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A STRUCTURE FOR THE BAYES ANALYSIS OF THE PERFORMANCE OF DETECTION SYSTEMS WITH MULTIPLE SENSORS AND SITES

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PREFACE

As part of the RAND Corporation's work under Project RAND in the area of signal processing, this Memorandum applies Bayes analysis to the passive detection of the presence or absence of a signal source by a variety of sensors located at different sites. More generally, the study provides a method of combining the costs of decision with the explicit costs of operation in the evaluation of expected system performance. It should therefore be of interest to those concerned with system analysis and design. Dr. Middleton is a consultant to The RAND Corporation.

SUMMARY

In this Memorandum, the usual Bayes analysis for optimum detection systems is extended to include the costs of equipment, location, maintenance, and other features of operation, in addition to the customary preassigned costs of correct and incorrect decisions, when multiple receiving sites (Q) and sensors ($\underset{\sim}{M}$) are employed for acquisition and ultimately joint processing of data for simple binary (i.e., "yes" or "no") decisions as to the presence or absence of a signal source.

A risk formalism is constructed which indicates how the expected overall costs (of both decision and operation) can be generally determined for a variety of realistic cost models. Since the expected cost (or average risk) of decision is a monotonically decreasing function of M and Q, while the associated costs of operation are reasonably described by monotonically increasing functions of M and Q, values of M and Q may also exist for which the total average cost (or risk) can be minimized, as well as the cost of decision itself (i.e., for a Bayes decision system). In any case, the physics of the particular detection situation (radar, radio, seismic, acoustic, etc.) is included in the usual way, under the important constraint of decision optimality (i.e., Bayes decisions). Thus, one purpose of this introductory study is to provide some possible models for system evaluation and comparison which specifically include the often controlling factors of operational costs (vis-à-vis those of decision). A simple analytic example is used to illustrate the approach, which is, however, capable of handling much more general examples, both conceptually and

quantitatively. In these more general cases it is expected that computer aids will be needed.

CONTENTS

I. INTRODUCTION

The central problem discussed in this Memorandum is the effective (passive) detection and classification of a signal source in the presence of interfering noise. To accomplish this, a variety of sensors is used, located at various ranges from the possible source. These sensors respond to different physical aspects of the source and are generally subject to different types of interference. It is necessary not only to detect the source, but also to locate it and to identify its characteristics. This study, however, is concerned solely with detection.

The approach is based on the following controlling factors: (1) the costs associated with both correct and incorrect decisions, (2) the cost of the equipment, and (3) the costs of location, maintenance, and operation. The physics of the observation process must also be included, through an adequate description of the signal and noise waveforms that are received, which involves the pertinent geography of sites and sensors. Finally, a practically close approach to optimum performance is sought. Here it is natural to choose as the criterion of optimality the minimum average cost (or risk) associated with the decision process of detection, in which the costs involved must include the types discussed above if there is to be a successful guide to expected system performance in real situations. Systems constructed on the basis of minimum average cost are known as Bayes systems, and to describe these, there is now a considerable body of established theory and practice.[1,2]

Closely associated with average and minimum average (i.e., Bayes) costs are the conditional error probabilities (α,β) and the conditional probabilities (1-α, 1-β) of correct decisions. In fact, in the usual theory, the former is a linear function of the latter. One of the questions to be studied is the improvement that may arise when more than one type of sensor is employed at an observation site. Another is the effect of increasing the number of sites from, say, one to several dozen. It is clear that increasing the number of different sensors at a site will improve the detection probability and lower the false-alarm rate, provided that the various background noises are essentially independent. There is then a larger statistical sample (containing the signal, if present) than with one sensor alone. Furthermore, increasing the number of sites acts to increase the statistical sample, with a consequent improvement in detection probabilities, etc. This occurs in a different way, of course, since now site location vis-à-vis the potential signal source plays a critical role, mainly through the effect of the distance from source to receiver on the level of the received signal. This may be called a "geographical" effect.

It is assumed here that the ultimate decision-making ("yes" or "no") is to be made at some central point, where the final processing of the data is to be carried out. Without this pooling of the data and common processing to an ultimate binary decision, the advantages gained by increasing the size of the effective statistical sample are generally lost. (In later studies, it may be desirable to examine what happens when central decision-making is not used, but for the purposes here--to gain a quantitative insight into the upper bounds on performance under limiting and often idealized conditions--ultimate centralized processing and decision-making will be assumed.)

While increasing the number of sites and sensors with proper handling leads to improved detection performance, the effective costs of doing so cannot be ignored. Practical limits to this improvement at once bound performance. For example, site and equipment costs, etc., can severely limit both the number and quality of the received data available for making decisions. Thus, as the quality of performance improves with the number of sensors, so does the cost, so that at some point a limit will occur, determined by what can be afforded. This limit can be established in a quantitative way by suitable choice of cost function, as shall be noted. Accordingly, Section II outlines the Bayes formation* for this problem, establishing the associated average risks and cost-function selections, while Section III illustrates the approach with a simple example (providing a strong upper bound on improvement) to show how many of the realistic features of the actual situation can be included simply in a first try at the problem.

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*Decision theory methods and their applications to detection problems are discussed in Refs. 1 and 2.

II. A BAYES FORMULATION

First, it is necessary to establish a general loss function, $\mathfrak{F}$, which is to combine in suitable fashion the data, the signal structure, the type and character of the system, and the decision rule. It is possible to distinguish a hierarchy of problems in which this must be done:

1. Single sites with a variable number of sensors, M. (Each sensor is a different type.)
2. Multiple sites with each site having a single sensor of the same type (Q = number of sites).
3. Combinations of 1 and 2, with multiple sites and sensors.

Within each member of this hierarchy, the more detailed physics of the sensing processes is included, along with geographical and other considerations. The loss function, $\mathfrak{F}$, has the general form

$$\mathfrak{F} = \mathfrak{F}\,(\text{system};\ \underset{\sim}{S},\ \underline{\gamma}(\underset{\sim}{V}))\ , \tag{1}$$

where $\underset{\sim}{S}$ and $\underset{\sim}{V}$ are respectively the signal and received data vectors, and $\underset{\sim}{S} = [S(t_1),\ldots,S(t_n)]$, etc., sampled discretely, for the moment, on a time interval $(t_o,\ t_o + T)$ in the usual way.* The $\gamma = [\gamma_0,\ldots,\gamma_k]$ are K+1 decisions, based on $\underset{\sim}{V}$, about the signal S. (In the present problem, K = 1, i.e., γ_0 and γ_1, are the decisions "no" (no signal) and "yes" (a signal) respectively.) "System" is included in $\mathfrak{F}$ as a reminder that it must specifically include the cost of the system, e.g., of single sites, multiple sensors, multiple sites, etc., as well as the costs associated with particular decisions.

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*See Ref. 2, Sections (1.2) and (1.3).

THE AVERAGE COST OR RISK

The average loss is

$$\mathcal{L}(\text{system}, \sigma,\delta) = \left\langle \int_{\Omega} d\underset{\sim}{S}\, \sigma(\underset{\sim}{S}) \int_{\Gamma} F_n(\underset{\sim}{V}|\underset{\sim}{S})\, d\underset{\sim}{V} \int_{\Delta} \delta(\underset{\sim}{\gamma}|\underset{\sim}{V})\, \mathfrak{F}(\text{system}, \underset{\sim}{S},\underset{\sim}{\gamma})\, d\underset{\sim}{\gamma} \right\rangle_{\text{system}} \tag{2}$$

where again Ω, Γ, and Δ are signal, data, and decision spaces respectively.* Here we define

$$\left.\begin{aligned} F_n(\underset{\sim}{V}|\underset{\sim}{S}) &= \text{conditional d.d.}^{**} \text{ of } \underset{\sim}{V}\ (= V(t_1),\ldots,V(t_n)),\ \text{given } \underset{\sim}{S} \\ \sigma(\underset{\sim}{S}) &= \text{a priori d.d. of } \underset{\sim}{S} \end{aligned}\right\} \tag{3}$$

and $\delta(\underset{\sim}{\gamma}|\underset{\sim}{V})$ is a decision rule, which for detection here is a probability, viz., the probability of making the decisions $\underset{\sim}{\gamma}$ on the basis of the data $\underset{\sim}{V}$. In effect, δ is the embodiment of the receiver structure. Here $\langle\ \rangle_{\text{system}}$ refers to appropriate statistical averages over the system itself, where needed.

At this point $\mathfrak{F}$ is chosen as the direct or explicit cost function

$$\mathfrak{F} \rightarrow \mathfrak{F}_1(\text{system}, \underset{\sim}{S},\underset{\sim}{\gamma}) = C(\text{system}, \underset{\sim}{S},\underset{\sim}{\gamma}) \tag{4}$$

where C is independent of the decision rule δ and associates a set of preassigned fixed costs to the various possible decisions $\underset{\sim}{\gamma}$ about $\underset{\sim}{S}$. Where $\mathfrak{F}_1 = C$ is used in Eq. (2) the result is the average cost or risk, i.e.,

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*See Ref. 2, Sections (1.2) and (1.3).

**Distribution density.

$$R(\text{system}, \sigma, \delta) = \left\langle \int_{\Omega} d\underset{\sim}{S}\, \sigma(\underset{\sim}{S}) \int_{\Gamma} F_n(\underset{\sim}{V}|\underset{\sim}{S})\, d\underset{\sim}{V} \int_{\Delta} \delta(\underset{\sim}{\gamma}|\underset{\sim}{V})\; C(\text{system}, \underset{\sim}{S}, \underset{\sim}{\gamma})\, d\underset{\sim}{\gamma} \right\rangle_{\text{system}} \tag{5}$$

which is the measure used here for evaluating system performance.

Equation (5) may be developed further. First, set

$$\underset{\sim}{V}_{k\ell} = \text{discrete data received at } k^{th} \text{ sensor, located at } \ell^{th} \text{ site} \tag{6a}$$

$$\therefore \underset{\sim}{V} \equiv \underset{\sim}{V}_{MQ} = \left[\underset{\sim}{V}_{k\ell}\right] = \text{complex of all received data.} \tag{6b}$$

Then we have

$$\left.\begin{aligned} F_n(\underset{\sim}{V}|\underset{\sim}{S}) &\rightarrow F_n(\underset{\sim}{V}_{11}, V_{21}, \ldots V_{M1}; \ldots; V_{1Q}, V_{2Q}, \ldots V_{MQ}|S)_{M,Q} \\ &\equiv F_n(\underset{\sim}{V}|\underset{\sim}{S}; M, Q) \equiv \text{joint (conditional) d.d. of all the received data, } \underset{\sim}{V}\text{, given } \underset{\sim}{S} \end{aligned}\right\} . \tag{7}$$

Once F_n is known or postulated, the various couplings and interrelations seen in the various data received at the different sensors and sites are automatically included. Note that F_n is also explicitly a parametric function of the number of sensors (M) at each site ($\ell = 1, \ldots, Q$). In the above formalism it is assumed that M is the same at each site. A more general condition is

$$F_n(\underset{\sim}{V}|\underset{\sim}{S};)_{\underset{\sim}{M},Q} \equiv F_n\left(V_{11}, \ldots, V_{M_1 1}; \ldots; V_{1\ell}, \ldots, V_{M_\ell \ell}; \ldots; V_{1N}, \ldots V_{M_N N}|\underset{\sim}{S}\right), \tag{7a}$$

which makes it possible to distinguish the number and kind of sensor at each site ($\ell = 1, \ldots, Q$).

OPTIMUM SYSTEMS

Next, Bayes, or optimum, systems, with minimum average risk, are considered. For a fixed number of sensors per site ($\underset{\sim}{M} = M_1, \ldots, M_\ell, \ldots, M_Q$ fixed) and a fixed number of sites we have

$$\min_{\delta \to \delta^*} R(\text{system}; \sigma, \delta)_{\underset{\sim}{M},Q} = R^*(\text{system}; \sigma, \delta^*)_{\underset{\sim}{M},Q}, \tag{8}$$

where δ^* is the optimizing decision rule, according to the left member of Eq. (8). Thus, $R^*_{\underset{\sim}{M},Q}$ is the Bayes risk for a fixed system. Next, for a fixed number of sites (Q fixed), a minimization over $\underset{\sim}{M}$ may be attempted:

$$\min_{\underset{\sim}{M}} R^*(\text{system}; \sigma, \delta^*)_{\underset{\sim}{M},Q} = \min_{\underset{\sim}{M} \to \underset{\sim}{M}^*} \min_{\delta \to \delta^*} R(\text{system}; \sigma, \delta) \equiv R^{**}(\text{system}; \sigma, \delta^*)_{\underset{\sim}{M}^*,Q}. \tag{9}$$

(Note that the order (min M) after (min δ) is essential: if the reverse, clearly (min $\underset{\sim}{M}$) first would set $\underset{\sim}{M}=0$ (to give R=0), which is a trivial result.) Similarly, for optimization over the number of sites (Q), (min Q) must follow (min δ). However, (min Q) and (min $\underset{\sim}{M}$) are interchangeable in the sense that site optimization per se need not preclude sensor optimization at each site. Of course, it can be seen that

$$R^{**|*} \equiv \min_{Q} \min_{\underset{\sim}{M}} R^*_{\underset{\sim}{M},Q} \neq \min_{\underset{\sim}{M}} \min_{Q} R^*_{\underset{\sim}{M},Q} \equiv R^{*|**} \tag{10}$$

since site selection may influence the choice of $\underset{\sim}{M}$ and vice-versa. Some sites may profitably (in the sense of minimum average cost, of course) employ fewer or more sensors than others, or site selection may be influenced by the need for and/or the number of sensors available, and so on. Because of the decreasing cost of decisions with increasing numbers of sites and sensors, along with the increasing cost of operation, maintenance, etc., there should be non-zero values of $\underset{\sim}{M}$ and Q which yield acceptable values of R^{**}, $R^{*|**}$, and $R^{**|*}$. Stationary values of average risk for selected $\underset{\sim}{M}$ and Q may also be expected. However, these stationary values depend rather critically on a proper choice of cost function C (see Eq. (4)).

CONSTRUCTION OF A COST FUNCTION

It is now significant to construct a reasonable cost function for the present problem. First, it is observed that the detection situation requires a binary decision ("yes" or "no") and that there are four possible costs of the consequences of the decisions to be associated with the possible decisions. With the direct cost function used here, the costs are constant and preassigned, yielding

$$\begin{aligned} C_\alpha &= \text{cost of incorrectly deciding a signal is present when noise alone exists,} \\ C_{1-\alpha} &= \text{cost of correctly deciding noise alone,} \\ C_\beta &= \text{cost of incorrectly deciding only noise occurs when both a signal and noise occur,} \\ C_{1-\beta} &= \text{cost of correctly deciding that a signal is present.} \end{aligned} \tag{10}$$

Clearly, by these definitions we have

$$C_\alpha > C_{1-\alpha}; \qquad C_\beta > C_{1-\beta}. \tag{11}$$

Note that these costs (of decision only) are assigned vis-à-vis the possible hypothesis states (H_0 = no signal; H_1 = signal or signal classes) and not with respect to any one signal in a signal class. It is emphasized here that costs of operation, maintenance, etc., are not included in the C_α, etc., which by definition are also independent of the number of sites and sensors.

Next, it is appropriate to postulate a constant cost function (the same* for all four possible outcomes) which specifically takes into account the maintenance and operational costs of sensors and sites. This is

$$C(\text{system}, \underset{\sim}{S}, \underset{\sim}{Y})_{\text{oper.}} = C_o(Q, \underset{\sim}{M}) \equiv C_{Q,M}, \tag{12}$$

where $C(Q,\underset{\sim}{M})$ is a monotonically increasing function of Q, the number of sites, and $\underset{\sim}{M}$, the number of sensors at each site. Just what functions are to be used needs to be explored. A simple example would be

$$C_o(Q,\underset{\sim}{M}) = \sum_{\ell=1}^{Q} a_\ell M_\ell \ . \tag{13}$$

If different costs for different sensors at site (ℓ) are distinguished, then $M_\ell \Rightarrow \left[M_{\ell 1}, M_{\ell 2}, \ldots, M_{\ell J}\right]$, where M_{53} is the number of sensors of type 3 at site 5, etc. Then, Eq. (13) is expanded to

$$C_o(Q,\underset{\sim}{M}) = \sum_{\ell=1}^{Q} \sum_{j=1}^{J} A_{\ell j} M_{\ell j} \ . \tag{14}$$

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*More generally, it is possible to assign different costs to the different outcomes, but this degree of refinement does not seem necessary at this stage.

In all cases the a_ℓ, $a_{\ell j} \geq 0$. The adjustable parameters $a_{\ell j}$ permit flexibility in assigning the weightings to the M's; a stronger dependence on the number of sensors might be

$$C_o(Q,\underline{M}) = \sum_{\ell=1}^{Q} \sum_{j=1}^{J} b_{\ell j} M_{\ell j}^2 , \tag{15}$$

and so on. As a start, the linear cost function of Eq. (14) appears reasonable. The chief feature to note here about $C_o(Q,\underline{M})$ is that it is the same for all decisions.

With the help of Eqs. (10) and (12) through (15) it is possible to construct a cost matrix $\underline{C}$(system, $\underline{S},\underline{\gamma}$) as follows:

$$\begin{aligned} \underline{C}(\text{system}, \underline{S},\underline{\gamma}) &= \begin{bmatrix} C_{1-\alpha} & C_\alpha \\ C_\beta & C_{1-\beta} \end{bmatrix} + \lambda \begin{bmatrix} C_o(Q,\underline{M}) & C_o(Q,\underline{M}) \\ C_o(Q,\underline{M}) & C_o(Q,\underline{M}) \end{bmatrix} \\ &= \begin{bmatrix} C_{1-\alpha} + \lambda\, C_{QM} & C_\alpha + \lambda\, C_{QM} \\ C_\beta + \lambda\, C_{QM} & C_{1-\beta} + \lambda\, C_{QM} \end{bmatrix} , \end{aligned} \tag{16}$$

where $\lambda(>0)$ represents a coupling factor, which embodies in some quantitative way (i.e., by its numerical magnitude) the respective weighting of the two types of cost: those of decision and its consequences, and those of site and sensor operation, etc. Thus, λ represents an additional degree of freedom at the system analyst's disposal, to adjust the various objective and subjective elements that necessarily go into the overall assignment of costs.

AVERAGE AND BAYES RISKS FOR BINARY DETECTION

Applying Eq. (16) to Eq. (5) and integrating over the two points (γ_0, γ_1) in decision space here, where γ_0 = the decision that only noise is present and γ_1 = the decision that a signal as well as noise occurs, gives in the usual way* for the average cost

$$R(\text{system}, \sigma, \delta)_{\underline{M},Q} = qC_{1-\alpha} + pC_{1-\beta} + q(C_\alpha - C_{1-\alpha})\alpha + p(C_\beta - C_{1-\beta})\beta + \lambda C_o(Q,\underline{M}), \tag{17}$$

where

$$\left.\begin{aligned} p = 1-q &= \text{a priori probability that the data sample received contains signal as well as noise} \\ q &= \text{a priori probability that the received data contains noise only} \end{aligned}\right\}, \tag{18}$$

and

$$\alpha = \int_\Gamma F_n(\underline{V}|\underline{S})_{\underline{M},Q}\, \delta(\gamma_1|\underline{V})d\underline{V} = \alpha(\underline{M},Q) : \quad \text{Type 1 error prob.} \tag{19a}$$

$$\beta = \int_\Gamma F_n(\underline{V}|\underline{S})_{\underline{M},Q}\, \delta(\gamma_0|\underline{V})d\underline{V} = \beta(\underline{M},Q;S) : \quad \text{Type 2 error prob.} \tag{19b}$$

Thus, α is the conditional error probability of saying that a signal is present when only noise occurs, while β is the conditional error probability that "noise" is decided, when a signal, as well as noise, is present. Note that α,β are implicit functions of $\underline{M}$,Q through the

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*See Ref. 2, pp. 19-21, 23-26.

structure of the general data distribution density F_n (see Eqs. (7) and (7a)). In general, it is expected that

$$\lim_{M \text{ or } Q \to \infty} \alpha,\beta \to 0 \ , \qquad (0 < S < \infty) \tag{20}$$

provided that the background noise is not strongly correlated for the sensors and sites.* Writing

$$R_o \equiv qC_{1-\alpha} + pC_{1-\beta} \ , \tag{21}$$

$$K \equiv \frac{C_\alpha - C_{1-\alpha}}{C_\beta - C_{1-\beta}} \ (\geq 0) \ , \tag{22}$$

with

$$\mu \equiv p/q \ , \tag{23}$$

for the irreducible risk and a threshold, respectively, Eq. (17) can now be expressed more compactly as

$$R(\text{system}, \sigma,\delta)_{\underline{M},Q} = R_o + p(C_\beta - C_{1-\beta})\left[\frac{K}{\mu}\,\alpha(\underline{M},Q) + \beta(\underline{M},Q)\right] + \lambda C_o(Q,\underline{M}) \ . \tag{24}$$

The key point here is that for min $\underline{M}$ (and Q), R may have a minimum value since α and β both decrease as $\underline{M}$ (and/or Q) become larger (see Eq. (20)), while $C_o(Q,\underline{M})$ increases. Thus, even though the detection system here, embodied in the decision rules $\delta(\gamma_1|\underline{V})$ and $\delta(\gamma_1|\underline{V})$, is not optimum (minimum average risk), it should often be possible to reduce, with reasonable choices of λ, K, μ, etc., this average risk by increasing the

*If this were the case, the statistical sample would not effectively increase for complete correlation.

number of sensors and sites, and, accordingly for this suboptimum system, achieve an average risk, minimized over the system $(Q,\underline{M})$.

More can be done to reduce the average cost of overall performance now by asking in addition that the decision system (the parts dependent on δ_0 and δ_1) yield a minimum average or Bayes risk, according to Eq. (8). Thus, applying Eqs. (9) and (10) yields the further "min-min" expressions $R^{**|*}$ or $R^{*|**}$, etc.

In the Bayes decision case, $\delta \to \delta^*$ (see Eq. (8)), with the well-known result that the decision process is given by

$$\begin{aligned} &\text{decide } H_0\text{: noise only, if} && \Lambda_n < K \\ &\text{decide } H_1\text{: signal and noise, if} && \Lambda_n \geq K\,, \end{aligned} \tag{25}$$

where Λ_n is the generalized likelihood ratio, with the help of Eqs. (7) and (7a) expressed as

$$\Lambda_n = \frac{p\left\langle F_n(\underline{V}|\underline{S})_{\underline{M},Q}\right\rangle_S}{q\ F_n(\underline{V}|0)_{\underline{M},Q}}\ . \tag{26}$$

Usually $\log \Lambda_n$ is used as the test statistic against the threshold $\log K$. Then, for $\delta \to \delta^*$, Eq. (24) becomes

$$R^*(\text{system}, \sigma,\delta^*)_{\underline{M},Q} = R_o + p(C_\beta - C_{1-\beta})\left(\frac{K}{\mu}\,\alpha^*_{M,Q} + \beta^*_{M,Q}\right) + \lambda\ C_o(Q,\underline{M}), \tag{25}$$

and α^* and β^* may be determined by standard techniques (see Eq. (2.4) of Ref. 2).

In the case of a single site (Q=1), it is necessary to find the $M^*_{\ell 1}$, $M^*_{12}, \ldots, M^*_{1J}$ that satisfy

$$\min_{\underset{\sim}{M}_1 \to \underset{\sim}{M}^*_1} R^*(\text{system}, \sigma, \delta^*)_{\underset{\sim}{M},1} = \min_{\underset{\sim}{M}_1} \left[p(C_\beta - C_{1-\beta})\left(\frac{K}{\mu}\alpha^* + \beta^*\right) + \lambda C_o \right] \quad (26)$$

and so on, for multiple sites, according to Eq. (10). Clearly, this can be a rather involved searching operation, as α^* and β^* are not simple functions of $\underset{\sim}{M}_1$, etc., so that it would be expected that at this point solutions would be aided by computers. Evaluation of performance may be described by considering R^* versus $|\underset{\sim}{M}| = M$, and $\alpha = P_F$ and $1-\beta = P_D$ versus $|\underset{\sim}{M}|$, as sketched in Fig. 1 for Q = 1. As suggested by the figure, there may be no value of M (> 1) that minimizes R^*. Different levels of acceptable upper bounds on overall expense are also indicated in Fig. 1. Note that although there may be no minimum (R^{**}) for M > 1, there may still be a useable value of R^* that gives an acceptable P_D and P_F and lies below an acceptable bound on overall expense.

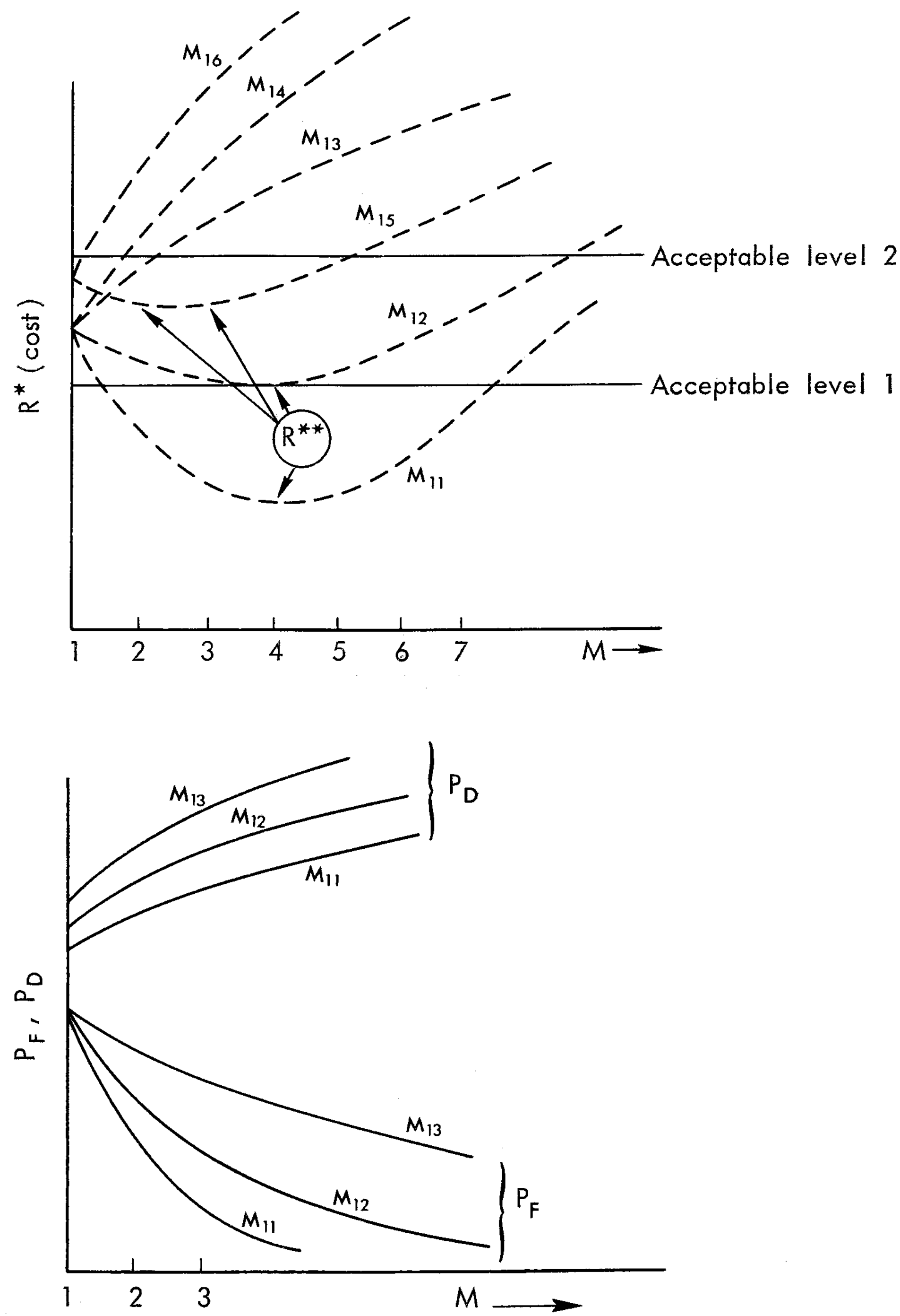

Fig. 1 — R^* versus $|\underset{\sim}{M}|$, and P_F and P_D versus $|\underset{\sim}{M}|$ for various types and combinations of sensors (with a chosen value of λ)

III. A SIMPLE EXAMPLE

It is assumed here that $M_{\ell} = 1$ (i.e., a single sensor at each site), and $Q \geq 1$, one or more sites, and that:

1. All noise processes are gaussian, have the same level, and are independent;
2. All sensors are the same;
3. All signals have the same strength at each site and, of course, come from a single source;
4. Reception is completely coherent; and
5. Equation (14) is used for the operational costs, with $a_{\ell j} = 1$.

Then, using the result of an earlier analysis,* a fixed number of sites Q is obtained for the Bayes error probabilities

$$\left\{\begin{matrix} \alpha^*_{1,Q} \\ \beta^*_{1,Q} \end{matrix}\right\} = \frac{1}{2}\left\{1 - \Theta\left[\frac{a_o\sqrt{\Phi_S Q}}{2\sqrt{2}} \pm \frac{\log \frac{K}{\mu}}{\sqrt{2}\, a_o \sqrt{\Phi_S Q}}\right]\right\} \tag{27}$$

where

$$a_o^2 = \left(\frac{1}{T}\int_o^T S(t)^2\, dt\right)\Big/ 2\psi_N = \text{normalized signal (power) level}$$

$$\Phi_S = 2B_eT = \text{effective processing gain; } B_e = \text{effective bandwidth of signal}$$

$$\sigma_o^2 = \text{output signal-to-noise (power) ratio} = a_o^2\, \Phi_S/2 = (S/N)^2\, \Phi_S/2 \tag{28}$$

$$\Theta(x) = \frac{2}{\sqrt{\pi}}\int_o^x e^{-y^2}\, dy = \text{error integral}$$

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*See Problem (23.4), p. 1045, Ref. 1.

Here

$$\alpha^*_{1,Q} = P_F(Q)_1;\ 1-\beta^*_{1,Q} = P_D(Q)_1 = \frac{1}{2}\left\{1 + \Theta\left[\sigma_o\sqrt{Q} - \Theta^{-1}(1-2P_F(Q)_1)\right]\right\} \quad (29a)$$

$$= \frac{1}{2}\left\{1 + \Theta\left[\frac{\sigma_o\sqrt{Q}}{2} - \frac{\log\frac{K}{\mu}}{2\sigma_o\sqrt{Q}}\right]\right\} \quad (29b)$$

The associated Bayes risk is

$$R^*(\text{system};\ \sigma,\delta_1^*)_{1,Q} = R_o + p(C_\beta - C_{1-\beta})\left[\frac{K}{\mu}\alpha^*_{1;Q} + \beta^*_{1,Q}\right] + \lambda Q; \quad (\lambda>0). \quad (30)$$

Now, it is possible to minimize the number of sites over Q, according to

$$\frac{dR^*}{dQ} = \frac{K}{\mu}\frac{d\alpha^*}{dQ} + \frac{d\beta^*}{dQ} + \lambda = 0 \quad (31)$$

for an extremal, with Eqs. (29):

$$\frac{d\alpha^*}{dQ} = \left(\frac{\sigma_o}{4\sqrt{Q}} - \frac{\log\frac{K}{\mu}}{4\sigma_o Q^{3/2}}\right)\frac{2}{\sqrt{\pi}}\ e^{-y^2(Q)}; \quad y = \frac{\sigma_o\sqrt{Q}}{2} + \frac{\log\frac{K}{\mu}}{2\sigma_o\sqrt{Q}} \quad (32a)$$

$$\frac{d\beta^*}{dQ} = \left(\frac{\sigma_o}{4\sqrt{Q}} + \frac{\log\frac{K}{\mu}}{4\sigma_o Q^{3/2}}\right)\frac{2}{\sqrt{\pi}}\ e^{-z^2(Q)}; \quad z = \frac{\sigma_o\sqrt{Q}}{2} - \frac{\log\frac{K}{\mu}}{2\sigma_o\sqrt{Q}} \quad (32b)$$

Substituting Eqs. (32a) and (32b) into Eq. (31) may then yield the desired value of Q. A simpler way to find Q is numerical: plot Eq. (30) versus Q and choose the Q that gives $\min_Q R^*$, if it exists, or, as in Fig. 1, choose the largest Q that will keep R^*_Q below an acceptable upper bound on cost, since increasing Q always decreases $\alpha^*_{1,Q}$ and $\beta^*_{1,Q}$, etc. (see Eq. (27)).

In the special case $\frac{K}{\mu} = 1$, Eqs. (32a) and (32b) simplify at once to give

$$\frac{dR^*}{dQ} = \frac{\sigma_o}{\sqrt{Q}} \frac{1}{\sqrt{\pi}} e^{-\sigma_o^2 Q/2} + \lambda = 0 \qquad (33)$$

for a possible extremal value of Q. However, since $\lambda > 0$, it is clear from Eq. (33) that in this case no such value exists ($Q > 0$, real, etc.). Returning to Eq. (30), we take instead the largest Q which will still keep $R^*_{1,Q}$ less than the acceptable upper bound.

A more realistic version relaxes assumption 3: now different signal levels can occur at the different sites, reflecting different distances from the source. The more general form of Problem (23.4) of Ref. 1 now gives for Eq. (27),

$$\left\{\begin{matrix} \alpha^*_{1,Q} \\ \beta^*_{1,Q} \end{matrix}\right\} = \frac{1}{2}\left\{1 - \Theta\left(\frac{\sqrt{\sum_{\ell=1}^{Q} \sigma^2_{o\ell}}}{2} \pm \frac{\log \frac{K}{\mu}}{2\sqrt{\sum_{\ell=1}^{Q} \sigma^2_{o\ell}}}\right)\right\}, \qquad (34)$$

where

$$\sigma^2_{o\ell} \equiv a^2_{o\ell}\, \Phi_{s\ell}/2 = (S/N)^2_{out-\ell} \qquad (\ell = 1, \ldots, Q;\ \text{site numbers}) \qquad (34a)$$

Substituting Eq. (34) into Eq. (30) for the overall cost, we can proceed as above to search for a minimizing value of Q, or more likely, a largest acceptable Q, in the sense that $R^*(Q_{max}) <$ the acceptable upper bound on total average cost.

This example is deliberately simple in order to illustrate compactly how to proceed with a given physical situation. The method, however, is in no way changed when more complicated types of sensing (coherent and incoherent, and with different modes of perception) and various locations, costs, etc., are imposed. In any case, this represents a possible model for handling the problem of determining site and sensor selection in conjunction with expected detection performance.

REFERENCES

1. Middleton, D., An Introduction to Communication Theory, McGraw-Hill Book Company, Inc., New York, 1960, Chapter 18, et seq.

2. Middleton, D., Topics in Communication Theory, McGraw-Hill Book Company, Inc., New York, 1965.