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A strong convergence theorem for generalized-Φ-strongly monotone maps, with applications

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Abstract

Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. In this paper, a Mann-type iterative algorithm that approximates the zero of a generalized-Φ-strongly monotone map is constructed. A strong convergence theorem for a sequence generated by the algorithm is proved. Furthermore, the theorem is applied to approximate the solution of a convex optimization problem, a Hammerstein integral equation, and a variational inequality problem. This theorem generalizes, improves, and complements some recent results. Finally, examples of generalized-Φ-strongly monotone maps are constructed and numerical experiments which illustrate the convergence of the sequence generated by our algorithm are presented.

MSC: 47H09; 47H05; 47J25; 47J05

Keywords: Generalized-Φ-strongly monotone map; Optimization problem; Hammerstein integral equation; Variational inequality problem; Strong convergence

1 Introduction

Let X be a real Banach space with dual space X^*. Let $A : D(A) \subset X \to X$ be a map, where $D(A)$ denotes the domain of A. The map A is called *accretive* if, for each $u, v \in D(A)$, there exists $j(u - v) \in J(u - v)$ such that

$$\langle Au - Av, j(u - v)\rangle \geq 0,$$

where $J : X \to 2^{X^*}$ is the *normalized duality* map defined, for each $u \in X$, by

$$J(u) = \{u^* \in X^* : \langle u, u^*\rangle = \|u\|\|u^*\|, \|u^*\| = \|u\|\}.$$

The map A is called *strongly accretive* if there exists $k > 0$ such that, for each $u, v \in D(A)$, there exists $j(u - v) \in J(u - v)$ such that

$$\langle Au - Av, j(u - v)\rangle \geq k\|u - v\|^2.$$

The map A is called *strongly-Φ-accretive* if, for each $u,v \in D(A)$, there exist $j(u-v) \in J(u-v)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Au - Av, j(u-v)\rangle \geq \Phi\big(\|u-v\|\big)\|u-v\|.$$

The map A is called *generalized-Φ-strongly accretive* if, for each $u,v \in D(A)$, there exist $j(u-v) \in J(u-v)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Au - Av, j(u-v)\rangle \geq \Phi\big(\|u-v\|\big).$$

The accretive map A is called *m-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$ (see, e.g., Guan and Kartsatos [35], Reich [47], and the references therein). It is known that the class of generalized-Φ-strongly accretive maps properly contains the class of Φ-strongly accretive maps which, in turn, contains the class of strongly accretive maps. In Hilbert spaces, accretive maps are called *monotone maps*. The accretive maps were introduced independently in 1967 by Browder [11] and Kato [38]. Interest in such maps stems mainly from their firm connection with *evolution equations* (see, e.g., Berinde [4], Chidume [17], Reich [48], and the references contained in them). A fundamental problem in the study of accretive maps in Banach spaces is the following:

$$\text{Find } u \in X \text{ such that } Au = 0. \tag{1.1}$$

Several existence theorems have been established for equation (1.1) (see, e.g., Browder [8–11], Martin [40, 41]). It is well known that the class of generalized-Φ-strongly accretive maps is the largest class of accretive-type maps for which, if a solution of equation (1.1) exists, it is always unique. Iterative algorithms for approximating solutions of equation (1.1) have been studied extensively by numerous authors. The first iterative method for approximating solutions of equation (1.1) in real Banach spaces more general than Hilbert spaces, as far as we know, was that by Chidume [16]. He proved that if $X = L_p, p \geq 2$, and $T : K \to K$ is a *Lipschitz strongly pseudo-contractive* map, then the Mann iteration process converges strongly to $u^* \in F(T)$, where K is a nonempty closed convex and bounded subset of X and $F(T) := \{x \in K : Tx = x\}$. This result signalled the return to extensive research on iterative methods for approximating solutions of equation (1.1) in more general Banach spaces. This theorem of Chidume has been generalized in various directions by numerous authors. It has been extended to more general real Banach spaces and more general classes of nonlinear operators. The literature on this abounds, and most of these extensions and their applications can be found in any of the following monographs and journal papers: Berinde [4], Chidume [17], Goebel and Reich [34, 49]. Let $A : D(A) \subset X \to X^*$ be a map, where $D(A)$ denotes the domain of A. The map A is called *monotone* if

$$\langle u - v, Au - Av\rangle \geq 0, \quad \forall u,v \in D(A). \tag{1.2}$$

The map A is called *strongly monotone* if there exists $k > 0$ such that

$$\langle u - v, Au - Av\rangle \geq k\|u-v\|^2, \quad \forall u,v \in D(A). \tag{1.3}$$

The map A is called *Φ-strongly monotone* if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle u - v, Au - Av\rangle \geq \Phi\bigl(\|u - v\|\bigr)\|u - v\|, \quad \forall u, v \in D(A). \tag{1.4}$$

The map A is called *generalized-Φ-strongly monotone* if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle u - v, Au - Av\rangle \geq \Phi\bigl(\|u - v\|\bigr), \quad \forall u, v \in D(A). \tag{1.5}$$

It is easy to see that the class of generalized-Φ-strongly monotone maps contains the class of Φ-strongly monotone maps and the class of strongly monotone maps.

Remark 1 The class of generalized-Φ-strongly monotone maps is the largest class of monotone maps for which, if a solution of equation (1.1) exists, it is always unique.

Interest in monotone maps stems mainly from their usefulness in numerous applications. Consider, for example, the following: Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi-continuous and convex function. The *subdifferential* of f, $\partial f : X \to 2^{X^*}$ is defined, for each $u \in X$, by

$$\partial f(u) = \bigl\{u^* \in X^* : f(v) - f(u) \geq \bigl\langle v - u, u^*\bigr\rangle, \forall v \in X\bigr\}.$$

It is easy to see that the ∂f is a *monotone map* on X and that $0 \in \partial f(u)$ if and only if u is a minimizer of f. Setting $\partial f \equiv A$, then solving the equation $Au = 0$ is equivalent to solving for a minimizer of f. Several existence theorems have been established for the equation $Au = 0$ when the map A is of monotone type (see, e.g., Deimling [32]; Pascali and Shurian [46], and the references contained in them). Iterative methods for approximating solutions of $Au = 0$, where $A : X \to X^*$ is of monotone type, have been studied by various authors. Unfortunately, not much has been achieved. Part of the problem is that in real Banach spaces more general than Hilbert spaces, since the map A maps X to X^*, the recursion formulas containing u_n and Au_n used for accretive-type maps may not be well defined in this setting. Several attempts have been made to overcome this difficulty in the recursion formulas for approximating zeros of monotone-type maps (see, e.g., Chidume *et al.* [26], Kamimura and Takahashi [37], Reich and Sabach [51], Reich [48], Chidume *et al.* [28], and the references contained in them). In 2015, Diop *et al.* [33] studied an iterative scheme of *Mann type* to approximate the zero of a *strongly monotone bounded* map in a 2-uniformly convex real Banach space with a uniformly Gâteaux differentiable norm. They proved the following theorem.

Theorem 1.1 (Diop *et al.* [33]) *Let X be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and X^* be its dual space. Let $A : X \to X^*$ be a bounded and k-strongly monotone map such that $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in X$, let $\{u_n\}$ be the sequence defined iteratively by*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n Au_n), \quad n \geq 1,$$

where J is the normalized duality map on X and $\{\alpha_n\} \subset (0,1)$ is a real sequence satisfying the following conditions: (i) $\sum \alpha_n = \infty$, (ii) $\sum \alpha_n^2 < \infty$. *Then there exists $\gamma_0 > 0$ such that $\alpha_n < \gamma_0$, the sequence $\{u_n\}$ converges strongly to the solution of the equation $Au = 0$.*

It is our purpose in this paper to first prove a *strong convergence theorem* for a generalized-Φ-strongly monotone map using a Mann-type iterative algorithm and without imposing the restriction that the operator be *bounded*. Then, the convergence theorem proved is applied to approximate the solution of a *convex minimization problem*, a *Hammerstein integral equation*, and a *variational inequality problem* over the set of common fixed points of a *finite family* of *quasi-Φ-nonexpansive maps*. Our theorems are improvements of the results of Diop *et al.* [33], Chidume and Bello [20], Chidume [18], Chidume *et al.* [24, 26], and a host of other results in the literature (see Remark 5 below). Finally, we construct examples of generalized-Φ-strongly monotone maps and also give numerical experiments to illustrate the convergence of the sequence generated by our algorithm.

2 Preliminaries

Let X be a smooth real Banach space with dual space X^*. The map $\psi : X \times X \to \mathbb{R}$, defined by $\psi(u,v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2, \forall u,v \in X$, will play a central role in what follows. The map ψ was introduced by Alber [1] and has been studied by Alber [1], Kamimura and Takahashi [37], Reich [51], Chidume [17], Berinde [4], Chidume and Monday [23], and a host of other authors. It is easy to see from the definition of the map ψ that

$$\bigl(\|u\| - \|v\|\bigr)^2 \leq \psi(u,v) \leq \bigl(\|u\| + \|v\|\bigr)^2, \quad \forall u,v \in X. \tag{2.1}$$

Let $V : X \times X^* \to \mathbb{R}$ be a map defined by $V(u,u^*) = \|u\|^2 - 2\langle u,u^*\rangle + \|u^*\|^2, \forall u \in X, u^* \in X$. Observe that $V(u,u^*) = \phi(u, J^{-1}(u^*)), \forall u \in X, u^* \in X^*$. The following lemmas will be needed in the sequel.

Lemma 2.1 (Alber [1]) *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then*

$$V\bigl(u,u^*\bigr) + 2\bigl\langle J^{-1}u^* - u, v^*\bigr\rangle \leq V\bigl(u, u^* + v^*\bigr) \quad \textit{for all } u \in X \textit{ and } u^*, v^* \in X^*.$$

Lemma 2.2 (Chidume [18]) *Let X be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{u \in X : \|u\| \leq r\}$. Then, for arbitrary $u,v \in B_r(0)$, the following inequality holds*:

$$\psi(u,v) \leq \|u - v\|^2 + \|u\|^2.$$

Lemma 2.3 (Tan and Xu [55]) *Let $\{a_n\}$ and $\{\sigma\}$ be sequences of nonnegative real numbers. For some $N_o \in \mathbb{N}$, the following relation holds*:

$$a_{n+1} \leq a_n + \sigma_n, \quad n \geq 0.$$

(a) *If $\sum \sigma_n < \infty$, then* $\lim a_n$ *exists.* (b) *If, in addition, the sequence $\{a_n\}$ has a subsequence that converges to* 0, *then $\{a_n\}$ converges to* 0.

Lemma 2.4 (Kamimura and Takahashi [37]) *Let X be a uniformly convex and uniformly smooth real Banach space and $\{u_n\},\{v_n\}$ be sequences in X such that either $\{u_n\} or \{v_n\}$ is bounded. If $\lim_{n\to\infty} \psi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Remark 2 It is easy to see that the converse of Lemma 2.4 is also true whenever $\{u_n\} and \{v_n\}$ are bounded.

Lemma 2.5 (Alber and Ryazantseva [2]) *Let X be a uniformly convex Banach space with dual space X^*. Then, for any $R > 0$ and for any $u, v \in X^*$ such that $\|u\| \le R$, $\|v\| \le R$, the following inequality holds*:

$$\|J^{-1}u - J^{-1}v\| \le c_2 \delta_X^{-1}\big(4RL\|u - v\|\big),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.6 (Alber and Ryazantseva [2]) *Let X be a uniformly convex Banach space with dual space X^*. Then, for any $R > 0$ and for any $u, v \in X$ such that $\|u\| \le R$, $\|v\| \le R$, the following inequality holds*:

$$\|Ju - Jv\| \le c_2 \delta_{X^*}^{-1}\big(4RL\|u - v\|\big),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.7 (Rockafellar [52], see also Pascali and Sburlin [46]) *A monotone map $A : X \to X^*$ is locally bounded at the interior points of its domain.*

Definition 2.8 A map $A : X \to X^*$ is *quasi-bounded* if, for every $\mu > 0$, there exists $\gamma > 0$ such that whenever $\langle v, Av\rangle \le \mu\|v\|$ and $\|v\| \le \mu$, then $\|Av\| \le \gamma$.

The following lemma has been proved. However, for completeness, we present the proof here (see, e.g., Pascali and Sburlan [46], chapter III, Lemma 3.6).

Lemma 2.9 *Let X be a real normed space with dual space X^*. Every monotone map $A : D(A) \subset X \to X^*$ with $0 \in \operatorname{Int} D(A)$ is quasi-bounded.*

Proof By Lemma 2.7, A is locally bounded at 0, i.e., there exists $r > 0$ such that

$$\|Au\| \le \mu, \quad \forall u \in B_r(0), \text{for some } \mu > 0.$$

Now, using this $\mu > 0$, suppose $\langle v, Av\rangle \le \mu\|v\|$ and $\|v\| \le \mu$. Then, by the monotonicity of A, we have that

$$\langle v, Av\rangle \ge \langle u, Av\rangle + \langle v - u, Au\rangle, \quad \forall u \in B_r(0).$$

Observe that

$$\langle v - u, Au\rangle \le \|Au\|\big(\|v\| + \|u\|\big) \le \mu\big(\|v\| + r\big).$$

Thus,

$$\langle u, Av\rangle \leq \langle v, Av\rangle + \langle u - v, Au\rangle$$
$$\leq \mu\|v\| + \mu\big(\|v\| + r\big) = \mu\big(2\|v\| + r\big), \quad \forall u \in B_r(0).$$

This implies that

$$\big|\langle u, Av\rangle\big| \leq \mu\big(2\|v\| + r\big), \quad \forall u \in B_r(0).$$

Thus,

$$\sup_{\|u\|\leq r}\big|\langle u, Av\rangle\big| \leq \mu\big(2\|v\| + r\big).$$

Therefore,

$$\|Av\| \leq \frac{\mu}{r}\big(2\|v\| + r\big).$$

Hence, A is quasi-bounded. □

3 Main result

In Theorem 3.1 below, the sequence $\{\beta_n\} \subset (0,1)$ is assumed to satisfy the following conditions: (C_1) $\sum \beta_n = \infty$, $\lim \beta_n = 0$; (C_2) $2\sum \delta_X^{-1}(\beta_n M)M < \infty$; (C_3) $2\delta_X^{-1}(\beta_n M) \leq \gamma_0$ for some $M > 0$, $\gamma_0 > 0$, where δ_X is the modulus of convexity (*see, e.g., Chidume* [17], *pp. 5, 6*).

Theorem 3.1 *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $A : D(A) = X \to X^*$ be a generalized-Φ-strongly monotone map, where $D(A)$ is the domain of A and $A^{-1}(0) \neq \emptyset$. For arbitrary $v_1 \in X$, let $\{v_n\}$ be a sequence generated iteratively by*

$$v_{n+1} = J^{-1}(Jv_n - \beta_n Av_n), \quad n \geq 1, \tag{3.1}$$

where J is the normalized duality map on X, and the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2, and C_3. Then the sequence $\{v_n\}$ converges strongly to $v^* \in A^{-1}(0)$.

Proof First, we observe that if the equation $Au = 0$ has a solution, it is necessarily unique. If y^* is a solution of the equation $Au = 0$, then, from inequality (1.5), we have that

$$\big\langle x - y^*, Ax\big\rangle \geq \Phi\big(\big\|x - y^*\big\|\big), \quad \forall x \in X. \tag{3.2}$$

Suppose that $u^* \neq y^*$ is another solution of the equation $Au = 0$, substituting u^* in inequality (3.2), we have

$$0 \geq \Phi\big(\big\|u^* - y^*\big\|\big),$$

which implies, by the properties of Φ, that $u^* = y^*$. This contradiction yields the uniqueness of the solution. The remainder of the proof is now in two steps.

Step 1. We show that the sequence $\{v_n\}$ is bounded. Let $v^* \in A^{-1}(0)$. Let $\mu > 0$ be arbitrary but fixed. Then there exists $r > 0$ such that

$$r > \max\{4\mu^2 + \|v^*\|^2, \psi(v^*, v_1)\}. \tag{3.3}$$

Define $B := \{v \in X : \psi(v^*, v) \leq r\}$. It suffices to show that $\{\psi(v^*, v_n)\}$ is bounded for each $n \in \mathbb{N}$. We proceed by induction. For $n = 1$, by construction, we have that $\psi(v^*, v_1) \leq r$. Assume that $\psi(v^*, v_n) \leq r$ for some $n \geq 1$. Using inequality (2.1), we have that $\|v_n\| \leq \|v^*\| + \sqrt{r}$. Now, we show that $\psi(v^*, v_{n+1}) \leq r$. Suppose by contradiction that $\psi(v^*, v_{n+1}) \leq r$ does not hold. Then $\psi(v^*, v_{n+1}) > r$. Since $A : X \to X^*$ is locally bounded at $v \in X$, there exist $r_v > 0$ and $m > 0$ such that

$$\|Ax\| \leq m, \quad \forall x \in B_{r_v}(v).$$

$$\text{In particular,} \quad \|Av\| \leq m.$$

$$\text{Therefore,} \quad \langle v, Av \rangle \leq m\|v\|.$$

Define $M_0 := \max\{m, \|v^*\| + \sqrt{r}\}$. Then $\langle v, Av \rangle \leq M_0\|v\|$ and $\|v\| \leq M_0$. By Lemma 2.9, there exists $M > 0$ such that $\|Av\| \leq M, \forall v \in B$. Define $\gamma_0 := \min\{1, \frac{\Phi(\mu)}{M}, \frac{\mu}{M}\}$. Using Lemma 2.1, we compute as follows:

$$\begin{aligned}
\psi(v^*, v_{n+1}) &= V(v^*, Jv_n - \beta_n Av_n) \\
&\leq V(v^*, Jv_n) - 2\beta_n \langle J^{-1}(Jv_n - \beta_n Av_n) - v^*, Av_n - Av^* \rangle \\
&= \psi(v^*, v_n) - 2\beta_n \langle v_n - v^*, Av_n - Av^* \rangle - 2\beta_n \langle v_{n+1} - v_n, Av_n \rangle.
\end{aligned} \tag{3.4}$$

Using the fact that A is a generalized-Φ-strongly monotone map and Lemma 2.5, it follows from inequality (3.4) that

$$\begin{aligned}
\psi(v^*, v_{n+1}) &\leq \psi(v^*, v_n) - 2\beta_n \Phi(\|v_n - v^*\|) + 2\beta_n \delta_X^{-1}(4RL\beta_n\|Av_n\|)\|Av_n\| \\
&\leq \psi(v^*, v_n) - 2\beta_n \Phi(\|v_n - v^*\|) + 2\beta_n \delta_X^{-1}(\beta_n M)M.
\end{aligned} \tag{3.5}$$

But from recursion formula (3.1), we have that

$$\|Jv_{n+1} - Jv_n\| = \beta_n\|Av_n\| \leq \beta_n M. \tag{3.6}$$

Applying Lemma 2.5 and inequality (3.6), we have that

$$\|v_{n+1} - v_n\| = \|J^{-1}(Jv_{n+1}) - J^{-1}(Jv_n)\| \leq 2\delta_X^{-1}(\beta_n M). \tag{3.7}$$

Thus, from inequality (3.7), we obtain that

$$\|v_n - v^*\| \geq \|v_{n+1} - v^*\| - 2\delta_X^{-1}(\beta_n M). \tag{3.8}$$

From Lemma 2.2, we have that

$$r < \psi(v^*, v_{n+1}) \leq \|v_{n+1} - v^*\|^2 + \|v^*\|^2. \tag{3.9}$$

Using inequality (3.3), we have that

$$4\mu^2 + \|v^*\|^2 - \|v^*\|^2 < r - \|v^*\|^2 \le \|v_{n+1} - v^*\|^2.$$

Hence,

$$2\mu \le \|v_{n+1} - v^*\|. \tag{3.10}$$

From inequalities (3.7), (3.8), and the definition of γ_0, we have that

$$\|v_n - v^*\| \ge 2\mu - 2\delta_X^{-1}(\beta_n M) \ge 2\mu - \mu = \mu. \tag{3.11}$$

Since Φ is strictly increasing, we have that

$$\Phi(\|v_n - v^*\|) \ge \Phi(\mu). \tag{3.12}$$

From inequality (3.5) and the definition of γ_0, we have that

$$r < \psi(v^*, v_{n+1}) \le \psi(v^*, v_n) - 2\beta_n\Phi(\mu) + 2\beta_n\delta_X^{-1}(\beta_n M)M \tag{3.13}$$

$$\le r - 2\beta_n\Phi(\mu) + \beta_n\Phi(\mu) < r. \tag{3.14}$$

This is a contradiction. Hence, $\{\psi(v^*, v_n)\}$ is bounded. Consequently, $\{v_n\}$ is bounded.

Step 2. We show that the sequence $\{v_n\}$ converges strongly to a point $v^* \in A^{-1}(0)$. Using inequality (3.5), we have that

$$\begin{aligned}\psi(v^*, v_{n+1}) &\le \psi(v^*, v_n) - 2\beta_n\Phi(\|v_n - v^*\|) + 2\beta_n\delta_X^{-1}(\beta_n M)M \\ &\le \psi(v^*, v_n) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \end{aligned} \tag{3.15}$$

By Lemma 2.3, we get that $\{\psi(v^*, v_n)\}$ is convergent. Furthermore, we have that

$$2\beta_n\Phi(\|v_n - v^*\|) \le \psi(v^*, v_n) - \psi(v^*, v_{n+1}) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \tag{3.16}$$

Claim. $\liminf \Phi(\|v_n - v^*\|) = 0$.

Suppose by contradiction that $\liminf \Phi(\|v_n - v^*\|) = 0$ does not hold. Then $\liminf \Phi(\|v_n - v^*\|) = s > 0$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$\Phi(\|v_n - v^*\|) > \frac{s}{2} \quad \text{for all } n \ge N_1. \tag{3.17}$$

Using inequality (3.17), conditions C_1 and C_2, we have that

$$s\sum_{n=1}^{\infty}\beta_n \le \sum_{n=1}^{\infty}\big(\psi(v^*, v_n) - \psi(v^*, v_{n+1})\big) + 2\sum_{n=1}^{\infty}\delta_X^{-1}(\beta_n M)M < \infty. \tag{3.18}$$

This is a contradiction. Hence, $\liminf \Phi(\|v_n - v^*\|) = 0$. Thus, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\lim_{k\to\infty}\Phi(\|v_{n_k} - v^*\|) = 0. \tag{3.19}$$

Using the property of Φ, it follows that $\lim_{k\to\infty}\|v_{n_k}-v^*\|=0$. By Remark 2, we have that

$$\lim_{k\to\infty}\psi\left(v^*,v_{n_k}\right)=0. \tag{3.20}$$

Consequently, by Lemma 2.3, we have that $\lim_{n\to\infty}\psi(v^*,v_n)=0$. Hence, by Lemma 2.4, we have that $\lim_{n\to\infty}\|v_n-v^*\|=0$.

This completes the proof. □

4 Application to convex optimization problem

In this section, we apply Theorem 3.1 in solving the problem of finding minimizers of convex functions defined on real Banach spaces. First, we begin with the following known results.

Lemma 4.1 (See, e.g., Diop *et al.* [33]) *Let X be a real Banach space and $g: X\to\mathbb{R}$ be a convex and differentiable function. Let $dg: X\to X^*$ denote the differential map associated with g. Then $v\in X$ is a minimizer of g if and only if $dg(v)=0$.*

Lemma 4.2 (Xu [56], see also Chidume [17], p. 43) *Let X be a uniformly convex real Banach space. For arbitrary $r>0$, let $B_r(0):=\{v\in X:\|v\|\le r\}$. Then there exists a continuous strictly increasing convex function $\Phi:[0,\infty)\to[0,\infty)$, $\Phi(0)=0$ such that, for every $u,v\in B_r(0)$, the following inequality holds*:

$$\langle u-v,Ju-Jv\rangle\ge\Phi\left(\|u-v\|\right),$$

where J is the single-valued normalized duality map on X.

Lemma 4.3 (Chidume *et al.* [26]) *Let X be a uniformly convex and uniformly smooth real Banach space. Let $g: X\to\mathbb{R}$ be a differentiable convex function. Then the differential map $dg: X\to X^*$ satisfies the following inequality*:

$$\left\langle u-v,dg(u)-dg(v)\right\rangle\ge\langle u-v,Ju-Jv\rangle,\quad\forall u,v\in X,$$

where J is the single-valued normalized duality map on X.

Remark 3 If for any $R>0$ and for any $u,v\in X$ such that $\|u\|\le R$, $\|v\|\le R$, then the map $dg: X\to X^*$ is generalized-Φ-strongly monotone. This can easily be seen from Lemmas 4.2 and 4.3.

We now prove the following theorem.

Theorem 4.4 *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $g: X\to\mathbb{R}\cup\{\infty\}$ be a differentiable, convex, proper, and coercive function such that $(dg)^{-1}(0)\neq\emptyset$. For arbitrary $v_1\in X$, let the sequence $\{v_n\}$ be generated by*

$$v_{n+1}=J^{-1}\left(Jv_n-\beta_n\,dg(v_n)\right),\quad n\ge1,$$

where J is the normalized duality map on X. Assume that $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2, and C_3 of Theorem 3.1. Then g has a unique minimizer $v^* \in X$ and the sequence $\{v_n\}$ converges strongly to v^*.

Proof Since g is a lower semi-continuous, convex, proper, and coercive function, then g has a minimizer $v^* \in X$. Furthermore, $dg : X \to X^*$ is generalized-Φ-strongly monotone. Hence, the conclusion follows from Theorem 3.1. □

5 Application to Hammerstein integral equation

Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation of Hammerstein type has the form

$$u(x) + \int_\Omega k(x,y)f\bigl(y,u(y)\bigr)\,dy = w(x), \tag{5.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space X of measurable real-valued functions. Define a linear map K by

$$Kv(x) = \int_\Omega k(x,y)v(y)\,dy \tag{5.2}$$

on Ω and denote by F the superposition or *Nemitskyi* operator corresponding to f, i.e., $Fu(y) = f(y,u(y))$. Then equation (5.1) can be put in the form

$$u + KFu = 0, \tag{5.3}$$

where, without loss of generality, we have taken $w \equiv 0$. Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule, be put in the form (5.1) (see, e.g., Pascali and Sburian [46], chapter p. 164). Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., Brezis and Browder [5, 6], Chepanovich [15], Browder and Gupta [12], De Figueiredo and Gupta [31], and the references contained in them). In general, equations of Hammerstein type are nonlinear and there is no known method to find closed form solutions for them. Consequently, methods for approximating solutions of such equations are of interest. For earlier and more recent works on approximation of solutions of equations of Hammerstein type, the reader may consult any of the following: Brezis and Browder [5, 6], Chidume and Shehu [27], Chidume and Ofoedu [25], Chidume and Zegeye [29], Chidume and Djitte [22], Ofoedu and Onyi [45], Ofoedu and Malonza [44], Zegeye and Malonza [58], Chidume and Bello [20], Minjibir and Mohammed [42], and the references contained in them. We now apply Theorem 3.1 to approximate a solution of equation (5.3). The following lemma would be needed in the proof of Theorem 5.2 below.

Lemma 5.1 *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^* and $E = X \times X^*$. Let $F : X \to X^*$ and $K : X^* \to X$ be generalized-Φ_1-strongly monotone and generalized-Φ_2-strongly monotone maps, respectively. Let $A : E \to E^*$ be defined by $A([u,v]) = [Fu - v, Kv + u]$. Then A is a generalized-Φ-strongly monotone map.*

Proof Let $[u_1, v_1], [u_2, v_2] \in E$. Then

$$\begin{aligned}
&\big\langle [u_1, v_1] - [u_2, v_2], A\big([u_1, v_1]\big) - A\big([u_2, v_2]\big)\big\rangle \\
&\quad = \big\langle [u_1 - u_2, v_1 - v_2], [Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2]\big\rangle \\
&\quad = \langle u_1 - u_2, Fu_1 - Fu_2\rangle + \langle v_1 - v_2, Kv_1 - Kv_2\rangle \\
&\quad \geq \Phi_1\big(\|u_1 - u_2\|\big) + \Phi_2\big(\|v_1 - v_2\|\big).
\end{aligned}$$

□

Remark 4 For A defined in Lemma 5.1, $[u^*, v^*]$ is a zero of A if and only if u^* solves (5.3), where $v^* = Fu$.

In Theorem 5.2 below, the sequence $\{\beta_n\} \subset (0,1)$ is assumed to satisfy the following conditions:

(C_1) $\sum \beta_n = \infty$; $\lim \beta_n = 0$.
(C_2) $2\sum(\delta_X^{-1}(\beta_n M_1)M_1 + \delta_X^{-1}(\beta_n M_2)M_2) < \infty$.
(C_3) $2\max\{\delta_X^{-1}(\beta_n M_1)M_1, \delta_{X^*}^{-1}(\beta_n M_2)M_2\} \leq \gamma_0$ for some $M_1 > 0$, M_2, $\gamma_0 > 0$.
(C_4) $\gamma_0 = \min\{1, \frac{\Phi(\mu)}{2M_1}, \frac{\Phi(\mu)}{2M_2}\}$, δ_X is the modulus of convexity (*see, e.g., Chidume* [17], *pp. 5, 6*). We now prove the following theorem.

Theorem 5.2 *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : D(F) = X \to X^*$ and $K : D(K) = X^* \to X$ be generalized-Φ_1-strongly monotone and generalized-Φ_2-strongly monotone maps, respectively, where $D(F)$ and $D(K)$ denote the domains of F and K, respectively, and such that equation* (5.3) *has a solution. For arbitrary $(u_1, v_1) \in X \times X^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ by*

$$u_{n+1} = J^{-1}\big(Ju_n - \beta_n(Fu_n - v_n)\big), \quad n \geq 1; \qquad v_{n+1} = J_*^{-1}\big(J_* v_n - \beta_n(Kv_n + u_n)\big), \quad n \geq 1.$$

Assume that the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2, and C_3 of Theorem 3.1. *Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of the equations $u + KFu = 0$ and $v^* = Fu^*$.*

Proof Set $E = X \times X^*$ and $A : E \to E^*$ by $A([u, v]) = [Fu - v, Kv + u]$. Then by Lemma 5.1, A is a generalized-Φ-strongly monotone map. Hence, by Theorem 3.1 and Remark 4, the result is immediate. □

6 Application to variational inequality problems

Let X be a real normed space with dual space X^*. Let $A : C \subset X \to X^*$ be a nonlinear map. The *classical variational inequality problem* is the following:

$$\text{find } u \in C \text{ such that } \langle u - v, Au\rangle \geq 0, \forall v \in C. \qquad (6.1)$$

The set of solutions of problem (6.1) is denoted by $VI(A, C)$. Variational inequality problems were first introduced and studied by Stampacchia [54] in 1964 and have been found to have numerous applications in the study of nonlinear analysis (see, e.g., Shi [53], Noor [43], Yao [57], Stampacchia [54], and the references contained in them). Several existence results for problem (6.1) have been proved when A is a monotone-type map defined on certain Banach spaces (see, e.g., Hartman and Stampacchia [36], Browder [7], Barbu and Precupanu [3], and the references contained in them). Iterative approximation of solutions of problem (6.1), assuming existence, has been studied extensively. For earlier and recent works on variational inequality problems, the reader may consult any of the following: Stampacchia [54], Korpelevich [39], Censor *et al.* [13], Chidume *et al.* [19, 21], and the references contained in them. We now prove the following theorem.

Theorem 6.1 *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*, and let C be a nonempty closed and convex subset of X. Let $A : D(A) = X \to X^*$ be a generalized-Φ-strongly monotone map, where $D(A)$ is the domain of A. Let $T_i : C \to X$, $i = 1, 2, \dots, N$, be a finite family of quasi-ϕ-nonexpansive maps such that $P := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. For arbitrary $v_1 \in X$, define the sequence $\{v_n\}$ generated by*

$$v_{n+1} = J^{-1}\big(J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)\big), \quad n \geq 1, \text{ where } T_{[n]} := T_n \mod N. \tag{6.2}$$

Assume that $VI(A,P) \neq \emptyset$, and the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2, and C_3 of Theorem 3.1. Then the sequence $\{v_n\}$ converges strongly to $v^* \in VI(A,P)$.

Proof The proof is in two steps.

Step 1. We show that the sequence $\{v_n\}$ is bounded.

Let $v^* \in G^{-1}(0)$. Let $\mu > 0$ be arbitrary but fixed. Then there exists $r > 0$ such that

$$r > \max\big\{4\mu^2 + \|v^*\|^2, \psi\big(v^*, v_1\big)\big\}. \tag{6.3}$$

Define $B = \{v \in X : \psi(v^*, v) \leq r\}$. It suffices to show that $\{\psi(v^*, v_n)\}$ is bounded for each $n \in \mathbb{N}$. We proceed by induction. For $n = 1$, by construction, $\psi(v^*, v_1) \leq r$. Assume that $\psi(v^*, v_n) \leq r$ for some $n \geq 1$. Applying the definition of the map ψ, we have that $\|v_n\| \leq \|v^*\| + \sqrt{r}$. Now, we show that $\psi(v^*, v_{n+1}) \leq r$. Suppose not, i.e., suppose $\psi(v^*, v_{n+1}) > r$. By Lemma 2.9, A is quasi-bounded. Thus, there exists $M > 0$ such that $\|Av\| \leq M, \forall v \in B$. Define $\gamma_0 := \min\{1, \frac{\Phi(\mu)}{M}, \frac{\mu}{M}\}$. Using Lemma 2.1, we compute as follows:

$$\begin{aligned}
\psi\big(v^*, v_{n+1}\big) &= V\big(v^*, J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)\big) \\
&\leq V\big(v^*, J(T_{[n]}v_n)\big) - 2\beta_n\big\langle J^{-1}\big(J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)\big) - v^*, A(T_{[n]}v_n)\big\rangle \\
&= \psi\big(v^*, T_{[n]}v_n\big) - 2\beta_n\big\langle T_{[n]}v_n - v^*, AT_{[n]}v_n\big\rangle - 2\beta_n\langle v_{n+1} - T_{[n]}v_n, AT_{[n]}v_n\rangle \\
&\leq \psi\big(v^*, v_n\big) - 2\beta_n\big\langle T_{[n]}v_n - v^*, AT_{[n]}v_n - Av^*\big\rangle - 2\beta_n\big\langle T_{[n]}v_n - v^*, Av^*\big\rangle \\
&\quad - 2\beta_n\big\langle v_{n+1} - T_{[n]}v_n, A(T_{[n]}v_n)\big\rangle \\
&\leq \psi\big(v^*, v_n\big) - 2\beta_n\big\langle T_{[n]}v_n - v^*, AT_{[n]}v_n - Av^*\big\rangle \\
&\quad - 2\beta_n\big\langle v_{n+1} - T_{[n]}v_n, A(T_{[n]}v_n)\big\rangle.
\end{aligned} \tag{6.4}$$

Using the fact that A is a generalized-Φ-strongly monotone map and Lemma 2.5, it follows from inequality (6.4) that

$$\begin{aligned}
\psi\big(v^*, v_{n+1}\big) &\leq \psi\big(v^*, v_n\big) - 2\beta_n\Phi\big(\big\|T_{[n]}v_n - v^*\big\|\big) + 2\beta_n\delta_X^{-1}\big(4RL\beta_n\|AT_{[n]}v_n\|\big)\|AT_{[n]}v_n\| \\
&\leq \psi\big(v^*, v_n\big) - 2\beta_n\Phi\big(\big\|v_n - v^*\big\|\big) + 2\beta_n\delta_X^{-1}(\beta_n M)M.
\end{aligned} \tag{6.5}$$

But from recursion formula (6.2), we have that

$$\|Jv_{n+1} - JT_{[n]}v_n\| = \beta_n \|Av_n\| \leq \beta_n M. \tag{6.6}$$

Applying Lemma 2.5 and inequality (6.6), we have that

$$\|v_{n+1} - T_{[n]}v_n\| = \left\|J^{-1}(Jv_{n+1}) - J^{-1}(JT_{[n]}v_n)\right\| \leq 2\delta_X^{-1}(\beta_n M). \tag{6.7}$$

Thus, from inequality (6.7), we obtain that

$$\left\|T_{[n]}v_n - v^*\right\| \geq \left\|v_{n+1} - v^*\right\| - 2\delta_X^{-1}(\beta_n M). \tag{6.8}$$

From Lemma 2.2, we have that

$$r < \psi\left(v^*, v_{n+1}\right) \leq \left\|v_{n+1} - v^*\right\|^2 + \left\|v^*\right\|^2. \tag{6.9}$$

Using inequality (6.3), we have that

$$4\mu^2 + \left\|v^*\right\|^2 - \left\|v^*\right\|^2 < r - \left\|v^*\right\|^2 \leq \left\|v_{n+1} - v^*\right\|^2.$$

Hence,

$$2\mu \leq \left\|v_{n+1} - v^*\right\|. \tag{6.10}$$

From inequalities (6.8), (6.10), and the definition of γ_0, we have that

$$\left\|T_{[n]}v_n - v^*\right\| \geq 2\mu - 2\delta_X^{-1}(\beta_n M) \geq 2\mu - \mu = \mu. \tag{6.11}$$

Since Φ is strictly increasing, we have that

$$\Phi\left(\left\|T_{[n]}v_n - v^*\right\|\right) \geq \Phi(\mu). \tag{6.12}$$

From inequality (6.5) and the definition of γ_0, we have that

$$r < \psi\left(v^*, v_{n+1}\right) \leq \psi\left(v^*, v_n\right) - 2\beta_n\Phi(\mu) + 2\beta_n\delta_X^{-1}(\beta_n M)M \tag{6.13}$$

$$\leq r - 2\beta_n\Phi(\mu) + \beta_n\Phi(\mu) < r. \tag{6.14}$$

This is a contradiction. Hence, $\{\psi(v^*, v_n)\}$ is bounded. Consequently, $\{v_n\}$ is bounded. The remaining part of the proof follows from the proof of Theorem 3.1. □

7 Examples

Example 1 Let $X = l_p$, $1 < p < 2$, and let $A : l_p \to l_p^*$ be a map defined by

$$Au = Ju, \quad \forall u \in l_p, u = (u_1, u_2, u_3, \ldots),$$

where J is the normalized duality map on X. Then

$$\begin{aligned}\langle u-v, Au-Av\rangle &= \langle u-v, Ju-Jv\rangle \\ &\geq (p-1)\|u-v\|^2, \quad \forall u,v \in X.\end{aligned}$$

Hence, A is generalized-Φ-strongly monotone map with $\Phi(t) = (p-1)t^2$ (see, e.g., Chidume [17], p. 55).

Example 2 Let $X = l_p, 2 \leq p < \infty$, and let $A : l_p \to l_p^*$ be a map defined by

$$Au = \frac{1}{2}J_p u, \quad \forall u \in l_p, u = (u_1, u_2, u_3, \ldots).$$

Then

$$\begin{aligned}\langle u-v, Au-Av\rangle &= \langle u-v, J_pu-J_pv\rangle \\ &\geq p^{-1}c_p\|u-v\|^p, \quad \forall u,v \in X, c_p > 0.\end{aligned}$$

Hence, A is a generalized-Φ-strongly monotone map with $\Phi(t) = p^{-1}c_pt^p$ (see, e.g., Chidume [17], p. 54).

8 Numerical illustration

In this section, we present numerical examples to illustrate the convergence of the sequence generated by our algorithm.

Example 3 In Theorem 3.1, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Av = \begin{pmatrix} 5 & -5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then it is easy to see that A is a generalized-Φ-strongly monotone map and the vector $v^* = (0,0)$ is the unique solution of the equation $Av = 0$. Take $\beta_n = \frac{1}{n+1}, n = 1, 2, \ldots$, as our parameter in Theorem 3.1. With this, we now give the following algorithm which is a specialized version of Theorem 3.1.

Algorithm.

Step 0: Choose any $v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set the maximum number of iterations, n.

Step 1: If $\|v_k\| \leq \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_n = \frac{1}{k+1}$.

Step 2: Compute

$$v_{k+1} = v_k - \beta_k Av_k.$$

Step 3: Set $k = k + 1$ and go to Step 1.

Table 1 gives our test results using 10^{-6} tolerance.

The numerical result for the initial point $(1, \frac{1}{2})$ is sketched below where the y-axis represents the values of $\|v_{n+1} - 0\|$ while the x-axis represents the number of iterations n (see Fig. 1).

Table 1 Numerical illustration for the zero of a generalized-ϕ-strongly monotone map

Initial points	Num. of iter	Approx. solution
$(1,0)$	88	9.6598×10^{-7}
$(0,1)$	95	9.3690×10^{-7}
$(2,1)$	103	9.9756×10^{-7}
$(1,4)$	120	9.5080×10^{-7}
$(\frac{1}{2},\frac{1}{2})$	86	9.3020×10^{-7}
$(1,\frac{1}{2})$	92	9.6662×10^{-7}

Figure 1 Convergence of the sequence $\{v_n\}$ with initial point $(1,\frac{1}{2})$

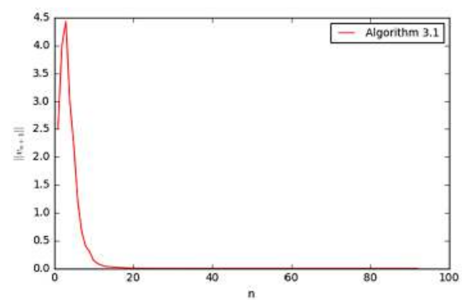


Table 2 Numerical illustration for the solution of Hammerstein integral equation

Initial points	Num. of iter	Approx. sol. ($\|u_{n+1}\|$)
$(1,0)$, $(0,1)$	45	9.7064×10^{-7}
$(1,1)$, $(2,3)$	49	9.4440×10^{-7}
$(2,3)$, $(1,1)$	49	9.9188×10^{-7}
$(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$	36	9.6055×10^{-7}
$(\frac{1}{2},1)$, $(\frac{1}{2},2)$	38	9.4539×10^{-7}
$(3,5)$, $(2,1)$	55	9.7373×10^{-7}

Example 4 In Theorem 5.2, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Fu = \begin{pmatrix} 3 & -1 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad Kv = \begin{pmatrix} 7 & 2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then it is easy to see that F and K are generalized-Φ-strongly monotone maps and the vector $u^* = (0,0)$ is the unique solution of the equation $u + KFu = 0$. Take $\beta_n = \frac{1}{(n+1)}, n = 1,2,\ldots,$ as our parameters in Theorem 5.2. With this, we now give the following algorithm which is a specialized version of Theorem 5.2.

Algorithm.

Step 0: Choose any $u_1, v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set the maximum number of iterations, n.

Step 1: If $\|u_k\| \le \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_k = \frac{1}{(k+1)}$.

Step 2: Compute

$$\begin{cases} u_{k+1} = u_k - \beta_k(Fu_k - v_k), \\ v_{k+1} = v_k - \beta_k(Kv_k + u_k). \end{cases}$$

Step 3: Set $k = k + 1$ and go to Step 1.

Table 2 gives our test results using 10^{-6} tolerance.

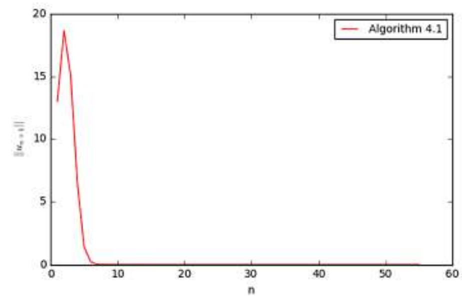


Figure 2 Convergence of the sequence $\{u_n\}$ with initial point $(3,5)$, $(2,1)$

Table 3 Numerical illustration for the solution of variational inequality problem

Initial points	Num. of iter	Approx. solution
$(1,0)$	24	8.2377×10^{-7}
$(1,1)$	24	9.6812×10^{-7}
$(2,3)$	25	9.6103×10^{-7}
$(-2,1)$	25	9.3095×10^{-7}
$(\frac{1}{2},\frac{1}{2})$	22	7.1434×10^{-7}
$(-\frac{1}{10},-1)$	92	9.6662×10^{-7}
$(5,8)$	27	8.3144×10^{-7}

The numerical result for the initial point $(3,5)$, $(2,1)$ is sketched below where the y-axis represents the values of $\|u_{n+1} - 0\|$, while the x-axis represents the number of iterations n (see Fig. 2).

Example 5 In Theorem 6.1, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Av = \begin{pmatrix} 5 & -5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad Tv = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then it is easy to see that A is a generalized-Φ-strongly monotone map, T is quasi-Φ-nonexpansive, and the vector $v^* = (0,0)$ is the common solution. We take $\beta_n = \frac{1}{n+1}, n = 1,2,\ldots$, as our parameter in Theorem 6.1. With this, we now give the following algorithm which is a specialized version of Theorem 6.1.

Algorithm.

Step 0: Choose any $v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set the maximum number of iterations, n.

Step 1: If $\|v_k\| \leq \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_n = \frac{1}{k+1}$.

Step 2: Compute

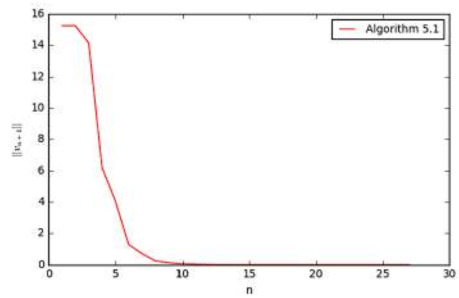
$$v_{k+1} = T_{[k]}v_k - \beta_k A(T_{[k]}v_k).$$

Step 3: Set $k = k + 1$ and go to Step 1.

Table 3 gives our test results using 10^{-6} tolerance.

The numerical result for the initial point $(5,8)$ is sketched below where the y-axis represents the values of $\|v_{n+1} - \mathbf{0}\|$, while the x-axis represents the number of iterations n (see Fig. 3).

Figure 3 Convergence of the sequence $\{v_n\}$ with initial point $(5,8)$



Remark 5 Our theorem is a significant improvement of the results of Diop *et al.* [33], Chidume and Bello [20], Chidume [18], Chidume *et al.* [26], and Chidume *et al.* [24] in the following sense:

(1) Theorems 3.1 and 5.2 are proved in a more general real Banach space which contains the space of 2-uniformly convex space and L_p spaces, $1 < p < \infty$.

(2) The class of *strongly monotone maps* studied in Diop *et al.* [33], Chidume and Bello [20] is extended to the more general class of *generalized-Φ-strongly monotone maps* in Theorems 3.1 and 5.2, respectively.

(3) The requirement that the maps A, K, and F be *bounded*, which is assumed in Theorems 1.1 and 3.1 of Diop *et al.* [33], Chidume and Bello [20], respectively, and in the theorem of Chidume *et al.* [24, 26] and Chidume [18], is dispensed with in our theorems.

9 Conclusion

In this paper, a *Mann-type* iterative algorithm that approximates the zero of a generalized-Φ-strongly monotone map is presented. A strong convergence theorem of the sequence generated by the algorithm is proved. Furthermore, the theorem proved is applied to approximate solutions of a convex minimization problem, a Hammerstein integral equation, and a variational inequality problem. The theorem proved generalizes, extends, and improves the results of Diop *et al.* [33], Chidume and Bello [20], Chidume [18], Chidume *et al.* [26], Chidume *et al.* [24], and other recent important related results in the literature. Finally, examples of generalized-Φ-strongly monotone maps are constructed and numerical experiments which illustrate the convergence of the sequence generated by our algorithm, are presented.

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Authors' contributions
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