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A STRONG COUPLING APPROXIMATION: COVARIANT PERTURBATION EXPANSION AROUND INDEPENDENT VALUED FIELD THEORIES *)

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ABSTRACT

A covariant perturbation expansion is developed around Klauder's independent valued field theories as zeroth approximation. A diagram technique is constructed and an analytic regularization is introduced to handle the superficially divergent terms of the expansion. For a wide class of interactions, the divergent terms can be either compensated by renormalization or vanish identically after the removal of the regularization. As a result only tree diagrams are found to contribute to the various Green's functions; nonetheless one finds non-trivial scattering amplitudes with correct clustering and positivity properties.

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1. INTRODUCTION

Strong coupling approximations [1] to quantum field theories are of utmost importance. Apart from practical questions (some coupling constants may be too large to warrant the application of ordinary perturbation theory), various model studies have suggested that the solutions of coupled field equations may be singular at vanishing coupling constants [2]. Thus expansions in increasing powers of a coupling constant may not be only impractical, but entirely meaningless from the mathematical point of view. While there appeared a large number of strong coupling schemes in the literature (starting with the pioneering work of Wentzel's strong coupling theory [1], there were only a few attempts [3),4] made in order to maintain manifest Lorentz covariance in a strong coupling perturbation treatment.

Intuitively, one wants to split a Lagrangian into kinetic and interaction parts, solve the field theory in zeroth approximation by entirely omitting the kinetic term which is afterwards treated as a perturbation. In order to formalize this idea, let us consider a self-coupled spinless field without internal symmetries as an example.

The generating functional of Green's functions in Euclidean metric may be written as the functional integral

$$Z[j] = \int \mathcal{D}[\phi]\, e^{i\int d^4x\, j(x)\phi(x)}\, e^{K[\phi]-V[\phi]} . \tag{1.1}$$

Here $K[\phi]$ stands for the kinetic part of the Lagrangian, viz.,

$$\begin{aligned} K[\phi] &\equiv \frac{1}{2}\int d^4x\, d^4y\; \phi(x)\,\Box_x \delta^{(4)}(x-y)\,\phi(y) \\ &\equiv \frac{1}{2}\int d^4x\, d^4y\; \phi(x)\,K(x-y)\,\phi(y) , \end{aligned} \tag{1.2}$$

whereas all non-derivative terms of the action are collected into $V[\phi] \equiv \int V(\phi)d^4x$. The external source is denoted by $j(x)$.

One can write:

$$\begin{aligned} Z[j] &= e^{K[\frac{1}{i}\frac{\delta}{\delta j}]}\, Z_0[j] \\ &= Z_0[j] + \frac{1}{1!}K[\frac{1}{i}\frac{\delta}{\delta j}]Z_0[j] + \dots , \end{aligned} \tag{1.3}$$

where -- at least formally --

$$Z_o[j] = \mathcal{K}\int \mathcal{D}[\phi]\, e^{\,i\int d^4x\, j(x)\phi(x)}\; e^{-V[\phi]} . \tag{1.4}$$

Equations (1.3) and (1.4) thus define a (formal) "strong coupling expansion", provided $Z_o[j]$ can be given a meaning.

Thus the task we are facing is twofold:

i) to define $Z_o[j]$;

ii) to deal with the perturbation expansion formally described by Eq. (1.3).

The key to the development of an independent valued (I.V.) theory has been provided by Klauder in a series of papers [5)-8)]. Klauder argues as follows.

i) The functional $Z_o[j]$ of Eq. (1.4) is a normalized positive definite functional by construction.

ii) Since it is independent valued at every space-time point, it can be written as

$$Z_o[j] = \exp \int F(j(x))\, d^4x \tag{1.5}$$

with some appropriate <u>function</u>, F of $j(x)$ with $F(0)=0$.

A continuous functional $Z_o[j] \neq 0$ with the above-mentioned properties has the form of a characteristic functional of a generalized random process and $\exp(+F(j))$ is the characteristic function of some infinitely divisible random variable [9)]. The most general form of such a functional is known and there are different representations available in the mathematical literature. The canonical representation, due to Lévy [10)], is as follows:

$$F(j) = iaj - bj^2 + \int_{|u|>0} \left(e^{iju} - 1 - ij\frac{u}{1+u^2}\right) d\sigma(u), \tag{1.6a}$$

where a, b, σ are real, $b \geq 0$ and $d\sigma(u)$ is a positive measure fulfilling the requirement that

$$\int \frac{u^2}{1+u^2}\, d\sigma(u) < \infty . \tag{1.6b}$$

For $F_E(j) = F_E(-j)$ we have

$$F_E(j) = -bj^2 - \int (1 - \cos j{\cdot}u)\, d\sigma(u); \qquad (1.6c)$$

this corresponds to an even potential in Eq. (1.4). From now on we write $d\sigma(u) = f(u)du$, where $f(u)$ may be a positive generalized function.

<u>The choice of a, b and $f(u)$ completely characterizes the interaction</u>. Using Eqs. (1.5) and (1.6) instead of Eq. (1.4) is equivalent to a "suitable" choice of the functional measure $D[\phi]$ and the interaction $V(\phi)$ [see, e.g., Ref. 6)]. Klauder has developed a method of connecting a naive independent valued Lagrangian with the continuous density $f(u)$ via a non-linear representation of the field operators in Fock space. For a wide class of interactions he finds

$$\begin{aligned} f(u) &= |u|^{-1} \exp(-V(u)), \\ a &= b = 0, \end{aligned} \qquad (1.7)$$

if $V(u) = V(-u)$. It is to be emphasized here that Klauder's result certainly corresponds to an unconventional functional measure in Eq. (1.4).

In order to illustrate this point one may consider an independent valued φ^4 theory $[V(\phi) = \frac{1}{2} m_0^2\phi^2 + \lambda_0\phi^4]$. The model was investigated by means of a perturbation expansion [11] and by the functional integral method [12] as well -- which concerns us at this point -- using Eq. (1.4) with the naive measure:

$$\int \mathcal{D}[\phi] \exp\left(-V[\phi] + i\int j\phi\, d^4x\right) \equiv \lim_{\epsilon \to 0} \int \prod_i d\phi_i \exp\left(-\tfrac{1}{2} m_0^2 \phi_i^2 - \lambda_0 \phi_i^4 + i j_i \phi_i\right)\epsilon^4 .$$

Without the renormalization of the coupling constant λ_0, $F(j)$ is simply zero here. Even after renormalization $(\lambda_{phys} = \lambda_0\epsilon^4)$ only a term proportional to j^2 - free or quasifree theory - can be "saved": $b \neq 0$, $f(u) = 0$. With Klauder's method we get Eq. (1.7) with $V(u) \equiv \frac{1}{2}\alpha_m^2 i^2 + \alpha_\lambda u^4$. (We prefer to write α_m and α_λ instead of the original mass and coupling constant, to indicate that they are not necessarily identical.) It is important to notice here that the limit $\alpha_\lambda \to 0$ does not reproduce the free theory [which would be determined by $b = (2m_0^2)^{-1}$, $a = f = 0$] but the so-called pseudo-free one. This construction strongly suggests that the measures appropriate for a free and an interacting theory, respectively, are inequivalent ones in the sense that there exists no non-singular mapping between the two measures [6),7)].

(This may be a reflection of Haag's theorem in a functional formulation of field theory.)

In this paper we develop a perturbation theory around the independent valued (strong coupling) limit of a field theory defined by Eqs. (1.3), (1.5) and (1.6). In the next Section diagram rules are established for the description of the terms of the formal expansion (1.3). It is found (not surprisingly) that most of the terms are meaningless due to the singular nature of the kernel $K(x,y)$. A suitable regularization procedure is introduced in Section 3, which permits the analysis of the (formal) divergences of the series (1.3). The outcome of that analysis is rather surprising: we find that all non-trivial loop diagrams vanish. Thus, as illustrated in Section 4, the perturbed IV theory is similar to a classical theory with correct propagation and cluster properties.

2. FORMAL STRONG COUPLING EXPANSION; DIAGRAM RULES

Our starting point is the expression of $Z[j]$ in Euclidean metric, given by Eqs. (1.3), (1.5) and (1.6).

There are some remarks worth making on the form of $F(j)$. For the moment we do not have to specify the function $f(u)$ beyond the requirement that all functional derivatives of Z_0 must exist. Hence, $f(u)$ is assumed to decrease at infinity faster than any power. Later we will show explicitly (Section 3) that a meaningful perturbation expansion can be derived only for theories in which $f(u)$ can be written as

$$f(u) = e^{-\alpha u^2} \bar{g}(u) , \qquad \alpha > 0, \tag{2.1}$$

where $\bar{g}$ does not contain any Gaussian. A necessary condition of the stability of the unperturbed ground state, viz., $(\delta Z_0/\delta j)_{j=0} = 0$ implies a relation among a, b and f.

Now with the expression (1.2) of K the perturbation series (1.3) becomes:

$$
\begin{aligned}
Z[j] = Z_0[j]\Big\{ 1 &+ \tfrac{1}{2}\int d^4x_1\, d^4y_1\, K(x_1,y_1)\big[F_2(x_1)\,\delta^{(4)}(x_1-y_1) \\
&+ F_1(x_1)\,F_1(y_1)\big] \\
&+ \tfrac{1}{4}\int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\, K(x_1,x_2)\,K(y_1,y_2)\big\{F_4(x_1)\,\delta^{(4)}(x_1-x_2)\,\delta^{(4)}(x_1-y_1)\,\delta^{(4)}(x_1-y_2) \\
&\quad + 4F_2(x_1)\,F_1(y_1)\,F_1(x_2)\,\delta^{(4)}(x_1-y_2) \\
&\quad + 4F_3(x_1)\,F_1(x_2)\,\delta^{(4)}(x_1-y_1)\,\delta^{(4)}(y_1-y_2) \\
&\quad + 2F_2(x_1)\,F_2(x_2)\,\delta^{(4)}(x_1-y_1)\,\delta^{(4)}(x_2-y_2) \\
&\quad + \big(F_2(y_1)\,\delta^{(4)}(y_1-y_2) + F_1(y_1)\,F_1(y_2)\big)\big(F_2(x_1)\,\delta^{(4)}(x_1-x_2) + F_1(x_1)\,F_1(x_2)\big)\big\} \\
&+ \ldots \Big\} .
\end{aligned}
\tag{2.2}
$$

In Eqs. (2.2) we used the notation:

$$
F_n(x) \equiv \frac{1}{i^n}\,\frac{d^n F(s)}{ds^n}\bigg|_{s=j(x)} .
$$

It is convenient to represent the terms in the series (2.2) graphically. The structure of the diagrams is similar to Feynman diagrams and the rules of correspondence between the diagrams and analytic expressions can be established by a careful study of the perturbation series.

Below we summarize the diagram rules:

i) each factor $K(x_1,x_2)$ is represented by a line joining the points x_1 and x_2 (in the scalar theory without internal symmetries considered here, the lines are not oriented);

ii) each factor $F_n(x)$ corresponds to a vertex at point x where n lines meet;

iii) every coordinate is integrated over. Each expression corresponding to a diagram Γ_L (where L is the number of lines in the graph) is multiplied by the numerical factor

$$N(\Gamma_L) = \frac{1}{L!} 2^{-N} C(\Gamma_L),$$

where (a) $(L!)^{-1}$ is coming from the expansion of the exponential; (b) 2^{-N} appears from the definition of $K[\phi]$ see Eq. (1.2); and (c) $C(\Gamma_L)$ is the combinatorial factor giving the number of different ways in which the L lines can be inserted between the vertices as if $K(x_1, x_2)$ were oriented. (It is worth remembering that for lines which begin and end at the same vertex, the factor of 2 coming from the orientedness of the lines does not appear.)

(These rules are the same as those applied in the counting of equivalent Feynman diagrams. The only -- unessential -- complication arises from the circumstance that in our expansion the number of lines entering a vertex is generally not limited.)

One readily verifies that the expansion of $S[j]$ is obtained simply by omitting disconnected diagrams, just as in ordinary perturbation theory. In Fig. 1 we give the expansion of $S[j]$ up to third order in the perturbation.

3. REGULARIZATION; ANALYSIS OF LOOP DIAGRAMS

Most of the terms in the expansion (2.2) are meaningless, due to the fact that the product of generalized functions is ambiguous. Superficially, it appears that the "divergences" arising in this perturbation expansion are disastrous and cannot be removed by renormalization. As an example consider the simple loop diagrams shown in Fig. 2.

The contribution of diagram (2a) is proportional to the following expression

$$(1a) \sim \int d^4x\, d^4y\, F_2(x) F_2(y) \left(\Box_x \delta^{(4)}(x-y)\right)^2 = \int d^4p\, \tilde{F}_2(p) \tilde{F}_2(-p) \Sigma_2(p)$$

where

$$\tilde{F}_2(p) = \int d^4x \; e^{ipx} F_2(x)$$

and

$$\Sigma_2(p) \sim \int d^4k \;\; k^2(k-p)^2 .$$

By introducing a high momentum cut-off, Λ, one finds by naive power counting that $\Sigma_2 \sim \Lambda^8$ and the contribution from (2b) diverges as Λ^{14}, etc. The situation becomes only worse if one considers more complicated loop diagrams. We argue, however, that a naive power-counting argument is misleading.

Indeed, any high momentum cut-off is equivalent to the introduction of a space-time smearing. (We work in Euclidean metric, thus one does not have to worry about the distinction between space-like and time-like directions.) As emphasized by Klauder and Narnhofer [8], a space-time smearing cannot be introduced into $Z_0[j]$, for it would destroy the I.V. character of our zeroth approximation. Thus, the kernel, $K(x-y)$ of the perturbing operator has to be regularized in such a way as to render <u>all the resulting diagrams</u> finite.

This task can be most simply and elegantly accomplished by observing that $K(x-y)$ is a generalized homogeneous function, which can be regularized in a standard fashion [13] (analytic regularization).

Indeed we may write

$$K(x) = \Box\delta^{(4)}(x) = \frac{1}{4\pi^2}\,\Box^2 \frac{1}{x^2 - i\epsilon} .$$

Thus, by regularizing the generalized homogeneous function $(x^2 - i\epsilon)^{-1}$, one achieves the desired regularization of K. Following the prescription of Ref. 13), we replace K by

$$K_{2r}(x) = \frac{1}{4\pi^2}\,\Box^2 \; \frac{(x^2 - i\epsilon)^{-2r}}{\Gamma(2-2r)} , \qquad (3.1)$$

the Γ function in the denominator supplies the normalization factor appropriate for four space-time dimensions.

By carrying out the differentiation in (3.1), we get an explicit expression for the regularized kernel:

$$K_{\nu}(x) = \frac{4}{\pi^2}\, \nu(\nu+1)\, \frac{(x^2 - i\epsilon)^{-\nu-2}}{\Gamma(-\nu)} \,. \tag{3.2}$$

One observes, of course, that the regularized kernel is defined as a causal boundary value of an analytic function. The regularized kernel, as a generalized function, is an entire function of ν, see Ref. 13).

This regularization, in particular, has the advantage that no space-time smearing has to be introduced at any stage of the calculation. In order to isolate the true divergences -- if any -- present in our perturbation series, we have to study the singularities of the regularized expressions in the variable ν. The renormalized expression of a diagram can then be defined as its finite part at $\nu = 1$, see, e.g., Speer [13].

Let us now return to the general two-point n line diagram of the type exhibited in Fig. 2. Apart from trivial factors, its regularized kernel, $\Sigma_{n,\nu}(p)$, is given by

$$\Sigma_{n,\nu}(p) \sim \int d^4x\, e^{ipx} \left(K_{\nu}(x)\right)^n .$$

Obviously, kernels of this type also occur as insertions in more complicated diagrams. Regularized homogeneous functions can be multiplied with impunity, so that we obtain

$$\Sigma_{n,\nu}(p) \sim \left(\frac{4\nu(\nu+1)}{\pi^2\,\Gamma(-\nu)}\right)^n \int d^4x\, e^{ipx} (x^2 - i\epsilon)^{-n(\nu+2)}$$

The Fourier transform can be computed with the help of the integral representation,

$$(x^2 - i\epsilon)^{-\alpha} = \frac{e^{i\alpha\frac{\pi}{2}}}{\Gamma(\alpha)} \int_0^\infty \frac{dt}{t}\, t^{\alpha}\, e^{-it(x^2 - i\epsilon)} \tag{3.3}$$

and the standard four-dimensional Gaussian integral:

$$\int d^4x\, e^{i(ax^2 \pm bx)} = \frac{\pi^2}{ia^2}\, e^{-i\frac{b^2}{4a}}, \qquad a>0. \tag{3.4}$$

Omitting again trivial factors, the final result is proportional to

$$\Sigma_{n,\nu}(p) \sim \frac{\Gamma(2-n(\nu+2))}{(\Gamma(-\nu))^n}\,(p^2+i\epsilon)^{n(\nu+2)-2} \tag{3.5}$$

Here, for $n \geq 2$ the numerator of Eq. (3.5) behaves around $\nu = 1$ as

$$\frac{(p^2+i\epsilon)^{3n-2}}{(1-\nu)}$$

whereas

$$\Gamma(-\nu)^{-n} \sim (1-\nu)^n.$$

Thus, near $\nu = 1$

$$\Sigma_{n,\nu}(p) \sim (1-\nu)^{n-1}.$$

Not only does $\Sigma_{n,\nu}$ not have a pole at $\nu = 1$, but it vanishes instead!

We now proceed to show that this result is true in general: the contribution of every closed loop is either zero or it can be absorbed into the Gaussian factor and the parameter a in Eq. (2.1).

First of all, we consider those loops which contain only one vertex ("petals", see Fig. 3). We show that their contribution only renormalizes the coefficients a and α, see Eqs. (1.6) and (2.1), respectively.

Indeed, consider a vertex at point x and begin to attach "petals" to it. Let us suppose that there are k lines joining this vertex to the rest of the diagram. The contribution of the uncorrected (= no petals on the pre-selected vertex) diagram can be written as follows:

$$N(\Gamma)\int G(y_1,\ldots,y_k)\,K_\nu(y_1-x)\ldots K_\nu(y_k-x)\,F_k(x)\,d^4y_1\ldots d^4y_k\,d^4x, \tag{3.6}$$

where $G(y_1, \ldots, y_K)$ stands for the contribution of the "box" in Fig. 3, and $N[\Gamma]$ is the numerical factor for the entire diagram. It is easy to see that the attachment of S petals changes (3.6) into

$$\frac{N[\Gamma]}{s!} \int G(y_1, \ldots, y_k) K_\nu(y_1 - x) \ldots K_\nu(y_k - x) F_{k+2s}(x) \times [K_\nu(0)]^s \, d^4y_1 \ldots d^4y_k \, d^4x . \tag{3.7}$$

Having used Eqs. (1.6) and (2.1), we have for $k \geq 2$

$$\sum_{s=0} \frac{1}{s!} [K_\nu(0)]^s F_{k+2s}(j(x)) + 2b\,\delta_{k,2} = $$
$$= \int_{|u|>0} du \quad u^k \sum_s \frac{(u^2 K_\nu(0))^s}{s!} \quad e^{-\alpha u^2} \bar{g}(u) e^{iju}$$
$$= \int_{|u|>0} du \; u^k e^{-(\alpha - K_\nu(0))u^2} \bar{g}(u) \; e^{iju}$$

and for $k = 1$

$$\sum_{s=0} \frac{1}{s!} (K_\nu(0))^s F_{1+2s}(j) = \left(a - \int_{|u|>0} \frac{u}{1+u^2} (e^{-(\alpha - K_\nu)u^2} - e^{-\alpha u^2}) \bar{g}(u) du \right)$$
$$+ i\,2b j(x) + \int_{|u|>0} (u e^{iju} - \frac{u}{1+u^2}) e^{-(\alpha - K_\nu(0))u^2} \bar{g}(u) du.$$

The expression of $K_\nu(0)$ is -- of course -- ambiguous, even after regularization. However, a redefinition of α and a removes the effect of the petals altogether.

Next, we observe that those loop diagrams in which some pair of vertices is joined by multiple lines need not be considered separately (Fig. 4). Indeed, the contributions of diagrams 4b, 4c, etc., differ from the contribution of 4a, essentially in the replacement of K_ν by $\Sigma_{n,\nu}$ $(n = 2, 3, \ldots)$. Hence, if the contribution of diagram 4a is shown to vanish at $\nu = 1$, the contributions of 4b, 4c, etc., will a fortiori vanish. Thus we are left with skeleton loops, containing no petals and no multiple lines between any two points.

Consider now a general skeleton loop with L lines and $v+1$ vertices of any kind (see, e.g., Fig. 5a). Since the diagram is connected and it is not a tree, we have $L \geq v+1$. The loop may be part of a larger diagram (see, e.g., Fig. 5b).

In order to isolate the contribution of the loop, we again go over to Fourier space as we did in the case of the two-point loops, $\Sigma_{n,v}$. The regularized kernel $G_{L,v}(p_1, \dots p_v)$ depends on v independent momenta only. Omitting unessential factors we have

$$G(p_1,\dots,p_v) \approx \left[\Gamma(-v)\right]^{-L}\left[v(v+1)\right]^{L} \times \int d^4x_1 \dots d^4x_{v+1}\, \delta^{(4)}(x_{v+1}) \exp\left(i\sum_{k=1}^{v}(p_k\cdot x_k)\right) \times \prod_{1\leq i\leq j\leq v+1}\left[(x_i-x_j)^2-i\epsilon\right]^{-v-2} . \tag{3.8}$$

Having used Eq. (3.3) we get

$$G \approx \left[\Gamma(v)\Gamma(-v)\right]^{-L}\int d^4x_1\dots d^4x_v \exp\left(i\sum_{k=1}^{v}(p_k\cdot x_k)\right) \times \int_0^\infty \prod_{i=1}^{L}\frac{d\alpha_i}{\alpha_i}\alpha_i^{v+2} \exp -i\sum_{k,\ell\leq v}(x_k\cdot x_\ell)A_{k\ell}(\alpha)$$

where $A_{K\ell}(\alpha)$ is a linear homogeneous function of $\alpha_1, \dots, \alpha_L$. The 4v-fold Gaussian integration can be readily performed with the result:

$$G \approx \left(\Gamma(v)\Gamma(-v)\right)^{-L}\left(\frac{\pi^2}{i}\right)^{v}\int\left(\prod_{i=1}^{L}\frac{d\alpha_i}{\alpha_i}\alpha_i^{v+2}\right)\left(\operatorname{Det} A\right)^{-2} \times \exp i\sum_{k,\ell\leq v}(p_k\cdot p_\ell)A^{-1}_{k\ell}(\alpha) .$$

Introduce new variables, λ, ξ_i $(1 \leq i < L,\ \Sigma\xi_i = 1)$, by the substitution $\alpha_i = \lambda^{-1}\xi_i$. Using the homogeneity of $A_{K\ell}$ in α_i, this gives:

$$G \approx \left[\Gamma(\nu)\Gamma(-\nu)\right]^{-L} \left(\frac{\pi^2}{i}\right)^{v} \int_0^\infty \frac{d\lambda}{\lambda}\, \lambda^{2v - L(\nu+2)}$$

$$\times \int_0^1 \prod_{i=1}^{L} d\xi_i\, \xi_i^{\nu+1} \left[\mathrm{Det}\, A(\xi)\right]^{-2} \delta\left(1-\sum \xi_i\right) \exp i\lambda \sum_{k,\ell} (p_k \cdot p_\ell) A^{-1}_{k\ell}(\xi)$$

$$= \left[\Gamma(\nu)\Gamma(-\nu)\right]^{-L} \left(\frac{\pi^2}{i}\right)^{v} i^{2v - L(\nu+2)}\, \Gamma\left(2v - L(\nu+2)\right)$$

$$\times \int_0^1 \prod_{i=1}^{L} d\xi_i\, \xi_i^{\nu+1}\, \delta\left(1-\sum \xi_i\right) \left[\mathrm{Det}\, A(\xi)\right]^{2v - L(\nu+2) - 2}$$

$$\times \left(\sum_{k,\ell} (p_k \cdot p_\ell)\, [A(\xi)]_{k\ell}\right)^{L(\nu+2) - 2v},$$

where we have used $A^{-1}_{K\ell} = [A]_{K\ell}\ (\mathrm{Det}\ A)^{-1}$.

Now as $\nu \to 1$

$$\left(\Gamma(\nu)\Gamma(-\nu)\right)^{-L}\ \Gamma\left(2v - L(\nu+2)\right) \sim (1-\nu)^{L-1}, \tag{3.10}$$

since $2v - 3L$ is certainly a non-positive integer in view of the inequality $L \geq v + 1$. Thus the contribution of the loop vanishes, unless the multiple integral over ξ_i is sufficiently singular at $\nu = 1$ so as to overcome the factor (3.10). We show that this is not the case. If $\mathrm{Det}\ A \neq 0$ in the L dimensional hypercube $0 \leq \xi_i \leq 1$, the integration cannot be singular over ξ_i as $\nu = 1$ since $L(\nu + 2) - 2v > 0$ and $[A(\xi)]_{K\ell}$ is a polynomial in ξ.

It is readily seen from the structure of the diagrams that $\mathrm{Det}\ A$ is a multilinear function of degree v of its arguments, hence it can vanish only if some of the ξ_i tend to zero. If $\mathrm{Det}\ A$ vanished when one of the ξ_i's tends to zero, one would get a factor from the integration over ξ_i proportional to $[2v - 2 - (L-1)(\nu+2)]^{-1}$. This may be "dangerous" as $\nu \to 1$ (depending on v and L), hence it may contribute a factor $(\nu - 1)^{-1}$. Without the restriction $\Sigma_{i=1}^{L}\ \xi_i = 1$, one would thus be able to pick up altogether at most $(\nu-1)^{-\nu}$, which, using $L \geq v+1$ gives the upper estimate for the degree of the singularity as $L-1$. However, due to the presence of the δ function under the integral, the degree of singularity can at most be equal to $L-2$. Thus, the singularity coming from the ξ integration can never compensate the Γ functions in the denominator.

Thus, we have shown that only tree diagrams survive in the expansion of $\ell n\, Z$ around its I.V. limit. In other words, a perturbed I.V. theory is similar to a classical field theory in the sense that no virtual pairs appear.

4. SCATTERING AMPLITUDES

One can collect now the surviving terms in the strong coupling expansion of $\ell n\, Z[j]$. As shown in the previous Section, all those terms are represented by tree diagrams with renormalized a and Gaussian factor.

It is convenient at this point to introduce a modified kernel, $g(x,y)$, which takes into account "F_2 insertions". Indeed, given any diagram which contains a kernel, K, running between points x and y, it is accompanied by an infinite set of diagrams with an increasing number of vertices F_2 inserted into the given line, see Fig. 6.

In order to take these insertions into account, one defines $g(x,y)$ through the Dyson equation

$$g(x,y) = K(x-y) + \int d^4z\, K(x-z)\, F_2(z)\, g(z,y). \tag{4.1}$$

In the limit $j(x) \to 0$, one has $F_n(x) \to F_n = \text{const.}$, so that

$$g(x,y)\Big|_{j=0} = \frac{1}{(2\pi)^4} \int d^4k\, e^{ik(x-y)} \frac{-k^2}{1+F_2 k^2} \quad \text{(Euclidean metric)}, \tag{4.2}$$

In terms of the modified kernel, the first few terms of the expansion of $\ell n\, Z[j]$ are exhibited in Fig. 7.

The connected Green's functions can now be calculated by repeated functional differentiation with respect to the source j. In doing so, one has to remember, of course, that the modified kernel, g, is itself a functional of the source. Its derivatives can be, however, easily calculated if one notices that (4.1) can be written symbolically as

$$g = K + KF_2(j)g \tag{4.3}$$

from which

$$g = (1 - KF_2(j))^{-1} K \, . \tag{4.4}$$

Differentiating (4.3) we get

$$\frac{\delta g}{\delta j} = K \, \frac{\delta F_2(j)}{\delta j} g + K \, F_2(j) \frac{\delta g}{\delta j} \, ,$$

hence, taking (4.4) into account

$$\frac{\delta g}{\delta j} = g \, \frac{\delta F_2(j)}{\delta j} g$$

or written out in detail,

$$\begin{aligned} \frac{\delta g(x,y)}{\delta j(z)} &= \int d^4t \;\; g(x,t) \, \frac{\delta F_2(t)}{\delta j(z)} \, g(t,y) \\ &= g(x,z) \, iF_3(z) \, g(z,y) \, . \end{aligned} \tag{4.5}$$

Higher derivatives are calculated similarly.

From now on, the calculation is entirely straightforward and we merely quote the final results for two-, four- and six-point functions of a symmetric theory in Minkowskian momentum space. Given the fact that nothing but tree diagrams survive, the transition to Minkowski space presents no problem. As a consequence of the symmetry requirement, all Green's functions of odd order vanish.

We find

$$G_2(p_1,p_2) = (2\pi)^4 \, \delta^{(4)}(p_1+p_2) \, \frac{1}{1 - F_2 \, p_1^2} \equiv (2\pi)^4 \, \delta^{(4)}(p_1+p_2) \, G(p_1) \, ,$$

$$G_4(p_1, \ldots p_4) = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{4} p_i\Big) \; F_4 \; \prod_{i=1}^{4} \; G(p_i) \, ,$$

$$G_6(p_1, \ldots, p_6) = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{6} p_i\Big) \prod_{i=1}^{6} \; G(p_i)$$

$$\times\left\{F_6 + F_4^2 \sum_{(ijk)} \frac{(p_i+p_j+p_k)^2}{1 - F_2(p_i+p_j+p_k)^2}\right\}, \tag{4.6}$$

where $F_n(j)\big|_{j=0} = F_n$. The summation in the expression of G_6 is extended over all independent reaction channels.

Without specifying the constants F_K any further, we see the following:

i) The resulting strong coupling theory possesses "normal" propagation properties. Since $f(u)$ is a positive definite function, F_2 is necessarily positive. Hence, G_2 describes the propagation of quanta of mass F_2^{-1} with a positive residue at the pole. It is assumed that there is at least one mass scale m_s, in the theory, thus the whole perturbation procedure can be conducted with dimensionless variables. Hence the mass of the quanta is $m^2 = m_s^2 F_2^{-1}$.

ii) The one-particle Green's functions are factorized out from the multipoint Green's functions corresponding to each external particle (clustering). The propagators of virtual quanta have poles at the correct position (see expression for G_6).

iii) By introducing a finite renormalization of the wave function ($\phi \to \sqrt{Z}\,\phi$; $Z = F_2^{-1}$), we can normalize the residue of the one-particle Green's function at the pole to unity. The rescaled Green's functions are

$$\tilde{G}_2(p_1,p_2) = (2\pi)^4 \delta^{(4)}(p_1+p_2) \frac{1}{F_2^{-1} - p_1^2} \equiv (2\pi)^4 \delta^4(p_1+p_2)\, \tilde{G}(p_1),$$

$$\tilde{G}_4(p_1,\dots,p_4) = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{4} p_i\Big) \prod_{i=1}^{4} \tilde{G}(p_i)\, \frac{F_4}{F_2^2},$$

$$\tilde{G}_6(p_1,\dots,p_6) = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{6} p_i\Big) \prod_{i=1}^{6} \tilde{G}(p_i)$$

$$\times\left\{\frac{F_6}{F_2^3} + \left(\frac{F_4}{F_2^2}\right)^2 \sum_{(ijk)} \frac{(p_i+p_j+p_k)^2}{F_2^{-1} - (p_i+p_j+p_k)^2}\right\}. \tag{4.7}$$

The two-point function has also been derived by Scarpetta [14] using the combinatorial method developed by the Naples group [11].

Finally, by way of illustration, we consider two examples here: a free theory and the φ^4 theory with the unconventional measure derived by Klauder [7] for the zeroth order, Eq. (1.7).

(a) Free theory: $Z_0[j]$ as defined by Eq. (1.4) exists with $V(\phi) = \frac{1}{2}m^2\phi^2$ and leads to $b = 1/2m^2$ and $a = f = 0$. Thus Eq. (4.7) reduces to the free propagator

$$\tilde{G}_2(p_1 p_2) = (2\pi)^4 \delta^4(p_1 + p_2) \frac{1}{m^2 - p^2} .$$

(b) Klauder's φ^4 theory: $a = b = 0$ and

$$f(u) = \frac{1}{|u|} e^{-V(u)} \qquad \text{with} \qquad V(u) = \frac{\alpha_m^2}{2} u^2 + \alpha_\lambda u^4 .$$

We find

$$m_{eff}^2 \equiv F_2^{-1} = \frac{\alpha_m^2}{4} \frac{e^{-t^2}}{\sqrt{\pi}\, t\, (1 - \Phi(t))} ,$$

$$\lambda_{eff} \equiv \frac{F_4}{F_2^2} = \frac{1}{2\sqrt{\pi}} \frac{2\exp(-t^2)/\sqrt{\pi} - 2t(1 - \Phi(t))}{\exp t^2 \; (1 - \Phi(t))^2} ,$$

$$\frac{F_6}{F_2^3} = \frac{1}{2\pi} \frac{\exp(-2t^2)}{(1 - \Phi(t))^2} - \frac{t}{\pi\sqrt{\pi}} \frac{\exp(-3t^2)}{(1 - \Phi(t))^3} \tag{4.8}$$

where $t \equiv \alpha_m^2 / 4\sqrt{\alpha_\lambda}$ and $\Phi(t)$ is the error function. It is interesting to notice that the effective coupling, λ_{eff}, depends only on t and not independently on α_m and α_λ.

In order to get a somewhat better understanding of this result we investigate the limits for m_{eff}^2 and λ_{eff}, when α_λ (coupling constant) goes to zero (weak coupling) and infinity (strong coupling), while the "mass", α_m, is kept fixed.

Weak coupling $(t \to \infty)$: $m_{eff}^2 \longrightarrow \frac{1}{4}\alpha_m^2 \left(1 + \frac{1}{2t^2} + \cdots\right)$,

$$\lambda_{eff} \longrightarrow \frac{1}{2}\left(1 - \frac{1}{2t^2} + \cdots\right).$$

Strong coupling $(t \to 0)$: $m_{eff}^2 \longrightarrow \sqrt{\frac{\alpha_\lambda}{\pi}}\left(1 + \frac{2t}{\sqrt{\pi}} + \cdots\right)$,

$$\lambda_{eff} \longrightarrow \frac{1}{\pi}\left(1 + t \cdot \frac{4-\pi}{\sqrt{\pi}} + \cdots\right).$$

It is remarkable that the effective interaction does not vanish even in the weak coupling limit as it was conjectured by Klauder. Although the weak coupling limit cannot be expected to be physically meaningful in this scheme, nevertheless it illustrates the inequivalence of a free and an interacting theory.

5. CONCLUSION

In this paper we have demonstrated the basic feasibility of a strong coupling expansion of Green's functions around the limit where all derivatives in the Lagrangian are completely neglected.

It is evident now that the key to the development of such a strong coupling scheme is the use of the representation (1.6) for the zeroth approximation. Indeed, this step permits one to circumvent the delicate (and largely unresolved) question of finding an "appropriate" measure in the functional integral (1.4). It is worth re-emphasizing here that by taking this crucial step, one has lost none of the physical insight provided by a "naive" (classical) Lagrangian. Either one constructs the density $f(u)$ according to Klauder's explicit prescription [in which case there is a direct connection between the "naive" Lagrangian and $f(u)$], or one may be even more "radical" and claim that $f(u)$ (and not a classical Lagrangian) is the function which determines the physical contents of a theory.

What comes as a surprise is the result that -- contrary to previous expectation -- not only is the theory renormalizable, but, in fact, it is more convergent than a conventional weak coupling expansion. We find, unfortunately, that all loop diagrams (representing the effects of quantum fluctuations of the field) vanish as well. We consider this result somewhat pathological even though the resulting "classical" field theory is far from being a trivial one: see, e.g., the dependence of the effective mass, m_{eff}, and interaction strength, λ_{eff}, on the parameters α_m and α_λ in Eq. (4.8).

There remains, of course, the possibility that by some appropriate resummation of the strong coupling expansion, at least some of the loop contributions can be "revived"; this question is under active investigation at present.

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FIGURE CAPTIONS

Fig. 1 : Graphical representation of the perturbed generating functional.

Fig. 2 : Examples of loop diagrams.

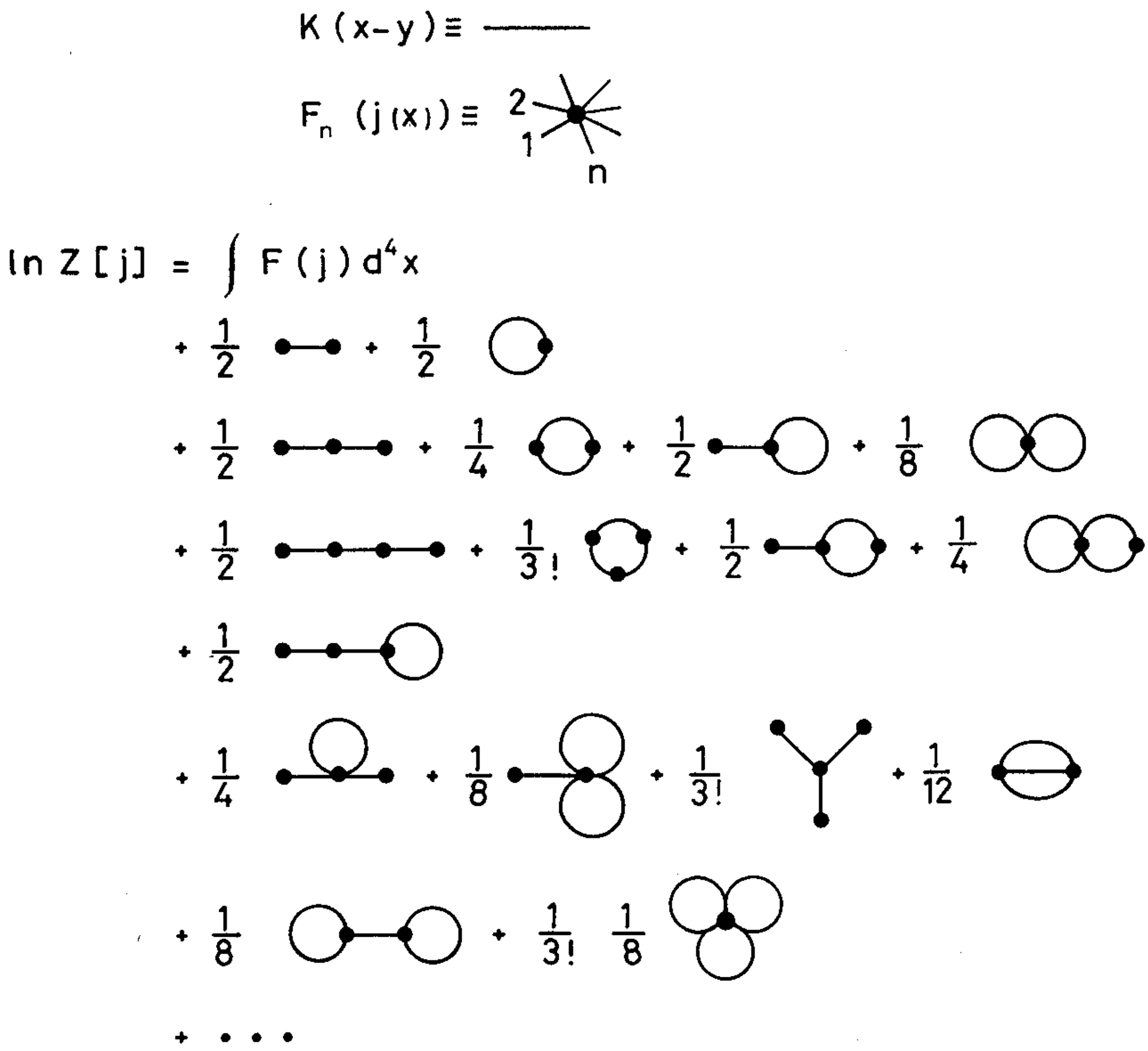
Fig. 3 : Petal corrections to vertices.

Fig. 4 : Example of pair of vertices joined by one, two, or three lines.

Fig. 5 : a) Skeleton loop diagram.
b) Same skeleton loop shown as part of a diagram.

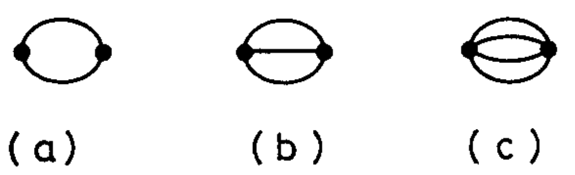
Fig. 6 : Graphical representation of the modified kernel, $g(x,y)$.

Fig. 7 : The final form of the logarithmic generating functional; dotted lines stand for the modified kernel, $g(x,y)$.



K (x−y) ≡
F_n (j(x)) ≡
2
1
n
ln Z [j] = ∫ F (j) d^4x
+ 1/2 + 1/2
+ 1/2 + 1/4 + 1/2 + 1/8
+ 1/2 + 1/3! + 1/2 + 1/4
+ 1/2
+ 1/4 + 1/8 + 1/3! + 1/12
+ 1/8 + 1/3! 1/8
+ • • •

FIG. 1



(a)
(b)
(c)

FIG. 2

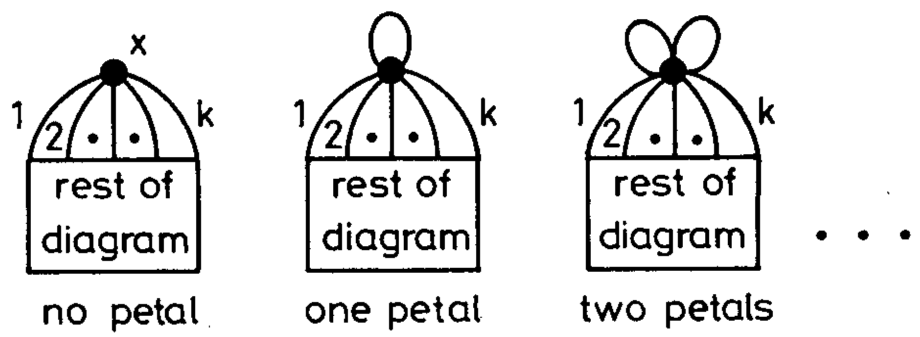


FIG. 3

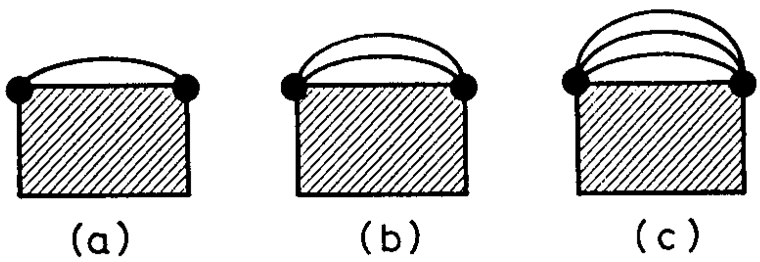


FIG. 4

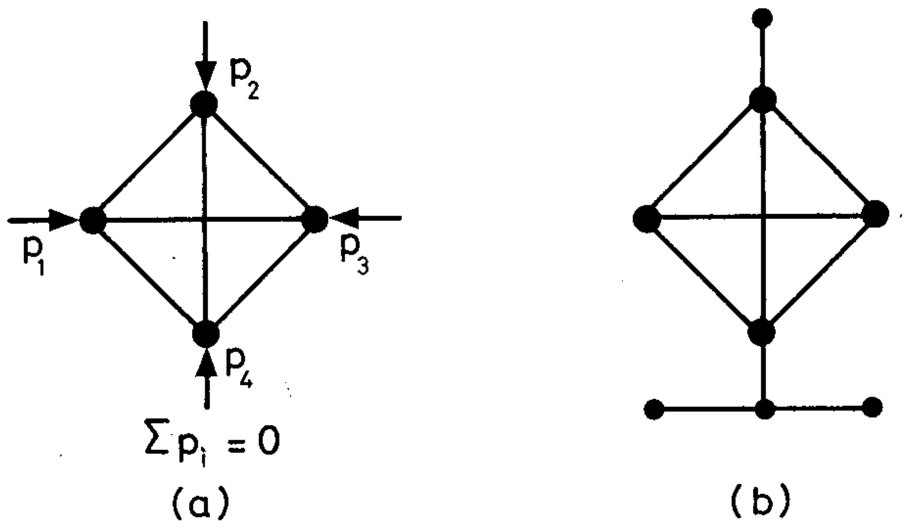


FIG. 5

FIG. 6

$g = - - -$

$$\ln Z[j] = \int F(j)\, d^4x$$

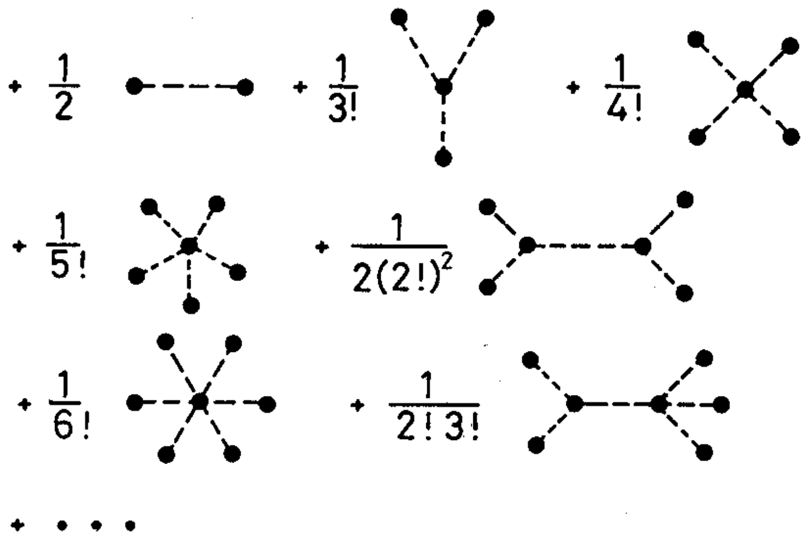


FIG. 7