A Stroll Through the Gaussian Primes

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THE MOAT PROBLEM. One cannot walk to infinity on the real line if one uses steps of bounded length and steps on the prime numbers. This is simply a restatement of the classic result that there are arbitrarily large gaps in the primes. The proof is simple: a gap of size k is given by $(k+1)!+2, (k+1)!+3, \ldots, (k+1)!+(k+1)$.

But the same problem in the complex realm is unsolved. More precisely, an analogous question asks whether one can walk to infinity in $\mathbf{Z}[i]$, the Gaussian integers, using the Gaussian primes (henceforth, G-primes) as stepping stones, and taking steps of bounded length. The Gaussian question is much more complex because of the additional dimension. For example, there are arbitrarily large disks in $\mathbf{Z}[i]$ that contain only Gaussian composites (see [**A**, p. 119]; this also follows from our Theorem 4.1), but that has little impact on a trek to infinity for a walker who can, with luck, simply walk around the obstacle.

This problem is sometimes called the Gaussian moat problem, since one way of establishing a walk's nonexistence is to present a sufficiently wide moat (region of composites) that completely surrounds the origin.

The literature has often attributed the Gaussian moat problem to Paul Erdős. But in fact, the question was first posed by Basil Gordon in 1962 at the International Congress of Mathematicians in Stockholm. There have been few published references to the problem since then: it seems to have been mentioned in only three books ([**G**], [**Mo**], [**W1**]) and two papers ([**JR**], [**H**]). A paper by J. H. Jordan and J. R. Rabung [**JR**] contains some computational results, and also the comment that Erdős conjectured that a walk to infinity does exist. Other authors have also attributed the problem to Erdős ([**G**], [**H**], [**Mo**]). But Erdős [**E1**] recently confirmed that the problem was not posed by him and offered the opinion that the sought-after walk does not exist. Jordan and Rabung constructed a $\sqrt{10}$-moat; thus, steps of size 3 will not get a Gaussian prime-walker to infinity. In this paper we present two larger moats (4 and $\sqrt{18}$), as well as a computational proof that a $\sqrt{26}$-moat exists. Thus, steps of length 5 are insufficient to reach infinity. The first author and Harold Stark [**GS**] have shown that, starting *anywhere* in the complex plane, and taking steps of length at most two, one cannot walk to infinity. Ilan Vardi [**V**] has shown that some reasonable probabalistic assumptions about the primes allow one to apply percolation theory to obtain heuristic reasons why walks to infinity using steps of bounded size should not exist.

In Section 2 we summarize some definitions and facts about the G-primes. Section 3 contains several new Gaussian moats, and Section 4 contains results that were inspired by William Duke and questions of Gaussian prime geometry.

2. BACKGROUND. The ring of Gaussian integers, denoted $\mathbf{Z}[i]$, consists of integers in the field $\mathbf{Q}(i)$; they have the form $a+bi$ where $a, b \in \mathbf{Z}$ and $i = \sqrt{-1}$.

Like other rings that enjoy unique factorization, $\mathbf{Z}[i]$ admits a well-defined notion of primality. And there is a simple characterization of the G-primes as

follows:

(1) If $a, b \neq 0$, then $a + bi$ is a G-prime if and only if $a^2 + b^2 = p$, where p is a prime.
(2) A Gaussian integer of the form a or ai, $a \in \mathbf{Z}$, is a G-prime if and only if a is a prime and $|a| \equiv 3 \pmod 4$.

Further, the *units* of $\mathbf{Z}[i]$ are ± 1 and $\pm i$. The *norm* of a Gaussian integer $x + iy$ is defined to be $N(x + iy) = x^2 + y^2$. Therefore, to restate (1), a Gaussian integer $a + bi$ $(a, b \neq 0)$ is a G-prime if and only if $N(a + bi)$ is a prime.

Two Gaussian integers v, w are *associates* if $v = uw$ where u is a unit. In such a case, $N(v) = N(w)$. It is well known, and not hard to prove, that a prime $p \equiv 1 \pmod 4$ can be written uniquely as the sum of two squares. For example, $5 = (\pm 1)^2 + (\pm 2)^2$. Thus, there are exactly eight G-primes corresponding to the prime 5, namely $\pm 1 \pm 2i$ and $\pm 2 \pm i$. Hence, up to associates, there are exactly two distinct G-primes corresponding to each prime $p \equiv 1 \pmod 4$. For proofs of these assertions, see [**R**, p. 188], [**HW**, ch. XV], and [**Z**].

Geometrically, what does all of this mean? Specifically, given a G-prime q satisfying $N(q) = p$, where p is a prime congruent to 1 (mod 4), consider the circle of radius $\sqrt{p}$ centered at the origin: the eight G-primes corresponding to p lie, two in each quadrant, on this circle. Similarly, if a Gaussian integer $x + iy$ is composite, then $\pm x \pm yi$ and $\pm y \pm xi$ are composite as well. Thus the geometric structure of the Gaussian integers has an induced eightfold symmetry. Figure 1 shows all G-primes of norm less than 1000, or alternatively, those G-primes that are within Euclidean distance $\sqrt{1000}$ of the origin. Suppose we wish to prove that one cannot walk to infinity using steps of length k or less. We can try, as did Jordan and Rabung, to find a *moat* of composite Gaussian integers, of minimum width k and completely traversing the first octant (the sector $0 \leq \theta \leq \pi/4$). More precisely, we want the moat to cut a swath from the positive x-axis to the line

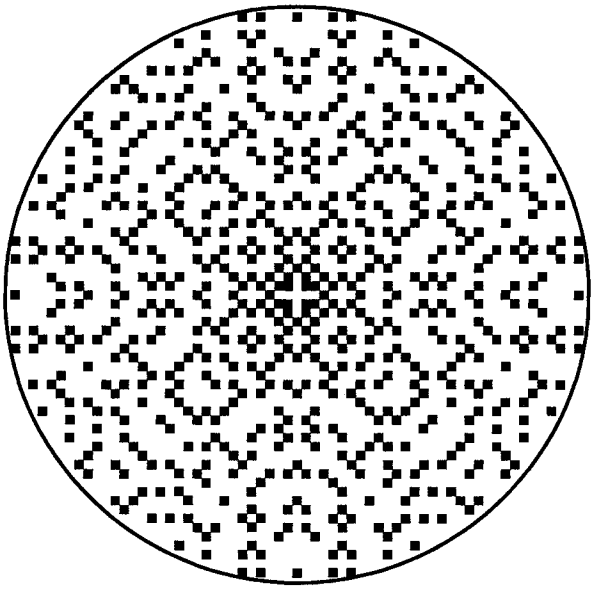


Figure 1. G-primes of norm less than 1000.

$y = x$. The eightfold symmetry allows us to reflect this moat across the $y = x$ line, across the y-axis, across the $y = -x$ line, and so on, until it completely surrounds the origin. Such a moat proves that a trek to infinity requires a step of length greater than k. To prove that one cannot walk to infinity it would therefore suffice to show that moats of width k exist for any k. This has not been done, but we offer evidence in support of the conjecture in the next two sections.

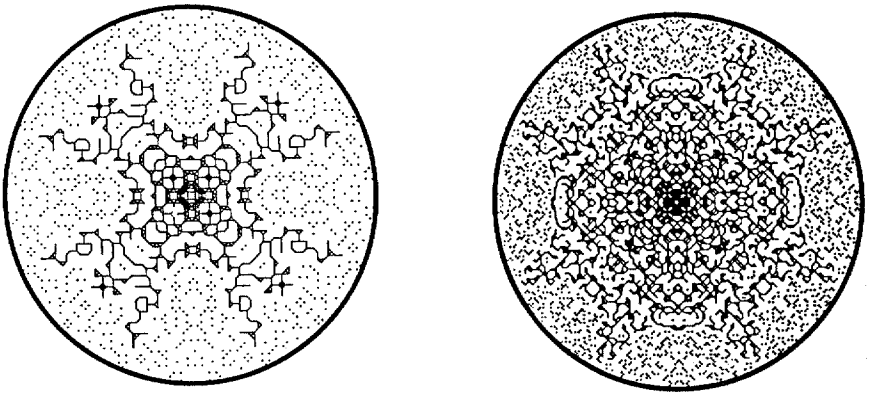
Finally, we mention another unsolved problem [**G**, A8] that has some bearing on the Gaussian moat problem. It has been conjectured that

$$\lim_{n\to\infty} \left(\sqrt{p_{n+1}} - \sqrt{p_n}\right) = 0,$$

where p_n denotes the nth prime. If the conjecture is true, then the circles upon which the G-primes lie become more crowded as one travels farther away from the origin in the complex plane. Thus there would be no chance of finding a truly annular moat (i.e., a moat that is the region between two circles) of composite Gaussian integers.

We next turn our attention to the construction and existence of new Gaussian moats.

3. NEW GAUSSIAN MOATS. Thanks to modern software (*Mathematica*®), we have been able to investigate the distance-k graph in the G-primes to much greater depths than has been done before. We are considering the graph whose vertices are the G-primes, and having edges that join all pairs of G-primes at distance k or less. By the k-component (of the origin) we mean the set of G-primes that are connected to $1 + i$ by a path in the distance-k graph.



(a) (b)

Figure 2. Two small moats: (a) shows the set of G-primes reachable using steps of size at most 2; (b) shows the reachability set in the case of steps of size at most $\sqrt{8}$. The dots show the G-primes that are not in the components. The left image spans the region of norm at most 60 while the right one goes to norm 115. These images, when rendered in color, are quite pretty; contact the second author if you are interested in a color image.

Note that the edges in these graphs can have limited shapes. Ignoring the eightfold symmetry, and also ignoring the $(1, 0)$ step that emanates only from several very small G-primes, the edges have vector representations that can be only (in order of length): $(1, 1)$, $(2, 0)$, $(2, 2)$, $(3, 1)$, $(4, 0)$, $(3, 3)$, $(4, 2)$, $(5, 1)$, $(4, 4)$, $(5, 3)$, and so on. If $a^2 + b^2$ is an odd prime then the number obtained by changing the

parity of one of a or b is necessarily composite, and this explains why the distances under discussion take on the values $\sqrt{2}$, $\sqrt{4}$, $\sqrt{8}$, and so on.

The finiteness of the $\sqrt{10}$-component was proved in 1970 by Jordan and Rabung [**JR**]. Figure 3 contains an image of the entire component (requires only a few minutes of CPU time on a *Macintosh PowerMac*®). The first octant part contains 31,121 primes, so we resort to a compression trick. Each pixel represents a grid (3×3 in this case), and the pixel is darkened if it contains at least one reachable G-prime. One can easily make a color image in which the color used is a function of the number of reachable G-primes in a given grid.

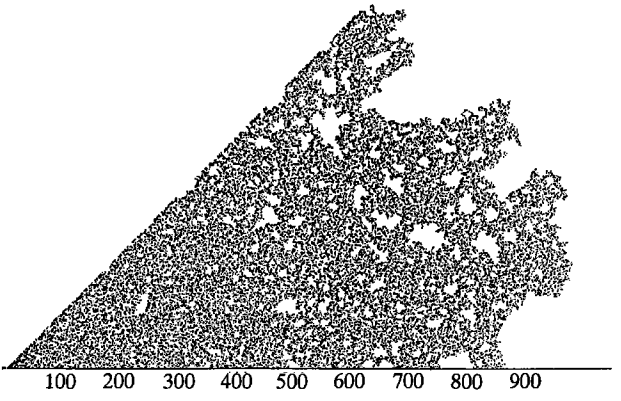


Figure 3. An image of the $\sqrt{10}$-component in the first octant; it contains 31,121 G-primes.

It takes much more computing effort to get the 4-component (347,638 G-primes) or $\sqrt{18}$-component (2,386,129 G-primes). Figures 4 and 5 show these. For more details on the use of *Mathematica* to get these images, see [**W1**, ch. 9] and [**W2**]. It is really quite simple. We proceed level-by-level, where the G-primes of level n are those that can be reached in n steps, but not fewer. To get level $n + 1$ one simply

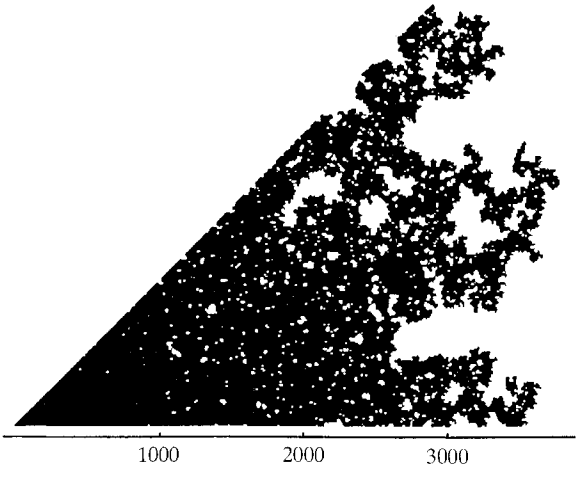


Figure 4. An image of the 4-component in the first octant; it contains 347,638 G-primes. Each pixel represents a 15×15 grid and is blackened if the grid contains a single reachable G-prime.

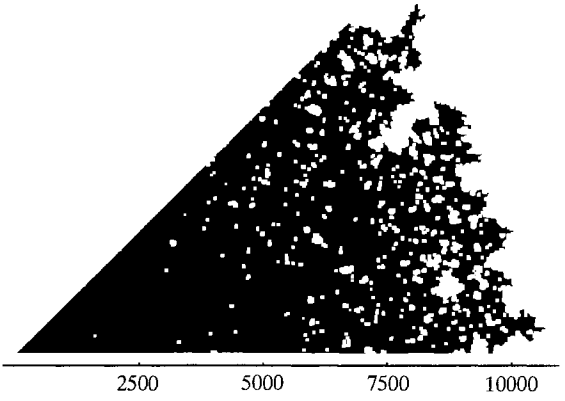


Figure 5. An image of the $\sqrt{18}$-component in the first octant; it contains 2,386,129 G-primes and each pixel represents a 25×25 grid.

looks at the neighbors of the primes in level n that are not in level n or level $n-1$. Thus it suffices to keep only two levels in memory, and that is why we can examine a graph with over 2,000,000 vertices without ever keeping more than several hundred primes in memory at once. Table 1 summarizes the data we obtained in getting these moats.

To deal with the $\sqrt{26}$-component, we adopted a completely different approach. We started at 4,000,000 on the real axis and moved up to $4{,}000{,}000 + 4{,}000{,}000i$ in small sections, showing by brute force that every path that comes into this vertical line from the left yields only dead ends on the right. We chose 4,000,000 after some preliminary investigations at 2,000,000 and 5,000,000; a low starting point will work in theory, but the large distances to the dead ends will slow down the computation too much. A typical part of the computation is shown in Figure 6. We do use some refinements to the naive method and are grateful to Larry Carter for some useful speedup suggestions. Once the 45° line is reached, the finiteness of the $\sqrt{26}$-component is proved; of course this implies that the $\sqrt{20}$-component is finite too. This sort of computation yields only upper bounds on the reachable distance, as opposed to the exact extent of the component of the origin.

TABLE 1. Data for diverse G-prime walks from the origin by walkers of various sizes. The data describe only the first-octant part of the components. For $\sqrt{26}$ we examined 9,631,177 primes and determined that a bound on the reachable distance is 5,586,757. For the $\sqrt{20}$ case we traced around the border of the component until it was completely surrounded.

Component of origin using distance	Farthest point reached	Farthest distance reached	Total size of the component
1	$2 + i$	2.23	2
$\sqrt{2}$	$11 + 4i$	11.70	14
2	$42 + 17i$	45.31	92
$\sqrt{8}$	$84 + 41i$	93.47	380
$\sqrt{10}$	$976 + 311i$	1024.35	31221
4	$3297 + 2780i$	4312.61	347638
$\sqrt{18}$	$8174 + 6981i$	10749.4	2386129
$\sqrt{20}$	$109677 + 64268i$	127120	Finite
$\sqrt{26}$		≤ 5586757	Finite

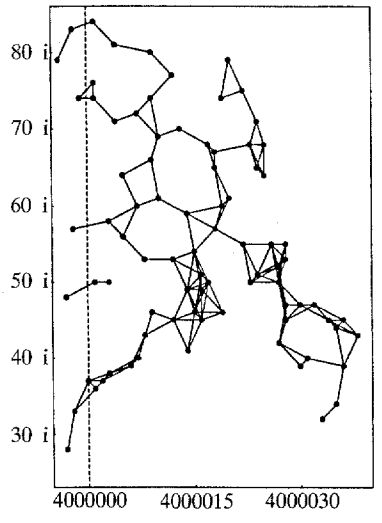


Figure 6. A typical dead-end computation working right from the $x = 4{,}000{,}000$ line. Many similar computations show that all such walks (that start near the line from 4,000,000 to 4,000,000 + 4,000,000i, and using steps of size $\sqrt{26}$ or less) die out fairly quickly. This proves that a $\sqrt{26}$-moat must exist.

Going straight up, as opposed to angling left, might seem inefficient given that we expect the density of primes to be roughly constant on circles. But it has the great advantage of being parallelizable: One computer can investigate the fourth million while the first computer chugs away on the first million. Thus each of us handled at least a million imaginary parts. In all, we examined 9,631,177 G-primes. This approach is also intriguing because one could imagine a much larger computation being done over the world-wide web, with dozens or hundreds of investigators looking at different regions simultaneously. One could surely prove that a $\sqrt{32}$-moat exists in this way, but it hardly seems worth the effort since it appears clear from these computations that moats of any size exist. A much more important step would be a heuristic justification of the conjecture that arbitrarily large moats exist, using reasonable assumptions about the distribution of the primes.

4. STROLLING ON GAUSSIAN LINES. We know that the Gaussian moat problem could be solved by finding moats of arbitrary width in the first octant. Analytic number theorists have raised what appears to be an equally hard question: Can one find *any* sector in which one cannot walk to infinity on G-primes in steps of bounded length? A variation on this is: Is there a line in the complex plane on which there is a G-prime walk to infinity? We prove here that the answer is NO.

Theorem 4.1. *Let L be a line that contains at least two distinct Gaussian integers and let k be a positive integer. There is a Gaussian integer w on this line such that all Gaussian integers within a distance k of w are composite.*

Three preliminary results are needed for the proof of Theorem 4.1, one of which is the following elementary lemma.

Lemma 4.2. *Let L be a line that contains at least two distinct Gaussian integers. There are Gaussian integers $m \neq 0$ and b such that the real and imaginary parts of m*

are relatively prime and the Gaussian integer z is on this line if and only if there is an $x \in \mathbf{Z}$ such that $z = mx + b$.

Proof: Let z_1 and z_2 be two Gaussian integers on the line L. Let $b = z_1$, let $m_0 = z_2 - z_1$, and let d be the greatest common divisor of the real and imaginary parts of m_0. Finally, let $m = m_0/d = u + vi$. It follows that the real and imaginary parts of m are relatively prime, and every point on the line is of the form $mx + b$ where x is a real number.

Let z be any Gaussian integer on the line L. There is a real number x such that $z = mx + b$. We need to show that x is an integer. We know that $mx = ux + vxi = z - b$ is a Gaussian integer; as a consequence, ux and vx are integers. Since u and v are relatively prime, there are integers r and s such that $ur + vs = 1$. Hence $uxr + vxs = x$ is an integer. It is straightforward to see that if x is an integer then $z = mx + b$ is a Gaussian integer. ■

We also need the Chinese remainder theorem (henceforth, CRT) for $\mathbf{Z}$, which we state next; see [**A**, p. 117] for a proof.

Theorem. *Suppose $a_i, m_i \in \mathbf{Z}$ are such that the m_i are relatively prime in pairs. Then there is a solution to the system of linear congruences*

$$X \equiv a_1 \pmod{m_1}, X \equiv a_2 \pmod{m_2}, \ldots, X \equiv a_s \pmod{m_s}.$$

Moreover, if a is a smallest such solution (in magnitude), then all solutions are given by $\{a + Mn : n \in \mathbf{Z}\}$, where M is the product of the moduli. In other words, the solution is unique modulo M.

Finally, we need to know that the CRT is valid for $\mathbf{Z}[i]$, which it is. One way of proving this is to make straightforward modifications to the proof in [**A**, p. 117]. A more general proof (also valid in other rings of integers) is given in [**ZS**, p. 279].

Proof of Theorem 4: Let m and b be two Gaussian integers that define the line L as per Lemma 4.2.

We consider three cases. First, assume that L is horizontal. Then m is real, and so we may take $m = 1$ because its real and imaginary parts are relatively prime. Let b_1 be the imaginary part of b and let $K = |b_1| + k$. Our goal will be to find an integer w_0 with the property that each Gaussian integer z such that $|z - w_0| \leq K$ is composite. From this it quickly follows that if $w = w_0 + b_1 i$, then w is on L, and if z is any Gaussian integer with the property that $|z - w| \leq k$, then z is composite.

Since the set of all Gaussian integers z with the property that $|z| \leq K$ is a finite set, say with N elements, we may index its elements so that $z_j = u_j + v_j i$ with u_j and v_j being integers for $j = 1, \ldots, N$. Assume that $z_1 = 0$.

Inductively define a system of linear congruences $x \equiv a_j \pmod{b_j}$, for $j = 1, \ldots, N$ so that

(1) each a_j and b_j is an integer,
(2) each b_j is larger than 1,
(3) the b_js are pairwise relatively prime, and
(4) $z_j + a_j$ and b_j are not relatively prime.

Start with $a_1 = 0$ and $b_1 = 4$. The four conditions are satisfied. Now suppose that $a_1, \ldots, a_{j-1}, b_1, \ldots, b_{j-1}$ have been defined. Recall that $z_j = u_j + v_j i$. Next, we must find an integer s_j so that $s_j + v_j i$ has a G-prime factor p_j with the

property that $p_j\overline{p_j}$ is larger than the product $b_1 \cdots b_{j-1}$. To accomplish this, let $s_j = v_j M$, where M is the product of all G-primes q such that $|q| \leq b_1 \cdots b_{j-1}$. Then we have $s_j + v_j i = v_j(M + i)$, and the magnitude of any prime factor of $M + i$ is larger than $b_1 \cdots b_{j-1}$, as desired. Choose one such factor p_j. Now let $b_j = p_j\overline{p_j}$ and $a_j = s_j - u_j$. Conditions (1), (2), and (3) are trivially satisfied. Condition (4) follows from the identity $z_j + a_j = s_j + v_j i$. This concludes the induction.

The CRT guarantees an infinite set of solutions to this system of linear congruences. Suppose w is one of these solutions and let $P = \prod b_j$. We may assume that w is larger than $P + K$. Let j be one of the integers $1, \ldots, N$. Since b_j and $z_j + a_j$ are not relatively prime, there is a G-prime q_j that divides them both. Since w is a solution to the system of congruences, it follows that $w + z_j \equiv a_j + z_j \pmod{b_j}$ and thus $w + z_j \equiv 0 \pmod{q_j}$. Since the magnitude of $w + z_j$ is larger than P ($|w + z_j| > |w| - |z_j| \geq (P + K) - K = P$) and since the magnitude of q_j is less than P (q_j is a prime factor of the composite number P), the quotient $(w + z_j)/q_j$ has magnitude larger than 1. Hence, $w + z_j$ is not a G-prime for $j = 1, \ldots, N$.

In the second case, we assume that the real part of m is 0. The argument is nearly identical to that of the first case.

In the final case, assume that both the real and imaginary parts of m are nonzero. As in the first case, let $z_j = u_j + iv_j$, $j = 1, \ldots, N$ index the Gaussian integers z with $|z| < k$. Assume that $z_1 = 0$.

Inductively define a system of linear congruences $x \equiv z_j \pmod{b_j}$ so that

(1) the b_js are distinct G-primes,
(2) $b_j = u_j + v_j i$ where $0 < v_j < u_j$ (i.e., the b_js are in the first octant and are not real)
(3) m and b_j are relatively prime.

Let b_1 be a G-prime that satisfies conditions (2) and (3), and let the first linear congruence be $x \equiv z_1 \pmod{b_1}$. Then the three conditions are satisfied. Suppose $z_1, \ldots, z_{j-1}, b_1, \ldots, b_{j-1}$ have been defined. Find a G-prime b_j with the property that $|b_j|$ is larger than the product $|mb_1 \cdots b_{j-1}|$ and satisfies condition (2). Condition (1) is satisfied because $|b_j| > |b_i|$ for $i = 1, \ldots, j - 1$, and condition (3) is satisfied because $|b_j| > |m|$. This concludes the induction.

The CRT for Gaussian integers guarantees an infinite set of solutions to the system of congruences. Let w_0 be one of these solutions and let $P = \prod b_j$. We claim that the real and imaginary parts of P are relatively prime. To see this, suppose p is a nontrivial prime divisor of the real and imaginary parts of P. Then p divides P. Now, p's G-prime factorization is either just p (if it is $3 \pmod 4$) or $(r + si)(r - si)$ (if p is $1 \pmod 4$). But this factorization must be a subset of the list of b_js, a contradiction since by condition (2) each b_j has positive imaginary part.

Also, P is relatively prime to m by condition (3). As a consequence, there are Gaussian integers r and s such that $mr - Ps = w_0 - b$. Hence $mr + b = Ps + w_0$. Let $r = r_0 + r_1 i$ and $P = P_0 + P_1 i$. Since P_0 and P_1 are relatively prime, there are integers t_0 and t_1 such that $P_0 t_1 + P_1 t_0 = r_1$. Let $t = t_0 + t_1 i$. It follows that for any integer n, $r - tP - nP\overline{P}$ is an integer and therefore that $w_n = P(s - tm - nm\overline{P}) + w_0 = m(r - tP - nP\overline{P}) + b$ is a point on the line L. Choose n so that $|w_n| > |P| + k$ to ensure that $|w_n + z_j| > |P|$ (because $|w_n + z_j| > |w_n| - |z_j| \geq |P| + k - k = |P|$). Note also that $w_n \equiv w_0 \pmod{P}$. Thus $w_n + z_j \equiv w_0 + z_j \pmod{b_j}$ for each $j = 1, \ldots, N$ and hence $(w_n + z_j)/b_j$ is a Gaussian integer. Moreover, the

magnitude of $(w_n + z_j)/b_j$ is larger than 1 since b_j is a divisor of P. It follows that $w_n + z_j$ is composite for each $j = 1, \ldots, N$. This concludes the final case. ■

Finally, we turn to a related problem in G-prime geometry. In **[E2]**, Erdős showed that

$$\limsup[\mathit{min}(d_{n+1}, d_{n+2})/\log n] = \infty,$$

where $d_n = p_{n+1} - p_n$. That is, d_n is the difference between the $(n + 1)$st and nth primes. This theorem was subsequently generalized by H. Maier **[Ma]**. A more simplistic statement of this result is that for any $k > 0$, there is a prime p that lies at the center of a neighborhood of radius k in which all integers except p are composite. These neighborhoods are intervals on the real axis. What about the real G-primes? Can we isolate real G-primes in 2-dimensional neighborhoods as well? The answer turns out to be YES, as we show next.

Definition 4.3. A G-prime q is *k-isolated* if all Gaussian integers in the disk of radius k around q are composite, with the exception of q.

Theorem 4.4. *For any $k > 0$ there is a real k-isolated G-prime.*

One final result is needed for the proof of this theorem, namely Dirichlet's theorem on primes in arithmetic progression, which can be found in **[A**, ch. 7].

Dirichlet's Theorem. *Suppose a and b are relatively prime integers. Then the arithmetic progression $\{a + bn : n \in \mathbf{Z}\}$ contains infinitely many primes.*

Proof of Theorem 4.4: We give only an outline of the proof since it is similar to the proof found in the first case of Theorem 4.1. I. Vardi **[V]** has proved this theorem independently.

We find a solution x_0 to a similar system of linear congruences except that we replace the first congruence with $x \equiv 3 \pmod 4$, and we require that a_i and b_i are relatively prime. Since all solutions to this system of congruences are of the form $x_0 + nP$, and since x_0 and P are relatively prime, we invoke Dirichlet's theorem to find a prime, p, as a solution. This prime is a G-prime because $p \equiv 3 \pmod 4$. ■

Using *Mathematica* to do a search, we found the smallest real k-isolated G-primes for $k = 3, 4, \ldots, 17$, as shown in Table 2. The 14-isolated G-prime, which is also 15-, 16-, and 17-isolated, is shown in Figure 7.

TABLE 2. Smallest real isolated G-primes.

k	Real k-isolated G-prime	Nearest G-prime to the left	Nearest G-prime to the right
3	79	71	83
4	523	503	547
5	563	547	571
6	7559	7547	7583
7	14243	14207	14251
8	35759	35747	35771
9	50023	49999	50047
10	849143	849131	849179
11	959207	959183	959219
12	4100479	4100443	4100527
13	16441543	16441519	16441583
14	20785207	20785187	20785267

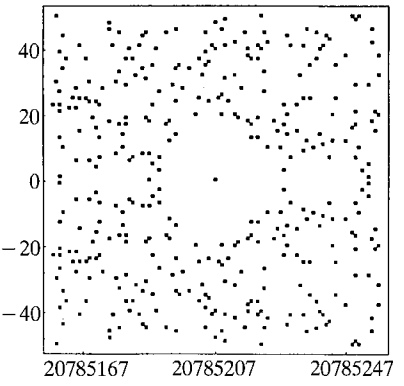


Figure 7. 20785207 is the smallest real 14-, 15-, 16-, and 17-isolated G-prime.

5. THE ROAD AHEAD. Having examined several walks in the Gaussian primes, the reader might be wondering about other venues for such trips. Similar questions about walks to infinity may be asked for the finitely many imaginary quadratic fields of class number 1; see [**H**]. These are the imaginary quadratic fields whose rings of integers are equipped with unique factorization; see [**S**, p. 295] and [**HW**, ch. XV]. Hence, because the geometric structure of such rings in the complex plane is similar to that of $\mathbf{Z}[i]$, most of the techniques of this paper should extend to those rings as well.

In cases where there is no unique factorization, one looks at prime ideals. What are the interesting moat questions for these rings? What answers would one expect? Can computational methods be applied?

While the fundamental question about walks to infinity may prove to be resistant to a rigorous proof, there are numerous other geometrical properties that one can investigate. The Gaussian primes are a visually appealing set, and we imagine that computation and visualization will play an important role in unraveling their mysteries.

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