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A STRONG LAW OF LARGE NUMBERS FOR GENERALIZED ALMOST SURE CENTRAL LIMIT THEOREMS

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Abstract. We prove a Strong Law of Large Numbers in which the variables are assumed to be asymptotically negligible and a generalized Almost Sure Central Limit Theorem is given. As an application we obtain a result about the so-called intersective ASCLT.

1. Introduction

In this paper we study the asymptotic behaviour of the "weighted" random variables

$$\frac{1}{\phi(n)}\sum_{k=1}^{n}\big(\phi(k)-\phi(k-1)\big)X_k, \tag{1.1}$$

where $(X_k)_{k\geq 1}$ is a sequence of mean square integrable random variables and ϕ is a function increasing to ∞. Results concerning random variables such as (1.1) are known as *Almost Sure Central Limit Theorems* (ASCLT) (see Berkes–Csáki [3] for an exhaustive list of references). In Section 3 we state and prove our main result; it may be listed among the most comprehensive ones about the ASCLT, and we get it as an application of a generalized Strong Law of Large Numbers, where covariances of the random variables involved are assumed to be *asymptotically negligible* (see Definition 2.2). We point out that our result takes into account also *almost orthogonal* random sequences studied by M. Weber. As an application of the main result, in Section 4 we prove a new version of the so-called *intersective law*, more general than the one recently obtained by Giuliano Antonini and Weber [8].

2. Preliminary results

In what follows we denote by $(X_n)_{n\geq 1}$ a sequence of L^2-random variables and by

$$\psi\colon \mathbb{R}^+\to\mathbb{R}^+$$

a decreasing function with $\int_1^\infty \psi(t)\,dt<\infty$. We start with two definitions.

Definition 2.1. A sequence $(X_n)_n$ is *mean square controlled by a random variable Z* if there exists a number C such that, for each n, we have

$$\mathsf{E}[X_n^2]\leq C\int_{\{|Z|\leq n\}} Z^2\,d\mathsf{P}.$$

If $\sup_n \mathsf{E}[X_n^2]$ is finite, $(X_n)_n$ is clearly mean square controlled by 1.

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Definition 2.2. A sequence $(X_n)_n$ has *ψ-negligible covariance* if, for every pair of integers p and q, with $p < q$, we have

$$|\mathrm{Cov}(X_p, X_q)| \leq \psi\big(\log(q-p+1)\big).$$

Notice that the condition of ψ-negligible covariance yields

$$\lim_{q-p\to\infty} \mathrm{Cov}(X_p, X_q) = 0.$$

The following result turns out to be a very useful strong law of large numbers.

Lemma 2.3. *Let $(X_n)_n$ be a sequence of positive random variables such that $\sup_n \mathsf{E}[X_n]$ is finite; assume moreover that $(X_n)_n$ has ψ-negligible covariance and is mean square controlled by an integrable random variable. Then the random variables*

$$Y_n = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mathsf{E}[X_k])$$

converge to 0 *in* L^2 *and almost surely.*

Proof. Let $a \geq 1$ be fixed and put $a_n = \lfloor a^n \rfloor$. We note that $(a_n)_n$ is an ultimately increasing sequence of integers. Moreover, since X_n is positive, for any integer m with $a_n < m < a_{n+1}$ we have

$$\frac{1}{m}\left(a_n Y_{a_n} - \sum_{k=a_n+1}^{m}\mathsf{E}[X_k]\right) \leq Y_m \leq \frac{1}{m}\left(a_{n+1}Y_{a_{n+1}} + \sum_{k=m+1}^{a_{n+1}}\mathsf{E}[X_k]\right).$$

If $b_m = (a_{n+1} - m)/m$ and $c_m = (m - a_n)/m$, from the above relation we get

$$\begin{aligned}
|Y_m| &\leq \left(|Y_{a_{n+1}}|\frac{a_{n+1}}{m} + b_m \sup_k \mathsf{E}[X_k]\right) \vee \left(|Y_{a_n}|\frac{a_n}{m} + c_m \sup_k \mathsf{E}[X_k]\right) \\
&\leq \left(|Y_{a_{n+1}}|a + 2(a-1)\sup_k \mathsf{E}[X_k]\right) \vee \left(|Y_{a_n}| + 2(a-1)\sup_k \mathsf{E}[X_k]\right) \\
&\leq |Y_{a_{n+1}}|a + |Y_{a_n}| + 2(a-1)\sup_k \mathsf{E}[X_k],
\end{aligned}$$

since $a_{n+1} \leq am$ and $b_m \vee c_m \leq 2(a-1)$ for $m = a_n + 1 + \cdots + a_{n+1}$. Hence it will be enough to prove the statement for the subsequence $(Y_{a_n})_n$; to this end we show that

$$\sum_{n\geq 1}\mathsf{E}\big[Y_{a_n}^2\big] < \infty.$$

First, note that

$$\sum_{n\geq 1}|Z|a_n^{-2}\sum_{k=1}^{a_n} I_{\{|Z|\leq k\}} = \sum_{k\geq 1}|Z| I_{\{|Z|\leq k\}}\sum_{a_n\geq k} a_n^{-2} \leq \frac{4a^2|Z|}{a^2-1}\sum_{k\geq 1} I_{\{|Z|\leq k\}} k^{-2} \leq 8 + \frac{4}{a-1}.$$

Since $(X_n)_n$ has ψ-negligible covariance and is mean square controlled by an integrable random variable Z, we get

$$\begin{aligned}
\sum_{n\geq 1}\mathsf{E}\big[Y_{a_n}^2\big] &\leq \sum_{n\geq 1} a_n^{-2}\sum_{k=1}^{a_n}\mathsf{E}\big[Z^2 I_{\{|Z|\leq k\}}\big] + 2\sum_{n\geq 1} a_n^{-2}\sum_{q=2}^{a_n}\sum_{p=1}^{q-1}\psi\big(\log(q-p+1)\big) \\
&\leq \mathsf{E}\left[|Z|\sum_{n\geq 1}|Z|a_n^{-2}\sum_{k=1}^{a_n} I_{\{|Z|\leq k\}}\right] + 2\sum_{n\geq 1} a_n^{-2}\sum_{q=2}^{a_n}\sum_{h=2}^{q}\psi\big(\log(h)\big) \\
&\leq \big(8 + 4(a-1)^{-1}\big)\,\mathsf{E}[|Z|] + 2\sum_{n\geq 1} a_n^{-1}\sum_{h=2}^{a_n}\psi\big(\log(h)\big).
\end{aligned}$$

Moreover, note that

$$\sum_{n\geq 1} a_n^{-1}\sum_{h=2}^{a_n}\psi\big(\log(h)\big)\sim\sum_{n\geq 1}a^{-n}\int_2^{a_n}\psi\big(\log(x)\big)\,dx\sim\sum_{n\geq 1}a^{-n}\int_{\log 2}^{n\log a}\psi(u)e^u\,du.$$

Thus, it suffices to prove that $\sum_{n\geq 1}a^{-n}\int_{\log 2}^{n\log a}\psi(u)e^u\,du<\infty$. We get the statement thanks to the integrability of ψ on $[1,\infty[$ since

$$\begin{aligned}\sum_{n\geq 1}a^{-n}\int_{\log 2}^{n\log a}\psi(u)e^u\,du&=\int_{\log 2}^{\infty}\psi(u)e^u\Big(\sum_{n\geq 1}a^{-n}I_{\{u\leq n\log a\}}\Big)\,du\\&=\int_{\log 2}^{\infty}\psi(u)e^u\Big(\sum_{n\geq u/\log a}a^{-n}\Big)\,du\\&\leq\int_{\log 2}^{\infty}\psi(u)e^u a^{-u/\log a}\frac{a}{a-1}\,du=\frac{a}{a-1}\int_{\log 2}^{\infty}\psi(u)\,du.\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.4. The above argument shows that the statement of Lemma 2.3 holds also under the condition

$$\sum_{q=2}^{n}\sum_{p=1}^{q-1}\mathrm{Cov}(X_p,X_q)\leq\psi_1\big(\log(n)\big),\tag{2.1}$$

where ψ_1 is a decreasing function with $\int_1^\infty\psi_1(t)/t^3\,dt<\infty$. This condition does not necessarily imply that $(X_n)_n$ has ψ-negligible covariance and is weaker than the condition of Peligrad and Shao [14], used in a recent paper by Dudziński [6]. Moreover, we could also weaken the assumption that the X_n are positive random variables by only requiring that they are bounded from below by a constant.

We get a stronger result than Lemma 2.3 when $(X_n-\mathsf{E}[X_n])_n$ is an *almost-orthogonal* sequence, namely when the quadratic form defined on $l^2(\mathbb{N})$ by

$$(x_n)_n\mapsto\sum_{h,k}\mathrm{Cov}(X_h,X_k)x_hx_k\tag{2.1}$$

is bounded. The condition of almost-orthogonality plays an important role in the study of random series such as $\sum_n c_n(X_n-\mathsf{E}[X_n])$ (see Kac, Salem, and Zygmund [10]) since, in this setting, the Rademacher–Menshov Theorem is still valid and $\sum_n c_n(X_n-\mathsf{E}[X_n])$ is almost surely convergent for any sequence $(c_n)_n$ of real numbers with

$$\sum_n c_n^2\log^2 n<\infty.$$

In particular, the almost-orthogonality of $(X_n-\mathsf{E}[X_n])_n$ assures that, for any $\beta>3$, the random variable $(n/\log^\beta n)^{1/2}Y_n$ converges to 0 almost surely. A condition which assures almost-orthogonality has been introduced by Weber [16]; it states that

$$\sup_{k\geq 1}\sum_{h=1}^{\infty}|\mathrm{Cov}(X_h,X_k)|<\infty\tag{2.2}$$

implies the almost-orthogonality of $(X_n)_n$. By assuming (2.3) we are able to prove the following result.

Lemma 2.5. *Let $(X_n)_n$ be a sequence of random variables with ψ-negligible covariance and $\sup_n \mathrm{Var}[X_n] < \infty$. If $\int_1^\infty \psi(u)\exp(u)\,du$ is finite, the sequence $(X_n - \mathsf{E}[X_n])_n$ is almost-orthogonal. In particular, for any $\beta > 3$, the random variable*

$$\frac{1}{\sqrt{n \log^\beta n}} \sum_{k=1}^{n} (X_k - \mathsf{E}[X_k]) \tag{2.3}$$

converges to 0 almost surely and in L^2.

Proof. The relation

$$\begin{aligned} \sup_k \sum_{h=1}^{\infty} |\mathrm{Cov}(X_h, X_k)| &\le \sup_k \mathrm{Var}(X_k) + 2 \sum_{l=2}^{\infty} \psi(\log l) \\ &\le \sup_k \mathrm{Var}(X_k) + 2\psi(\log 2) + 2 \int_2^\infty \psi\big(\log(u)\big)\,du \\ &\le \sup_k \mathrm{Var}(X_k) + 4\psi(\log 2) + 2 \int_1^\infty \psi(u) e^u\,du \end{aligned}$$

yields immediately that the sequence $(X_n - \mathsf{E}[X_n])_n$ is almost-orthogonal and the random variable defined in (2.4) converges in L^2. Thus, the lemma is proven by the above remarks on almost-orthogonality. □

3. The main result

With the notation of the previous section, let $(X_n)_n$ be a sequence of positive and L^2-random variables. We prove the following result:

Theorem 3.1. *Let ϕ be an increasing positive function on $\mathbb{R}^+$, with $\sup \phi = \infty$. Moreover, let ψ be a decreasing positive function on $\mathbb{R}^+$, with $\int_0^\infty \psi(u)\,du < \infty$. For every pair p and q of integers, with $p < q$, suppose*

$$|\mathrm{Cov}(X_p, X_q)| \le \psi\big(\log(\phi(q) - \phi(p) + 1)\big).$$

Then

$$U_n = \frac{1}{\phi(n)} \sum_{k=1}^{n} \big(\phi(k) - \phi(k-1)\big)\big(X_k - \mathsf{E}[X_k]\big)$$

converges to 0 in L^2 and almost surely. Moreover, if $\int_0^\infty \psi(u)\exp(u)\,du$ is finite, then for any $\beta > 3$, $U_n[\phi(n)/\log^\beta(\phi(n))]^{1/2}$ converges to 0 in L^2 and almost surely.

Proof. For every $t > 0$, put

$$V_t = \frac{1}{t} \sum_{k=1}^{\lfloor \phi^{-1}(t) \rfloor} \big(\phi(k) - \phi(k-1)\big)\big(X_k - \mathsf{E}[X_k]\big).$$

Since $U_n = V_{\phi(n)}$, it suffices to prove that, as t goes to ∞, the random variable V_t converges to 0 in L^2 and almost surely. Since, for any real number $t \in \,]m, m+1]$, $m \in \mathbb{N}$, we have

$$\frac{1}{t}\Bigg(mV_m - \sum_{k=\lfloor \phi^{-1}(m) \rfloor + 1}^{\lfloor \phi^{-1}(t) \rfloor} b_k\, \mathsf{E}[X_k]\Bigg) \le V_t \le \frac{1}{t}\Bigg((m+1)V_{m+1} - \sum_{k=\lfloor \phi^{-1}(t) \rfloor + 1}^{\lfloor \phi^{-1}(m+1) \rfloor} b_k\, \mathsf{E}[X_k]\Bigg),$$

where $b_k = \phi(k) - \phi(k-1)$, it will be enough to prove the theorem for $(V_m)_m$. To this end, for $k \geq 1$, put

$$\tilde{X}_k = \sum_{j=\lfloor \phi^{-1}(k-1)\rfloor+1}^{\lfloor \phi^{-1}(k)\rfloor} (\phi(j) - \phi(j-1))X_j$$

and $c_k = \lfloor \phi^{-1}(k)\rfloor$. (If $c_{k-1}+1 > c_k$, we mean $\tilde{X}_k \equiv 0$.) Clearly $\tilde{X}_k$ is a positive random variable that satisfies

$$\|\tilde{X}_k\|_{L^2} \leq \sum_{j=c_{k-1}+1}^{c_k} (\phi(j) - \phi(j-1))\|X_j\|_{L^2} \leq \left(\sup_n \mathsf{E}[X_n^2]\right)^{1/2}.$$

Moreover, for every pair p and q, with $p < q$, we have

$$\begin{aligned}\left|\mathrm{Cov}\big(\tilde{X}_p, \tilde{X}_q\big)\right| &\leq \sum_{i=c_{p-1}+1}^{c_p} \sum_{j=c_{q-1}+1}^{c_q} b_i b_j \psi\big(\log(\phi(j) - \phi(i) + 1)\big) \\ &\leq \psi\big(\log(\phi(c_{q-1}+1) - \phi(c_p) + 1)\big) \leq \psi\big(\log(q - \phi(c_p))\big) \\ &\leq \psi\big(\log(q-p)\big) \leq \tilde{\psi}\big(\log(q-p+1)\big),\end{aligned}$$

where $\tilde{\psi}$ is the mapping $x \mapsto \psi((x - \log 2)^+)$; hence the sequence $(\tilde{X}_n)_n$ has $\tilde{\psi}$-negligible covariance. Since V_m is equivalent to $(\tilde{X}_1 + \cdots + \tilde{X}_m)/m$, the statement follows from Lemmas 2.3 and 2.5 by simply remarking that $\tilde{\psi}$ has the same integrability properties as ψ. □

Remark 3.1. It may happen that the condition of ψ-negligibility is not satisfied by the whole sequence $(X_n)_n$, but by suitable subsequences $(X_{n_k})_k$ only. This is the case, for instance, for sequences $(X_n)_n$ such that $\liminf_n |\mathrm{Cov}(X_n, X_{2n})| > 0$ but

$$|\mathrm{Cov}(X_p, X_q)| \leq r\left(\frac{g(p)}{g(q)}\right), \tag{3.1}$$

where g is an increasing positive function on $\mathbb{N}$, with $\sup g = \infty$, and r is an increasing function vanishing at 0. More specifically, if for a subsequence $(n_k)_k$ (induced by an increasing map f defined on $\mathbb{R}^+$) there exists a constant $c > 0$ such that, for every pair p and q, with $p \leq q$,

$$cg(n_q) \geq g(n_{q-p+1})g(n_p), \tag{3.2}$$

then $(X_{n_k})_k$ has ψ-negligible covariance where $\psi = r \circ \frac{c}{g\circ f} \circ \exp$. So conditions (3.1) and (3.2) yield

$$\left|\mathrm{Cov}(X_{n_p}, X_{n_q})\right| \leq \psi\big(\log(q-p+1)\big).$$

In what concerns the integrability of the function ψ, we remark that the condition suggested by Weber, namely

$$\sup_m \sum_{h=1}^m r\left(\frac{g(h_h)}{g(n_m)}\right) + \sum_{h=m+1}^{\infty} r\left(\frac{g(n_m)}{g(n_h)}\right) < \infty, \tag{3.3}$$

assures even the integrability of the function $\psi \circ \log$ on $[e, \infty[$ (i.e., the integrability of $u \mapsto \psi(u)e^u$ on $[1, \infty[$). As a matter of fact, we have

$$\begin{aligned}\int_e^\infty \psi(\log t)\, dt = \int_e^\infty r\left(\frac{c}{g(f(t))}\right) dt &\leq \sum_{h=1}^\infty r\left(\frac{c}{g(n_h)}\right) \\ &\leq \overline{m} r\left(\frac{c}{g(n_1)}\right) + \sum_{h=\overline{m}+1}^\infty r\left(\frac{g(n_{\overline{m}})}{g(n_h)}\right),\end{aligned}$$

where $\overline{m}$ is an integer such that $g(n_{\overline{m}}) \geq c$. In other words, if the variances of $(X_{n_k})_k$ are bounded, conditions (3.1), (3.2), and (3.3) guarantee that $(X_{n_k} - \mathsf{E}[X_{n_k}])_k$ is an almost-orthogonal sequence. Some examples of almost-orthogonal subsequences will be given in Section 4. We remark also that the integrability of ψ is assured by the condition

$$\int_1^\infty \frac{dx}{x^2} \int_1^x r\left(\frac{g(f(t))}{g(f(x))}\right) dt < \infty, \tag{3.4}$$

which is weaker than (3.3). Finally if, instead of (3.2), the condition

$$c\, g(q) \geq g(p) e^{C\,(\phi(q)-\phi(p)+1)} \tag{3.5}$$

holds for a pair c and C of positive numbers and for an increasing positive function ϕ, with $\sup \phi = \infty$, then we get

$$r(g(p)/g(q)) \leq \tilde{\psi}(\log(\phi(q) - \phi(p) + 1)),$$

where $\tilde{\psi} = r \circ c \cdot \exp(-C \exp)$. Hence conditions (3.1) and (3.5) yield that

$$U_n \left[\phi(n)/\log^\beta(\phi(n))\right]^{1/2}$$

converge almost surely to 0, for every $\beta > 3$, if $\int_0^\infty r(ce^{-Cx})\, dx < \infty$.

4. Applications

Many situations in which the above results can be applied are listed in Berkes and Csáki [4]. In this section, we study an example which generalizes the classical *Almost Sure Central Limit Theorem.* More specifically, let $(W_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables, with $\mathsf{E}\, W_1^2 = 1$ and $\mathsf{E}\, W_1 = 0$, and put

$$U_n = \frac{W_1 + \cdots + W_n}{\sqrt{n}}.$$

Moreover, consider the random variables

$$X_n = \mathbf{1}_{\bigcap_{k=a_n}^{b_n} \{U_k \in J\}},$$

where J is an interval and (a_n) and (b_n) are two increasing sequences of integers, with $a_n \leq b_n$. In the case $a_n = b_n = n$, the classical ASCLT states that $\sum_{j=1}^n j^{-1} X_j / \log n$ converges almost surely to $\mathcal{N}(0,1)(J)$. In particular, for any $M > 1$, it follows that $(X_{\lfloor M^1 \rfloor} + \cdots + X_{\lfloor M^n \rfloor})/n$ converges almost surely to $\mathcal{N}(0,1)(J)$. In Giuliano Antonini and Weber [8] the classical ASCLT is generalized for $a_n = n$, $b_n = \lfloor Ln \rfloor$, and $L \geq 1$.

In the present situation, we remark that there exists a constant C, depending only on the law of W_1, such that, for every pair p and q of integers, with $p \leq q$, we have

$$|\mathrm{Cov}(X_p, X_q)| \leq C \left(\frac{b_p}{a_q}\right)^{1/4}. \tag{4.1}$$

(See Giuliano Antonini [7] for a proof.) The above inequality does not guarantee that the sequence (X_n) has ψ-negligible covariance for some function ψ. Nevertheless, by Remark 3.2, if there exist two constants $c_1, c_2 > 0$ such that

$$b_p \leq c_1 a_q e^{-c_2(\phi(q)-\phi(p))} \tag{4.2}$$

(i.e., condition (3.5) is verified), we deduce that

$$U_n = \frac{1}{\sqrt{\phi(n)} \log^{\beta/2}\left(\phi(n)\right)} \sum_{k=1}^n \left(\phi(k) - \phi(k-1)\right)\left(X_k - \mathsf{E}[X_k]\right)$$

converges to 0 almost surely, for any $\beta > 3$. In particular, we get the following result:

Generalized ASCLT. *Assume that condition* (4.2) *holds and*

$$\lim_n b_n/a_n = L, \qquad \phi(k) - \phi(k-1) \sim \phi'(k).$$

Then

$$\frac{1}{\phi(n)} \sum_{k=1}^{n} \phi'(k) \mathbf{1}_{\bigcap_{j=a_k}^{b_k} \{U_j \in J\}}$$

almost surely converges to $\mathsf{P}(\bigcap_{1 \le t \le L} \{B_t/\sqrt{t} \in J\})$, *where* $(B_t)_t$ *denotes the standard Brownian motion. In particular, if* $\phi = \log$ *and* $(a_n/n^c)_n$ *is an increasing sequence for some positive constant* c, *condition* (4.2) *is verified and* $\lim_n b_n/a_n = L$ *implies that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\bigcap_{j=a_k}^{b_k} \{U_j \in J\}}$$

converges to $\mathsf{P}(\bigcap_{1 \le t \le L} \{B_t/\sqrt{t} \in J\})$ *almost surely; moreover, the random measure*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\sup_{a_k \le j \le b_k} U_j}$$

strictly converges to μ *almost surely, where* μ *denotes the distribution of*

$$\sup_{1 \le t \le L} \{B_t/\sqrt{t}\}.$$

Proof. It is immediate to note that $\phi = \log$ satisfies condition (4.2) if $(a_n/n^c)_n$ is an increasing sequence. Hence, by the remarks of the previous section, it is enough to prove that

$$\lim_n \mathsf{P}\Bigg(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\Bigg) = \mathsf{P}\Bigg(\bigcap_{1 \le t \le L} \{B_t/\sqrt{t} \in J\}\Bigg). \tag{4.3}$$

To this end, for every real number $t \ge 1$, let

$$V_n(t) = \frac{1}{\sqrt{\lfloor a_n t \rfloor}} \sum_{k=1}^{\lfloor a_n t \rfloor} X_k.$$

By Donsker's functional central limit theorem, $(V_n(t))_{t \ge 1}$ converges in distribution (according to Skorokhod's distance) to the Gaussian process

$$V(t) = \frac{B_t}{\sqrt{t}}.$$

Let ϵ be a fixed positive number, with $\epsilon < L$, and consider the functions g_1 and g_2, defined on Skorokhod's space by

$$g_1(x) = I_{\{\sup_{t \in [1, L-\epsilon]}(-x(t)) \le -c,\ \sup_{t \in [1, L-\epsilon]} x(t) \le d\}},$$
$$g_2(x) = I_{\{\sup_{t \in [1, L+\epsilon]}(-x(t)) \le -c,\ \sup_{t \in [1, L+\epsilon]} x(t) \le d\}},$$

where c and d are the endpoints of the interval J. Put $Y = \sup_{1 \le t \le L+\epsilon} V(t)$. Then Y has a density with respect to Lebesgue measure (see Nualart [12], Proposition 2.1.4), hence g_1 and g_2 are almost surely continuous with respect to $Y(P)$. From Donsker's functional central limit theorem we deduce that

$$\lim_n \mathsf{E}[g_1(V_n)] = \mathsf{E}[g_1(V)], \qquad \lim_n \mathsf{E}[g_2(V_n)] = \mathsf{E}[g_2(V)],$$

so that

$$\mathsf{E}[g_1(V)] = \liminf_n \mathsf{E}[g_1(V_n)] \le \liminf_n \mathsf{P}\Big(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\Big) \le \limsup_n \mathsf{P}\Big(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\Big)$$
$$\le \limsup_n \mathsf{E}[g_2(V_n)] = \mathsf{E}[g_2(V)].$$

Since the difference $\mathsf{E}[g_2(V)] - \mathsf{E}[g_1(V)]$ converges to 0 as ϵ goes to 0 (the sample paths of V are continuous), from the above relations we get (4.3), hence the theorem. □

Remark 4.1. Let $(n_k)_k$ be the subsequence defined by $n_k = \lfloor a_k \rfloor$ and assume that b_n/a_n converges to a real number L. From inequality (4.1) and relations (3.1) and (3.2), it is not difficult to deduce that $(X_{n_k} - \mathsf{E}[X_{n_k}])_k$ is an almost-orthogonal sequence if there exists a constant $c > 0$ such that, for any pair p and q, with $p \le q$,

$$ca_q \ge a_{q-p+1}a_p$$

and

$$\sup_m \sum_{h=1}^{m} \sqrt[4]{\frac{a_h}{a_m}} + \sum_{h=m+1}^{\infty} \sqrt[4]{\frac{a_m}{a_h}} < \infty.$$

In particular, if $a_n = M^n$ and $b_n = LM^n$, with $L \ge 1$ and $M > 1$, the above conditions are verified and

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\bigcap_{j=a_k}^{b_k}\{U_j \in J\}}$$

converges to $\mathsf{P}(\bigcap_{1\le t\le L}\{B_t/\sqrt{t} \in J\})$ almost surely. We can analogously get the result for the random measures $n^{-1}\sum_{k=1}^{n} \delta_{\sup_{a_k \le j \le b_k} U_j}$.

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