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Abstract

We present new algorithmic results for the class of Helly graphs, *i.e.*, for the discrete analogues of hyperconvex metric spaces. Specifically, an undirected unweighted graph is Helly if every family of pairwise intersecting balls has a nonempty common intersection. It is known that every graph isometrically embeds into a Helly graph, that makes of the latter an important class of graphs in Metric Graph Theory. We study diameter and radius computations within the Helly graphs, and related graph classes. This is in part motivated by a conjecture on the fine-grained complexity of these two distance problems within the graph classes of bounded *fractional* Helly number — that contain as particular cases the proper minor-closed graph classes and the bounded clique-width graphs. Note that under plausible complexity assumptions, neither the diameter nor the radius can be computed in truly subquadratic time on general graphs.

- In contrast to these negative results, we first present algorithms which given an n-vertex m-edge Helly graph G as input, compute with high probability (w.h.p.) its radius and its diameter in $\tilde{\mathcal{O}}(m\sqrt{n})$ time (*i.e.*, subquadratic in $n+m$). Our algorithms are based on the Helly property and on the unimodality of the eccentricity function in Helly graphs: every vertex of locally minimum eccentricity is a central vertex.
- Then, we improve our results for the C_4-free Helly graphs, that are exactly the Helly graphs whose balls are convex. For this subclass, we present linear-time algorithms for computing the eccentricity of all vertices. Doing so, we generalize previous results on strongly chordal graphs to a much larger subclass, that includes, among others, all the bridged Helly graphs and the hereditary Helly graphs.
- Lastly, we derive approximate versions of our results for the class of chordal graphs: with the latter satisfying an almost-Helly-type property, and a stronger (induced-path) convexity property than the C_4-free Helly graphs. For the chordal graphs, we can compute in quasi linear time the eccentricity of all vertices with an additive one-sided error of at most one, which is best possible under the Strong Exponential-Time Hypothesis (SETH). This answers an open question of [Dragan, IPL 2019]. In fact, we obtain this last result as a byproduct from a more general reduction: from diameter computation on chordal graphs to the DISJOINT SETS problem. Roughly, it implies that the split graphs are the *only* hard instances for diameter computation on chordal graphs. We also get from our reduction that on any subclass of chordal graphs with constant VC-dimension(and so, for undirected path graphs), the diameter can be computed in truly subquadratic time.

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1 Introduction

For any undefined graph terminology, see [9]. Given an undirected unweighted graph G, the distance $dist_G(u,v)$ between two vertices u and v is the minimum number of edges on a uv-path. The eccentricity $e_G(u)$ of a vertex u is the maximum distance from u to every other vertex. The radius and the diameter of G, denoted $rad(G)$ and $diam(G)$, respectively, are the smallest and the largest eccentricities of a vertex in G, respectively. We study the fundamental problems of computing the diameter and the radius. There is a textbook algorithm for both problems, running in $\mathcal{O}(nm)$ time on n-vertex m-edge graphs. However, it is a direct reduction to All-Pairs Shortest-Paths (APSP), that is a seemingly more complex problem with a much larger (quadratic-size) output than for the diameter and radius problems. As a continuous attempt to break this quadratic barrier (in the size $n+m$ of the input), there has been a long line of work presenting more efficient algorithms for computing the diameter and/or the radius on some special graph classes, by exploiting their geometric representations and/or some forbidden pattern (*e.g.*, excluding a minor, or a family of induced subgraphs). A typical such example is the class of interval graphs, *a.k.a.* the intersection graphs of intervals on the real line [40, 58]. See [14, 27, 32, 44, 46, 48] for other examples.

We here study the *Helly graphs* as a broad generalization of interval graphs. Recall that a graph is Helly if every family of pairwise intersecting balls has a non-empty common intersection. This latter property on the balls will be simply referred as the Helly property in what follows. We believe that studying which of the many nice properties of the interval graphs leads to fast diameter and radius computations is an interesting research topic on its own. This is especially so because the interval graphs are less structured than most "easy" graph classes for diameter and radius computations (*i.e.*, they have unbounded treewidth and clique-width, and they do not exclude any fixed minor). The Helly graphs, sometimes called absolute retracts or disk-Helly graphs (by opposition to other Helly-type properties on graphs [33]) are well studied in Metric Graph Theory. *E.g.*, see the survey [2] and the papers cited therein. This is partly because every graph is an isometric subgraph of some Helly graph, thereby making of the latter the discrete equivalent of hyperconvex metric spaces [42, 54]. In particular, let us consider the family of cycles. We denote by C_k the cycle of length k. Although there are very simple graphs that are not Helly, such as (to be provoking) C_6, *every* graph that contains a C_6 is a distance-preserving subgraph of some Helly graph. It implies that the Helly graphs cannot be characterized via some forbidden structure — unlike the subclass of the interval graphs, that do exclude some infinite family of induced subgraphs. Polynomial-time recognition algorithms for the Helly graphs were presented in [4, 34, 56], as well as several structural properties of these graphs in [4, 5, 6, 21, 34, 35, 38, 59, 60]. The dually chordal graphs are exactly the Helly graphs in which the intersection graph of balls is chordal, and they were studied independently from the general Helly graphs [14, 16, 41, 36, 38]. In particular, the diameter and the radius of a dually chordal graph can be computed in linear time, that is optimal. However, to the best of our knowledge it was open until this paper whether there are truly subquadratic-time algorithms for these two problems on Helly graphs.

Related work Chordal graphs (graphs with no induced cycles of length > 3) are another important generalization of interval graphs, with several nice geometric characterizations. *E.g.*, they are exactly the graphs whose balls are m-convex, *i.e.*, such that every induced path between two vertices in a ball is fully contained in this ball [47]. Furthermore, while not all chordal graphs are Helly, they satisfy a very similar property: for any family of pairwise intersecting balls in a chordal graph, there exists a vertex at distance at most some constant to all balls in the family (this result holds, more generally, for every class of graphs with bounded hyperbolicity [28]). Chepoi and Dragan proposed an elegant linear-time algorithm for computing a central vertex in a chordal graph [23]. However, they observed that already for split graphs (a subclass of chordal graphs), computing the diameter is roughly equivalent to DISJOINT SETS, *a.k.a.*, the monochromatic ORTHOGONAL VECTOR problem [22]. Under the Strong Exponential-Time Hypothesis (SETH), we cannot solve DISJOINT SETS in truly subquadratic time [68], and so neither we can compute the diameter of split graphs [10] (see also [61] whose authors were the first to prove such a SETH-hardness result for general graphs). It was also observed in [26] that assuming the so-called HITTING SET Conjecture (HS), the reduction from [1] implies that the radius of a 2-hyperbolic graph cannot be computed in truly subquadratic time. Therefore, the naive algorithm for radius and diameter computations is conditionally optimal already for very simple graph classes from a metric point of view. On the positive side, there exist linear-time algorithms for computing the diameter on various subclasses of chordal graphs, *e.g.,* interval graphs, directed path graphs and strongly chordal graphs [14, 30, 40]. Most of these special cases, including the three aforementioned examples, are strict subclasses of chordal Helly graphs. As a result, our work pushes forward the tractability border for diameter computation on chordal graphs and beyond.

More generally, this paper is part of a recent series of articles, with co-authors, where we try to understand the role of abstract geometric properties, and of tools and concepts from Computational Geometry, in the fast computation of metric graph invariants [43, 44, 45]. In this respect, Cabello and Knauer showed in [20] how to use a standard data-structure for orthogonal range searching in order to compactly represent the distances in a bounded-treewidth graph. Their approach became the cornerstone of conditionally optimal algorithms for diameter computation on bounded-treewidth graphs and other graph classes [1, 17, 43, 45]. Then in another seminal paper [19], Cabello introduced a new framework based on abstract Voronoi diagrams in order to compute the diameter of planar graphs in truly subquadratic time. See also [50] for improvements upon his work. Perhaps surprisingly, both planar graphs and bounded-treewidth graphs are particular cases of a large family of geometric graph classes, namely the graphs of bounded distance VC-dimension (with the latter parameter being defined, for any graph G, as the VC-dimension of its ball hypergraph). This was first proved by Chepoi et al. for the superclass of K_h-minor free graphs, which combined with a fractional Helly property of all spaces of constant VC-dimension [57], allowed them to cover any planar graph of diameter $2R$ with a constant number of balls of radius R [29]. – Recall that a class $\mathcal{H}$ of hypergraphs has *fractional Helly number* at most k if for any positive α there is some positive β such that, in any subfamily of hyperedges in a hypergraph of $\mathcal{H}$, if there is at least a fraction α of all the k-tuples of hyperedges with a non-empty common intersection, then there exists an element that is contained in a fraction at least β of all hyperedges in this subfamily. Then, the fractional Helly number of a graph class $\mathcal{G}$ is the fractional Helly number of the family of the ball hypergraphs of all graphs in $\mathcal{G}$. – Later the results of Chepoi et al. were extended to all hereditary graph classes of constant distance VC-dimension [11, 13]. From the algorithmic side, it was proved in [45] that on all proper minor-closed graph classes, and several other classes of bounded distance VC-dimension, the diameter can be computed in truly subquadratic time. Seeking for a common property of these graphs and of dually chordal graphs, we ask whether the diameter can be computed in truly subquadratic time on every graph class with a fractional Helly property. As a first step toward resolving this difficult question, our current research focuses on the simpler class of Helly graphs.

Our Contributions We present truly subquadratic-time algorithms for computing both the radius and the diameter of Helly graphs (Theorem 4). In fact, for the Helly graphs, it is sufficient to compute the diameter in order to derive the radius [34]. Note that such a property is not known to hold for general graphs. Nevertheless, we present separate algorithms for diameter and radius computations. Indeed, our approach for computing the radius can be applied to a broader class than the Helly graphs, both as an exact and approximation algorithm. Our algorithms run in time $\tilde{\mathcal{O}}(m\sqrt{n})$ w.h.p., and they use as their main ingredients several consequences of the unimodality of the eccentricity function in Helly graphs [34]: every local minimum of the eccentricity function in a Helly graph is a global minimum.

Then, in light of the gap between our running times and the known linear-time algorithms for the dually chordal graphs, we studied whether stronger complexity results could hold true on more restricted subclasses, such as *chordal* Helly graphs or, more generally, C_4-free Helly graphs. This latter choice was also partly motivated by a nice characterization of *hereditary* Helly graphs: indeed, they are exactly the 3-sun-free chordal graphs [34]. We stress that the C_4-free Helly graphs have been studied on their own and that they have more interesting convexity properties than general Helly graphs [34, 35]. In particular, the center $C(G)$ of a C_4-free Helly graph G is convex and it has diameter at most 3 and radius at most 2 [34, 35]. In contrast, the center $C(G)$ of a general Helly graph G is isometric but it can have arbitrarily large diameter; in fact, any Helly graph H is the center of some other Helly graph G [34]. For the C_4-free Helly graphs, we are able to compute the eccentricity of all vertices in linear time (Theorem 7), which for Helly graphs can be reduced to computing the graph center. Our starting point for that is the well-known multi-sweep heuristic of Corneil et al. [30], in order to compute vertices of provably large eccentricity. – As notified to us by Chepoi (private communication), this heuristic can be arbitrarily bad for general Helly graphs, that can be deduced from a careful analysis of the embedding method of arbitrary graphs into Helly graphs [42, 54]. – Our approach is partly inspired by the algorithms of Chepoi and Dragan [23] and Dragan and Nicolai [39], in order to compute a central vertex in chordal graphs and a diametral pair in distance-hereditary graphs, respectively. Nevertheless, extending such ideas to the C_4-free Helly graphs required us to prove several new nontrivial properties for this class. In particular, it led us to a reduction from finding a diametral pair under some technical assumptions to the same problem on split Helly graphs.

Finally, in light of this above reduction, we studied whether there are other graph classes where the diameter can be efficiently computed from a subfamily of split graphs. In particular, can we reduce diameter computation on *general* chordal graphs to the same problem on split graphs? This would imply that the subclass of split graphs is, in some sense, the sole hard case for diameter computation on chordal graphs. Furthermore, beyond the case of chordal Helly graphs, this could help in finding new subclasses of chordal graphs for which we can compute the diameter faster than in $\mathcal{O}(nm)$ time. We prove a slightly weaker result, namely, a reduction from diameter computation on chordal graphs to DISJOINT SETS (Theorem 8). As a byproduct of our reduction, we prove that the diameter can be computed in truly subquadratic time on any subclass of chordal graphs with constant *VC-dimension* (Theorem 10). This nicely complements the results from [45], which mostly apply to sparse graph classes of constant distance VC-dimension or assuming a bounded (sublinear) diameter. In particular, our result in this paper can be applied to *undirected path graphs* since they have VC-dimension at most 3 [12]. Before our work, only a linear-time algorithm for the subclass of directed path graphs was known [30]. Another application of our reduction is the approximate computation in quasi linear time of the eccentricity of all vertices in a chordal graph with an additive one-sided error of at most 1. The latter result answers an open question from [37].

Notations. We recall that we denote by $dist_G(u,v)$ the distance between vertices u and v. The metric interval $I_G(u,v)$ between u and v is defined as $\{w \in V \mid dist_G(u,w) + dist_G(w,v) = dist_G(u,v)\}$. For any $k \leq dist_G(u,v)$, we can also define the slice $L(u,k,v) := \{w \in I_G(u,v) \mid dist_G(u,w) = k\}$. The ball of radius r and center v is defined as $\{u \in V \mid dist_G(u,v) \leq r\}$, and denoted $N^r_G[v]$. In particular, $N_G[v] := N^1_G[v]$ and $N_G(v) := N_G[v] \setminus \{v\}$ denote the closed and open neighbourhoods of a vertex v, respectively. More generally, for any vertex-subset S we define $dist_G(u,S) := \min_{v \in S} dist_G(u,v)$, $N^r_G[S] := \bigcup_{v\in S} N^r_G[v]$, $N_G[S] := N^1_G[S]$ and $N_G(S) := N_G[S] \setminus S$. The metric projection of a vertex u on S, denoted $Pr_G(u,S)$, is defined as $\{v \in S \mid dist_G(u,v) = dist_G(u,S)\}$. Recall that the eccentricity of a vertex u is defined as $\max_{v\in V} dist_G(u,v)$ and denoted by $e_G(u)$. We define the set $F_G(u) := \{v \in V \mid dist_G(u,v) = e_G(u)\}$ of all the farthest vertices from vertex u. – Note that we will omit the subscript if the graph G is clear from the context. – The radius and the diameter of a graph G are denoted by $rad(G)$ and $diam(G)$, respectively. Finally, $C(G) := \{v \in V \mid e(v) = rad(G)\}$ is the center of G, *a.k.a.* the set of all the central vertices of G.

2 Fast Computations within Helly graphs

The main result in this section is a truly subquadratic algorithm for computing the diameter of a Helly graph (Theorem 4). By the following Lemma 1, it can be easily turned into a subquadratic algorithm for radius computation:

Lemma 1 ([34])**.** *If G is a Helly graph then $2rad(G) \geq diam(G) \geq 2rad(G) - 1$. In particular, $rad(G) = \lceil diam(G)/2 \rceil$.*

Section 2.1 is devoted to a subquadratic-time randomized algorithm for radius computation, that can be turned to an exact or approximation algorithm for larger classes than the Helly graphs (namely, in every class where the diameter equals twice the radius, up to some additive constant). Then, we combine this approach with several other technical arguments in order to prove our Theorem 4 (cf. Section 2.2).

2.1 Radius computation

We start this section with a simple randomized test, which is inspired from previous works on adaptive greedy set cover algorithms [64].

Lemma 2. *Let $G = (V,E)$ be a graph, let r be a positive integer and let $\varepsilon \in (0;1)$. There is an algorithm that w.h.p. computes a set $D\langle G;r;\varepsilon\rangle$ in $\tilde{\mathcal{O}}(m/\varepsilon)$ time with the following two properties:*

- *if $e(v) \leq r$ then $v \in D\langle G;r;\varepsilon\rangle$;*
- *conversely, if $v \in D\langle G;r;\varepsilon\rangle$ then $|N^r[v]| \geq (1-\varepsilon)\cdot n$.*

Proof. Let $p = c \cdot \frac{\log n}{\varepsilon n}$ for some arbitrary large constant c. If $p \geq 1$ then $n \leq c\frac{\log n}{\varepsilon}$, and so we can compute the set of all the vertices of eccentricity at most r in time $\tilde{\mathcal{O}}(m/\varepsilon)$ by running a BFS from every vertex. From now on we assume that $p < 1$. By $U(p)$ we mean a subset in which every vertex was added independently at random with probability p. Observe that we have $\mathbb{E}[|U(p)|] = c\frac{\log n}{\varepsilon} > c \cdot \log n$. By Chernoff bounds we get $|U(p)| = \tilde{\mathcal{O}}(\varepsilon^{-1})$ with probability

$\geq 1-n^{-c}$. Then, for every $v \in V$, we compute $N^r[v] \cap U(p)$. For that, we run a BFS from every vertex of $U(p)$, that takes total time $\tilde{\mathcal{O}}(m/\varepsilon)$. We divide our analysis in two cases. First let us assume that $e(v) \leq r$. Then, with probability 1 we have $U(p) \subseteq N^r[v]$. Second, let us assume that $|N^r[v]| < (1-\varepsilon) \cdot n$. We get $Prob[U(p) \subseteq N^r[v]] < (1-p)^{\varepsilon n} = (1-p)^{\frac{1}{p} \cdot c \log n} \leq n^{-c}$. Overall, let $D\langle G; r; \varepsilon \rangle$ contain all the vertices v such that $U(p) \subseteq N^r[v]$. By a union bound over at most n vertices, the set $D\langle G; r; \varepsilon \rangle$ satisfies our two above-stated properties with probability $\geq 1 - n^{-(c-1)}$. □

We derive from this simple test above an approximation algorithm for computing the radius and the diameter, namely:

Lemma 3. *Let $G = (V, E)$ be a graph and r be a positive integer. There is an algorithm that w.h.p. runs in $\tilde{\mathcal{O}}(m\sqrt{n})$ time and such that:*

- *If the algorithm accepts then $diam(G) \leq 2r$;*
- *If the algorithm rejects then $rad(G) > r$.*

Note that since $diam(G) \leq 2rad(G)$, this algorithm rejects any graph G with $diam(G) > 2r$. However, it might also reject some graphs G such that $diam(G) \leq 2r$ but $rad(G) > r$.

Proof. For some ε to be defined later, we construct a set $D\langle G; r; \varepsilon \rangle$ as in Lemma 2. W.h.p. it takes time $\tilde{\mathcal{O}}(m/\varepsilon)$. There are two cases. If $D\langle G; r; \varepsilon \rangle = \emptyset$ then we know that $rad(G) > r$ and we stop. Otherwise, we pick any vertex $c \in D\langle G; r; \varepsilon \rangle$ and we compute $N^r[c]$. Here it is important to observe that all the vertices of $N^r[c]$ are pairwise at a distance $\leq 2r$. Furthermore, w.h.p. we have $|V \setminus N^r[c]| \leq \varepsilon \cdot n$. We end up computing a BFS from every vertex of $V \setminus N^r[c]$, accepting in the end if and only if all these vertices have eccentricity $\leq 2r$. By setting $\varepsilon = n^{-1/2}$, the total running time is w.h.p. in $\tilde{\mathcal{O}}(m\sqrt{n})$. □

An important consequence of Lemma 3 is that the hard instances for diameter and radius approximations are those for which the difference $2rad(G) - diam(G)$ is large, namely:

Corollary 1. *If $2rad(G) - diam(G) \leq k$ then, w.h.p., we can compute an additive $+\lfloor k/2 \rfloor$-approximation of $rad(G)$ and an additive $+k$-approximation of $diam(G)$ in total $\tilde{\mathcal{O}}(m\sqrt{n})$ time.*

Proof. We compute by dichotomic search the smallest r such that the algorithm of Lemma 3 accepts. Note that w.h.p. $r \geq \lceil diam(G)/2 \rceil$, and so $r \geq \left\lceil \frac{2rad(G)-k}{2} \right\rceil = rad(G) - \lfloor k/2 \rfloor$. Furthermore, we have w.h.p. $r \leq rad(G)$, and so, $2r \leq diam(G) + k$. We output r and $2r$ as approximations of $rad(G)$ and $diam(G)$, respectively. □

Application to Helly graphs. By combining Lemma 1 with Corollary 1, we finally obtain the main result of this section, namely:

Theorem 1. *If G is a Helly graph then, w.h.p., we can compute $rad(G)$ and an additive $+1$-approximation of $diam(G)$ in time $\tilde{\mathcal{O}}(m\sqrt{n})$.*

We observe that one *cannot* compute a central vertex in a Helly graph by using our approach in this section. This issue is resolved, in Sec. 3.1, for the subclass of C_4-free Helly graphs.

2.2 Diameter computation

We continue with a parameterized algorithm for computing all eccentricities up to some threshold value in a Helly graph G. We recall that we denote by $N^r_G[v]$ the ball of radius r and center v.

Problem 1 (Small Eccentricities).

Input: *a graph $G = (V, E)$; a vertex-subset A; a positive integer k.*

Output: *the set $B_k := \{b \in V \mid A \subseteq N^k[b]\}$.*

We note that already for $k = 2$, Problem 1 is unlikely to be solvable in truly subquadratic time. Indeed, this special case is somewhat related to the Hitting Set problem [1]. We explain next how to solve this problem in parameterized linear time when G is a Helly graph.

Theorem 2. *If G is Helly then, for every subset A and every positive integer k, we can solve* Small Eccentricities *in $\mathcal{O}(km)$ time.*

Proof. We reduce the problem to the construction of a partition $\mathcal{P}^k_A = (A^k_1, A^k_2, \ldots, A^k_{p_k})$ of A, for some arbitrary $p_k =^{def} |\mathcal{P}^k_A| \leq |A|$, with the following two properties:

- for every $1 \leq j \leq p_k$, we have $\cap\{N^k[a] \mid a \in A^k_j\} = V^k_j \neq \emptyset$;
- furthermore, the sets $V^k_1, V^k_2, \ldots, V^k_{p_k}$ are pairwise disjoint.

Indeed, observe that we have $B_k \neq \emptyset$ if and only if $p_k = 1$, and in such a case $B_k = V^k_1$. It now remains to prove that we can construct the partition $\mathcal{P}^k_A$, and the associated sets V^k_j, in $\mathcal{O}(km)$ time. If $k = 0$, then we set $\mathcal{P}^0_A = A$ and we are done (notice that for every $a_j \in A$, the corresponding set V^0_j is exactly the singleton $\{a_j\}$). Otherwise, we show how to construct $\mathcal{P}^k_A$ and its associated sets from $\mathcal{P}^{k-1}_A$ and the V^{k-1}_j's, in linear time. For that, let us define for every j the new subset $W^k_j := N[V^{k-1}_j]$. Notice that constructing the sets W^k_j takes total linear time as by the hypothesis, the sets V^{k-1}_j are pairwise disjoint. However, we may have $W^k_j \cap W^k_{j'} \neq \emptyset$ for some $j \neq j'$.

Claim 1. $W^k_j = \bigcap\{N^k[a] \mid a \in A^{k-1}_j\}$.

Proof. Since $V^{k-1}_j = \bigcap\{N^{k-1}[a] \mid a \in A^{k-1}_j\}$, $W^k_j \subseteq \bigcap\{N^k[a] \mid a \in A^{k-1}_j\}$. Conversely, let $v \in \bigcap\{N^k[a] \mid a \in A^{k-1}_j\}$ be arbitrary. Since $N[v]$ and $N^{k-1}[a], \forall a \in A^{k-1}_j$ pairwise intersect, by the Helly property, $N[v] \cap V^{k-1}_j \neq \emptyset$, proving that $v \in W^k_j$. ⋄

We are left with computing the V^k_j's from the W^k_j's. For that, we need the following additional result:

Claim 2. *Let $v \in V$ be such that $\#\{j \mid v \in W^k_j\}$ is maximized. For any j' such that $v \notin W^k_{j'}$, we have $W^k_{j'} \cap \bigcap\{W^k_j \mid v \in W^k_j\} = \emptyset$.*

Proof. Suppose by contradiction that there is a vertex $u \in W^k_{j'} \cap \bigcap\{W^k_j \mid v \in W^k_j\}$. Then, $\{j \mid v \in W^k_j\} \subset \{j' \mid u \in W^k_{j'}\}$. However, this contradicts the maximality of vertex v. ⋄

We now proceed as follows in order to compute $\mathcal{P}^k_A$ and the V^k_j's. Let $\mathcal{F}_k$ be an empty family of sets (we shall get $\mathcal{F}_k = (V^k_1, V^k_2, \ldots, V^k_{p_k})$ at the end of this sub-procedure below). While

$\mathcal{P}_A^{k-1} \neq \emptyset$ we pick a vertex $v \in V$ such that $\#\{j \mid v \in W_j^k\}$ is maximized. We add the new sets $A_v := \bigcup\{A_j^{k-1} \mid v \in W_j^k\}$ and $\bigcap\{W_j^k \mid v \in W_j^k\}$, in the families $\mathcal{P}_A^k$ and $\mathcal{F}_k$, respectively. Indeed, by Claim 1, $\bigcap\{N^k[a] \mid a \in A_v\} = \bigcap\{W_j^k \mid v \in W_j^k\}$. Then, we remove from $\mathcal{P}_A^{k-1}$ every A_j^{k-1} such that $v \in W_j^k$. By Claim 2, the sets in $\mathcal{F}_k$ are pairwise disjoint. Finally, in order to construct $\mathcal{F}_k$ efficiently, during a pre-processing step we compute $\#\{j \mid v \in W_j^k\}$ for every vertex v. Since the W_j^k's can be constructed in total linear time, this pre-processing also takes linear time. We create an array of $|\mathcal{P}_A^{k-1}|$ lists, where $\forall i$ the i^{th} list contains all the vertices that are in exactly i subsets W_j^k (these lists are dynamically updated throughout the algorithm). We also need that, for every vertex v, we can enumerate all the subsets W_j^k such that $v \in W_j^k$ in time linear in the size of the output (*i.e.*, in $\mathcal{O}(\sum\{|W_j^k| \mid v \in W_j^k\})$ time). For that, it suffices to maintain the incidence graph between V and the groups W_j^k. Then, starting from $i = |\mathcal{P}_A^{k-1}|$, if the i^{th} list is empty then $i := i-1$. Otherwise, we can pick any vertex v of this list as it maximizes $\#\{j \mid v \in W_j^k\}$. In this latter case the total running time of the step (including the lists updates) is in $\mathcal{O}(\sum_{j|v\in W_j^k} |A_j^{k-1}| + |W_j^k|)$. Since all the subsets A_j^{k-1}, such that $v \in W_j^k$, are subsequently removed from $\mathcal{P}_A^{k-1}$, after this step v is no more contained in a group W_j^k, for any $A_j^{k-1} \in \mathcal{P}_A^{k-1}$ and so it will never be used again during the sub-procedure. Overall, the running time is in $\mathcal{O}(\sum_j |A_j^{k-1}| + |W_j^k|) = \mathcal{O}(n+m)$. □

Corollary 2. *For any Helly graph G and positive integer k, we can compute the set of all the vertices of eccentricity at most k in $\mathcal{O}(km)$ time.*

Proof. It suffices to apply Theorem 2 with $A = V$. □

If the diameter of a Helly graph is sublinear in the number of nodes, then by the above the eccentricity of all vertices can be computed in subquadratic time. However, there exist very simple Helly graphs, such as paths, for which the diameter is linear in the number of nodes. For such "giant-diameter" Helly graphs, we next adapt a well-known sampling technique for distance oracles [8]. Recall that a function is called *unimodal* if every its local minimum is global. It was proved in [34] that the eccentricity function in Helly graphs is unimodal, and that the latter implies the following interesting property:

Lemma 4 ([34])**.** *If G is Helly then, for every vertex v, $e(v) = dist(v, C(G)) + rad(G)$.*

Theorem 3. *Let G be a Helly graph such that $rad(G) > 3k = \omega(\log n)$. Then w.h.p. in $\tilde{\mathcal{O}}(mn/k)$ time, we can compute a diametral pair for G.*

Proof. Let $p = c\frac{\log n}{k}$ for some sufficiently large constant c. Since we assume $k = \omega(\log n)$, for sufficiently large n we have $p < 1$. We construct a subset $U(p)$ where every vertex is included independently with probability p. By Chernoff bounds we have $|U(p)| = \tilde{\mathcal{O}}(n/k)$ w.h.p., and we assume from now on that it is indeed the case. We perform a BFS from every vertex in $U(p)$, in total $\tilde{\mathcal{O}}(mn/k)$ time. Then, we define for every vertex $v \in V$: $\bar{e}(v) := \min_{u \in U(p) | dist(u,v) \leq k} dist(u,v) + e(u)$ (by convention, $\bar{e}(v) = 0$ if every vertex of $U(p)$ is at distance $> k$ from v). We now divide our analysis in two cases:

- Case $e(v) < rad(G) + k$. Then, for any $u \in U(p)$ such that $dist(u,v) \leq k$, we get $e(u) \leq e(v) + k < rad(G) + 2k$. Hence, $\bar{e}(v) < rad(G) + 3k \leq 2 \cdot rad(G) - 1$. By Lemma 1, we get that $\bar{e}(v) < diam(G)$ with probability 1.

- Case $e(v) \geq rad(G) + k$. Note that in particular, we always fall in this case if v is an end of a diametral path. Let us consider the set S_v of the k first vertices on a fixed shortest path between v and a closest vertex of $C(G)$. By Lemma 4, $e(v) = dist(v, C(G)) + rad(G)$. In particular, for every $u \in S_v$ we have $e(v) = dist(u,v) + e(u)$. If $U(p) \cap S_v \neq \emptyset$, then this implies $\bar{e}(v) = e(v)$. Therefore, we are left proving that w.h.p., $U(p) \cap S_v \neq \emptyset$. That is indeed the case since $Prob[U(p) \cap S_v = \emptyset] = (1-p)^{|S_v|} = (1-p)^k = (1-p)^{\frac{c \log n}{p}} \leq n^{-c}$.

Altogether combined, w.h.p., a vertex v maximizing $\bar{e}(v)$ is an end of a diametral path. □

Combining Theorem 2 and Theorem 3, we finally obtain our main result, namely:

Theorem 4. *A diametral pair in a Helly graph can be computed w.h.p. in $\tilde{\mathcal{O}}(m\sqrt{n})$ time.*

Proof. Let $k = 6\lceil\sqrt{n}\rceil$. By Corollary 2 we can compute the set B_k of all the vertices of eccentricity at most k in $\mathcal{O}(mk) = \mathcal{O}(m\sqrt{n})$ time. There are now two cases. First assume that $B_k = V$. We can compute by dichotomic search the smallest $d \leq k$ such that $B_d = V$, which is exactly the diameter, in $\tilde{\mathcal{O}}(m\sqrt{n})$ time. Then, a vertex is an end of a diametral path if and only if it is in $V \setminus B_{d-1}$, and by Corollary 2 we can enumerate all such vertices in $\mathcal{O}(md) = \mathcal{O}(m\sqrt{n})$ time. Otherwise, $B_k \neq V$, and so, $diam(G) > k$. Note that it implies $rad(G) > k/2 \geq 3\sqrt{n}$. By Theorem 3, we can compute a diametral pair of G w.h.p. in time $\tilde{\mathcal{O}}(mn/\sqrt{n}) = \tilde{\mathcal{O}}(m\sqrt{n})$. □

3 Journey to the Center of C_4-free Helly graphs

We now improve our results from Sec. 2, for the class of C_4-free Helly graphs.

3.1 Computing a central vertex

We start with general properties of Helly graphs and C_4-free Helly graphs which we will then use in our analysis. The first such property is a consequence of the unimodality of the eccentricity function in Helly graphs (see [34]). In what follows, recall that the metric interval $I_G(u,v)$ between u and v is defined as $\{w \in V \mid dist_G(u,w) + dist_G(w,v) = dist_G(u,v)\}$. For any $k \leq dist_G(u,v)$, let $L(u,k,v) := \{w \in I_G(u,v) \mid dist_G(u,w) = k\}$. Finally, let $F_G(u) := \{v \in V \mid dist_G(u,v) = e_G(u)\}$ contain all the farthest vertices from vertex u.

Lemma 5 ([34])**.** *Let G be a Helly graph. Then, for any vertex v of G and any farthest vertex $u \in F(v)$ we have $L(u, rad(G), v) \cap C(G) \neq \emptyset$.*

Pseudo-modular graphs are exactly the graphs where each family of three pairwise intersecting balls has a common intersection [3]. Clearly, Helly graphs is a subclass of pseudo-modular graphs.

Lemma 6 ([3])**.** *For every three vertices x, y, z of a pseudo-modular graph G there exist three shortest paths $P(x,y)$, $P(x,z)$, $P(y,z)$ connecting them such that either (1) there is a common vertex v in $P(z,y) \cap P(x,z) \cap P(x,y)$ or (2) there is a triangle $\triangle(x',y',z')$ in G with edge $z'y'$ on $P(z,y)$, edge $x'z'$ on $P(x,z)$ and edge $x'y'$ on $P(x,y)$ (see Fig. 1). Furthermore, (1) is true if and only if $d(x,y) = p+q$, $d(x,z) = p+k$ and $d(y,z) = q+k$, for some $k,p,q \in \mathbb{N}$, and (2) is true if and only if $d(x,y) = p+q+1$, $d(x,z) = p+k+1$ and $d(y,z) = q+k+1$, for some $k,p,q \in \mathbb{N}$.*

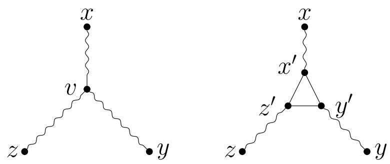

Figure 1: Vertices x, y, z and three shortest paths connecting them in pseudo-modular graphs.

The next properties are specific to C_4-free Helly graphs. A set $S \subseteq V$ of a graph $G = (V, E)$ is called convex if for every $x, y \in S$, $I(x, y) \subseteq S$ holds.

Lemma 7 ([35])**.** *Every ball of a C_4-free Helly graph is convex.*

Lemma 8. *For every vertices v and u of a C_4-free Helly graph G and any integer $k \leq dist(u, v)$, the set $L(u, k, v)$ is a clique.*

Proof. Consider any two vertices $x, y \in L(u, k, v)$ and assume that they are not adjacent. Let $\ell = dist(x, y) \geq 2$. Consider balls $N^1[x], N^{\ell-1}[y]$ and $N^{k-1}[u]$ in G. Since we have $dist(x, y) = \ell = 1 + (\ell - 1)$, $dist(x, u) = k = 1 + (k - 1)$ and $dist(y, u) = k \leq (\ell - 1) + (k - 1)$ (*i.e.*, because we assume $\ell \geq 2$), these balls pairwise intersect. By the Helly property, there must exist a vertex z on a shortest path from x to y which is at distance at most $k - 1$ from u. As by Lemma 7 the ball $N^{dist(v,x)}[v]$ is convex, z must belong to $N^{dist(v,x)}[v]$. Thus, $dist(u, v) \leq dist(u, z) + dist(z, v) \leq k - 1 + dist(v, x) = dist(u, v) - 1$, and a contradiction arises. □

The *multi-sweep* heuristic consists in performing a BFS [31] from an arbitrary vertex v, then from a farthest vertex u from v, and finally to output $e(u)$ as an estimate of $diam(G)$. On many graph classes it gives us a *constant additive approximation* of the diameter [25, 30, 31, 34]. We prove that it is the case for C_4-free Helly graphs.

Lemma 9. *Let G be a C_4-free Helly graph with diameter d and radius r. Let s be an arbitrary vertex, v be a vertex most distant from s, and (x, y) be a diametral pair of G. Then, $e(v) \geq d - 2$. Furthermore, if $e(v) = d - 2$, then $e(v) = 2r - 3 = dist(v, x) = dist(v, y)$ and $d = 2r - 1$. So, in particular, if $e(v)$ is even, then $e(v) \geq d - 1$.*

Proof. By Lemma 1, d is either $2r$ or $2r - 1$. Let $\ell = e(s) = dist(s, v)$. For vertices s, v, x, y of G, we have $dist(x, y) = d$, $dist(s, x) \leq dist(s, v) = \ell$, $dist(s, y) \leq dist(s, v) = \ell$. Furthermore, the three of $dist(v, x)$, $dist(v, y)$ and $dist(s, v) = \ell$ are at most $e(v)$.

First we show that, if $\max\{dist(v, y), dist(v, x)\} \leq 2k$ for some integer k, then $d \leq 2k + 1$. By the triangular inequality, we may assume that we have $2k \leq 2e(s) = 2\ell$. Consider balls $N^{\ell-k}[s]$, $N^k[v]$, $N^k[y]$ in G. As $dist(v, y) \leq 2k$ and $dist(s, y) \leq \ell$, those balls pairwise intersect. By the Helly property, there is a vertex a in G belonging to all three balls. Necessarily, $dist(a, s) = \ell - k$, $dist(a, v) = k$ and $dist(a, y) \leq k$. Similarly, we can get a vertex b in G such that $dist(b, s) = \ell - k$, $dist(b, v) = k$ and $dist(b, x) \leq k$. As both a and b are in $L(v, k, s)$, by Lemma 8, $dist(a, b) \leq 1$. Thus, $d = dist(x, y) \leq dist(x, b) + dist(b, a) + dist(a, y) \leq 2k + 1$.

Now, if $e(v) = 2k$ for some integer k, then $\max\{dist(v, y), dist(v, x)\} \leq e(v) = 2k$ and, therefore, $d \leq 2k + 1 = e(v) + 1$. If $e(v) = 2k + 1$ for some integer k, then either $\max\{dist(v, y), dist(v, x)\} < e(v) = 2k + 1$ and hence $d \leq 2k + 1 = e(v)$ or $\max\{dist(v, y), dist(v, x)\} = e(v) = 2k + 1$. As in the latter case $\max\{dist(v, y), dist(v, x)\} < 2k + 2$, we also get $d \leq 2k + 3 = e(v) + 2$.

In what follows, we consider this case, when $e(v) = 2k+1 = \max\{dist(v,y), dist(v,x)\}$, in more details. If d is even (i.e., $d = 2r$), then $d \leq 2k+2$ and therefore $d \leq e(v)+1$. Assume now that d is odd (i.e., $d = 2r-1$) and that $d = e(v)+2 = 2k+3 = 2r-1$. That is, $r = k+2$. We will show that, under these conditions, $dist(v,y) = dist(v,x) = 2k+1$ must hold. For that assume w.l.o.g. that $dist(v,y) = \max\{dist(v,y), dist(v,x)\} = 2k+1$. Since $v \in F(s)$, we have that $dist(s,y) \leq \ell = dist(s,v)$. Furthermore, by the triangular inequality, we have $2k+1 = dist(v,y) \leq dist(v,s) + dist(s,y) \leq 2e(s) = 2\ell$, and so $\ell \geq k+1$. We shall use the following intermediate results:

- If $dist(s,y) = \ell$ then, by Lemma 6, there is a triangle $\triangle(v',s',y')$ in G such that $dist(s,s') = \ell-k-1$, $dist(v',v) = dist(y,y') = k$. Necessarily, $v' \in L(v,k,s)$ and $s' \in L(v,k+1,s)$.
- If $dist(s,y) \leq \ell-1$, consider balls $N^{\ell-k-1}[s]$, $N^{k+1}[v]$, $N^{k}[y]$ in G. As these balls pairwise intersect, by the Helly property, there is a vertex a in G with $dist(a,s) = \ell-k-1$, $dist(a,v) = k+1$ and $dist(a,y) = k$. That is, $a \in L(v,k+1,s)$.
- If $dist(v,x) \leq 2k$ then, as before, we can get a vertex b in G with $dist(b,s) = \ell-k$, $dist(b,v) = k$ and $dist(b,x) \leq k$. Necessarily, $b \in L(v,k,s)$.

Summarizing, we get the following combinations. If $dist(v,y) = 2k+1$, $dist(v,x) \leq 2k$ and $dist(s,y) = \ell$, then $d = dist(x,y) \leq dist(x,b) + dist(b,v') + dist(v',y') + dist(y',y) \leq k+1+1+k = 2k+2$ (notice that, by Lemma 8, $dist(b,v') \leq 1$), contradicting with $d = 2k+3$. If $dist(v,y) = 2k+1$, $dist(v,x) \leq 2k$ and $dist(s,y) \leq \ell-1$, then $d = dist(x,y) \leq dist(x,b) + dist(b,a) + dist(a,y) \leq k+2+k = 2k+2$ (notice that, by Lemma 8, $dist(b,a) \leq 2$ as $a \in L(v,k+1,s)$ and $b \in L(v,k,s)$), contradicting with $d = 2k+3$.

Hence, $dist(v,x) = 2k+1 = dist(v,y)$ must hold. □

We left open whether the lower-bound of Lemma 9 can be refined to $e(v) \geq d-1$. Note that this would be best possible. Indeed, although in some cases of interest, *e.g.,* interval graphs, the output of the multi-sweep heuristic always *equals* the diameter [40], this nice property does not hold for strongly chordal graphs and so for C_4-free Helly graphs [30].

Then, before finally proving the main result of this subsection, we need the following gated property of Helly graphs. The (weak) diameter of a set S is equal to $diam(S) := \max_{x,y \in S} dist(x,y)$. We recall that the metric projection of a vertex u on S, denoted $Pr(u,S)$, is defined as $\{v \in S \mid dist(u,v) = dist(u,S)\}$.

Lemma 10. *Let G be a Helly graph and S be a subset of weak diameter at most two. Then, for any $v \notin S$ there exists a vertex $g_S(v) \in N^{dist(v,S)-1}[v] \cap \bigcap\{N(x) \mid x \in Pr(v,S)\}$.*

Proof. Since S has weak diameter at most two the balls $N^{dist(v,S)-1}[v]$ and $N[x], \forall x \in Pr(v,S)$ pairwise intersect. Therefore, the result follows from the Helly property. □

As it is standard [23] we call such a vertex a *gate* of v, and we denote it by $g_S(v)$.

Remark 1. *A variant of Maximum Cardinality Search was proposed in [23, 24] in order to compute all the gates in linear time. For that we first run a breadth-first search from S. Then, we recursively assign a gate to every vertex of $V \setminus S$, as follows. If $v \in N(S)$, then v is its own gate and we set $p(v) = |N(v) \cap S|$. Otherwise, we choose for every vertex v a father u, one step closer to S, that maximizes $p(u)$. Indeed, by induction, $p(u) = |Pr(u,S)|$. Then, we choose for v the same gate as for its father u, and we set $p(v) = p(u)$.*

We observe that, more generally, if S is an arbitrary *vertex-subset (possibly, of weak diameter larger than two), then for every vertex v with a gate in $N(S)$ this above procedure correctly computes such a gate. Indeed, let us define $I(v,S) := \bigcup\{I(v,s) \mid s \in Pr(v,S)\}$. This above procedure associates to every vertex v a vertex $v^* \in N(S) \cap I(v,S)$ which maximizes $|N(v^*) \cap S|$. If v has a gate with respect to S (w.r.t.), then it follows that $g_S(v) := v^*$ must be a gate of v. This observation is crucial in our proof of Theorem 7.*

Theorem 5. *If G is a C_4-free Helly graph then we can compute a central vertex and so $rad(G)$ in linear time.*

Proof. Let v be an arbitrary vertex, let $u \in F(v)$ and let $w \in F(u)$. By Lemma 9, $e(u) = dist(u,w) \geq diam(G) - 2$. Therefore, by Lemma 1, $rad(G) \in \{\lceil e(u)/2 \rceil, \lceil (e(u)+1)/2 \rceil, 1 + \lceil e(u)/2 \rceil\}$ (two of these numbers being equal, it gives us two possibilities). In order to decide in which case we are, we use the following result from [34]: if $rad(G) = r$, then $L(w,r,u) \cap C(G) \neq \emptyset$. By Lemma 8, for any r the set $C = L(w,r,u)$ is a clique. We compute, for every $x \notin C$, its distance $dist(x,C)$ and a corresponding gate $g(x)$ – which exists by Lemma 10. It takes linear time. Then, $rad(G) = r$ implies $\max_{x \in V} dist(x,C) = r$. If so then note that a vertex of C has eccentricity r if and only if it is adjacent to the gate of every vertex at a distance exactly r from C. Overall, in order to compute $rad(G)$ we pick the smallest r such that a vertex of eccentricity r can be extracted from $L(w,r,u)$. □

3.2 Computing a diametral pair

We base on the results from Section 3.1 so as to prove the following theorem:

Theorem 6. *If G is a C_4-free Helly graph then we can compute a diametral pair and so $diam(G)$ in linear time.*

Digression: an application to chordal Helly graphs

Our results in the paper are proved valid assuming the input graph to be Helly. However, the best-known recognition algorithms for this class of graphs run in quadratic time [56]. In what follows, we first explain an interesting application of Theorem 6 to general chordal graphs. We recall that it can be decided in linear time whether a given graph is chordal [62].

The Lexicographic Breadth-First-Search (LexBFS) [62], of which a description can be found in Fig. 2, is a standard algorithmic procedure that runs in linear time [53].

We use the following results on LexBFS in our analysis:

Lemma 11 ([40])**.** *Let v be the vertex visited last by an arbitrary LexBFS. If the graph is chordal, then the eccentricity of v is within 1 of the diameter.*

Lemma 12 ([30])**.** *If the vertex u of a chordal graph G last visited by a LexBFS has odd eccentricity, then $e(u) = diam(G)$.*

Altogether combined with Theorem 6 we obtain that:

Remark 2. *Consider an* arbitrary *chordal graph G. If we assume G to be Helly then, by Theorem 6, there exists a linear-time algorithm for computing a diametral pair of G. Note that, we can apply this algorithm to G without the knowledge that it is Helly, and either the algorithm will detect that*

```
Input: A graph G = (V,E)
Output: An ordering σ of the vertices of V
begin
  assign the label ∅ to each vertex ;
  for i = n to 1 do
      pick an unnumbered vertex x with the largest label in the lexicographic order ;
      for each unnumbered neighbour y of x do
          add i to label(y) ;
      σ(i) ← x   /* number x by i */ ;
end
```

Figure 2: Algorithm LexBFS [62].

G is not Helly (e.g., because some property of Helly graphs does not hold for G) or it will output some pair of vertices (x, y). Furthermore, if G is chordal Helly, then (x, y) is a diametral pair. Let $d = dist(x, y)$. We can check for a chordal graph G whether $diam(G) = d$, or G is not Helly, as follows:

- *Let u be the vertex visited last by a LexBFS. We may assume, by Lemma 11, that $e(u) \in \{d-1, d\}$ (otherwise, $d \neq diam(G)$, and so G is not Helly) and, by Lemma 12, that $e(u)$ is even. Then, we compute $rad(G)$, which takes linear time [23]. By Lemma 11, $diam(G) \in \{e(u), e(u)+1\}$, and so either G is* not *Helly or, by Lemma 1, $rad(G) \in \{e(u)/2, e(u)/2+1\}$.*
- *If $e(u) = d$, d is even and $rad(G) = d/2$, then this certifies that $diam(G) = e(u)$. Else, either G is* not *Helly or we have $e(u) = d - 1$, d is odd and $rad(G) = e(u)/2 + 1$. Since $dist(x, y) = d = e(u) + 1$, we get $diam(G) = d$ by Lemma 11.*

Proof of Theorem 6

The remainder of this subsection is devoted to the proof of Theorem 6. We first compute $r = rad(G)$, which by Theorem 5 can be done in linear time. We also apply the multi-sweep heuristic, *i.e.*, we pick an arbitrary vertex v and we perform a BFS from a vertex $u \in F(v)$. There are two main cases depending on the parity of $e(u)$.

Case $e(u)$ is even. By Lemma 9, $e(u) \geq diam(G) - 1$. Since by Lemma 1 we have $diam(G) \geq 2r - 1$, it follows that $e(u) \in \{2r-2, 2r\}$. In particular, if $e(u) = 2r - 2$ then $diam(G) = 2r - 1$, otherwise $diam(G) = e(u) = 2r$. Therefore, the difficulty here is not to compute the diameter, but rather to compute a diametral pair. Let us assume that we have $diam(G) = e(u) + 1 = 2r - 1$ (else, $e(u) = 2r = diam(G)$, and we are done). Let $w \in F(u)$. W.l.o.g., $e(w) = 2r - 2$ (else, $e(w) = diam(G)$ and we are done). Therefore, $dist(u, w) = 2r - 2$ and u, w are mutually far apart. The next result is a cornerstone of our algorithm:

Lemma 13. *Let u, w be mutually far apart vertices in a C_4-free Helly graph G such that $dist(u, w) = diam(G) - 1 = 2r - 2$ is even, and let $C = L(u, r-1, w)$. Then, (x, y) is a diametral pair of G if and only if $dist(x, C) = dist(y, C) = r - 1$ and $Pr(x, C) \cap Pr(y, C) = \emptyset$.*

Proof. Since $e(u) = e(w) = 2r - 2$, for any $x \in V$, the balls of radius $r - 1$ and with centers u, w, x, respectively, pairwise intersect. The Helly property implies the existence of a vertex $c \in V$ such that $\max\{dist(u, c), dist(w, c), dist(x, c)\} \leq r - 1$. Since we also have $dist(u, w) = 2r - 2$, we conclude that $c \in L(u, r-1, w) = C$ and $dist(x, C) \leq dist(x, c) \leq r-1$. Now on one direction, let (x, y) be a diametral pair. By Lemma 8, C is a clique, implying $dist(x, y) \leq dist(x, C) + 1 + dist(y, C) \leq 2r - 1 = diam(G)$. Therefore, $dist(x, C) = dist(y, C) = r - 1$. For similar reasons, we must have $Pr(x, C) \cap Pr(y, C) = \emptyset$ (otherwise, $dist(x, y) \leq dist(x, C) + dist(y, C) \leq 2r - 2 < diam(G)$, a contradiction). Conversely, let (x, y) be such that $dist(x, C) = dist(y, C) = r - 1$ and $Pr(x, C) \cap Pr(y, C) = \emptyset$. Suppose by contradiction $dist(x, y) < diam(G)$. In particular, the balls of radius $r - 1$ and respective centers u, v, x, y pairwise intersect. By the Helly property, there exists a $c \in C$ such that $\max\{dist(x, c), dist(y, c)\} \leq r - 1 = dist(x, C) = dist(y, C)$. But then, $c \in Pr(x, C) \cap Pr(y, C) = \emptyset$, a contradiction. Hence, (x, y) is a diametral pair. □

Our strategy now consists in computing a pair (x, y) that satisfies the condition of this above Lemma 13. We do so by using the "gated property" of Lemma 10. Indeed, let $C = L(u, r-1, w)$ be as above defined, and let $S = \{x^* \mid \exists x \in V \text{ such that } g_C(x) = x^* \wedge dist(x, C) = r - 1\}$. Since by Lemma 8 C is a clique, this set S is well-defined, and it can be computed in linear time. In order to compute a diametral pair of G, by Lemma 13 it is sufficient to compute a pair $x^*, y^* \in S$ such that $N(x^*) \cap N(y^*) \cap C = \emptyset$. At first glance this approach does not look that promising since it is a particular case of the Disjoint Sets problem (sometimes called the monochromatic Orthogonal Vector), that cannot be solved in truly subquadratic time under SETH [68]. Before presenting our solution to this special Disjoint Sets problem (*i.e.*, Lemma 16) we introduce a pruning rule in order to discard some vertices from S:

Lemma 14. *In a C_4-free Helly graph G, for any clique C and adjacent vertices $s, t \in N(C)$, the metric projections $Pr(s, C)$ and $Pr(t, C)$ are comparable, i.e., either $Pr(s, C) \subseteq Pr(t, C)$ or $Pr(t, C) \subseteq Pr(s, C)$.*

Proof. Let $s, t \in N(C)$ be adjacent and suppose for the sake of contradiction that there exist $s^* \in Pr(s, C) \setminus Pr(t, C)$ and $t^* \in Pr(t, C) \setminus Pr(s, C)$. Then, (s, t, t^*, s^*, s) induces a C_4. □

If $x^*, y^* \in S$ are adjacent, then by Lemma 14 either $N(x^*) \cap C \subseteq N(y^*) \cap C$ or $N(y^*) \cap C \subseteq N(x^*) \cap C$. Therefore, we can discard a vertex of $\{x^*, y^*\}$ with maximum number of neighbours in C from S. Doing so, we only need to consider a subset of gates that are pairwise non-adjacent.

Lemma 15. *Let u, w be two vertices in a C_4-free Helly graph G such that $dist(u, w) = 2r - 2$, let $C = L(u, r-1, w)$ and let $S \subseteq N(C)$ be a stable set. Then, $H = G[C \cup S]$ is a split Helly graph.*

Proof. By Lemma 8, the subset C is a clique, hence H is a split graph. Furthermore, let us consider a family of pairwise intersecting balls in H. We may assume w.l.o.g. that no such a ball is contained in $\mathcal{F} = \{N_H[c] \mid c \in C\} \cup \{N_H^2[z] \mid z \in C \cup S\}$. Indeed, all of these balls fully contain C, and so, there is a common intersection for all the balls in a family if and only if there is a common intersection for the sub-family of all these balls that are not in $\mathcal{F}$. In particular, we may consider the family of pairwise intersecting balls to be a collection of closed neighbourhoods $N_H[z]$, for all $z \in S'$, where S' is a subset of S. Since S (and so, S') is a stable set, the subsets $N_G(z) \cap C,\ z \in S'$ pairwise intersect. Then, we have that the balls of radius $r - 1$ and with centers in u, w and the balls of radius 1 and with centers in the vertices of S' pairwise intersect in G. By the Helly property (applied to G), there exists a vertex c at a distance $\leq r - 1$ from both u and w, and at a distance ≤ 1 from all of S'. Since $dist(u, w) = 2r - 2$, we get $c \in L(u, r-1, w) = C$ and so, $c \in \bigcap\{N_G(z) \cap C \mid z \in S'\}$. Consequently, H is Helly. □

Since every split graph has constant diameter (at most three), then by Corollary 2 the eccentricity of all vertices in a split Helly graph can be computed in total linear time. Nevertheless, in what follows, we propose a different approach for computing the diameter of a split Helly graph than the one we presented in Corollary 2. Interestingly, this approach also works for other Helly-type properties, *e.g.* for split open-neighbourhood-Helly graphs [55].

Lemma 16. *A diametral pair in a split Helly graph can be computed in linear time.*

Proof. Let $G = (C \cup S, E)$ be a split Helly graph with clique C and stable set S (note that if C and S are not given then they can be computed in linear time [52]). Assume G to be connected and $diam(G) > 1$ (otherwise, we are done). By the Helly property, $diam(G) = 2$ if and only if G contains a universal vertex. Furthermore, if it is the case then any pair x, y of non-adjacent vertices is diametral. Hence, from now on we assume that $diam(G) = 3$. Let $G_0 := G$ and let $(x_1, x_2, \ldots, x_{|S|})$ be an arbitrary total order of S. For every $1 \leq i \leq |S|$, we define $G_i := G[\bigcap\{N_G(x_j) \mid 1 \leq j \leq i\} \cup S]$. Our algorithm proceeds the vertices $x_i \in S$ sequentially, for $i = 1 \ldots |S|$, and does the following: If x_i has eccentricity 3 in G_{i-1}, then we compute a diametral pair in this subgraph which contains x_i and we stop.

We claim that our algorithm above is correct. For that we prove by finite induction that for any $i \geq 0$, if the algorithm did not stop in less than i steps then: `(i)` G_i is connected; and `(ii)` (x_p, x_q) is a diametral pair of G_i if and only if it is a diametral pair of G. Since $G_0 = G$, this is true for $i = 0$. From now on we assume $i > 0$. If the algorithm did not stop at step i then (since in addition G_{i-1} is connected by the induction hypothesis), x_i has eccentricity two in G_{i-1}. In particular, every vertex has a common neighbour with x_i, implying that there can be no isolated vertex in G_i. We so obtain that G_i is connected. Furthermore, if (x_p, x_q) is a diametral pair of G then, necessarily, it is also a diametral pair of the connected subgraph G_i (*i.e.*, because x_p and x_q have no common neighbour in this subgraph, and so they are at distance 3 to each other). Conversely, let (x_p, x_q) be a diametral pair of G_i. Suppose, by contradiction, that (x_p, x_q) is not a diametral pair of G, or equivalently $N_G(x_p) \cap N_G(x_q) \neq \emptyset$. Since the neighbour sets $N_G(x_j),\ j \in \{1, 2, \ldots, i\} \cup \{p, q\}$ pairwise intersect, by the Helly property, there exists a vertex $w \in \bigcap\{N_G(x_j) \mid j \in \{1, 2, \ldots, i\} \cup \{p, q\}\}$. But then, (x_p, x_q) is not a diametral pair of G_i (as $diam(G_i) \geq diam(G) = 3$), a contradiction. As a result, our above algorithm for computing a diametral pair of G is correct.

We still have to explain how to execute this algorithm in linear time. For that, we maintain a partition of the clique, initialized to $\mathcal{P}_0 := (C)$. At step i we refine the former partition $\mathcal{P}_{i-1} = (C_1^{i-1}, C_2^{i-1}, \ldots, C_{k_{i-1}}^{i-1})$ into a new partition $\mathcal{P}_i = (C_1^{i-1} \cap N_G(x_i), C_1^{i-1} \setminus N_G(x_i), C_2^{i-1} \cap N_G(x_i), C_2^{i-1} \setminus N_G(x_i), \ldots, C_{k_{i-1}}^{i-1} \cap N_G(x_i), C_{k_{i-1}}^{i-1} \setminus N_G(x_i))$. This partition refinement can be done in time $\mathcal{O}(N_G(x_i))$ (up to some initial pre-processing in $\mathcal{O}(|C|)$ time) [53]. Furthermore, an easy induction proves that for any $i \geq 0$ the first group of $\mathcal{P}_i$ is exactly $C_1^i = \bigcap\{N_G(x_j) \mid 1 \leq j \leq i\}$ *i.e.*, the clique of G_i. We finally explain how we use this partition in order to decide, at step i, whether x_i has eccentricity equal to 3 in G_{i-1}. At the beginning of the algorithm we compute the degree of every vertex in S. Then, at step i we consider all the vertices in $C_2^i = \bigcap\{N_G(x_j) \mid 1 \leq j < i\} \setminus N_G(x_i)$ sequentially (second group of the partition $\mathcal{P}_i$). For every $w \in C_2^i$ we enumerate all its neighbours in S and we decrease their respective degrees by one. In particular, if during step i the degree of some vertex $x_p \in S$ falls to 0, then x_p has no common neighbour with x_i in G_{i-1}. Equivalently, (x_i, x_p) is a diametral pair of G_{i-1} and the eccentricity of x_i, x_p in this subgraph is 3. We observe that the sets C_2^i on which we iterate are pairwise disjoint. As a result, the total complexity of the algorithm is linear. $\square$

This above Lemma 16 achieves proving Theorem 6 in the case when $e(u)$ is even.

Case $e(u)$ **is odd.** By Lemma 9, $e(u) \geq diam(G) - 2$. Therefore, by Lemma 1, $e(u) \in \{2r-3, 2r-1\}$. In the first subcase, we deduce from Lemma 1 that $diam(G) = 2r-1$. Let $w \in F(u)$, and assume $e(w) = e(u) = dist(u,w) = 2r-3$ (otherwise, either $e(w) = 2r-2$ is even and we are back to the former case, or $e(w) = 2r-1$ and then we are done since w is an end of a diametral path).

Lemma 17. *Let u, w be mutually far apart vertices in a C_4-free Helly graph G such that $dist(u,w) = diam(G) - 2 = 2r-3$ is odd, and let $A = L(w, r-2, u)$. Then, (x,y) is a diametral pair of G if and only if $dist(x,A) = dist(y,A) = r-1$, $Pr(x,A) \cap Pr(y,A) = \emptyset$, and in addition $dist(y,u) = dist(y,w) = dist(x,u) = dist(x,w) = 2r-3$.*

Proof. We first prove that for every vertex x we have $dist(x,A) \leq r-1$. Indeed, since w, u are mutually far apart, we have $dist(x,u) \leq dist(w,u) = 2r-3$ and $dist(x,w) \leq dist(w,u) = 2r-3$. Hence, balls $N^{r-1}[u]$, $N^{r-2}[w]$ and $N^{r-1}[x]$ pairwise intersect. By the Helly property, there is a vertex $z \in A$ with $dist(x,z) \leq r-1$.

For any pair (x,y) we have $dist(x,y) \leq dist(x,A) + dist(Pr(x,A), Pr(y,A)) + dist(y,A) \leq 2r-2 + dist(Pr(x,A), Pr(y,A))$. By Lemma 8, A is a clique, which implies that we have $dist(Pr(x,A), Pr(y,A)) \leq 1$. As a result, if (x,y) is diametral, we get $dist(x,A) = dist(y,A) = r-1$ and $Pr(x,A) \cap Pr(y,A) = \emptyset$. We also get $dist(y,u) = dist(y,w) = dist(x,u) = dist(x,w) = 2r-3$ by Lemma 9. Conversely let (x,y) be any pair that satisfies all these above properties, and suppose for the sake of contradiction that we have $dist(x,y) \leq 2r-2$. Consider balls $N^{r-1}[x], N^{r-1}[y], N^{r-1}[u]$ and $N^{r-2}[w]$. By distance requirements, these balls pairwise intersect. The Helly property implies that a vertex $z \in A$ exists such that $dist(x,z) = r-1$ and $dist(y,z) = r-1$. However, the latter contradicts with $Pr(x,A) \cap Pr(y,A) = \emptyset$. □

Overall, we can apply the same approach as in Lemma 13 in order to compute a diametral pair in this subcase.

The most difficult subcase is when $e(u) = 2r-1$. By Lemma 1, either $diam(G) = 2r-1$ or $diam(G) = 2r$. We explain below how, assuming $diam(G) = 2r$, we can compute in linear time *all* central vertices. Then, if $diam(G) = 2r$, by Lemma 4, a pair (x,y) is diametral if and only if both x and y are at a distance exactly r from all central vertices. We test a posteriori whether $dist(x,y) = 2r$, and if not $diam(G) = e(u) = 2r-1$. W.l.o.g. $r \geq 2$.

Lemma 18. *If G is a C_4-free Helly graph of diameter $2r \geq 4$, then $C(G)$ is a clique.*

Proof. Every central vertex is at a distance exactly r from both ends x, y of any diametral path. In particular, $C(G) \subseteq L(x,r,y)$, which is a clique by Lemma 8. □

Therefore, if $diam(G) = 2r$, by Lemma 18, for any central vertex c, we have $C(G) \subseteq N[c] = S$. Note that, by Theorem 5, we can compute such a central vertex c in linear time. Furthermore, $diam(S) \leq 2$. For every $x \notin S$, we compute $dist(x,S)$ and a corresponding gate $g(x)$, which exists by Lemma 10. By construction, $\max_x dist(x,S) \leq \max_x dist(x,c) \leq r$. Furthermore, every vertex at a distance $\leq r-2$ from S is at a distance $\leq r$ from every vertex of S. As a result, we only need to consider the vertices at a distance $\geq r-1$ from S.

In fact, and as already observed in the proof of Theorem 5, for a vertex of S to be central it needs to be adjacent to the gates of all the vertices at a distance exactly r from S. All the vertices which satisfy this necessary condition can be computed in linear time. Hence, we can restrict ourselves to the vertices that are at distance exactly $r-1$ from S.

Lemma 19. *Let G be a C_4-free Helly graph and let S be such that $diam(S) \leq 2$. If $dist(x, S) = r-1$, then there exists a vertex $pg_S(x) \in N^{r-1}[x]$ such that $S \cap N^r[x] \subseteq N[pg_S(x)]$. Moreover, $pg_S(x)$ is in the closed neighbourhood of some gate of x.*

We call such a vertex a *pseudo-gate* of x.

Proof. The existence of such a pseudo-gate follows from the fact that the balls $N^{r-1}[x]$ and $N[s]$, $s \in S \cap N^r[x]$ pairwise intersect, and from the Helly property. Now, let $pg(x)$ and $g(x)$ be arbitrary pseudo-gate and gate of x, respectively, and assume $pg(x) \neq g(x)$ and $pg(x)g(x) \notin E$ (else, we are done). In particular, $pg(x) \notin Pr(x, S)$. Note that, since we have $Pr(x, S) \subseteq N(g(x)) \cap N(pg(x))$, $dist(g(x), pg(x)) = 2$. Then, the balls $N^{r-2}[x], N[g(x)]$ and $N[pg(x)]$ pairwise intersect. By the Helly property, there exists a vertex x^* in their common intersection. We claim that x^* is a gate of x, that will prove the lemma. Indeed, for every $s \in Pr(x, S)$ we get a cycle $(s, pg(x), x^*, g(x), s)$. Since G is C_4-free, this implies $sx^* \in E$. □

By taking advantage of the property that $pg(x)$ is either equal or adjacent to some gate $g(x)$, we next explain how we can adapt the technique from [23, 24] in order to compute all pseudo-gates in total linear time.

Remark 3. *For every $x \in N(S)$, we can choose as its pseudo-gate any vertex of $N[x]$ that maximizes the intersection of S with its closed neighbourhood (possibly, x itself). Then, when we compute a gate for every vertex, we break ties by choosing one such a gate whose pseudo-gate maximizes its intersection with S. In doing so, we can compute a pseudo-gate for every vertex at distance $r - 1$ from S, in total linear time.*

Finally, if $diam(G) = 2r$, then the central vertices are exactly those adjacent to the gates of all the vertices at distance r from S *and* either equal or adjacent to the pseudo-gates of all the vertices at distance $r - 1$ from S. □

3.3 Computing all eccentricities

We are now ready to present the main result of this section, namely:

Theorem 7. *If G is a C_4-free Helly graph then we can compute the eccentricity of all vertices in linear time.*

Proof. By Lemma 4, we only need to compute $C(G)$. Our main tool for that is our parameterized linear-time algorithm for Small Eccentricities, for $k = 2$ (Theorem 2). In particular, by Corollary 2, we may assume that $rad(G) = r \geq 3$. Let $diam(G) = d$ and let (x, y) be a diametral pair. By Theorem 6, it can be computed in linear time. We assume in what follows $d = 2r - 1$ (else, by Lemma 1, $d = 2r$, and we already explained in Sec. 3.2 how to compute $C(G)$). Obviously, $C(G) \subseteq N^r[x] \cap N^r[y]$. Since $dist(x, y) = 2r-1$ we get $N^r[x] \cap N^r[y] = L(x, r-1, y) \cup L(y, r-1, x) \cup Z$, where $Z := \{z \in V \mid dist(z, x) = dist(z, y) = r\}$. Furthermore, for a C_4-free Helly graph, by Lemma 8 the slices $L(x, r - 1, y)$ and $L(y, r - 1, x)$ are cliques; we can reuse the same idea as for Theorem 5 in order to compute $C(G) \cap (L(x, r - 1, y) \cup L(y, r - 1, x))$ in linear time. From now on we focus on Z.

Claim 3. *Every $z \in Z$ has adjacent neighbours in $L(x, r - 1, y)$ and $L(y, r - 1, x)$, resp.*

Proof. Let $p = q = r-1$ and $k = 0$. We have $dist(x, y) = p+q+1$, $dist(x, z) = p+k+1$, $dist(y, z) = q + k + 1$. Since G is pseudo-modular, by Lemma 6, z is adjacent to the two ends of an edge in the middle of some shortest xy-path. $\diamond$

Since both $L(x, r-1, y)$ and $L(y, r-1, x)$ are cliques, the above implies that Z has weak diameter at most $3 \leq r$. More generally, every vertex at a distance $\leq r-2$ from $L(x, r-1, y) \cup L(y, r-1, x)$ is at a distance $\leq r$ from every vertex of Z. Therefore, we further restrict our study to the vertices w such that $dist(w, L(x, r-1, y) \cup L(y, r-1, x)) \geq r-1$.

Subcase $dist(w, L(x, r-1, y) \cup L(y, r-1, x)) = r-1$**.** Let $C := L(x, r-1, y)$ and consider the set W_x of all vertices w such that $dist(w, L(x, r-1, y) \cup L(y, r-1, x)) = dist(w, C) = r-1$. Let A_x contain a gate $g_C(w)$ for every $w \in W_x$, which exists by Lemma 10.

Claim 4. *For every $z \in Z$, we have $\max_{w \in W_x} dist(z, w) \leq r$ if and only if $A_x \subseteq N^2[z]$.*

Proof. First assume $A_x \subseteq N^2[z]$, or equivalently $\forall a \in A_x,\ N[a] \cap N[z] \neq \emptyset$. Let $w \in W_x$ and let $a := g_C(w) \in A_x$. Then, $dist(z, w) \leq dist(z, a) + dist(a, w) \leq 2 + (r-2) = r$. Conversely, let us assume $\max_{w \in W_x} dist(z, w) \leq r$, and let $w \in W_x$ be arbitrary. The balls $N^{r-1}[x], N^{r-1}[w], N^r[y], N[z]$ pairwise intersect. Therefore, by the Helly property, $N[z] \cap Pr(w, C) \neq \emptyset$. For any gate $g_C(w) \in A_x$, it implies $N[z] \cap N[g_C(w)] \supseteq N[z] \cap Pr(w, C) \neq \emptyset$. $\diamond$

Overall with this above claim we are reduced to Small Eccentricities, with $k = 2$, which by Theorem 2 can be solved in linear time. Our approach also works for $C' := L(y, r-1, x)$, up to reversing the respective roles of x and y.

Subcase $dist(w, L(x, r-1, y) \cup L(y, r-1, x)) > r-1$**.** Let W^* contain all vertices w such that $dist(w, L(x, r-1, y) \cup L(y, r-1, x)) \geq r$.

Claim 5. *For every $w \in W^*$ we have $dist(x, w) = dist(y, w) = 2r-1$ and $dist(w, Z) = r-1$. Moreover, $Pr(w, Z) = L(w, r-1, x) \cap L(w, r-1, y)$ is a clique.*

Proof. We first prove that $dist(x, w) = dist(y, w) = 2r-1$. Indeed, suppose for the sake of contradiction that $dist(x, w) \leq 2r-2$. Since the balls $N^{r-1}[x], N^{r-1}[w], N^r[y]$ pairwise intersect, by the Helly property, we get $dist(w, L(x, r-1, y)) \leq r-1$, a contradiction. Hence, the balls $N^{r-1}[w], N^r[x], N^r[y]$ pairwise intersect, which implies, by the Helly property, $dist(w, Z) = r-1$. In this situation, we have $Pr(w, Z) = L(w, r-1, x) \cap L(w, r-1, y)$. By Lemma 8, we get that $Pr(w, Z)$ is a clique. $\diamond$

The combination of this above claim with Lemma 10 implies the existence of a gate $g_Z(w) \in N^{r-2}[w] \cap \bigcap\{N(v) \mid v \in Pr(w, Z)\}$ for every vertex $w \in W^*$. Note that we can compute the gate of all such vertices w in linear time, by adapting the techniques from [23, 24]. So, let A^* contain a gate $g_Z(w)$ for every vertex $w \in W^*$. We prove as before:

Claim 6. *For every $z \in Z$, we have $\max_{w \in W^*} dist(z, w) \leq r$ if and only if $A^* \subseteq N^2[z]$.*

Proof. We can prove this above condition to be sufficient for having $\max_{w \in W^*} dist(z, w) \leq r$ in the exact same way as we did for Claim 4. Conversely, let us assume that we have $\max_{w \in W^*} dist(z, w) \leq r$, and let $w \in W^*$ be arbitrary. The balls $N[z], N^{r-1}[w], N^r[x], N^r[y]$ pairwise intersect, and so, by the Helly property, $N[z] \cap Pr(w, Z) = N[z] \cap (L(w, r-1, x) \cap L(w, r-1, y)) \neq \emptyset$. It implies that $N[z] \cap N[g_Z(w)] \supseteq N[z] \cap Pr(w, Z) \neq \emptyset$. Equivalently, $A^* \subseteq N^2[z]$. $\diamond$

We are done by reducing a final time to Small Eccentricities, with $k = 2$, which by Theorem 2 can be solved in linear time. $\square$

4 More reductions to split graphs

We conclude by considering diameter computation within another class than Helly graphs, namely chordal graphs. This is motivated by our results in Section 3.2 where we reduced the problem of computing a diametral pair on C_4-free Helly graphs (and so, on chordal Helly graphs) to the same problem on split Helly graphs. We prove next that there exists a (randomized) reduction from diameter computation on *general* chordal graphs to the same problem on split graphs. Specifically, a sparse representation of a split graph is the list of the closed neighbourhoods of vertices in its stable set [44]. – Such a representation may not be unique, since it depends on a specific bipartition of the split graph into a clique and a stable set. – Then, the Disjoint Sets problem consists in computing the diameter of a split graph given by one of its sparse representations. For a split graph H with stable set U we define $\ell(H) := \sum_{u \in U} deg_H(u)$, *a.k.a.* the size of its sparse representation.

Theorem 8. *For any chordal graph G, we can compute in $\tilde{\mathcal{O}}(m)$-time the sparse representations of a family (H_i) of split graphs such that, if for every i we can compute $diam(H_i)$ in time $\mathcal{O}(\ell(H_i) \cdot |V(H_i)|^b)$, then we can compute $diam(G)$ in time $\tilde{\mathcal{O}}(mn^b)$.*

The OV problem is given as inputs the sparse representation of a split graph G and two subsets A, B in its stable set, and it asks what the largest distance in G is between a vertex of A and a vertex of B. We highlight the following variant of our Theorem 8. Although it is not clear whether the two statements are equivalent, our proofs for both results are nearly identical.

Theorem 8′. *For a subclass $\mathcal{C}$ of chordal graphs, let $\mathcal{S}$ contain all split graphs that are induced subgraphs of a graph in $\mathcal{C}$. If for every $G \in \mathcal{S}$, for every bipartition A, B of its stable set, we can solve* OV *in $\mathcal{O}(\ell^b)$ time, for some $b \geq 1$, then there is a randomized $\mathcal{O}(m^b \log^2 n)$-time algorithm for computing w.h.p. the diameter of chordal graphs in $\mathcal{C}$.*

4.1 Preliminaries

We shall use the following metric properties of chordal graphs. We stress that these are quite similar to some metric properties of C_4-free Helly graphs that we proved in Section 3.

Lemma 20 ([23]). *For any clique C in a chordal graph G, we can compute in linear time the distance $dist(x, C)$ and a gate $g(x) \in \bigcap\{I(x, w) \mid w \in Pr(x, C)\}$ adjacent to all vertices from $Pr(x, C)$, for all vertices x.*

Lemma 21 ([23]). *In a chordal graph G, for any clique C and adjacent vertices $s, t \notin C$ the metric projections $Pr(s, C)$ and $Pr(t, C)$ are comparable, i.e., either $Pr(s, C) \subseteq Pr(t, C)$ or $Pr(t, C) \subseteq Pr(s, C)$.*

4.2 The reduction

Our reduction is one-to-many. We recall that a clique-tree of a graph G is a tree T of which the nodes are the maximal cliques of G, and such that for every vertex v the set of all the maximal cliques that contain v induces a connected subtree. It is known that G is chordal if and only if it has a clique-tree [18, 49, 67] and, furthermore, a clique-tree can be computed in linear time [66]. – See also [7, 15] and the references therein –. We may see a clique-tree T as a node-weighted tree where, for any maximal clique C, $w(C) = |C|$. Then, let $w(T) := \sum_C w(C)$. For a chordal graph, $w(T) = \mathcal{O}(n + m)$ [7]. We will use a standard result on weighted centroids in trees, namely:

Lemma 22 ([51]). *Every node-weighted tree T has at least one weighted centroid, that is, a node v whose removal leaves components of maximum weight $\leq w(T)/2$. Moreover, a weighted centroid can be computed in linear time.*

Let T be a fixed clique-tree of G. If T is reduced to a single node, or to exactly two nodes, respectively (equivalently, either G is a complete graph or it is the union of two crossing complete subgraphs), then we output $diam(G) = 1$, or $diam(G) = 2$, respectively (base case of our reduction). Otherwise, let $S \in V(T)$ be a weighted centroid of T. By Lemma 22, the clique S can be computed in $\mathcal{O}(|V(T)|) = \mathcal{O}(n)$ time if we are given T in advance. Furthermore, let $T_1, T_2, \ldots, T_\ell$ be the components of $T \setminus \{S\}$, and for every $1 \leq i \leq \ell$ let $V_i := (\bigcup V(T_i)) \setminus S$. Since $\forall i, N_G(V_i) \subseteq S$ is a clique, the closed neighbourhoods $N_G[V_i], 1 \leq i \leq \ell$ induce distance-preserving subgraphs of G, which we denote by $G_1, G_2, \ldots, G_\ell$. We apply our reduction recursively on each of these subgraphs G_i. Then, let $d_S := \max_{v_i \in V_i, v_j \in V_j | i \neq j} dist(v_i, v_j)$. We have: $diam(G) = \max\{d_S, \max\{diam(G_i) \mid 1 \leq i \leq \ell\}\}$. So, we are left with computing d_S. For that, we define $\forall i, d_i := \max_{v_i \in V_i} dist(v_i, S)$. We order the sets V_i by non-increasing value of d_i. Since S is a clique, we get $d_1 + d_2 \leq d_S \leq d_1 + d_2 + 1$. In order to decide in which case we are, we proceed as follows:

- We discard all sets V_i s.t. $d_i < d_2$. Doing so we are left with sets $V_1, V_2, \ldots, V_k,\ k \leq \ell$.
- Then, for every $1 \leq i \leq k$ and $v \in V_i$, if $dist(v, S) = d_i$ then we compute a gate for vertex v, which exists by Lemma 20. Furthermore, if two such gate vertices are adjacent then they must be in the same connected component of $G \setminus S$, and by Lemma 21 their respective metric projections on S are comparable. It implies that we can remove any of these two vertices with largest metric projection on S. Thus, from now on, we assume all selected gate vertices to be pairwise non-adjacent.
- Finally, let $a, b \notin V$ be fresh new vertices which we make adjacent to each other and to all vertices of S. There are two subcases:
  - Case $d_1 \neq d_2$. We make vertex a adjacent to all gates of the vertices in V_1, while we make vertex b adjacent to all gates of the vertices in $\bigcup_{i=2}^{k} V_i$.
  - Case $d_1 = d_2$. For every $1 \leq i \leq k$, with probability $1/2$ we make vertex a adjacent to all gates of the vertices in V_i (otherwise, we do so with vertex b).

Doing as above we get a split graph H whose clique and stable set are $S \cup \{a, b\}$ and the selected gate vertices, respectively. We stress that deciding whether $diam(H) = 3$ is here equivalent to deciding whether the largest distance in $H \setminus \{a, b\}$ between a vertex of $A = N_H(a) \setminus (S \cup \{b\})$ and a vertex of $B = N_H(b) \setminus (S \cup \{a\})$ is equal to 3. Furthermore, $H \setminus \{a, b\}$ is an induced split subgraph of G by construction.

Claim 7. *If $diam(H) = 3$ then $d_S = d_1 + d_2 + 1$. Conversely, if $d_S = d_1 + d_2 + 1$ then $diam(H) = 3$ with probability $\geq 1/2$.*

Proof. Let x^* and y^* be two gates such that $dist_H(x^*, y^*) = 3$. By construction of H, x^*, y^* are the respective gates of two vertices x, y such that $\min\{dist_G(x, S), dist_G(y, S)\} \geq d_2$. Furthermore, since $\{a, b\} \cap (N_H(x^*) \cap N_H(y^*)) = \emptyset$, we get that x, y are in different components of $G \setminus S$, and $\{dist_G(x, S), dist_G(y, S)\} = \{d_1, d_2\}$. Hence, $d_S \geq dist(x, y) \geq d_1 + d_2 + 1$. Conversely, let us assume the existence of a pair (x, y) such that: x and y are in different components of $G \setminus S$, and

$dist(x,y) = d_1 + d_2 + 1$. W.l.o.g., let $dist(x,S) = d_1$ and $dist(y,S) = d_2$. Let also x^*, y^* be the two gates computed for x and y, respectively (we may assume, w.l.o.g., that x^*, y^* are indeed in the stable set of H). We must have $N_H(x^*) \cap N_H(y^*) \subseteq \{a,b\}$. In particular, if $d_1 \neq d_2$ then $N_H(x^*) \cap N_H(y^*) = \emptyset$, else $Prob[N_H(x^*) \cap N_H(y^*) = \emptyset] \geq 1/2$. ⋄

Overall, we may repeat the construction of this above split graph H up to $\mathcal{O}(\log n)$ times in order to compute d_S with high probability.

4.3 Analysis

Since at every step of our reduction we pick a weighted centroid in the clique-tree of every subgraph G_i considered, there are $\mathcal{O}(\log w(T)) = \mathcal{O}(\log n)$ recursion levels. Therefore, up to polylogarithmic factors, the total running-time of the reduction is of the same order of magnitude as the worst-case running time of a single step. Furthermore, it is not hard to prove that the first step, when we only consider the full input graph G, runs in linear time (*i.e.*, omitting the computation of the diameter for the related split graph H). However, during the next steps of our reduction we may need to consider pairwise overlapping subgraphs G_i, thereby making the analysis more delicate. The key insight here is that the clique-trees of all these subgraphs form a family of pairwise disjoint subtrees of T. We next explain how to perform the first step, and so all subsequent ones, in time $\mathcal{O}(w(T))$. Since $w(T) = \mathcal{O}(n+m)$ [7], doing so we can compute all the desired split graphs H throughout our reduction in total quasi linear time $\tilde{\mathcal{O}}(n+m)$.

Lemma 23. *Let S be any clique of a chordal graph G. If a clique-tree T is given, then in time $\mathcal{O}(w(T))$ we can compute $\forall v \notin S, dist(v,S)$ and a corresponding gate $g(v)$.*

Proof. Let $\mathcal{C} := V(T) \cup \{S\}$ be the set of all maximal cliques of G, to which we also add the clique S if it is not maximal. We define a set of fresh new vertices indexed by $\mathcal{C}$, namely let $X_{\mathcal{C}} := \{x_C \mid C \in \mathcal{C}\}$. Then, let $J := (V \cup X_{\mathcal{C}}, \{vx_C \mid v \in C\})$ be the vertex-clique incidence graph of G. Note that we can construct J by scanning once the clique S and all the maximal cliques of the graph G.

We prove as a subclaim that for every vertex v we have $dist_J(v,x_S) = 2 \cdot dist_G(v,S) + 1$. Indeed, since by construction $N_J(x_S) = S$, we get $dist_J(v,x_S) = 1 + \min_{u \in S} dist_J(v,u) = 1 + dist_J(v,S)$. Furthermore, in every vS-path of J, that is in every path between v and a closest vertex of S, half of the internal vertices must be in $X_{\mathcal{C}}$. Since in addition two vertices that are adjacent to a same maximal clique in J are adjacent in G, it allows us to transform such a path in J to a vS-path in G that is twice shorter. Conversely, any vS-path of G can be transformed into a vS-path of J that is twice longer, simply by adding between every two consecutive vertices a maximal clique which contains both of them. – Note that combining the two constructions, the latter exactly characterizes the shortest vS-paths in J. – As a result, we proved as claimed that $dist_J(v,S) = 2 \cdot dist_G(v,S)$. It implies that after a BFS in J rooted at x_S we get $dist_G(v,S), \forall v \notin S$.

Then, we recursively define $p(\alpha), \forall \alpha \in V(J) \setminus N_J[x_S]$ as follows (recall that $N_J[x_S] = \{x_S\} \cup S$): if $\alpha = x_C$ for some $C \in \mathcal{C} \setminus \{S\}$, and $C \cap S \neq \emptyset$ (equivalently, $dist_J(x_S,\alpha) = 2$), then $p(\alpha) = |C \cap S|$ (number of neighbours at distance one from x_S); otherwise, $p(\alpha) = \max\{p(\beta) \mid \beta \in N_J(\alpha) \cap I_J(x_S,\alpha)\}$. Note that, we can compute all those values $p(\alpha)$ during a BFS with no significant computational overhead. Furthermore, we claim that $\forall v \in V, p(v) = |Pr_G(v,S)|$. Indeed, by induction, $p(v) = \max\{|C \cap S| \mid x_C \in X_{\mathcal{C}} \cap I_J(v,x_S),\ C \neq S\}$. We recall our earlier characterization of the shortest vS-paths in J as those obtained from the shortest vS-paths in G by adding a maximal clique between every two consecutive vertices. It implies that for every clique C such that $x_C \in X_{\mathcal{C}} \cap I_J(v,S)$, we have $C \cap S \subseteq Pr_G(v,S)$. In particular, if we start from a shortest vS-path in G that contains a gate $g_S(v)$, there is a corresponding vS-path in J that contains a clique $C \supseteq Pr(v,S) \cup \{g_S(v)\}$. As a result, $p(v)$ is exactly $|Pr_G(v,S)|$.

Finally, we recursively define $g(\alpha), \forall \alpha \in V(J) \setminus N_J^2[x_S]$ as follows (recall that $N_J^2[x_S] = S \cup \{x_C \mid C \cap S \neq \emptyset\}$): if $\alpha \in V$ and $\alpha \in N_G(S)$ (equivalently, $dist_J(x_S,\alpha) = 3$), then $g(\alpha) = \alpha$; otherwise, $g(\alpha) \in \{g(\beta) \mid \beta \in N_J(\alpha) \cap I_J(x_S,\alpha) \wedge p(\alpha) = p(\beta)\}$. Again, we can compute all those values $g(\alpha)$ during a BFS with no significant computational overhead. Furthermore, it also follows from our above characterization of shortest vS-paths in J that we have, $\forall v \notin S$, $g(v)$ is a gate of v. □

Lemma 24. *For a clique-tree T of a given chordal graph G, let H be the split graph constructed as in Section 4.2. A sparse representation of the split graph H can be computed in $\mathcal{O}(w(T))$ time. Furthermore, if $U \subseteq V(H)$ is the stable set of H, then $|U| = \mathcal{O}(|V(T)|)$.*

Proof. After Lemma 23, we need to select a subset of pairwise non-adjacent gates in order to construct U. For that, we consider all the maximal cliques C sequentially. If C contains at least two gates, then we suppress all the gates in C but one with minimum metric projection on S. Overall, this post-processing also takes time $\mathcal{O}(w(T))$. Doing so there is at most one gate selected per maximal clique, *i.e.*, $|U| \leq |V(T)|$. Furthermore, we have $\ell(H) = \sum_{u \in U} deg_H(u) \leq \sum_C |C| = w(T)$. We end up adding a, b and the edges incident to these two vertices and the stable set U, that takes total time $\mathcal{O}(|U|) = \mathcal{O}(|V(T)|)$. □

Finally, in order to complete the proof of Theorem 8, let (H_i) be the family of split graphs considered for a given recursive step of the reduction. Let us assume that for every i, we can compute $diam(H_i)$ in $\mathcal{O}(\ell(H_i) \cdot |V(H_i)|^b)$ time. By Lemma 24, $\sum_i \ell(H_i) = \mathcal{O}(w(T)) = \mathcal{O}(n+m)$. Furthermore, $\max_i |V(H_i)| \leq n$. As a result, computing $diam(H_i)$ for all i takes total time $\mathcal{O}(mn^b)$. In the same way, if we can solve $diam(H_i)$ in $\mathcal{O}(\ell(H_i)^b)$ time, for some $b \geq 1$, then computing $diam(H_i)$ for all i takes total time $\mathcal{O}(m^b)$ (as explained in Sec. 4.2, the latter reduces to OV on a $H_i' \subseteq H_i$, where H_i' is an induced split subgraph of G). Note that we need to repeat our probabilistic construction of the H_i's at most $\mathcal{O}(\log n)$ times for each step, and that there are $\mathcal{O}(\log n)$ steps in total. As a result, the running time is in $\mathcal{O}(m^b \log^2 n)$. □

4.4 Applications

It was proved in [37] that for all chordal graphs, an additive +2-approximation of all eccentricities can be computed in total linear time. Using our previous reduction from Section 4.2, we improve this result to an additive +1-approximation, but at the price of a logarithmic overhead in the running time.

Theorem 9. *For every n-vertex m-edge chordal graph, we can compute an additive +1-approximation of all eccentricities in total $\mathcal{O}(m \log n)$ time.*

Roughly, Theorem 9 follows from the observation that, in a *split* graph, if we set $e(v) = 2$ for every vertex v, then we get an additive +1-approximation of all the eccentricities.

Proof. In what follows, let G be chordal and let T be a fixed clique-tree of G. We can assume that G has at least two vertices.

- If T is reduced to a single node, or equivalently, G is a complete graph, every vertex has eccentricity equal to 1. In the same way if T is reduced to exactly two nodes, then G is the union of two crossing complete subgraphs C_1 and C_2. Furthermore, every vertex of $C_1 \cap C_2$ has eccentricity equal to 1, whereas every vertex of the symmetric difference $C_1 \Delta C_2$ has eccentricity equal to 2.
- Otherwise, let $S \in V(T)$ be a weighted centroid of T. By Lemma 22, the clique S can be computed in time $\mathcal{O}(|V(T)|) = \mathcal{O}(w(T))$ if we are given T in advance.
  - Let $T_1, T_2, \ldots, T_\ell$ be the components of $T \setminus \{S\}$, and for every $1 \leq i \leq \ell$ let $V_i := (\bigcup V(T_i)) \setminus S$. We recall that the closed neighbourhoods $N_G[V_i], 1 \leq i \leq \ell$ induce distance-preserving subgraphs of G. As in Section 4.2, we denote these subgraphs by $G_1, G_2, \ldots, G_\ell$. We apply our reduction recursively on each of the G_i's. Doing so, for every i and every vertex $v_i \in V_i$ we get an additive +1-approximation of $e_{G_i}(v_i)$.
  - Then, for every vertex $v \notin S$, we compute $dist_G(v, S)$ and a gate $g_S(v)$. By Lemma 23 this can be done in total $\mathcal{O}(w(T))$ time if T is given in advance. Furthermore, notice that since S is a clique, for every $s \in S$ we have that $\max_{v \in V} dist_G(v, S)$ is an additive +1-approximation of $e_G(s)$.
  - Finally, we observe that for every i and every vertex $v_i \in V_i$ we have:

    $$e_G(v_i) = \max\{e_{G_i}(v_i), \max_{u \notin V_i} dist_G(v_i, u)\}.$$

    As in Section 4.2 we define $\forall i, d_i := \max_{v_i \in V_i} dist(v_i, S)$. If we compute the two largest values amongst the d_i's, then we can compute $\forall i, e_i := \max_{j \neq i} d_j$. We are done as for every $v_i \in V_i$, $dist_G(v_i, S) + e_i$ is an additive +1-approximation of $\max_{u \notin V_i} dist_G(v_i, u)$.

As already observed in Section 4.3, there are $\mathcal{O}(\log w(T)) = \mathcal{O}(\log n)$ recursion levels. Since at any level, the clique-trees of all the subgraphs considered form a family of pairwise disjoint subtrees of T, each step can be done in $\mathcal{O}(w(T))$ time. Therefore, the total running time is in $\mathcal{O}(w(T) \log n) = \mathcal{O}(m \log n)$ [7]. □

Finally, the VC-dimension of a graph G is the largest cardinality of a subset S such that $\{N[v] \cap S \mid v \in V\} = 2^S$ (we say that S is shattered by G). For instance, interval graphs have VC-dimension at most two [12].

Theorem 10. *For every $d > 0$, there exists a constant $\eta_d \in (0; 1)$ such that in $\mathcal{O}(mn^{1-\eta_d})$ time, we can compute the diameter of any chordal graph of VC-dimension at most d.*

Proof. If a split graph H has VC-dimension at most d', then we can compute its diameter in truly subquadratic time $\mathcal{O}(\ell(H) \cdot |V(H)|^{1-\varepsilon_{d'}})$, for some $\varepsilon_{d'} \in (0; 1)$ [45, Theorem 1]. As a result, it is sufficient to prove that all the split graphs H_i, which are output by the reduction of Theorem 8, have a VC-dimension upper bounded by a function of d. We observe that every such H_i is obtained from an induced subgraph H_i' of G by adding two new vertices a and b (see Section 4.2). Since G has VC-dimension at most d, so does H_i'. Then, let S be a largest subset shattered by H_i. We can extract from S a maximal shattered subset $S' \subseteq V(H_i')$ (*i.e.*, not containing a and b). In particular, $|S'| \geq |S| - 2$. Furthermore, since S' is shattered by H_i, $\{N[v] \cap S' \mid v \in V(H_i)\} = 2^{S'}$ holds. It implies that $|\{N[v] \cap S' \mid v \in V(H_i') = V(H_i) \setminus \{a, b\}\}| \geq 2^{|S'|} - 2$. By the Sauer-Shelah-Perles Lemma [63, 65], we also have $|\{N[v] \cap S' \mid v \in V(H_i')\}| = \mathcal{O}(|S'|^d)$, which implies $|S'| = \mathcal{O}(d \log d)$. Consequently, every H_i has VC-dimension in $\mathcal{O}(d \log d)$. □

An *undirected path graph* is the intersection graphs of paths in a tree. Since the undirected path graphs have VC-dimension at most 3 [12], the following result is a direct consequence of our Theorem 10:

Corollary 3. *There is a truly subquadratic-time algorithm for computing the diameter of undirected path graphs.*

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