j \rho_{ij}\sigma_i\sigma_j](s-t)\}$$

Recognizing that $E_t \int_t^\infty G_s e^{-\lambda(s-t)} ds = \int_t^\infty E_t(G_s) e^{-\lambda(s-t)} ds$, equation (B4) immediately implies (17a). The equivalence of (17a) and (17b) follows from noting that $\frac{1}{dt}E_t(d \ln G_t) = \frac{1}{dt}E_t \sum_{i=0}^{n+1} c_i d \ln w_{it} = \sum_{i=0}^{n+1} c_i(\pi_i - \frac{1}{2}\sigma_i^2)$ and that $\frac{1}{dt}Var_t(d \ln G_t) = \frac{1}{dt}Var_t[\sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_i c_j (d \ln w_{it}) d \ln w_{jt})] = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_i c_j \rho_{ij}\sigma_i\sigma_j$.

Appendix C

Proof of Lemma 2

It will be convenient to define $x = (x_1, \ldots, x_m)'$, $i = (1,\ldots,1)'$ and $\Sigma_0 = (\rho_{01}\sigma_0\sigma_1, \ldots, \rho_{0m}\sigma_0\sigma_m)'$. Let Σ be the mxm matrix with (i, j) element equal to $\mathrm{cov}(\sigma_i dZ_i, \sigma_j dZ_j)$ and let $\mathrm{diag}(\Sigma)$ be the mx1 vector with ith element equal to σ_i^2. Observe that S( ) may be written as

(C1) $$S(x_0, x) = x_0\sigma_0^2 + x'\mathrm{diag}\Sigma + x_0^2\sigma_0^2 + 2x_0x'\Sigma_0 + x'\Sigma x$$

The constraint $\sum_{i=0}^{m} x_i = -1$ can be written as $x_0 = -(1 + x'i)$. Substituting this expression for x_0 into (C1) allows us to express the value of $S(x_0, x)$ subject to this constraint as a function $S^*(x)$

(C2) $$S^*(x) = (1+x'i)(x'i)\sigma_0^2 - 2(1+x'i)\Sigma_0'x + x'\mathrm{diag}(\Sigma) + x'\Sigma x$$

Combining the linear terms in x together and the quadratic terms together we obtain

(C3) $$S^*(x) = x'[\sigma_0^2 i - 2\Sigma_0 + \mathrm{diag}(\Sigma)] + x'[\sigma_0^2 ii' - i\Sigma_0' - \Sigma_0 i' + \Sigma]x$$

Let Ω denote the mxm covariance matrix with (i, j) element equal to $\mathrm{cov}(\sigma_i dZ_i - \sigma_0 dZ_0, \sigma_j dZ_j - \sigma_0 dZ_0)$. Therefore

(C4) $$\Omega = \Sigma - i\Sigma_0' - \Sigma_0 i' + \sigma_0^2 ii'$$

Substituting (C4) into (C3) yields

(C5) $$S^*(x) = x'\mathrm{diag}(\Omega) + x'\Omega x$$

If $\mathrm{var}(\sigma_i dZ_i - \sigma_0 dZ_0) = 0$ for $i = 1,\ldots,m$, then $\Omega = 0$ and $S^*(x) = 0$ for all $x \geq 0$. If $\mathrm{var}(\sigma_i dZ_i - \sigma_0 dZ_0) \neq 0$ for any i, then Ω has at least one strictly positive element on its diagonal. In this case, if $x > 0$ then $S^*(x) > 0$ (since Ω is nonnegative definite).

Appendix D

<u>Proof of Lemma 3</u>

$$\text{Define } A_j = \begin{bmatrix} a_1-\lambda & a_2 & \cdots & a_j \\ a_2 & -\lambda & & 0 \\ \vdots & & \ddots & \\ a_j & 0 & & -\lambda \end{bmatrix}$$

and observe that the eigenvalues of the mxm matrix A satisfy det $A_m = 0$. Also observe that det $A_1 = a_1-\lambda$ and det $A_2 = -\lambda$ det $A_1 - a_2^2$. In general, expanding around the last row of A_j, we have

(D1) $$\det A_j = -\lambda \det A_{j-1} + (-1)^{j-1} a_j \det \begin{bmatrix} a_2 & a_3 & \cdots & a_j \\ -\lambda & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & & -\lambda & 0 \end{bmatrix} , \quad j = 2,3,\ldots$$

Expanding the second determinant on the right hand side of (D1) around its last column we obtain

(D2) $$\det A_j = -\lambda \det A_{j-1} - a_j^2(-\lambda)^{j-2} \qquad j = 2,3,\ldots$$

Equation (D2) is a first-order difference equation with initial condition det $A_2 = \lambda^2-a_1\lambda-a_2^2$. The solution to the difference equation is

(D3) $$\det A_j = (-\lambda)^{j-2}[\lambda^2-a_1\lambda-\sum_{i=2}^{j} a_i^2]$$

Therefore, the eigenvalues of A are the m roots of

(D4) $$(-\lambda)^{m-2}[\lambda^2-a_1\lambda-\sum_{i=2}^{m} a_i^2] = 0$$

By inspection, $m-2$ roots are equal to zero. The remaining two roots satisfy $\lambda^2 - a_1\lambda - \sum_{i=2}^{m} a_i^2 = 0$ implying that these two roots have a sum of a_1 and a product of $-\sum_{i=2}^{m} a_i^2$.

q.e.d.