BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 1, January 1977

A STRUCTURAL GENERALIZATION OF THE RAMSEY THEOREM

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Communicated by Solomon Feferman, August 30, 1976

ABSTRACT. A generalization of the Ramsey theorem is stated. This solves a problem of P. Erdős and others. The result has recent applications in the theory of ultrafilters and model theory.

The Ramsey theorem [3] states:

For all positive integers k, m, p *there exists an* n *such that for every coloring* $c\colon [n]^p \longrightarrow k$, *there exists a homogeneous* m *set*, $M \subseteq n$, $|M| = m$, *with* $|c([M]^p)| = 1$.

This can be generalized to set systems of a given type and to set systems without forbidden subsystems. The purpose of this note is to announce this result.

A family $\Delta = (\delta_i;\ i \in I)$, $\delta_i \geqslant 1$, is called a type. $(X, \mathcal{M}) = (X, (\mathcal{M}_i;\ i \in I))$ is a set system of type Δ if $\mathcal{M}_i \subseteq [X]^{\delta_i}$ and X is a finite ordered set. $f\colon (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N}) = (Y, (\mathcal{N}_i;\ i \in I))$ is called an embedding if $f\colon X \longrightarrow Y$ is a monotone 1-1 mapping and $f(M) \in \mathcal{N}_i \Longleftrightarrow M \in \mathcal{M}_i$ for every $i \in I$. $(X, \mathcal{M})$ is a subsystem of $(Y, \mathcal{N})$ if the inclusion $X \subseteq Y$ is an embedding. Denote by $\mathrm{Emb}(A, B)$ the set of all embeddings $A \longrightarrow B$ and by $\mathrm{Set}(\Delta)$ the category of all set systems of type Δ and all embeddings.

The following holds:

THEOREM. *Let a type* Δ *be fixed. Let* k *be a positive integer and* $A \in \mathrm{Set}(\Delta)$. *Then for every* $B \in \mathrm{Set}(\Delta)$ *there exists* $C \in \mathrm{Set}(\Delta)$ *such that the following holds*: *for every coloring* $c\colon \mathrm{Emb}(A, C) \longrightarrow k$ *there exists a subsystem* B' *of* C *which is isomorphic to* B *such that* $|c(\mathrm{Emb}(A, B'))| = 1$. *Moreover, if* B *does not contain a fundamental set system* D, *then* C *may be chosen with the same property. Here* $D = (X, \mathcal{M})$ *is fundamental if for every* $i \in I$ *either* $\mathcal{M}_i = \varnothing$ *or* $\mathcal{M}_i = [X]^{\delta_i}$.

This generalizes the Ramsey theorem and has the following consequences:

COROLLARY 1. *For every graph* $G = (V, E)$ *without a complete graph with k-vertices, there exists a graph* $H = (W, F)$ *without a complete subgraph*

AMS (MOS) subject classifications (1970). Primary 05A99; Secondary 04A20, 02H05.
Key words and phrases. Ramsey theorem, set systems, partition.

with k vertices such that for every partition $[W]^2 = F_1 \cup F_2$, *there exists an induced subgraph* $G' = (V', E')$ *of* H, $G \simeq G'$, *such that* $E' \subseteq F_i$ *and* $[V']^2 \backslash E' \subseteq F_j$ *for suitable* i, j.

(This is a generalization of the authors' solution of the Erdös-Folkman-Galvin-Hajnal problem, see [1]. This result was used by F. Galvin (private communication) in the theory of ultrafilters.)

COROLLARY 2. *For every p-uniform set system* $B = (X, M)$ (*i.e.* $M \subseteq [X]^p$) *there exists a p-uniform set system* $C = (Y, N)$ *such that for every partition* $[Y]^p = N_1 \cup N_2$, *there exists a subsystem* $B' = (X', M')$ *of* C, $B \simeq B'$, *such that* $M' \subseteq N_i$, $[X']^p \backslash M' \subseteq N_j$ *for suitable* i, j. *Moreover, if B does not contain a subsystem isomorphic to* $(Z, [Z]^p)$, *then C may be chosen with the same property.*

This solves a problem of P. Erdös and others.

COROLLARY 3. *Let k be a positive integer,* $\Delta = (\delta_i; i \in I)$ *a fixed type with* $\delta_0 = p$ (*hence we assume* $0 \in I$). *Then for every set system* (X, M) *of type* Δ, *there exists a set system* (Y, N) *of type* Δ *such that for every coloring* c: $[Y]^p \longrightarrow k$, *there exists a subsystem* $(X', M') = (X', (M'_i; i \in I))$ *of* (Y, N), $(X', M') \simeq (X, M)$, *with the property that the color* $c(P)$ *of a* $P \in [X']^p$ *depends on the isomorphism type of* $(P, N|_P)$ *only.*

(In an another setting this result was recently obtained independently by F. G. Abramson and L. A. Harrington who discovered this in a model theoretical context–models of Peano arithmetic without indiscernibles.)

The proof of the above Theorem will appear in J. Combinatorial Theory A. Intuitively, the main difficulty is caused by the fact that a Ramsey type theorem needs very complex objects while (in the above Theorem) the desirable objects are (locally) meager. This follows from two demands: we consider embeddings (rather than monomorphisms) and we do not admit "forbidden" subsystems.

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