A STRUCTURE THEORY OF LIE TRIPLE SYSTEMS(1)

BY

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If in an associative algebra $\mathfrak{A}$ a new composition is introduced by putting $[ab]=ab-ba$, $\mathfrak{A}$ becomes a nonassociative algebra $\mathfrak{A}_L$ with a skew-symmetric product satisfying the Jacobi identity $[[ab]c]+[[bc]a]+[[ca]b]=0$. Furthermore, any abstract algebra having these properties is isomorphic to a subalgebra of some $\mathfrak{A}_L$. These algebras, called Lie algebras, have been extensively studied, and in this paper we shall often refer to some of the many known results. If, on the other hand, we consider the system denoted by $\mathfrak{A}_J$ comprising $\mathfrak{A}$ and the product $\{ab\}=ab+ba$, we observe that $\mathfrak{A}_J$ is a commutative, nonassociative algebra such that $\{\{\{aa\}b\}a\}=\{\{aa\},\{ba\}\}$. Such algebras and their subalgebras are *special Jordan algebras*. This terminology is necessary since they are not characterized by the above properties. Abstract algebras defined by these properties are simply *Jordan algebras*.

The starting point for our discussion is the observation that in an associative algebra $[[ab]c]=\{\{cb\}a\}-\{\{ca\}b\}$, so that any special Jordan algebra may be treated as a subspace of an associative algebra closed under triple Lie products. Jacobson has characterized subspaces of Lie algebras closed under triple Lie products and called them *Lie triple systems*. Further, if in any Jordan algebra $\mathfrak{A}$ we introduce the ternary composition $[abc]=\{\{cb\}a\}-\{\{ca\}b\}$, $\mathfrak{A}$ becomes a Lie triple system, the associator Lie triple system of $\mathfrak{A}$. Also, the mappings $xR_a=\{xa\}$ in a Jordan algebra constitute a Lie triple system which is a homomorph of the associator Lie triple system and is called the multiplication Lie triple system of the Jordan algebra. Suggested by these relations is the possibility of using Lie algebra methods and results in the study of Jordan algebras. Jacobson [7](3) has successfully done this, and it is not our present purpose to pursue further such applications of Lie triple systems. We shall, rather, regard such connections as motivation for the abstract study of the structure and classification of these systems.

In section I we introduce some fundamental concepts and note some results of Jacobson concerning imbeddings of Lie triple systems in Lie algebras. Section II develops notions of the radical, semi-simplicity, and solvability as defined for Lie triple systems including proofs of the existence of a semi-simple subsystem complementary to the radical and of the decomposi-

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tion of a semi-simple system into the direct sum of simple ideals. Section III is somewhat of a digression concerned with the extension to Lie triple systems of linear transformations certain Lie algebra criteria for complete reducibility. In section IV we determine all simple Lie triple systems over an algebraically closed field.

In the sequel we shall presume well known results concerning Lie algebras and suppose throughout that each vector space is finite-dimensional over a field of characteristic 0.

I. Fundamental concepts and imbeddings

Definition 1.1. *A Lie triple system* (L.t.s.) is a vector space $\mathfrak{T}$ over a field Φ and a mapping denoted by $[xyz]$ of $\mathfrak{T}\times\mathfrak{T}\times\mathfrak{T}$ into $\mathfrak{T}$ satisfying:

(i) $[\ ]$ is trilinear.

(ii) $\alpha[xyz]=[(\alpha x)yz]=[x(\alpha y)z]=[xy(\alpha z)]$ for all α in Φ.

(iii) $[xyz]=-[yxz]$.

(1.1) (iv) $[[xyz]vw]+[[yxv]zw]+[yx[zvw]]+[zv[xyw]]=0$.

(v) $[xyz]+[yzx]+[zxy]=0$.

(vi) $[[xyz]vw]+[[yxv]zw]+[[vzy]xw]+[[zvx]yw]=0$.

(vii) $[[[xyz]vw]tu]+[[[yxz]vt]wu]+[[[yxv]zw]tu]$
$+[[[xyv]zt]wu]+R+S=0$,

where R and S are obtained by cyclically permuting the pairs (x, y), (z, v), (w, t) in the displayed terms.

Definition 1.2. If $\mathfrak{T}$ and $\mathfrak{U}$ are L.t.s. over Φ, a homomorphism of $\mathfrak{T}$ into $\mathfrak{U}$ is a linear transformation H of $\mathfrak{T}$ into $\mathfrak{U}$ satisfying:

$$[xyz]H = [(xH)(yH)(zH)].$$

The meaning of isomorphism is now clear. If $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are subsets of a L.t.s. $\mathfrak{T}$, by $[\mathfrak{X}\mathfrak{Y}\mathfrak{Z}]$ we mean the set whose elements have the form $\sum_i[x_iy_iz_i]$ where x_i is in $\mathfrak{X}$, y_i in $\mathfrak{Y}$, and z_i in $\mathfrak{Z}$. Thus a *subsystem* of $\mathfrak{T}$ is a subspace $\mathfrak{U}$ such that $[\mathfrak{U}\mathfrak{U}\mathfrak{U}]\subseteq\mathfrak{U}$.

Definition 1.3. An *ideal* of a L.t.s. $\mathfrak{T}$ is a subspace $\mathfrak{B}$ for which $[\mathfrak{B}\mathfrak{T}\mathfrak{T}]\subseteq\mathfrak{B}$.

If $\mathfrak{B}$ is an ideal, consider the collection $\{x+\mathfrak{B}\}$ of cosets and denote it by $\mathfrak{T}-\mathfrak{B}$. If we define a product by $[(x+\mathfrak{B})(y+\mathfrak{B})(z+\mathfrak{B})]=[xyz]+\mathfrak{B}$, $\mathfrak{T}-\mathfrak{B}$ becomes a L.t.s., the difference L.t.s. of $\mathfrak{T}$ modulo $\mathfrak{B}$. The mapping $x\rightarrow x+\mathfrak{B}$ is a homomorphism of $\mathfrak{T}$ onto $\mathfrak{T}-\mathfrak{B}$ with kernel $\mathfrak{B}$. Conversely, let $\mathfrak{B}$ be the kernel of a homomorphism of $\mathfrak{T}$ onto a L.t.s. $\mathfrak{U}$. $[vxy]H=[(vH)(xH)(yH)]=0$ if v is in $\mathfrak{B}$, so that $\mathfrak{B}$ is an ideal. Ideals, then, are precisely kernels of homomorphisms. Notice also that the ideals of $\mathfrak{T}-\mathfrak{V}$ may be written as $\mathfrak{W}-\mathfrak{V}$, where $\mathfrak{W}$ is an ideal of $\mathfrak{T}$ containing $\mathfrak{V}$.

Among the linear transformations in a L.t.s. those of the following type will be of particular interest:

DEFINITION 1.4. A *derivation* of a L.t.s. $\mathfrak{T}$ is a linear transformation D of $\mathfrak{T}$ into $\mathfrak{T}$ such that $[xyz]D=[(xD)yz]+[x(yD)z]+[xy(zD)]$.

It is readily verified that the set $\mathfrak{D}(\mathfrak{T})$ of derivations of $\mathfrak{T}$ is a Lie algebra of linear transformation acting in $\mathfrak{T}$, the *derivation algebra* of $\mathfrak{T}$. Further, if a_i, b_i, $i=1, \cdots, n$, are arbitrary in $\mathfrak{T}$, the identities (1.1) show that $x\rightarrow\sum_i [a_ib_ix]$ is a derivation in $\mathfrak{T}$. Derivations of this type form a subalgebra $\mathfrak{D}_0(\mathfrak{T})$ of $\mathfrak{D}(\mathfrak{T})$, the algebra of *inner derivations* of $\mathfrak{T}$.

In any associative algebra $\mathfrak{A}$, the subspaces closed under $[[ab]c]=abc-bac-cab+cba$ are L.t.s. In fact, as can be verified from (1.1), any subspace of a Lie algebra closed under $[[ab]c]$ is a L.t.s.; and, as we shall indicate, the identities (1.1) completely characterize such systems.

DEFINITION 1.5. An *imbedding* of a L.t.s. $\mathfrak{T}$ in a Lie algebra $\mathfrak{L}$ is a linear transformation $x\rightarrow x^R$ of $\mathfrak{T}$ into $\mathfrak{L}$ such that

(i) $[xyz]^R=[[x^Ry^R]z^R]$ and

(ii) the enveloping Lie algebra of the image set $\mathfrak{T}^R$ is $\mathfrak{L}$.

It is immediate that if $\mathfrak{T}$ is a L.t.s. contained in a Lie algebra $\mathfrak{L}$, and $[xyz]=[[xy]z]$, then $[\mathfrak{T}\mathfrak{T}]$ is a subalgebra of $\mathfrak{L}$ and $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$ is also. Thus if $\mathfrak{T}$ is an abstract L.t.s. and R is any imbedding, the enveloping algebra of $\mathfrak{T}^R$ is $\mathfrak{T}^R+[\mathfrak{T}^R\mathfrak{T}^R]$. These facts derive from the relations in a Lie algebra which may be written

$$(1.2)\quad \begin{aligned} &\text{(i)}\ \ [[\mathfrak{M}\mathfrak{N}],[\mathfrak{P}\mathfrak{Q}]]\subseteq[[[\mathfrak{M}\mathfrak{N}]\mathfrak{P}]\mathfrak{Q}]+[[[\mathfrak{M}\mathfrak{N}]\mathfrak{Q}]\mathfrak{P}],\\ &\text{(ii)}\ [[[\mathfrak{M}\mathfrak{N}]\mathfrak{P}]\mathfrak{Q}]\subseteq[[\mathfrak{M}\mathfrak{N}],[\mathfrak{P}\mathfrak{Q}]]+[[[\mathfrak{M}\mathfrak{N}]\mathfrak{Q}]\mathfrak{P}],\end{aligned}$$

for any subspaces $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$ of the algebra.

In [7] Jacobson has settled the problem of the existence of a 1-to-1 imbedding by constructing for any L.t.s. $\mathfrak{T}$ a particular imbedding S such that $\sum_i[a_i^Sb_i^S]=0$, a_i, b_i in $\mathfrak{T}$, if and only if $\sum_i [a_ib_ix]=0$ for every x in $\mathfrak{T}$, and such that $\mathfrak{L}_S=T^S\oplus[\mathfrak{T}^S\mathfrak{T}^S]$. Since this construction will be of later use, we shall call it the *standard imbedding* of $\mathfrak{T}$, and hereafter, when confusion will not result, we may suppose an abstract L.t.s. $\mathfrak{T}$ already imbedded in a 1-to-1 manner in a Lie algebra and simply write $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$.

DEFINITION 1.6. An imbedding U of a L.t.s. $\mathfrak{T}$ is called universal, and $\mathfrak{L}_U=\mathfrak{T}^U+[\mathfrak{T}^U\mathfrak{T}^U]$ is a universal Lie algebra of $\mathfrak{T}$ if and only if, for any imbedding R of $\mathfrak{T}$, the mapping $x^U\rightarrow x^R$ is single-valued and can be extended to a Lie algebra homomorphism of $\mathfrak{L}_U$ onto $\mathfrak{L}_R$.

The existence and, in the obvious sense, uniqueness of a universal imbedding U is proved in [7], and the standard imbedding shows that U is 1-to-1 and $\mathfrak{T}^U\cap[\mathfrak{T}^U\mathfrak{T}^U]=0$. If R is any imbedding, $\mathfrak{L}_R$ is finite-dimensional since, if dim $\mathfrak{T}=n$, dim $\mathfrak{L}_R\leqq n+n(n-1)/2$.

The standard imbedding is not necessarily universal. Consider an n-dimensional L.t.s. $\mathfrak{T}$ in which all products are 0. We verify that $\mathfrak{L}_S$ is the n-dimensional zero Lie algebra and that $\mathfrak{T}$ has a 1-to-1 imbedding U_0 in which $\dim \mathfrak{L}_{U_0}=n(n+1)/2$. Theorem 7.3 of [7] shows, however, that in the important case where $\mathfrak{L}_S$ is semi-simple, S is universal.

We shall later investigate more fully the connection between L.t.s. and automorphisms of Lie algebras. For the present it suffices to verify the following assertion:

THEOREM 1.1. *If $\mathfrak{T}$ is a L.t.s. imbedded in a Lie algebra $\mathfrak{L}$ such that $\mathfrak{L}=\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$, then $\mathfrak{T}$ is the set of skew-symmetric elements relative to a unique automorphism of period 2 in $\mathfrak{L}$.*

Proof. For x in $\mathfrak{L}$ write $x=x_1+x_2$, x_1 in $\mathfrak{T}$ and x_2 in $[\mathfrak{T}\mathfrak{T}]$. Let $xA=x_2-x_1$. A is a linear transformation such that $A^2=I$, and

$$[(xA),(yA)] = [x_1y_1]+[x_2y_2]-[x_1y_2]-[x_2y_1],$$

while

$$\begin{aligned}[xy]A &= [x_1y_1]A+[x_2y_2]A+[x_1y_2]A+[x_2y_1]A\\ &= [x_1y_1]+[x_2y_2]-[x_1y_2]-[x_2y_1].\end{aligned}$$

Since $\mathfrak{T}$ generates $\mathfrak{L}$, A is unique. We shall say that $\mathfrak{T}$ is the L.t.s. *determined by* A.

We may also observe that if $\mathfrak{L}_U$ is universal for $\mathfrak{T}$, and A is the automorphism of $\mathfrak{L}_U$ determining $\mathfrak{T}$, then every automorphism of $\mathfrak{T}$ is the contraction of a unique automorphism of $\mathfrak{L}_U$ commuting with A.

II. THE RADICAL

A. **Solvable and semi-simple Lie triple systems.** As in the study of algebras we shall introduce a "radical" for L.t.s., from this obtain the notion of semi-simplicity, and discuss the implications of these definitions. We begin by noting a relation between ideals in a L.t.s. and those in an enveloping Lie algebra. In [7] it is shown by using (1.2) that if $\mathfrak{B}$ is an ideal in the L.t.s. $\mathfrak{T}$, and if $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, then $\mathfrak{B}+[\mathfrak{B}\mathfrak{B}]$ is an ideal in $\mathfrak{B}+[\mathfrak{B}\mathfrak{T}]$ which is an ideal in $\mathfrak{L}$, the ideal generated by $\mathfrak{B}$.

If $\mathfrak{B}$ is an ideal in a L.t.s. $\mathfrak{T}$, put $\mathfrak{B}^{(1)}=[\mathfrak{T}\mathfrak{B}\mathfrak{B}]$ and $\mathfrak{B}^{(k)}=[\mathfrak{T}\mathfrak{B}^{(k-1)}\mathfrak{B}^{(k-1)}]$.

LEMMA 2.1. *For each k, $\mathfrak{B}^{(k)}$ is an ideal in $\mathfrak{T}$. Thus $\mathfrak{B}\supseteq\mathfrak{B}^{(1)}\supseteq\cdots\supseteq\mathfrak{B}^{(k)}$.*

Proof. By (1.2), $[\mathfrak{B}^{(1)}\mathfrak{T}\mathfrak{T}]=[[\mathfrak{T}\mathfrak{B}\mathfrak{B}]\mathfrak{T}\mathfrak{T}]\subseteq[[\mathfrak{T}\mathfrak{B}\mathfrak{T}]\mathfrak{B}\mathfrak{T}]+[[[\mathfrak{T}\mathfrak{B}],[\mathfrak{T}\mathfrak{B}]]\mathfrak{T}]$. By (1.1) (iii) and (v), $[\mathfrak{B}^{(1)}\mathfrak{T}\mathfrak{T}]\subseteq[\mathfrak{B}\mathfrak{B}\mathfrak{T}]+[[\mathfrak{T}\mathfrak{B}\mathfrak{T}],[\mathfrak{T}\mathfrak{B}]]\subseteq[\mathfrak{T}\mathfrak{B}\mathfrak{B}]+[\mathfrak{T}\mathfrak{B}\mathfrak{B}]=\mathfrak{B}^{(1)}$. Since $\mathfrak{B}^{(k)}=(\mathfrak{B}^{(k-1)})^{(1)}$, each $\mathfrak{B}^{(i)}$ is an ideal.

DEFINITION 2.1. An ideal $\mathfrak{B}$ in a L.t.s. $\mathfrak{T}$ is *solvable in* $\mathfrak{T}$ if there is a positive integer k for which $\mathfrak{B}^{(k)}=0$.

One justification for the adoption of this definition is the observation that if $\mathfrak{T}$ is a L.t.s., $\mathfrak{L}$ a Lie algebra such that $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, and $\mathfrak{R}$ a solvable ideal in $\mathfrak{L}$, then $\mathfrak{B}=\mathfrak{R}\cap\mathfrak{T}$ is a solvable ideal in $\mathfrak{T}$. For $[\mathfrak{T}\mathfrak{B}]\subseteq[\mathfrak{T}\mathfrak{R}]\subseteq\mathfrak{R}$, and $\mathfrak{B}^{(1)}=[\mathfrak{T}\mathfrak{B}\mathfrak{B}]\subseteq[\mathfrak{R}\mathfrak{B}]\subseteq[\mathfrak{R}\mathfrak{R}]=\mathfrak{R}^{(1)}$. Since $\mathfrak{R}^{(k)}=(\mathfrak{R}^{(k-1)})^{(1)}$, we get $\mathfrak{B}^{(k)}\subseteq\mathfrak{R}^{(k)}$ for each k. Another corollary of this relation is that if $\mathfrak{R}$ is a solvable ideal in a Lie algebra $\mathfrak{L}$, then $\mathfrak{R}$ is a solvable L.t.s. ideal in the L.t.s. $\mathfrak{L}$. Before pursuing the idea further it is well to remark that if $\mathfrak{U}$ and $\mathfrak{B}$ are ideals in a L.t.s., $\mathfrak{U}\subseteq\mathfrak{B}$, and $\mathfrak{B}$ is solvable, then $\mathfrak{U}$ is also; and if $\mathfrak{B}^{(k)}$ is solvable, so is $\mathfrak{B}$.

LEMMA 2.2. *If $\mathfrak{U}$ and $\mathfrak{B}$ are solvable ideals in a L.t.s. $\mathfrak{T}$, so is $\mathfrak{U}+\mathfrak{B}=\{u+v\}$, u in $\mathfrak{U}$, v in $\mathfrak{B}$.*

Proof. $(\mathfrak{U}+\mathfrak{B})^{(1)}\subseteq[\mathfrak{T}\mathfrak{U}\mathfrak{U}]+[\mathfrak{T}\mathfrak{B}\mathfrak{B}]+[\mathfrak{T}\mathfrak{U}\mathfrak{B}]+[\mathfrak{T}\mathfrak{B}\mathfrak{U}]\subseteq\mathfrak{U}^{(1)}+\mathfrak{B}^{(1)}+\mathfrak{U}\cap\mathfrak{B}$. Suppose $(\mathfrak{U}+\mathfrak{B})^{(k)}\subseteq\mathfrak{U}^{(k)}+\mathfrak{B}^{(k)}+\mathfrak{U}\cap\mathfrak{B}$.

$$\begin{aligned}(\mathfrak{U}+\mathfrak{B})^{(k+1)} &= [\mathfrak{T}(\mathfrak{U}+\mathfrak{B})^{(k)}(\mathfrak{U}+\mathfrak{B})^{(k)}]\\ &\subseteq [\mathfrak{T}(\mathfrak{U}^{(k)}+\mathfrak{B}^{(k)}+\mathfrak{U}\cap\mathfrak{B})(\mathfrak{U}+\mathfrak{B})^{(k)}]\\ &\subseteq \mathfrak{U}^{(k+1)}+\mathfrak{B}^{(k+1)}+\mathfrak{U}\cap\mathfrak{B}.\end{aligned}$$

There must, therefore, be an integer n such that $(\mathfrak{U}+\mathfrak{B})^{(n)}\subseteq\mathfrak{U}\cap\mathfrak{B}$, which is solvable by the remark above. Applying the same remark twice more, we see that $\mathfrak{U}+\mathfrak{B}$ is solvable. Now form $\mathfrak{R}(\mathfrak{T})=\sum_\alpha \mathfrak{B}_\alpha$, $\mathfrak{B}_\alpha$ solvable in $\mathfrak{T}$. Since $\mathfrak{R}(\mathfrak{T})$ is solvable, it is the unique maximal solvable ideal in $\mathfrak{T}$.

DEFINITION 2.2. The *radical* $\mathfrak{R}(\mathfrak{T})$ of a L.t.s. $\mathfrak{T}$ is the unique maximal solvable ideal whose existence is established in Lemma 2.2 In case $\mathfrak{R}(\mathfrak{T})=0$, $\mathfrak{T}$ is *semi-simple.*

THEOREM 2.3. *If $\mathfrak{R}$ is the radical of a L.t.s. $\mathfrak{T}$, $\mathfrak{T}-\mathfrak{R}$ is semi-simple, and if $\mathfrak{B}$ is an ideal in $\mathfrak{T}$ such that $\mathfrak{T}-\mathfrak{B}$ is semi-simple, then $\mathfrak{B}\supseteq\mathfrak{R}$.*

Proof. The ideals of $\mathfrak{T}-\mathfrak{R}$ have the form $\mathfrak{U}-\mathfrak{R}$ where $\mathfrak{U}$ is an ideal containing $\mathfrak{R}$. Since $(\mathfrak{U}-\mathfrak{R})^{(k)}=(\mathfrak{U}^{(k)}+\mathfrak{R})-\mathfrak{R}$, if $\mathfrak{U}-\mathfrak{R}$ is solvable, then, for some n, $\mathfrak{U}^{(n)}\subseteq\mathfrak{R}$. This implies that $\mathfrak{U}$ is solvable in $\mathfrak{T}$, hence $\mathfrak{U}=\mathfrak{R}$, and $\mathfrak{T}-\mathfrak{R}$ is semi-simple. Suppose $\mathfrak{T}-\mathfrak{B}$ is semi-simple and consider the solvable ideal of $\mathfrak{T}-\mathfrak{B}$, $\overline{\mathfrak{R}}=\{r+\mathfrak{B}\}$, r in $\mathfrak{R}$. Since $\overline{\mathfrak{R}}=0$, $\mathfrak{R}\subseteq\mathfrak{B}$.

The succeeding theorems are intended to establish the connection between the radical $\mathfrak{R}(\mathfrak{T})$ of a L.t.s. $\mathfrak{T}$ and the radical $\mathfrak{R}(\mathfrak{L})$ of an enveloping Lie algebra and to determine the general structure of a semi-simple L.t.s.

LEMMA 2.4. *If $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, then $[\mathfrak{T}\mathfrak{T}]^{(n)}\subseteq\sum_{n\geqq k\geqq n/2}[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-k)}]$. ($\mathfrak{T}^0=\mathfrak{T}$.)*

Proof. $[\mathfrak{T}\mathfrak{T}]^{(1)}=[[\mathfrak{T}\mathfrak{T}],[\mathfrak{T}\mathfrak{T}]]\subseteq[[\mathfrak{T}\mathfrak{T}\mathfrak{T}]\mathfrak{T}]=[\mathfrak{T}^{(1)}\mathfrak{T}]$. Suppose $[\mathfrak{T}\mathfrak{T}]^{(n-1)}\subseteq\sum_{n-1\geqq k\geqq(n-1)/2}[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}]$.

$$\begin{aligned}[\mathfrak{T}\mathfrak{T}]^{(n)} &= [[\mathfrak{T}\mathfrak{T}]^{(n-1)},[\mathfrak{T}\mathfrak{T}]^{(n-1)}]\\ &\subseteq \sum_{n-1\geqq k\geqq(n-1)/2,\ n-1\geqq i\geqq(n-1)/2}[[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}],[\mathfrak{T}^{(i)}\mathfrak{T}^{(n-1-i)}]].\end{aligned}$$

We examine an arbitrary term in this expansion. By the skew symmetry of products it is necessary to consider only the case $k \geqq i$.

$$
\begin{aligned}
&[[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}], [\mathfrak{T}^{(i)}\mathfrak{T}^{(n-1-i)}]] \\
&\qquad \subseteq [[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}\mathfrak{T}^{(i)}]\mathfrak{T}^{(n-1-i)}] + [[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}\mathfrak{T}^{(n-1-i)}]\mathfrak{T}^{(i)}] \\
&\qquad \subseteq [[\mathfrak{T}\mathfrak{T}^{(i)}\mathfrak{T}^{(i)}]\mathfrak{T}^{(n-1-i)}] + [[\mathfrak{T}\mathfrak{T}^{(n-1-i)}\mathfrak{T}^{(n-1-i)}]\mathfrak{T}^{(i)}]
\end{aligned}
$$

because $n-k \leqq n-i \leqq i+1$, and therefore $k \geqq n-i-1$. From this follows $[[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-1-k)}], [\mathfrak{T}^{(i)}\mathfrak{T}^{(n-1-i)}]] \subseteq [\mathfrak{T}^{(i+1)}\mathfrak{T}^{(n-1-i)}] + [\mathfrak{T}^{(n-i)}\mathfrak{T}^{(i)}]$, and we see that each term in the expansion above is contained in a term of the required form. This lemma shows that if $\mathfrak{T}$ is a solvable L.t.s., $[\mathfrak{T}\mathfrak{T}]$ is a solvable subalgebra of any enveloping algebra of $\mathfrak{T}$.

THEOREM 2.5. *If $\mathfrak{T}$ is a L.t.s. such that $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, then*

$$
\mathfrak{L}^{(n)} \subseteq \sum_{n-1 \geqq k \geqq (n-1)/2} [\mathfrak{T}^{(k)}\mathfrak{T}^{(n-k-1)}] + \begin{cases} \mathfrak{T}^{(n/2)} & \text{if } n \text{ is even}, \\ \mathfrak{T}^{((n+1)/2)} & \text{if } n \text{ is odd}. \end{cases}
$$

Proof. $\mathfrak{L}^{(1)} = [\mathfrak{L}\mathfrak{L}] = \mathfrak{T}^{(1)} + [\mathfrak{T}\mathfrak{T}]$. $\mathfrak{L}^{(2)} \subseteq \mathfrak{T}^{(1)} + [\mathfrak{T}^{(1)}\mathfrak{T}^{(1)}] + [\mathfrak{T}^{(1)}\mathfrak{T}] \subseteq \mathfrak{T}^{(1)} + [\mathfrak{T}^{(1)}\mathfrak{T}]$. Suppose the asserted result holds for an even integer n. In $\mathfrak{L}^{(n+1)} = [\mathfrak{L}^{(n)}\mathfrak{L}^{(n)}]$, by the proof of the preceding lemma any product of terms not involving the final term is included in $\sum_{n \geqq k \geqq n/2} [\mathfrak{T}^{(k)}\mathfrak{T}^{(n-k)}]$. $[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-k-1)}\mathfrak{T}^{(n/2)}] \subseteq [\mathfrak{T}\mathfrak{T}^{(n/2)}\mathfrak{T}^{(n/2)}] \subseteq \mathfrak{T}^{(n/2+1)}$ since $k \geqq n/2$, and $[\mathfrak{T}^{(n/2)}\mathfrak{T}^{(n/2)}] = [\mathfrak{T}^{((n+1-1)/2)}\mathfrak{T}^{((n+1-1)/2)}]$. If n is odd, the argument is similar, and we conclude by calculating $[\mathfrak{T}^{(k)}\mathfrak{T}^{(n-k-1)}\mathfrak{T}^{((n+1)/2)}] \subseteq \mathfrak{T}^{((n+1)/2)}$. From this result and the discussion following Definition 2.1 derives the following assertion:

COROLLARY 2.6. *Any enveloping Lie algebra of a solvable L.t.s. is solvable, and if a L.t.s. has some solvable enveloping Lie algebra, it is solvable.*

We turn now to semi-simple L.t.s. and their enveloping Lie algebras.

THEOREM 2.7. *If $\mathfrak{T}$ is a semi-simple L.t.s., then the universal Lie algebra $\mathfrak{L}_U$ of $\mathfrak{T}$ is semi-simple.*

Proof. Let S be the standard imbedding of $\mathfrak{T}$ and $\mathfrak{L}_S$ the enveloping Lie algebra. Recall that if $\mathfrak{L}_S$ is semi-simple, $\mathfrak{L}_S \cong \mathfrak{L}_U$. If $\mathfrak{R} = \mathfrak{R}(\mathfrak{L}_S)$, then $\mathfrak{R} \cap \mathfrak{T}^S = 0$ since it is a solvable ideal in $\mathfrak{T}^S$. Let A be the automorphism of $\mathfrak{L}_S$ which determines $\mathfrak{T}^S$. Since $\mathfrak{R}A$ is solvable, $\mathfrak{R}A = \mathfrak{R}$. This implies that $\mathfrak{R} \subseteq [\mathfrak{T}^S\mathfrak{T}^S]$, and therefore $[\mathfrak{R}\mathfrak{T}^S] \subseteq \mathfrak{R} \cap \mathfrak{T}^S = 0$. In the standard imbedding we must then have $\mathfrak{R} = 0$. Since homomorphic images of semi-simple Lie algebras are semi-simple, an immediate consequence of this theorem is that any enveloping Lie algebra of a semi-simple L.t.s. is semi-simple.

THEOREM 2.8. *Let A be an automorphism of period 2 in a Lie algebra $\mathfrak{L}$ such that either* (i) *$\mathfrak{L}$ is simple or* (ii) *$\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$, $\mathfrak{L}_i$ a simple ideal, $\mathfrak{L}_1 A = \mathfrak{L}_2$. If $\mathfrak{T}$*

is the L.t.s. of elements of $\mathfrak{L}$ skew-symmetric relative to A, then $\mathfrak{T}$ is simple. (Dim $\mathfrak{T}\neq 1$, *and T has no proper ideals.*)

Proof. Let $\mathfrak{L}=\mathfrak{T}\oplus\mathfrak{F}$, where $\mathfrak{F}$ is the set of elements fixed relative to A. It is easily verified that $\mathfrak{T}$ is a L.t.s., $[\mathfrak{T}\mathfrak{T}]\subseteq\mathfrak{F}$, and $[\mathfrak{T}\mathfrak{F}]\subseteq\mathfrak{T}$. $\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$ is an ideal in $\mathfrak{L}$ since

$$\begin{aligned}[(\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}])(\mathfrak{T}\oplus\mathfrak{F})] &\subseteq [\mathfrak{T}\mathfrak{T}]+\mathfrak{T}+[\mathfrak{T}\mathfrak{F}]+[\mathfrak{T}\mathfrak{T}\mathfrak{F}]\\ &\subseteq \mathfrak{T}+[\mathfrak{T}\mathfrak{T}]+[\mathfrak{F}\mathfrak{T}\mathfrak{T}]\\ &\subseteq \mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}].\end{aligned}$$

In case (i), $\mathfrak{L}=\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$, and if $\mathfrak{B}\neq 0$ is an ideal in $\mathfrak{T}$, $\mathfrak{B}\oplus[\mathfrak{B}\mathfrak{T}]$ is an ideal in $\mathfrak{L}$, so that $\mathfrak{B}\oplus[\mathfrak{B}\mathfrak{T}]=\mathfrak{L}$, hence $\mathfrak{B}=\mathfrak{T}$ and $\mathfrak{T}$ is simple.

In case (ii), $\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$ is invariant with respect to A and therefore different from $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Again $\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$ and any ideal $\mathfrak{B}$ in $\mathfrak{T}$ generates an invariant ideal $\mathfrak{B}\oplus[\mathfrak{B}\mathfrak{T}]$ in $\mathfrak{L}$. But $\mathfrak{L}$ has no proper ideals invariant with respect to A.

THEOREM 2.9. *If $\mathfrak{T}$ is a semi-simple L.t.s., then $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}_2\oplus\cdots\oplus\mathfrak{T}_s$ where $\mathfrak{T}_i$ is a simple ideal in $\mathfrak{T}$, and, conversely, any L.t.s. with this structure is semi-simple.*

Proof. Let $\mathfrak{T}$ be imbedded in its universal algebra $\mathfrak{L}_U$, and write $\mathfrak{L}_U=\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$. By Theorem 2.7, $\mathfrak{L}_U$ is semi-simple and has the decomposition $\mathfrak{L}_U=\mathfrak{L}_1\oplus\mathfrak{L}_2\oplus\cdots\oplus L_n$ where $\mathfrak{L}_i$ is a simple ideal in $\mathfrak{L}_U$. Let A be the automorphism in $\mathfrak{L}_U$ determining $\mathfrak{T}$: Since $A^2=I$, by renumbering the components of $\mathfrak{L}_U$ we get $\mathfrak{L}_U=\mathfrak{L}_1\oplus\mathfrak{L}_1A\oplus\cdots\oplus\mathfrak{L}_q\oplus\mathfrak{L}_qA\oplus\mathfrak{L}_{q+1}\oplus\cdots\oplus\mathfrak{L}_r$ where, for $i>q$, $\mathfrak{L}_iA=\mathfrak{L}_i$. Let $\mathfrak{M}_i=\mathfrak{L}_i+\mathfrak{L}_iA$ for $i=1,\cdots,r$. We now verify that if x is in $\mathfrak{T}$ and $x=\sum_i x_i$, x_i in $\mathfrak{M}_i$, then $x_iA=-x_i$; and, in fact, $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}_2\oplus\cdots\oplus\mathfrak{T}_r$ where $\mathfrak{T}_i=\mathfrak{T}\cap\mathfrak{M}_i$. By Theorem 2.8, $\mathfrak{T}_i$ is simple. Since $[\mathfrak{T}_i\mathfrak{T}_j]=0$ if $i\neq j$, $[\mathfrak{T}_i\mathfrak{T}\mathfrak{T}]=[\mathfrak{T}_i\mathfrak{T}_i\mathfrak{T}_i]\subseteq\mathfrak{T}_i$ and $\mathfrak{T}_i$ is an ideal in $\mathfrak{T}$. It also follows immediately that the only ideals in $\mathfrak{T}$ are sums of the components $\mathfrak{T}_i$. Since $\mathfrak{T}_i^{(1)}$ is an ideal in $\mathfrak{T}_i$, the converse is established.

COROLLARY 2.10. *If $\mathfrak{T}$ is semi-simple, $\mathfrak{T}^{(1)}=\mathfrak{T}$.*

THEOREM 2.11. *If $\mathfrak{T}$ is semi-simple, every derivation is inner.*

Proof. Let $\mathfrak{L}_U=\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$, and let D be a derivation in $\mathfrak{T}$. The mapping in $[\mathfrak{T}\mathfrak{T}]$ $[xy]\overline{D}=[(xD),\ y]+[x,\ (yD)]$ is single-valued since $\mathfrak{L}_U\cong\mathfrak{L}_S$ and therefore determines a derivation $\overline{D}$ in $\mathfrak{L}_U$ extending D. Since $\mathfrak{L}_U$ is semi-simple, $\overline{D}$ is inner. Thus if we denote for a in $\mathfrak{L}_U$ the derivation $x\rightarrow[xa]$ by Ad a, then for some a in $\mathfrak{L}_U$, $\overline{D}=\mathrm{Ad}\ a$. But since $\mathfrak{T}(\mathrm{Ad}\ a)\subseteq\mathfrak{T}$, a must be in $[\mathfrak{T}\mathfrak{T}]$, which is to say D is inner.

It follows from the argument above that any derivation of a semi-simple L.t.s. has a unique extension to a derivation of its universal Lie algebra.

Next we discuss the universal Lie algebras of simple L.t.s.

THEOREM 2.12. *Let $\mathfrak{L}$ be a Lie algebra such that $\mathfrak{L}^{(1)}=\mathfrak{L}$; then as a L.t.s, $\mathfrak{L}$ has a 1-to-1 imbedding in the Lie algebra $\mathfrak{M}=\mathfrak{L}\oplus\overline{\mathfrak{L}}$, $\overline{\mathfrak{L}}$ anti-isomorphic to $\mathfrak{L}$.*

Proof. Let $x\rightarrow\bar{x}$ be an anti-isomorphism of $\mathfrak{L}$ onto $\overline{\mathfrak{L}}$. Consider the subset of $\mathfrak{M}$ given by $\mathfrak{T}=\{x+\bar{x}\}$, for x in $\mathfrak{L}$, and the mapping $x\rightarrow\bar{x}+x$ of $\mathfrak{L}$ onto $\mathfrak{T}$. This is a L.t.s. isomorphism and $[\mathfrak{T}\mathfrak{T}]\subseteq\{x-\bar{x}\}$. But every element of $\mathfrak{L}$ may be written $x=\sum[x_1x_2]$. In $\mathfrak{M}$ we have $x-\bar{x}=[(x_1+\bar{x}_1)(x_2+\bar{x}_2)]$, so that $[\mathfrak{T}\mathfrak{T}]=\{x-\bar{x}\}$, and $\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]=\mathfrak{M}$.

If $\mathfrak{L}$ is a Lie algebra such that $\mathfrak{L}^{(1)}=\mathfrak{L}$, then every L.t.s. ideal of $\mathfrak{L}$ is a Lie algebra ideal(4), for $[\mathfrak{B}\mathfrak{L}\mathfrak{L}]\subseteq\mathfrak{B}$ implies that $[\mathfrak{L}\mathfrak{B}]=[\mathfrak{L}\mathfrak{L}\mathfrak{B}]\subseteq[\mathfrak{B}\mathfrak{L}\mathfrak{L}]\subseteq\mathfrak{B}$. In particular, we shall have occasion to use the fact that a simple Lie algebra is a simple L.t.s.

Return to the proof of Theorem 2.9 and notice that in order for $\mathfrak{L}_U$ to be universal for $\mathfrak{T}$ it is necessary and sufficient that each $\mathfrak{M}_i$ be universal for each $\mathfrak{T}_i$. Thus, if $\mathfrak{T}$ is simple, either (i) $\mathfrak{L}_U$ is simple or (ii) $\mathfrak{L}_U=\mathfrak{L}_1\oplus\mathfrak{L}_1A$, A an automorphism of period 2 in $\mathfrak{L}_U$, and $\mathfrak{T}$ the skew-symmetric elements relative to A. In case there is a L.t.s. isomorphism of a L.t.s. $\mathfrak{T}$ onto a Lie algebra $\mathfrak{L}$, we shall call $\mathfrak{T}$ the L.t.s. of the Lie algebra $\mathfrak{L}$. By Theorem 2.12, if $\mathfrak{T}$ is the L.t.s. of a simple Lie algebra, (ii) holds. If (ii) holds, and $x=x_1-x_1A$ is in $\mathfrak{T}$, $x\rightarrow x_1$ is a L.t.s. isomorphism of $\mathfrak{T}$ into $\mathfrak{L}_1$. But the set $\{x_1\}$ is a L.t.s. ideal in $\mathfrak{L}_1$, so that the mapping $x\rightarrow x_1$ is onto $\mathfrak{L}_1$. We have proved, then, the following result:

THEOREM 2.13. *If $\mathfrak{T}$ is a simple L.t.s. imbedded in its universal Lie algebra $\mathfrak{L}_U$, then either*

(i) *$\mathfrak{T}$ is the L.t.s. of a (simple) Lie algebra, and $\mathfrak{L}_U=\mathfrak{L}_1\oplus\mathfrak{L}_2$, where the $\mathfrak{L}_i$ are ideals isomorphic to the Lie algebra $\mathfrak{T}$ or*

(ii) *$\mathfrak{T}$ is not the L.t.s. of a Lie algebra and $\mathfrak{L}_U$ is simple.*

Theorems 1.1, 2.8, and 2.13 prove that the following comprise all simple L.t.s.:

(i) the L.t.s. of simple Lie algebras,

(ii) the L.t.s. of elements skew-symmetric relative to automorphisms of period 2 in simple Lie algebras.

This is the characterization which will be used to determine the simple L.t.s.

If $\mathfrak{L}$ is a Lie algebra, any automorphism or anti-automorphism is a L.t.s. automorphism of $\mathfrak{L}$. If $\mathfrak{L}$ is simple, the properties of the universal algebra given in Theorem 2.13 can be used to get the converse. We omit the argument since a proof of the more general statement which follows has already been given(5).

THEOREM 2.14. *Every L.t.s. homomorphism of a semi-simple Lie algebra maps each simple component isomorphically, anti-isomorphically, or into* 0.

Next we establish the general connection between the radical of a L.t.s. and that of an enveloping Lie algebra.

(4) For this observation the author is indebted to the referee.

(5) Theorem 14 in [4].

LEMMA 2.15. *If* $\mathfrak{T}$ *is a L.t.s., and* $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$, *then* $\mathfrak{R}(\mathfrak{L})\cap\mathfrak{T}=\mathfrak{R}(\mathfrak{T})$.

Proof. $\mathfrak{R}(\mathfrak{L})\cap\mathfrak{T}$ is an ideal solvable in $\mathfrak{T}$, hence $\mathfrak{R}(\mathfrak{L})\cap\mathfrak{T}\subseteq\mathfrak{R}(\mathfrak{T})$. If $\mathfrak{U}$ is an ideal solvable in $\mathfrak{T}$, $\mathfrak{U}$ is solvable in $\mathfrak{U}$, so that, by Corollary 2.6, $\mathfrak{U}+[\mathfrak{U}\mathfrak{U}]$ is a solvable subalgebra of $\mathfrak{L}$. But since, as we remarked at the beginning of this section, $\mathfrak{U}+[\mathfrak{U}\mathfrak{U}]$ is subinvariant in $\mathfrak{L}$, and Schenkman [8] has shown that every subinvariant subalgebra of $\mathfrak{L}$ is in $\mathfrak{R}(\mathfrak{L})$, $\mathfrak{U}\subseteq\mathfrak{R}(\mathfrak{L})$ and $\mathfrak{R}(\mathfrak{T})\subseteq\mathfrak{R}(\mathfrak{L})$.

THEOREM 2.16. *If* $\mathfrak{T}$ *is a L.t.s., and* $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$, *then* $\mathfrak{R}(\mathfrak{L})=\mathfrak{R}(\mathfrak{T})+[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}]$.

Proof. By Lemma 2.15, $\mathfrak{R}(\mathfrak{L})\supseteq\mathfrak{R}(\mathfrak{T})+[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}]=\mathfrak{R}_0$. In $\mathfrak{L}-\mathfrak{R}_0$ consider the collection $\overline{\mathfrak{T}}=(\mathfrak{T}+\mathfrak{R}_0)-\mathfrak{R}_0$ of cosets determined by elements of $\mathfrak{T}$. $\overline{\mathfrak{T}}$ is a L.t.s., $\overline{\mathfrak{T}}+[\overline{\mathfrak{T}}\overline{\mathfrak{T}}]=\mathfrak{L}-\mathfrak{R}_0$, and Lemma 2.15 shows that under the natural mapping $\overline{\mathfrak{T}}\cong\mathfrak{T}-\mathfrak{R}(\mathfrak{T})$. Thus $\overline{\mathfrak{T}}$ and hence also $\mathfrak{L}-\mathfrak{R}_0$ are semi-simple, and the latter implies that $\mathfrak{R}_0\supseteq\mathfrak{R}(\mathfrak{L})$.

B. **The Levi decomposition.** In this section we shall deduce the analogue for L.t.s. of Levi's theorem, which states that in any Lie algebra $\mathfrak{L}$ (over a field of characteristic 0) there is a subalgebra $\mathfrak{S}$ each that $\mathfrak{L}-\mathfrak{R}(\mathfrak{L})\cong\mathfrak{S}$. In the theory of Lie algebras this theorem is derived from a more abstract formulation, the so-called second Whitehead lemma(6). Although we have been unable to prove a second Whitehead lemma for L.t.s., it is possible to use the corresponding result for Lie algebras to get a Levi theorem. As for Lie algebras there is a first Whitehead lemma for L.t.s., from which Theorems 2.17 and 2.18 can be derived. This is not done here since they also follow easily from the connections between L.t.s. and Lie algebras.

A L.t.s. of linear transformations is a subspace $\mathfrak{S}$ of the associative algebra of linear transformations on a vector space, such that, if S_i is in $\mathfrak{S}$, $i=1, 2, 3$, then $[[S_1S_2]S_3]=[S_1S_2S_3]$ is also in $\mathfrak{S}$.

DEFINITION 2.3. A *representation* of a L.t.s. $\mathfrak{T}$ is a homomorphism of $\mathfrak{T}$ into a L.t.s. of linear transformations. If $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, the mapping $x\rightarrow\mathrm{Ad}\, x$ acting in $\mathfrak{L}$ is a representation of $\mathfrak{T}$, an *adjoint representation.* In general, if $\mathfrak{M}$ is a Lie algebra containing a L.t.s. $\mathfrak{T}$, $x\rightarrow\mathrm{Ad}\, x$ acting in $\mathfrak{M}$ is a representation of $\mathfrak{T}$.

THEOREM 2.17. *Any representation of a semi-simple L.t.s. is completely reducible.*

Proof. Let $\mathfrak{T}$ be semi-simple and S be a representation of $\mathfrak{T}$ in the space X. $\mathfrak{S}(\mathfrak{T})=\{S(t)\}$ is a semi-simple L.t.s. of linear transformations, so that $\mathfrak{S}(\mathfrak{T})+[\mathfrak{S}(\mathfrak{T}), \mathfrak{S}(\mathfrak{T})]=\mathfrak{L}$ is a semi-simple, hence completely reducible, Lie algebra. Finally, $X_1\subseteq X$ is invariant with respect to $\mathfrak{S}(\mathfrak{T})$ if and only if $X_1\mathfrak{L}\subseteq X_1$.

THEOREM 2.18. *If* $\mathfrak{T}$ *is a semi-simple subsystem of the L.t.s.* $\mathfrak{U}$, *then any*

(6) A discussion of the first and second Whitehead lemmas can be found in [3].

derivation of $\mathfrak{T}$ *into* $\mathfrak{U}$ *can be extended to an inner derivation in* $\mathfrak{U}$.

Proof. Suppose $\mathfrak{U}$ already imbedded, by the standard imbedding, in a Lie algebra $\mathfrak{M}$. Let $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, and let D be a derivation of $\mathfrak{T}$ into $\mathfrak{U}$. First we prove that D can be extended to a derivation $\overline{D}$ of $\mathfrak{L}$ into $\mathfrak{M}$. To accomplish this define $\overline{D}$ on $[\mathfrak{T}\mathfrak{T}]$ by setting, for x_i, y_i in $\mathfrak{T}$,

$$\sum_i [x_iy_i]\overline{D} = \sum_i [(x_iD)y_i] + \sum_i [x_i(y_iD)].$$

If $\overline{D}$ is single-valued, it is clearly a derivation. Suppose $\sum_i [x_iy_i]=0$. $\sum_i [x_iy_it]D=0$, so that

$$\sum_i [(x_iD)y_it] + \sum_i [x_i(y_iD)t] = 0. \tag{2.1}$$

We have to show $y_0=\sum[(x_iD)y_i]+\sum_i[x_i(y_iD)]=0$. Consider the representation $t\rightarrow\mathrm{Ad}\ t=S(t)$ acting in $\mathfrak{M}$. By the preceding theorem this representation is completely reducible. Suppose, for some element y in $\mathfrak{M}$ of the form $y=\sum_i z_iS(t_i)$, we have $yS(t)=0$ for all t in $\mathfrak{T}$. $\mathfrak{M}=(y)\oplus\mathfrak{N}$ where $\mathfrak{M}S(t)\subseteq\mathfrak{N}$ for all t in $\mathfrak{T}$. Therefore, y is in $\mathfrak{N}$, and $y=0$. Taking $y=y_0$ and applying (2.1) proves that $\overline{D}$ is uniquely defined. Since $\mathfrak{L}$ is semi-simple, $\overline{D}$ can be extended to an inner derivation D_0 in $\mathfrak{M}$(7). Suppose $D_0=\mathrm{Ad}\ (u+a)$ for u in $\mathfrak{U}$ and a in $[\mathfrak{U}\mathfrak{U}]$, and let $D_1=\mathrm{Ad}\ a$. Since $\mathfrak{T}D=\mathfrak{T}D_0\subseteq\mathfrak{U}$, $tD_0=tD_1$ for each t in $\mathfrak{T}$, and D_1 is an inner derivation in $\mathfrak{U}$.

For the remainder of this section we suppose, unless otherwise indicated, that

(i) $\mathfrak{U}$ is a L.t.s. imbedded by the standard imbedding in a Lie algebra $\mathfrak{M}$,
(ii) $\mathfrak{R}$ is the radical of $\mathfrak{M}$, $\mathfrak{R}(\mathfrak{U})$ the radical of $\mathfrak{U}$,
(iii) $\mathfrak{R}(\mathfrak{U})^{(1)}=[\mathfrak{U},\ \mathfrak{R}(\mathfrak{U}),\ \mathfrak{R}(\mathfrak{U})]=0$,
(iv) $\mathfrak{L}=\mathfrak{M}-\mathfrak{R}$, $\mathfrak{T}=(\mathfrak{U}+\mathfrak{R})-\mathfrak{R}$,
(v) $X=X_1\oplus X_2$, where $X_1=\mathfrak{R}(\mathfrak{U})$ and $X_2=[\mathfrak{U},\ \mathfrak{R}(\mathfrak{U})]$. By (i), (iii), and Theorem 2.16, $[\mathfrak{R}\mathfrak{R}]=0$. By Lemma 2.15, under the natural mapping, $\mathfrak{T}\cong\mathfrak{U}-\mathfrak{R}(\mathfrak{U})$. Thus $\mathfrak{T}$ is semi-simple, so is $\mathfrak{L}$, and $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$.

Now let $\mathfrak{B}$ be a subspace of $\mathfrak{U}$ complementary to $\mathfrak{R}(\mathfrak{U})$. Let $u(t)$ be any element of $\mathfrak{U}$ such that the coset $\bar{u}(t)$ determined by $u(t)$ in L is t. If $u(t)=r(t)+v(t)$ corresponds to the above decomposition of $\mathfrak{U}$, $\bar{v}(t)=t$, and $v(t)$ is uniquely determined by t. In fact $v(t)$ is a 1-to-1 linear transformation of $\mathfrak{T}$ onto $\mathfrak{B}$. $[\bar{v}(t_1),\ \bar{v}(t_2),\ \bar{v}(t_3)]=[t_1t_2t_3]=\bar{v}([t_1t_2t_3])$ so that

$$[v(t_1),\ v(t_2),\ v(t_3)] - v([t_1t_2t_3]) = r(t_1t_2t_3) \tag{2.2}$$

is in $\mathfrak{R}$, therefore in $\mathfrak{R}(\mathfrak{U})$. Besides this trilinear function on $\mathfrak{T}\times\mathfrak{T}\times\mathfrak{T}$ to $\mathfrak{R}(\mathfrak{U})$, we introduce for each t in $\mathfrak{T}$ the linear transformation in $X=\mathfrak{R}$ defined by $xS(t)=[x,\ v(t)]$. Observe that $t\rightarrow S(t)$ is a representation of $\mathfrak{T}$ in X independent of $\mathfrak{B}$, for if $\mathfrak{B}_0$ replaces $\mathfrak{B}$, then $t=\bar{v}_0(t)=\bar{v}(t)$ and $v_0(t)-v(t)$ is in

(7) This is a consequence of the first Whitehead lemma for Lie algebras.

$\mathfrak{R}(\mathfrak{U})$. Thus $[x, v_0(t)] = [x, v(t)]$ since $[\mathfrak{R}\mathfrak{R}] = 0$. That S is a representation follows from (2.2). In addition we have

$$X_1 S(t) \subseteq X_2, \quad \text{and} \quad X_2 S(t) \subseteq X_1 \text{ for all } t \text{ in } \mathfrak{T}. \tag{2.3}$$

LEMMA 2.19. *A necessary and sufficient condition that there exist a semi-simple subsystem* $\mathfrak{V}_0$ *of* $\mathfrak{U}$ *such that* $\mathfrak{U} = \mathfrak{R}(\mathfrak{U}) \oplus \mathfrak{V}_0$ *is that there exist a linear transformation* f *of* $\mathfrak{T}$ *into* X_1 *such that*

$$r(t_1t_2t_3) = f([t_1t_2t_3]) - f(t_1)S(t_2)S(t_3) + f(t_2)S(t_1)S(t_3) + f(t_3)[S(t_1), S(t_2)].$$

Proof. Let $\mathfrak{U} = \mathfrak{R}(\mathfrak{U}) \oplus \mathfrak{V} = \mathfrak{R}(\mathfrak{U}) \oplus \mathfrak{V}_0$. Let $f(t) = v_0(t) - v(t)$. Since $\bar{v}_0(t) = \bar{v}(t) = t$, $f(t)$ takes $\mathfrak{T}$ into $\mathfrak{R}(\mathfrak{U}) = X_1$. $\mathfrak{V}_0$ is a subsystem if and only if the function $r_0(t_1t_2t_3)$ associated with $\mathfrak{V}_0$ is identically 0.

$$[v_0(t_1), v_0(t_2), v_0(t_3)] = v_0([t_1t_2t_3]) + r_0(t_1t_2t_3).$$

$$[v(t_1) + f(t_1), v(t_2) + f(t_2), v(t_3) + f(t_3)] = v([t_1t_2t_3]) + f([t_1t_2t_3]) + r_0(t_1t_2t_3).$$

$$\begin{aligned} r_0(t_1t_2t_3) &= r(t_1t_2t_3) - f([t_1t_2t_3]) - [f(t_3), v(t_1), v(t_2)] \\ &\quad + [f(t_3), v(t_2), v(t_1)] - [f(t_2), v(t_1), v(t_3)] \\ &\quad + [f(t_1), v(t_2), v(t_3)] \\ &= r(t_1t_2t_3) - f([t_1t_2t_3]) - f(t_3)S(t_1)S(t_2) \\ &\quad + f(t_3)S(t_2)S(t_1) - f(t_2)S(t_1)S(t_3) + f(t_1)S(t_2)S(t_3). \end{aligned}$$

THEOREM 2.20. *If* $\mathfrak{U}$ *is a L.t.s. such that* $\mathfrak{R}(\mathfrak{U})^{(1)} = 0$, *then there is a semi-simple subsystem* $\mathfrak{V}_0$ *of* $\mathfrak{U}$ *such that* $\mathfrak{U} = \mathfrak{R}(\mathfrak{U}) \oplus \mathfrak{V}_0$.

Proof. By Lemma 2.19 it is sufficient to produce a function f satisfying the relation stated in that lemma. Let $\mathfrak{N}$ be a subspace complementary to $\mathfrak{R}$ in $\mathfrak{M}$. Since if $\mathfrak{U} = \mathfrak{R}(\mathfrak{U}) \oplus \mathfrak{V}$, $\mathfrak{V} \cap \mathfrak{R} = 0$, we may suppose that $\mathfrak{V} \subseteq \mathfrak{N}$. Again we may consider the unique element $n(l)$ in $\mathfrak{N}$ such that $\bar{n}(l)$ (in $\mathfrak{M} - \mathfrak{R}$) is l. The condition that $\mathfrak{V} \subseteq \mathfrak{N}$ is equivalent to the statement that if t is in $\mathfrak{T}$, $n(t) = v(t)$. $x \rightarrow [x, n(l)]$ is therefore the extension of the representation of (2.2) to $\mathfrak{L}$ and will also be denoted by S. The second Whitehead lemma for Lie algebras then asserts that if $q(l_1l_2) = [n(l_1), n(l_2)] - n([l_1l_2])$, there is a linear transformation g of $\mathfrak{L}$ into X satisfying $q(l_1l_2) = g([l_1l_2]) - g(l_1)S(l_2) + g(l_2)S(l_1)$. Next we observe the following relation between the functions r and q.

$$\begin{aligned} &[v(t_1), v(t_2), v(t_3)] - [n([t_1t_2]), v(t_3)] + [n([t_1t_2]), v(t_3)] - v([t_1t_2t_3]) \\ &\qquad = r(t_1t_2t_3) = q([t_1t_2], t_3) + q(t_1t_2)S(t_3). \end{aligned}$$

In terms of g we now have

$$\begin{aligned} r(t_1t_2t_3) &= g([t_1t_2t_3]) - g([t_1t_2])S(t_3) + g(t_3)[S(t_1), S(t_2)] \\ &\quad + g([t_1t_2])S(t_3) - g(t_1)S(t_2)S(t_3) + g(t_2)S(t_1)S(t_3) \\ &= g([t_1t_2t_3]) - g(t_1)S(t_2)S(t_3) + g(t_2)S(t_1)S(t_3) + g(t_3)[S(t_1), S(t_2)]. \end{aligned}$$

Decompose $g(t)$ by setting $g(t)=f(t)+h(t)$, where $f(t)$ is in X_1 and $h(t)$ is in X_2. f is a linear transformation of $\mathfrak{T}$ into X_1, and

$$\begin{aligned} r(t_1t_2t_3) = f([t_1t_2t_3]) &- f(t_1)S(t_2)S(t_3) + f(t_2)S(t_1)S(t_3) + f(t_3)[S(t_1), S(t_2)] \\ &+ h([t_1t_2t_3]) - h(t_1)S(t_2)S(t_3) + h(t_2)S(t_1)S(t_3) \\ &+ h(t_3)[S(t_1), S(t_2)]. \end{aligned}$$

But since $r(t_1t_2t_3)$ and $f(t_1)S(t_2)S(t_3)$ are in X_1, while $h(t_1)S(t_2)S(t_3)$ is in X_2 by (2.3), the sum of the last four terms in the expression above is 0, and the function f satisfies the condition of Lemma 2.19.

THEOREM 2.21. *If* $\mathfrak{U}$ *is any L.t.s., then there is a semi-simple subsystem* $\mathfrak{B}_0$ *such that* $\mathfrak{U}=\mathfrak{R}(\mathfrak{U})\oplus\mathfrak{B}_0$.

Proof. Let $\overline{\mathfrak{U}}=\mathfrak{U}-\mathfrak{R}^{(1)}$. We reduce the proof of this theorem to the case for which it is known from Theorem 2.20. Our result holds whenever the dimension of $\mathfrak{U}$ is 1. We may suppose therefore that a decomposition is possible for all L.t.s. of lower dimension than $\mathfrak{U}$. We may also suppose that $\mathfrak{R}^{(1)}\neq 0$ so that $\dim \overline{\mathfrak{U}}<\dim \mathfrak{U}$. Let $\overline{\mathfrak{R}}_0$ be the radical of $\overline{\mathfrak{U}}$. If $\overline{\mathfrak{R}}$ $(\mathfrak{U})$is the set of elements of $\overline{\mathfrak{U}}$ determined by $\mathfrak{R}(\mathfrak{U})$, $\overline{\mathfrak{R}}(\mathfrak{U})\subseteq\overline{\mathfrak{R}}_0$, and if $\overline{\mathfrak{R}}_0^{(k)}=0$, $\mathfrak{R}_0^{(k)}\subseteq\mathfrak{R}^{(1)}\subseteq\mathfrak{R}$ so that $\mathfrak{R}_0$ is solvable and $\overline{\mathfrak{R}}_0=\overline{\mathfrak{R}}(\mathfrak{U})$. By the induction hypothesis, for some semi-simple subsystem $\overline{\mathfrak{B}}_0$ of $\overline{\mathfrak{U}}$, $\overline{\mathfrak{U}}=\overline{\mathfrak{R}}(\mathfrak{U})\oplus\overline{\mathfrak{B}}_0$. But $\overline{\mathfrak{B}}_0$ has the form $\overline{\mathfrak{B}}_0=\mathfrak{B}_0-\mathfrak{R}^{(1)}$ for some subsystem $\mathfrak{B}_0\supseteq\mathfrak{R}^{(1)}$. Since $\overline{\mathfrak{B}}_0$ is semi-simple, $\mathfrak{R}^{(1)}$ is the radical of $\mathfrak{B}_0$. Since $\mathfrak{B}_0\neq\mathfrak{U}$, we may again assume $\mathfrak{B}_0=\mathfrak{R}^{(1)}\oplus\mathfrak{B}$ for some semi-simple subsystem $\mathfrak{B}$ of $\mathfrak{B}_0$. Because $\mathfrak{B}\cap\mathfrak{R}(\mathfrak{U})=0$, $\mathfrak{U}=\mathfrak{R}+\mathfrak{B}_0=\mathfrak{R}\oplus\mathfrak{B}$.

III. COMPLETELY REDUCIBLE LIE TRIPLE SYSTEMS

Relations between the structure of a Lie algebra of linear transformations and its complete reducibility are given in the following theorems, all of which may be found in Jacobson [5]:

THEOREM A. *A Lie algebra* $\mathfrak{L}$ *of linear transformations is completely reducible if and only if* $\mathfrak{L}=\mathfrak{L}^{(1)}\oplus\mathfrak{C}(\mathfrak{L})$ *where* $\mathfrak{L}^{(1)}$ *is semi-simple and the elements of the center* $\mathfrak{C}(\mathfrak{L})$ *have simple elementary divisors.*

THEOREM B. *If* $\mathfrak{L}$ *is a completely reducible Lie algebra, then any nilpotent element* d *of* $\mathfrak{L}$ *is contained in a* 3*-dimensional simple subalgebra* $\{d, e, f\}$ *of* $\mathfrak{L}$, *where* $[de]=f$, $[df]=d$, $[ef]=-e$.

THEOREM C. *If* $\mathfrak{L}$ *is a Lie algebra such that every nilpotent element* d *is contained in a* 3*-dimensional simple subalgebra, and* $\mathfrak{C}(\mathfrak{L})$ *is splittable*[8], *then* $\mathfrak{L}$ *is completely reducible.*

(8) Any linear transformation has a unique decomposition as the sum of two commuting transformations, one of which is nilpotent, and one of which has simple elementary divisors. If, for each element in an algebra, these components also belong to the algebra, the algebra is *splittable*.

THEOREM D. *If $\mathfrak{L}$ is a completely reducible Lie algebra, and $\mathfrak{M}$ a completely reducible subalgebra, then the centralizer*[9] *of $\mathfrak{M}$ in $\mathfrak{L}$ is completely reducible.*

Using these results we can establish analogues for L.t.s.

DEFINITION 3.1. The center $\mathfrak{C}(\mathfrak{T})$ of a L.t.s. $\mathfrak{T}$ is the set of elements c in $\mathfrak{T}$ such that $[c\mathfrak{T}\mathfrak{T}]=0$. $\mathfrak{C}(\mathfrak{T})$ is an ideal in $\mathfrak{T}$ and $\mathfrak{C}(\mathfrak{T})\subseteq\mathfrak{R}(\mathfrak{T})$.

THEOREM 3.1. *A L.t.s. of linear transformations $\mathfrak{T}$ is completely reducible if and only if $\mathfrak{T}=\mathfrak{T}^{(1)}\oplus\mathfrak{C}(\mathfrak{T})$, the elements of $\mathfrak{C}(\mathfrak{T})$ have simple elementary divisors, and $\mathfrak{T}^{(1)}$ is semi-simple.*

Proof. Let $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$. Suppose $\mathfrak{T}$ is completely reducible. Then by Theorem A, $\mathfrak{L}=\mathfrak{L}^{(1)}\oplus\mathfrak{C}(\mathfrak{L})$, and by Theorem 2.16, $\mathfrak{C}(\mathfrak{L})=\mathfrak{R}(\mathfrak{L})=\mathfrak{R}(\mathfrak{T})+[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}]$. Thus $\mathfrak{R}(\mathfrak{T})\subseteq\mathfrak{C}(\mathfrak{L})$, and $[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}]=0$, so that $\mathfrak{R}(\mathfrak{T})=\mathfrak{C}(\mathfrak{L})=\mathfrak{C}(\mathfrak{T})$. By Theorem 2.21, $\mathfrak{T}=\mathfrak{T}^{(1)}\oplus\mathfrak{C}(\mathfrak{T})$ and $\mathfrak{T}^{(1)}$ must be semi-simple. Now let $\mathfrak{T}=\mathfrak{T}^{(1)}\oplus\mathfrak{C}(\mathfrak{T})$, $\mathfrak{T}^{(1)}$ be semi-simple, and the elements of $\mathfrak{C}(\mathfrak{T})$ have simple elementary divisors. The last assertion implies that the adjoint mappings determined by elements of $\mathfrak{C}(\mathfrak{T})$ acting in the full algebra of linear transformations in $\mathfrak{T}$ have simple elementary divisors. $[\mathfrak{C}(\mathfrak{T}), \mathfrak{T}^{(1)}]\subseteq[[\mathfrak{C}(\mathfrak{T}), \mathfrak{T}\mathfrak{T}]\mathfrak{T}]=0$ so that $\mathfrak{R}(\mathfrak{L})=\mathfrak{C}(\mathfrak{T})+[\mathfrak{C}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T})]$. Since $[\mathfrak{C}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T})]=0$, if $\mathfrak{C}$ is in $\mathfrak{C}(\mathfrak{T})$, Ad C restricted to $\mathfrak{R}(\mathfrak{L})$ satisfies $(\text{Ad } C)^2=0$. By the remark above, Ad $C=0$ and $[\mathfrak{C}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T})]=0$. It is immediate that $\mathfrak{C}(\mathfrak{T})=\mathfrak{C}(\mathfrak{L})$ and $\mathfrak{L}=\mathfrak{T}^{(1)}+[\mathfrak{T}^{(1)}\mathfrak{T}^{(1)}]\oplus\mathfrak{C}(\mathfrak{T})$ where $\mathfrak{L}^{(1)}=\mathfrak{T}^{(1)}+[\mathfrak{T}^{(1)}\mathfrak{T}^{(1)}]$ is semi-simple. Theorem A shows $\mathfrak{L}$ is completely reducible, therefore T is also. Notice that the condition $\mathfrak{T}=\mathfrak{T}^{(1)}\oplus\mathfrak{C}(\mathfrak{T})$, $\mathfrak{T}^{(1)}$ semi-simple, is equivalent to $\mathfrak{R}(\mathfrak{T})=\mathfrak{C}(\mathfrak{T})$ and to the statement $\mathfrak{T}^{(1)}$ is semi-simple.

COROLLARY 3.2. *If $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$ is complelely reducible, then $\mathfrak{L}=(\mathfrak{T}^{(1)}+[\mathfrak{T}^{(1)}\mathfrak{T}^{(1)}])\oplus\mathfrak{C}(\mathfrak{T})$, $\mathfrak{C}(\mathfrak{T})=\mathfrak{C}(\mathfrak{L})$, $\mathfrak{L}^{(1)}=\mathfrak{T}^{(1)}+[\mathfrak{T}^{(1)}\mathfrak{T}^{(1)}]$.*

Also of use will be the following decomposition of a completely reducible L.t.s. obtained by decomposing $\mathfrak{T}^{(1)}$ according to Theorem 2.9: $\mathfrak{T}=\mathfrak{T}_0\oplus T_1\oplus\mathfrak{C}(\mathfrak{T})$, where $\mathfrak{T}_0$ and $\mathfrak{T}_1$ are semi-simple ideals, $\mathfrak{T}_0\cap[\mathfrak{T}_0\mathfrak{T}_0]=0$, and $\mathfrak{T}_1$ is a Lie algebra.

In what follows, by $\{d, e\}$ we shall mean the 2-dimensional, simple L.t.s. with basis elements d and e and the multiplication table $[dee]=e$, $[edd]=d$.

THEOREM 3.3. *Let $\mathfrak{T}$ be a L.t.s. such that every nilpotent element d is contained in a 2-dimensional simple subsystem and such that $\mathfrak{C}(\mathfrak{T})$ is splittable, then $\mathfrak{T}$ is completely reducible.*

Proof. In a Lie algebra $\mathfrak{L}$ of matrices, $[\mathfrak{L}, \mathfrak{R}(\mathfrak{L})]$ is contained in the radical of the enveloping associative algebra[10]. Thus, if $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$, $[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}\mathfrak{T}]\subseteq[\mathfrak{R}(\mathfrak{L}), \mathfrak{L}]$, and every element $d\neq 0$ in $[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}\mathfrak{T}]$ is nilpotent. If $\{d, e\}=\mathfrak{B}$, $\mathfrak{B}\subseteq\mathfrak{R}(\mathfrak{T})$, and $\mathfrak{B}$ is solvable in itself, which is impossible since $\mathfrak{B}$ is

[9] Those elements of $\mathfrak{L}$ commuting with each element of $\mathfrak{M}$.

[10] Theorem 2 in [5].

simple. We have, then, $[\mathfrak{N}(\mathfrak{T}), \mathfrak{T}\mathfrak{T}]=0$, or $\mathfrak{N}(\mathfrak{T})=\mathfrak{C}(\mathfrak{T})$. For the same reason $\mathfrak{C}(\mathfrak{T})$ contains no nonzero nilpotent elements. Since $\mathfrak{C}(\mathfrak{T})$ is splittable, the elements of $\mathfrak{C}(\mathfrak{T})$ have simple elementary divisors. By Theorem 3.1, $\mathfrak{T}$ is completely reducible.

THEOREM 3.4. *If $\mathfrak{T}$ is a completely reducible L.t.s., then any nilpotent element d is contained in a 2-dimensional simple subsystem $\{d, e\}$.*

Proof. Let $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$, and, using Corollary 3.2, set $\mathfrak{L}=\mathfrak{L}_1\oplus\mathfrak{L}_0$, $\mathfrak{L}_1=\mathfrak{T}_1$, $\mathfrak{L}_0=\mathfrak{T}_0\oplus\mathfrak{C}(\mathfrak{T})\oplus[\mathfrak{T}_0\mathfrak{T}_0]$. The result has been proved(11) in case $\mathfrak{L}_1=0$; i.e., $\mathfrak{T}\cap[\mathfrak{T}\mathfrak{T}]=0$. In particular the assertion holds in $\mathfrak{T}_0$. Theorem B shows the theorem is valid in $\mathfrak{T}_1$. Let d in $\mathfrak{T}$ be nilpotent and $d=d_0+d_1$, d_i in $\mathfrak{L}_i$. If a and b are linear transformations satisfying $[ab]=a$, then a is nilpotent. Let $\{d, g, h\}$ be a 3-dimensional simple subalgebra of $\mathfrak{L}$. $[dh]=d=[d_1h]+[d_0h]$ so that $[d_1h]=d_1$, $[d_0h]=d_0$. d_1 and d_0 are therefore nilpotent, and d_0 is in $\mathfrak{T}_0$. Let $\{d_1, e_1\}$ and $\{d_0, e_0\}$ be 2-dimensional simple subsystems of $\mathfrak{T}_1$ and $\mathfrak{T}_0$ respectively. If $e=e_1+e_0$, then e is in $\mathfrak{T}$ and $[dee]=[d_0+d_1, e_0+e_1, e_0+e_1]$ $=[d_0e_0e_0]+[d_1e_1e_1]=e_0+e_1=e$. Also $[edd]=d$.

DEFINITION 3.2. *If* $\mathfrak{U}$ is a subset of a L.t.s. $\mathfrak{T}$, the *centralizer* $\mathfrak{V}$ of $\mathfrak{U}$ in $\mathfrak{T}$ is the set of elements v of $\mathfrak{T}$ such that $[v\mathfrak{U}\mathfrak{T}]=0$.

THEOREM 3.5. *If $\mathfrak{U}$ is a completely reducible subsystem of a completely reducible L.t.s. $\mathfrak{T}$, then the centralizer $\mathfrak{V}$ of $\mathfrak{U}$ in $\mathfrak{T}$ is completely reducible.*

Proof. Let $\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]=\mathfrak{L}$ and $\mathfrak{U}+[\mathfrak{U}\mathfrak{U}]=\mathfrak{M}$. $\mathfrak{L}\supseteq\mathfrak{M}$, and both $\mathfrak{L}$ and $\mathfrak{M}$ are completely reducible Lie algebras. Corollary 3.2 shows that $[\mathfrak{U}\mathfrak{V}]=0$. Clearly $\mathfrak{V}+[\mathfrak{V}\mathfrak{V}]$ is contained in the centralizer $\mathfrak{N}$ of $\mathfrak{M}$ in $\mathfrak{L}$, which is completely reducible by Theorem D. As in Corollary 3.2, write $\mathfrak{L}=\mathfrak{L}_0\oplus\mathfrak{L}_1\oplus\mathfrak{C}(\mathfrak{T})$, where $\mathfrak{L}_0=\mathfrak{T}_0\oplus[\mathfrak{T}_0\mathfrak{T}_0]$, and $\mathfrak{L}_1=\mathfrak{T}_1$. Corresponding to this decomposition represent an element v in $\mathfrak{V}$ by $v=v_0+v_1+v_c$. If $u=u_0+u_1+u_c$, $[vu]=[v_0u_0]$ $+[v_1u_1]=0$, and $[v_0u_0]=[v_1u_1]=0$ so that $[v_0\mathfrak{U}]=[v_1\mathfrak{U}]=0$, proving that the components of v are again in $\mathfrak{V}$. For n in $\mathfrak{N}$ let $n=n_0+n_1+n_c+n'$ where n' is in $[\mathfrak{T}_0\mathfrak{T}_0]$. By the argument above n_0, n_1, and n_c are in $\mathfrak{V}$. Thus $[nv]$ $=[n_0v_0]+[n_1v_1]+[n'v_0]$ is in $\mathfrak{V}+[\mathfrak{V}\mathfrak{V}]$ since $[n'v_0]$ is in $\mathfrak{V}$. Now

$$\begin{aligned}[\mathfrak{N}, (\mathfrak{V}+[\mathfrak{V}\mathfrak{V}])] &\subseteq \mathfrak{V}+[\mathfrak{V}\mathfrak{V}]+[[\mathfrak{V}\mathfrak{V}]\mathfrak{N}] \\ &\subseteq \mathfrak{V}+[\mathfrak{V}\mathfrak{V}]+[[\mathfrak{N}\mathfrak{V}]\mathfrak{V}]\subseteq\mathfrak{V}+[\mathfrak{V}\mathfrak{V}].\end{aligned}$$

As an ideal of a completely reducible Lie algebra, $\mathfrak{V}+[\mathfrak{V}\mathfrak{V}]$ is again completely reducible, for the ideals of such an algebra are sums of the simple ideals of $\mathfrak{N}^{(1)}$ and arbitrary subspaces of $\mathfrak{C}(\mathfrak{N})$ and therefore satisfy Theorem A.

IV. SIMPLE LIE TRIPLE SYSTEMS

Theorem 2.9 reduces the study of the structure of semi-simple L.t.s. and

(11) Lemma 4 in [5].

the determination of semi-simple L.t.s. to an investigation of simple L.t.s.

A. **Derivations of simple Lie triple systems.** Again, since every derivation of a semi-simple L.t.s. is inner, Theorem 2.9 reduces the study of derivations to simple systems. If $\mathfrak{T}$ is simple, then in any imbedding $[\mathfrak{T}\mathfrak{T}]\cong\mathfrak{D}(\mathfrak{T})$, the derivation Lie algebra of $\mathfrak{T}$. We now determine the structure of $\mathfrak{D}(\mathfrak{T})$ and the way in which these transformations act in $\mathfrak{T}$.

To begin, we observe that Jacobson's proof in [5] of the complete reducibility of the derivation algebra of a semi-simple Jordan algebra is readily translated into a proof of the following theorem:

THEOREM 4.1. *A simple L.t.s. $\mathfrak{T}$ is completely reducible with respect to $\mathfrak{D}(\mathfrak{T})$.*

Theorem A, section III now enables us to assert:

COROLLARY 4.2. *If $\mathfrak{T}$ is simple, $[\mathfrak{T}\mathfrak{T}]$ is the direct sum of its center and a semi-simple ideal.*

LEMMA 4.3. *Let $\mathfrak{T}$ be a simple L.t.s. Either $\mathfrak{T}$ is irreducible with respect to $\mathfrak{D}(\mathfrak{T})$, or $\mathfrak{T}=\mathfrak{U}_1\oplus\mathfrak{U}_2$, $\mathfrak{U}_i$ irreducible and $[\mathfrak{U}_i\mathfrak{U}_i]=0$.*

Proof. Suppose $\mathfrak{T}$ is not irreducible and $\mathfrak{U}_1$ is a proper subspace such that $\mathfrak{U}_1\mathfrak{D}(\mathfrak{T})\subseteq\mathfrak{U}_1$. If we suppose $\mathfrak{T}$ imbedded in a Lie algebra $\mathfrak{L}$, then subspaces $\mathfrak{U}$ of $\mathfrak{T}$ invariant with respect to $\mathfrak{D}(\mathfrak{T})$ are characterized by $[\mathfrak{T}\mathfrak{T}\mathfrak{U}]\subseteq\mathfrak{U}$.

(a) $[\mathfrak{U}_1\mathfrak{U}_1]$ is an ideal in $[\mathfrak{T}\mathfrak{T}]$, for $[[\mathfrak{T}\mathfrak{T}], [\mathfrak{U}_1\mathfrak{U}_1]]\subseteq[[\mathfrak{T}\mathfrak{T}\mathfrak{U}_1]\mathfrak{U}_1]\subseteq[\mathfrak{U}_1\mathfrak{U}_1]$.

(b) If $\mathfrak{M}$ is an ideal in $[\mathfrak{T}\mathfrak{T}]$, then $[\mathfrak{M}\mathfrak{T}]$ is an invariant subspace of $\mathfrak{T}$, for $[\mathfrak{T}\mathfrak{T}[\mathfrak{M}\mathfrak{T}]]\subseteq[[\mathfrak{T}\mathfrak{T}\mathfrak{M}]\mathfrak{T}]+[[\mathfrak{T}\mathfrak{T}\mathfrak{T}]\mathfrak{M}]\subseteq[\mathfrak{M}\mathfrak{T}]+[\mathfrak{M}\mathfrak{T}]$.

(c) If $\mathfrak{U}_2$ is an invariant subspace such that $\mathfrak{T}=\mathfrak{U}_1\oplus\mathfrak{U}_2$, then $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2]=0$. On the one hand $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2]\subseteq[\mathfrak{T}\mathfrak{T}\mathfrak{U}_2]\subseteq\mathfrak{U}_2$, while on the other, $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2]\subseteq[\mathfrak{U}_2\mathfrak{U}_1\mathfrak{U}_1]\subseteq\mathfrak{U}_1$.

(d) $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1]$ is an ideal in $\mathfrak{T}$. We see this by getting

$$
\begin{aligned}
[[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1]\mathfrak{T}\mathfrak{T}] &\subseteq [[\mathfrak{U}_1\mathfrak{U}_1], [\mathfrak{U}_1\mathfrak{T}]\mathfrak{T}] + [[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}]\mathfrak{U}_1\mathfrak{T}] && \text{(by (1.2))}\\
&\subseteq [\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}] + [\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}] && \text{(by (a))}\\
&\subseteq [\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1] && \text{(by (c))}.
\end{aligned}
$$

(e) $[\mathfrak{U}_1\mathfrak{U}_1]=0$ since $\mathfrak{L}$ is a standard enveloping algebra and $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}]=[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1]=0$.

(f) $\mathfrak{U}_1$ is irreducible. Suppose $\mathfrak{T}=\mathfrak{V}_1\oplus\mathfrak{V}_2\oplus\cdots\oplus\mathfrak{V}_r$, each $\mathfrak{V}_i$ is irreducible, and $r>2$. $[\mathfrak{V}_1\oplus\mathfrak{V}_i, \mathfrak{V}_1\oplus\mathfrak{V}_i]=0=[\mathfrak{V}_1\mathfrak{V}_i]$ by (e). But this means that $[\mathfrak{V}_1\mathfrak{T}]=0=[\mathfrak{V}_1\mathfrak{T}\mathfrak{T}]$, which is impossible. We conclude then that $r\leqq 2$, and that $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are irreducible.

LEMMA 4.4. *If $\mathfrak{T}$ is not irreducible with respect to $\mathfrak{D}(\mathfrak{T})$, the center $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])\neq 0$, and $\mathfrak{T}=\mathfrak{U}_1\oplus\mathfrak{U}_2$, $\mathfrak{U}_i$ irreducible isomorphic zero subsystems of $\mathfrak{T}$.*

Proof. From Lemma 4.3 we know that $\mathfrak{T}=\mathfrak{U}_1\oplus\mathfrak{U}_2$ where each $\mathfrak{U}_i$ is an irreducible zero subsystem of $\mathfrak{T}$. Let D be the linear transformation in $\mathfrak{L}$ defined by $u_1D=u_1$ for u_1 in $\mathfrak{U}_1$, $u_2D=-u_2$ for u_2 in $\mathfrak{U}_2$, and $[\mathfrak{T}\mathfrak{T}]D=0$. Since $[u_1u_2]D=0=[u_1u_2]-[u_1u_2]$, it can be verified that D is a derivation in $\mathfrak{L}$. Since $\mathfrak{T}D\subseteq\mathfrak{T}$, $D=\operatorname{Ad} c$ for some c in $[\mathfrak{T}\mathfrak{T}]$, and from the definition of D, c is in $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])$. Because $\operatorname{Ad} c$ is in $[\operatorname{Ad}\mathfrak{T}, \operatorname{Ad}\mathfrak{T}]$, it is a sum of commutators and $\operatorname{tr}(\operatorname{Ad} c)=0$. Since $\operatorname{Ad} c$ is the zero transformation in $[\mathfrak{T}\mathfrak{T}]$, the same must be true of the restriction of $\operatorname{Ad} c$ to $\mathfrak{T}$. An immediate consequence is that $\dim \mathfrak{U}_1=\dim \mathfrak{U}_2$, which is another way of saying $\mathfrak{U}_1\cong\mathfrak{U}_2$.

THEOREM 4.5. *If $\mathfrak{T}$ is a simple L.t.s. over an algebraically closed field, then either*

(i) *$\mathfrak{T}$ is irreducible with respect to $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{T})$ is semi-simple, or*

(ii) *$\mathfrak{T}$ is the direct sum of two irreducible, isomorphic, zero subsystems, and the center of $\mathfrak{D}(\mathfrak{T})$ is 1-dimensional.*

Proof. If $\mathfrak{T}$ is not irreducible, then, by Lemma 4.4, $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])\neq 0$. If $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])\neq 0$, then $\mathfrak{T}$ cannot be irreducible, for, as is well known, in an algebraically closed field the only transformations commuting with an irreducible Lie algebra of linear transformations are the scalar multiples of the identity, and since every element of $\operatorname{Ad}\mathfrak{C}([\mathfrak{T}\mathfrak{T}])$ is a sum of commutators, it has trace 0 and cannot be the identity. It remains to prove, then, that if $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])\neq 0$, it is 1-dimensional.

In case $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])\neq 0$, $\mathfrak{T}=\mathfrak{U}_1\oplus\mathfrak{U}_2$, $\mathfrak{U}_i$ irreducible. If $c\neq 0$ is in $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])$, because the base field is algebraically closed, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ can be decomposed relative to $\operatorname{Ad}\mathfrak{C}([\mathfrak{T}\mathfrak{T}])$ into direct sums of 1-dimensional invariant subspaces (Theorem D, section III). Let u_1 in $\mathfrak{U}_1$ and u_2 in $\mathfrak{U}_2$ be elements chosen from such a basis. $[[u_1u_2]c]=[[u_1c]u_2]+[u_1[u_2c]]=\alpha_1[u_1u_2]+\alpha_2[u_1u_2]=0$. Thus $\alpha_1=-\alpha_2$, and there is a scalar $\alpha\neq 0$ such that $[x_1c]=\alpha x_1$ for all x_1 in $\mathfrak{U}_1$, and $[x_2c]=-\alpha x_2$ for all x_2 in $\mathfrak{U}_2$. It is immediate from this that any element of $\mathfrak{C}([\mathfrak{T}\mathfrak{T}])$ is a scalar multiple of any other.

B. **The determination of simple Lie triple systems.** Section IIA provides the basis for the determination of all simple L.t.s. by reducing the problem to the study of automorphisms of period 2 in simple Lie algebras. In this section we shall carry out this project in detail for the case of an algebraically closed base field, a property we assume henceforth. The results given below on the automorphisms of the infinite classes of simple Lie algebras can be found in Jacobson [6]. The exceptional algebras will be treated separately later. These infinite classes are as follows:

Class A: The algebras $\mathfrak{A}_n$ of $n+1$ by $n+1$ matrices of trace zero.

Class B: The algebras $\mathfrak{B}_n$ of $2n+1$ by $2n+1$ skew-symmetric matrices.

Class C: The algebras $\mathfrak{C}_n$ of $2n$ by $2n$ matrices skew-symmetric relative to the involution $A\to Q^{-1}A'Q$ where

$$Q=\left[\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right].$$

Class D: The algebras $\mathfrak{D}_n$, $n>2$, of $2n$ by $2n$ skew-symmetric matrices.

Let Φ_n be the full n by n matrix algebra. The involution used in defining class C will be denoted by $X \rightarrow \overline{X}$, and will be called the symplectic involution in Φ_n. We have associated with each simple L.t.s. a unique simple Lie algebra. We shall say that the L.t.s. *belongs to* the Lie algebra and denote the class of L.t.s. belonging to the Lie algebras in class A by class A, and so on. If class 0 denotes the L.t.s. isomorphic to simple Lie algebras, we may consider this class completely determined and will not discuss it further. Every L.t.s. in one of the infinite classes is then determined by an automorphism in a simple Lie algebra of the corresponding class. Clearly two distinct automorphisms of a Lie algebra may determine isomorphic L.t.s.

DEFINITION 4.1. Automorphisms G and H of a Lie algebra $\mathfrak{L}$ will be called *similar* if there is an automorphism K of $\mathfrak{L}$ such that $G=K^{-1}HK$.

It is an easy matter to observe that automorphisms determine isomorphic L.t.s. if and only if they are similar. We are thus required to determine the similarity classes among the automorphisms of period 2 in the simple Lie algebras. In the sequel superscript capitals denote automorphisms, capital letters denote matrices or linear transformations, while small letters are elements of the base vector space. The automorphisms of the algebras in the infinite classes may be given as follows:

Class A: (i) $A^G=G^{-1}AG$;

(ii) $A^H=-H^{-1}A'H$.

Class B: D: $A^G=G^{-1}AG$, where $GG'=\gamma I\neq 0$, γ in Φ. ($\mathfrak{D}_4$ has additional automorphisms and will be discussed with the exceptional algebras.)

Class C: $A^G=G^{-1}AG$, where $GQ^{-1}G'Q=\gamma I\neq 0$.

Thus, associated with each automorphism are matrices which determine it. These will often be denoted by the same letter.

LEMMA 4.6. *Automorphisms G and H are similar if and only if there is a matrix J and a scalar ρ such that for matrices determining G and H we have*:

Class A: (i) $G=\rho J^{-1}HJ$ *or* $G=\rho J^{-1}(H')^{-1}J$;

(ii) $G=\rho J'(H')^{-1}J$ *or* $G=\rho J'HJ$.

Class B: D: $G=\rho J^{-1}HJ$, *where* $JJ'=\gamma I\neq 0$.

Class C: $G=\rho J^{-1}HJ$, *where* $JQ^{-1}J'Q=\gamma I\neq 0$.

Proof. Class A(i); suppose for the automorphisms G and H we have $G=J^{-1}HJ$ where J is also of type (i). $G^{-1}AG=J^{-1}H^{-1}JAJ^{-1}HJ$ and $A(J^{-1}HJ)G^{-1}=(J^{-1}HJ)G^{-1}A$ for each n by n matrix of trace 0. This implies that $J^{-1}HJG^{-1}=\rho^{-1}I$, and the converse is obtained by reversing the argument. Suppose now we have for the automorphisms G and H, $G=J^{-1}HJ$ where J is of type (ii). $G^{-1}AG=-(H^{-1}(J^{-1})'A'J^{-1}H)^J=J^{-1}H'JAJ^{-1}(H')^{-1}J$ so that again we get $G=\rho J^{-1}(H')^{-1}J$.

Class A(ii); suppose the automorphism of similarity J is of type (i). $-G^{-1}A'G=-J^{-1}H^{-1}(J')^{-1}A'J'HJ$ for every n by n matrix A of trace 0.

As before this is equivalent to $G=\rho J'HJ$. If J is also of type (ii), we obtain $-G^{-1}A'G=J^{-1}H'(J^{-1})'A'J'(H^{-1})'J$ so that $G=\rho J'(H')^{-1}J$. The calculations for classes B, C, and D are similar to those in the first part of class A(i).

LEMMA 4.7. *An automorphism G is of period 2 if and only if G is determined by a matrix G such that in*:

Class A(i) $G^2=I$, (ii) $G=G'$ *or* $G=-G'$,

Class B $G^2=I$ *and* $G=G'$,

Class C $G^2=I$ *and* $G=\overline{G}$ *or* $G^2=-I$ *and* $G=-\overline{G}$,

Class D $G^2=I$ *and* $G=G'$ *or* $G^2=-I$ *and* $G=-G'$.

Proof. If the matrix G_1 determines the automorphism G of type (i) in class A, G_1^2 commutes with all n by n matrices of trace 0, hence $G_1^2=\rho I$. If $G_2=\rho^{-1/2}G_1$, G_2 also determines G and $G_2^2=I$. If G_1 determines an automorphism of period 2 of type (ii), $G_1(G_1')^{-1}=\gamma I$, or $G_1=\gamma G_1'$. But then $G_1'=\gamma G_1$, $G_1=\gamma^2G_1$, $\gamma^2=1$, and $\gamma=\pm 1$.

If G is an automorphism in a Lie algebra of class B, applying the same procedure as above, we select a matrix G determining G such that $G^2=\gamma I$, $GG'=I$. Then $\gamma G'=G$, $G'=\gamma G$, and $\gamma=\pm 1$. If $G^2=I$, $G'=G$, and when $G^2=-I$, $G'=-G$, which is impossible when the number of rows is odd as in this case. The same method yields the results for class D and when applied to $\overline{G}$ instead of G' includes class C. In what follows $P_{n,r}$ is the diagonal n by n matrix:

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & -I_{n-r} \end{array}\right].$$

THEOREM 4.8. *Every L.t.s. of class* A *is isomorphic to one of the following*:

(i) *the matrices of trace* 0 *skew-symmetric relative to the automorphism* $A^{P_{n,r}}=P_{n,r}^{-1}AP_{n,r}$,

(ii) *the symmetric matrices of trace* 0,

(iii) *the symplectic symmetric matrices of trace* 0 (*n even*).

Proof. (i) First we prove that every automorphism of period 2 of type (i) is similar to an automorphism determined by a matrix $P_{n,r}$. By Lemma 4.6 any matrix "similar" in the usual sense to one determining such an automorphism G determines a similar automorphism. Let R be the underlying n-dimensional vector space, and let G_0 be a linear transformation in R with matrix G, $G^2=I$. Any matrix for G_0 obtained by a change of basis in R will therefore determine an automorphism similar to the one determined by G. $G_0^2=I$, $(I=G_0)(I+G_0)=0$. Let $E_0=1/2(I-G_0)$ and $E_0^*=1/2(I+G_0)$. These transformations form a complete set of orthogonal projections, so that $R=RE_0\oplus RE_0^*$. $xG_0=x$ for x in RE_0, and $xG_0=-x$ for x in RE_0^*. Relative to any basis $e_1, \cdots, e_n$ where $e_1, \cdots, e_r$ are in RE_0 and $e_{r+1}, \cdots, e_n$ are in RE_0^*, G_0 will have a matrix of the form $P_{n,r}$.

(ii) Consider now an automorphism in $\mathfrak{A}_{n-1}$ determined in type (ii) by a matrix for which $G'=G$. Lemma 4.6 shows that any cogredient matrix will determine a similar automorphism. It is well known that any nonsingular symmetric matrix is cogredient to the identity matrix, which determines the automorphism $A\to -A'$.

(iii) Suppose for an automorphism of type (ii) we have a matrix for which $G'=-G$. This implies that R is even-dimensional. Thinking of G as the matrix of an alternate nondegenerate bilinear form over R, and recalling the fact that any such matrix is cogredient to the matrix Q, we conclude that the automorphism determined by G is similar to the one determined by Q, which is $A^Q=-Q^{-1}A'Q=-\overline{A}$.

In the proof of the next two theorems we make further use of the terminology and properties of bilinear forms.

THEOREM 4.9. *Every L.t.s. of classes* B *and* D *is isomorphic to one of the following*:

(i) *the L.t.s. of skew-symmetric matrices which are, in addition, skew-symmetric relative to an automorphism* $P_{n,r}$,

(ii) *the L.t.s. of skew-symmetric matrices which are, in addition, symplectic symmetric* (*Class* D *only*).

Proof. (i) Let (x, y) be a symmetric, nondegenerate bilinear form over R. Let G be a matrix determining an automorphism of a Lie algebra of class B or D. Let $G^2=I$, and suppose, using Lemma 4.7, that $G=G'$. To establish (i) it is sufficient to prove that if G is the matrix of a linear transformation G_0 relative to a Cartesian basis for R, then there is a Cartesian basis for R made up of bases for RE_0 and RE_0^*. The change of basis matrix J will then be orthogonal, and by Lemma 4.6 some $P_{n,r}$ matrix will determine an automorphism similar to the given one.

Now assume that G_0 is a linear transformation which has matrix G relative to a basis $f_1, \cdots, f_n$, Cartesian for (x, y). RE_0 is nonisotropic (i.e., (x, y) restricted to RE_0 is nondegenerate), for if $(x(I-G_0), y_0(I-G_0))=0$ for some y_0 and all x in R, since G_0 is orthogonal, $(x, y_0(I-G_0))=0$ for all x in R. Thus $y_0(I-G_0)=0$, and y_0 is in RE_0^*. It is clear now that if $S^\perp$ is the orthogonal complement of a subspace S in R, we have $(RE_0)^\perp=RE_0^*$. Since any nonisotropic subspace has a Cartesian basis, we select such a basis $e_1, \cdots, e_r$ for RE_0 and another $e_{r+1}, \cdots, e_n$ for RE_0^*. The remark above on orthogonal complements proves that $e_1, \cdots, e_n$ is a Cartesian basis for R.

(ii) If the dimension n of R is even, there are automorphisms of period 2 having matrices for which $G'=-G$. Assume again that G_0 is a transformation having G as a matrix relative to some Cartesian basis. Again it is sufficient to prove that there is a Cartesian basis for R with respect to which G_0 has Q as a matrix. First choose a vector e_1' for which $(e_1', e_1')=1$. Let $e_2'=e_1'G_0$. $(e_2', e_2')=(e_2'G_0, e_2'G_0)=(e_1', e_1')=1$ and $(e_1', e_2')=(e_1', e_1'G_0)=-(e_1'G_0, e_1')$

$=-(e_1', e_1'G_0)=0$. Let $S_1=(e_1')\oplus(e_2')$. If $(\alpha_1 e_1'+\alpha_2 e_2', S_1)=0$, $(\alpha_1 e_1'+\alpha_2 e_2', e_1')=0$ and $\alpha_1=0$; also $(\alpha_1 e_1'+\alpha_2 e_2', e_2')=0$ and $\alpha_2=0$. S_1 is therefore nonisotropic, $R=S_1\oplus S_1^\perp$, and this implies $S_1^\perp$ is also nonisotropic. Working in $S_1^\perp$, we may continue this procedure until a basis $e_1', \cdots, e_n'$ is chosen. This basis is Cartesian and the form Q for the matrix of G_0 is achieved if we make the further orthogonal change of basis:

$$e_1 = e_1', e_{v+1} = e_2', e_2 = e_3', e_{v+2} = e_4', \text{ etc., where } v = n/2.$$

In the next theorem $P_{n,n/2,r}$ stands for the n by n matrix

$$\left[\begin{array}{c|c} \begin{array}{c|c} I_r & 0 \\ \hline 0 & -I_{(n/2)-r} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} I_r & 0 \\ \hline 0 & -I_{(n/2)-r} \end{array} \end{array}\right].$$

THEOREM 4.10. *Every L.t.s. of class* C *is isomorphic to one of the following*:

(i) *the L.t.s. of symplectic skew-symmetric matrices which are, in addition, skew-symmetric relative to an automorphism* $P_{n,n/2,r}$,

(ii) *the L.t.s. of symplectic skew-symmetric matrices which are, in addition, symmetric.*

Proof. Let (x, y) be an alternate nondegenerate bilinear form over R, and let G, a matrix determining an automorphism of period 2 in $\mathfrak{C}_{n/2}$, be the matrix of a transformation G_0 relative to a symplectic basis for R. The theorem will be proved if it can be shown that when (i) $\overline{G}=G$, there is a symplectic basis with respect to which G_0 has matrix $P_{n,n/2,r}$, and that when (ii) $\overline{G}=-G$, a symplectic basis exists relative to which G_0 has matrix Q. This final condition on symplectic skew-symmetric matrices holds if and only if they are symmetric.

(i) $\overline{G}=G$. Since we may assume $G^2=I$, the calculation of Theorem 4.8 applies and gives $(RE_0)^\perp=RE_0^*$, and a similar argument establishes, in this instance, a symplectic basis $e_1', \cdots, e_s'$ for RE_0 and another $e_{s+1}', \cdots, e_n'$ for RE_0^*. s must therefore be even. Let $r=s/2$ and $v=n/2$. To form a symplectic basis for R take $e_i=e_i'$, $i=1, \cdots, r$, $e_{v+i}=e_{r+i}'$, $i=1, \cdots, r$, $e_{r+i}=e_{s+i}'$, $i=1, \cdots, v-r$, $e_{v+r+i}=e_{v+r+i}'$, $i=1, \cdots, v-r$. $e_1, \cdots, e_n$ is symplectic and G_0 has a matrix $P_{n,n/2,r}$ relative to it.

(ii) $\overline{G}=-G$. In this case there is a vector e_1' such that $(e_1', e_1'G_0)\neq 0$, so that an e_1 can be selected for which $(e_1, e_1G_0)=1$. Let $e_{v+1}=e_1G_0$. $e_{v+1}G_0=-e_1$, and $(e_1, e_{v+1})=1$. $S_1=(e_1)\oplus(e_{v+1})$ is nonisotropic, $S_1\oplus S_1^\perp=R$, $S_1^\perp$ is nonisotropic, and $S_1^\perp$ contains a vector e_2 for which $(e_2, e_2G_0)\neq 0$. Again we may assume $(e_2, e_2G_0)=1$ and set $e_{v+2}=e_2G_0$. Continuing, we obtain the required symplectic basis $e_1, \cdots, e_n$.

Using the representations provided by Theorems 4.8, 4.9, and 4.10, the form of the matrices representing each simple L.t.s. $\mathfrak{T}$ belonging to one of the infinite classes can be described. From these $[\mathfrak{T}\mathfrak{T}]$ can be determined and from this the structure of $\mathfrak{D}(\mathfrak{T})$ deduced. We confine ourselves to the list below of the structures of the algebras $\mathfrak{D}(\mathfrak{T})$. $C_{n,r}$ stands for the n by n matrix

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & -r/(n-r)I_{n-r} \end{array}\right].$$

	LIE ALGEBRA L.T.S.	DERIVATION ALGEBRA
$\mathfrak{A}_n$	(i) $\mathfrak{T}_i$, $i=0, \cdots, n-1$,	$\mathfrak{D}(\mathfrak{T}_i)\cong\mathfrak{A}_i\oplus\mathfrak{A}_{n-i-1}\oplus(C_{n+1,i+1})$
	(ii) $\mathfrak{U}$	$\mathfrak{D}(\mathfrak{U})\cong\mathfrak{D}_{(n+1)/2}$ (n odd)
		$\mathfrak{D}(\mathfrak{U})\cong\mathfrak{B}_{n/2}$ (n even)
	(iii) $\mathfrak{V}$	$\mathfrak{D}(\mathfrak{V})\cong\mathfrak{C}_{(n+1)/2}$ (n odd)
$\mathfrak{B}_n$	(i) $\mathfrak{T}_i$, $i=1, \cdots, n-1$,	$\mathfrak{D}(\mathfrak{T}_i)\cong\mathfrak{B}_i\oplus\mathfrak{D}_{n-i}$
$\mathfrak{C}_n$	(i) $\mathfrak{T}_i$, $i=1, \cdots, n-1$,	$\mathfrak{D}(\mathfrak{T}_i)\cong\mathfrak{C}_i\oplus\mathfrak{C}_{n-i}$
	(ii) $\mathfrak{U}$	$\mathfrak{D}(\mathfrak{U})\cong\mathfrak{A}_{n-1}\oplus(Q)$
$\mathfrak{D}_n$	(i) $\mathfrak{T}_i$, $i=1, \cdots, n-1$,	$\mathfrak{D}(\mathfrak{T}_i)\cong\mathfrak{D}_i\oplus\mathfrak{D}_{n-i}$
	(ii) $\mathfrak{U}_i$, $i=1, \cdots, n-1$,	$\mathfrak{D}(\mathfrak{U}_i)\cong\mathfrak{B}_i\oplus\mathfrak{B}_{n-1-i}$
	(iii) $\mathfrak{V}$	$\mathfrak{D}(\mathfrak{V})\cong\mathfrak{A}_{n-1}\oplus(Q)$

In order to determine the simple L.t.s. which belong to the exceptional Lie algebras $\mathfrak{G}_2$, $\mathfrak{F}_4$, $\mathfrak{E}_6$, $\mathfrak{E}_7$, $\mathfrak{E}_8$, which, together with the infinite classes, exhaust simple Lie algebras (over an algebraically closed field of characteristic 0), it will be necessary to employ the structure theory for semi-simple Lie algebras developed by Cartan, Weyl, and others. We therefore summarize briefly some of these results.

Let $\mathfrak{L}$ be a semi-simple Lie algebra and x an element of $\mathfrak{L}$. R_x^α is the set of elements y of $\mathfrak{L}$ such that, for some n, $(\alpha I - \text{Ad } x)^n$ annihilates y. An element x is *regular* if $\dim R_x^0 = \min \{\dim R_y^0\}$, y in $\mathfrak{L}$. A *Cartan subalgebra* $\mathfrak{H}$ of $\mathfrak{L}$ is a maximal commutative subalgebra containing a regular element. If x is regular, R_x^0 is a Cartan subalgebra. Relative to Ad $\mathfrak{H}$, $\mathfrak{L}$ is completely reducible, and the matrices of Ad $\mathfrak{H}$ may be simultaneously diagonalized relative to a "canonical" basis $h_1, \cdots, h_r, e_1, \cdots, e_s$, where $h_1, \cdots, h_r$ is a basis for $\mathfrak{H}$. This gives $[e_ih]=\alpha(h)e_i$ for some linear function α defined on $\mathfrak{H}$. It turns out that R_h^λ is 1-dimensional so that we may indicate these functions by writing $e_\alpha, \cdots, e_\sigma$ instead of $e_1, \cdots, e_s$. The functions α are the *roots* of $\mathfrak{H}$, and the e_α are *root vectors*. If α is a root so is $-\alpha$, but no other nonzero rational multiple of α is. Among the roots there are precisely r linearly independent. If $\alpha+\beta$ is a root not zero, $[e_\alpha e_\beta]=N_{\alpha,\beta}e_{\alpha+\beta}$ and $[e_\alpha e_{-\alpha}]=h_\alpha$ is in $\mathfrak{H}$, while $[e_\alpha e_\beta]=0$ if $\alpha+\beta$ is not a root. Further, the $N_{\alpha,\beta}$ may be taken to be rational, and each h_α a rational combination of the h_i. The mapping $\alpha\rightarrow h_\alpha$ is 1-to-1 and linear so that we may consider the roots imbedded in $\mathfrak{H}$. In particular the "roots" h_α span $\mathfrak{H}$. It is possible to order the roots linearly and among the positive roots call those which are not sums of other positive roots, *simple*. The set π of simple roots (also a *fundamental root basis*) is linearly independent, and any root may be obtained from π by the operations of addition (within the set of roots) and multiplication by -1. Any positive root ϕ then has the unique expression $\phi = \sum p_i\alpha_i$, α_i in π, where p_i is a non-negative integer. It follows from this that the set $\{e_\alpha\}$, α or $-\alpha$ a simple root, generate $\mathfrak{L}$.

In case Φ is the field of complex numbers, Gantmacher [2] has determined the form of the automorphisms with simple elementary divisors in any simple Lie algebra. His results lead to the determination of all complex simple L.t.s., and this information, we shall show, is enough to do the same for any algebraically closed field. As applied to our situation, Gantmacher's results establish the facts given in the next three theorems.

Let A be an automorphism with simple elementary divisors in a complex simple Lie algebra $\mathfrak{L}$. For some Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$, $\mathfrak{H}A = \mathfrak{H}$. A is *inner* if, for some $\mathfrak{H}$, $hA = h$ for all h in $\mathfrak{H}$. A is *outer* otherwise. Automorphisms A and B with simple elementary divisors are *inner similar* if for some inner automorphism C, $A = C^{-1}BC$. Now fix a Cartan subalgebra $\mathfrak{H}$ and a canonical basis.

THEOREM G. *Any automorphism A of period* 2 *in $\mathfrak{L}$ is inner similar to an automorphism A_0 such that $\mathfrak{H}A_0 = \mathfrak{H}$. A_0 can be chosen so that $hA_0 = h$ for all h in $\mathfrak{H}$ if and only if A is inner.*

Let S be the class of automorphisms of period 2 in $\mathfrak{L}$. If T_i is a linear transformation in $\mathfrak{H}$, let S_i be the elements of S which are inner similar to an automorphism inducing in $\mathfrak{H}$ the transformation T_i.

THEOREM H. *There exist liner transformations $T_0, T_1, \cdots, T_k$ of $\mathfrak{H}$ such that $S = S_0 \cup S_1 \cup \cdots \cup S_k$ and $S_i \cap S_j = 0$ if $i \neq j$. $\{T_i\}$ can be chosen so that T_0 is the identity and hence, by Theorem* G, *S_0 is the set of inner automorphisms of period* 2.

Any automorphism A such that $\mathfrak{H}A = \mathfrak{H}$ sends roots into roots; that is, $h_\alpha A = h_{\alpha^*}$ for some root α^*. These images determine completely a linear transformation in $\mathfrak{H}$. The next assertion is then immediate from Theorem H.

THEOREM I. (i) *For A in S_0, A is innear similar to an automorphism A_0 such that $hA_0 = h$ for all h in $\mathfrak{H}$ and $e_\alpha A_0 = \pm e_\alpha$.*

(ii) *For A in S_i, $i \neq 0$, A is inner similar to an automorphism A_0 such that $hA_0 = hT_i$ and $e_\alpha A_0 = \pm e_{\alpha^*}$ if $h_\alpha T_i = h_{\alpha^*}$.*

The only simple exceptional Lie algebra with outer automorphisms is $\mathfrak{E}_6$, and $S = S_0 \cup S_1$ at most, except for $\mathfrak{D}_4$, in which case $S = S_0 \cup S_1 \cup S_2 \cup S_3$. Using Theorem I we can give an explicit method for determining a kind of canonical basis for every L.t.s. determined by an inner automorphism of period 2.

As before fix a Cartan subalgebra $\mathfrak{H}$ and a canonical basis of a complex simple Lie algebra $\mathfrak{L}$. Let $\pi = \{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots.

THEOREM 4.11. *Let $A(e_{\alpha_1}, \cdots, e_{\alpha_r})$ be any function on $e_{\alpha_1}, \cdots, e_{\alpha_r}$ such that $e_{\alpha_i}A = \pm e_{\alpha_i}$; then $A(e_{\alpha_1}, \cdots, e_{\alpha_r})$ can be extended uniquely to an inner automorphism of period 2 in $\mathfrak{L}$, and any such automorphism is inner similar to an automorphism inducing such a function.*

Proof. The final statement of the theorem is true by Theorem I. To establish the first part define $hA = h$ for all h in $\mathfrak{H}$. Let $\pi(-)$ be the subset of π consisting of roots α_i such that if A represents the given function, $e_{\alpha_i}A = -e_{\alpha_i}$. If α is a positive root, $\alpha = \sum a_i\alpha_i$ where $a_i \geqq 0$ is an integer. Let $N(\alpha) = (-1)^{\Sigma a_i}$ and define $e_\alpha A = N(\alpha)e_\alpha$ if $\alpha \geqq 0$. If $\alpha < 0$, set $e_\alpha A = N(-\alpha)e_\alpha$. This defines a linear transformation A in $\mathfrak{L}$. A will be an automorphism if $[he_\alpha]A = [h(e_\alpha A)]$ and $[e_\alpha e_\beta]A = [(e_\alpha A)(e_\beta A)]$. The only case where this is not immediate is when $\alpha + \beta = \gamma$ is a root. Let $\alpha = \sum a_i\alpha_i$, $\beta = \sum b_i\alpha_i$, and $\gamma = \sum c_i\alpha_i$. $c_i = a_i + b_i$. $[(e_\alpha A)(e_\beta A)] = N_{\alpha\beta}e_\gamma$ if and only if $N(\alpha)N(\beta) = 1$. $[e_\alpha e_\beta]A = N_{\alpha\beta}e_\gamma A = N_{\alpha\beta}e_\gamma$ if and only if $N(\gamma) = 1$. We have however the relation $N(\gamma) = (-1)^{\Sigma c_i} = (-1)^{\Sigma a_i}(-1)^{\Sigma b_i} = N(\alpha)N(\beta)$. Moreover A is uniquely determined by its effect on $e_{\alpha_1}, \cdots, e_{\alpha_r}$ since this uniquely determines the effect on $e_{-\alpha_1}, \cdots, e_{-\alpha_r}$, and the subalgebra generated by these elements is $\mathfrak{L}$.

It is now an easy matter, if we use a Cartan subalgebra and a canonical basis, to determine multiplication tables for all simple L.t.s. belonging to a given simple Lie algebra via an inner automorphism of period 2.

As for L.t.s. which arise from outer automorphisms, the only cases which have not already been discussed are $\mathfrak{D}_4$ and $\mathfrak{E}_6$. In $\mathfrak{D}_4$ there are four inner similarity classes of automorphisms of period 2. Two of these, including S_0, determine L.t.s. listed in the preceding discussion of the infinite classes. Referring to [2, p. 134], we may verify that although not inner similar to an element of S_1, the automorphisms of S_2 and S_3 are in fact *similar* to those in S_1. Thus $\mathfrak{D}_4$ possesses no L.t.s. not already mentioned.

In $\mathfrak{E}_6$ choose a canonical basis $h_1, \cdots, h_6, e_\alpha, \cdots$ relative to a Cartan subalgebra H. $S = S_0 \cup S_1$ and T_1 can be taken to be the following transformation(12):

$$h_1T_1 = -h_2 + 1/3 \sum_{i=1}^{6} h_i,$$

$$h_2T_1 = -h_1 + 1/3 \sum_{i=1}^{6} h_i,$$

$$h_3T_1 = -h_4 + 1/3 \sum_{i=1}^{6} h_i,$$

(12) [2, p. 141].

$$h_4T_1 = - h_3 + 1/3 \sum_{i=1}^{6} h_i, \text{ etc.}$$

The roots take the form $\alpha_{pq}(h) = \alpha_{pq}(\sum \lambda_i h_i) = \lambda_p - \lambda_q$ for $p \neq q$, and $p, q = 1, \cdots, 6$. There are also roots $\alpha_{pqs}(h) = \lambda_p + \lambda_q + \lambda_s, -\alpha_{pqs}(h)$, where p, q, and s are all distinct, $p, q, s = 1, \cdots, 6$; and finally $\alpha_0(h) = \sum_{i=1}^{6} \lambda_i$ and $-\alpha_0(h)$. Denote the root vectors corresponding to these roots by e_{pq}, e_{pqs}, e'_{pqs}, e_0, and e'_0 respectively. $\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}$, and α_{135} form a set of simple roots. It follows from Theorem I and the form of T_1 that any outer automorphism of period 2 is inner similar to an automorphism A such that

$$\begin{aligned} &e_{12}A = \pm e_{12}, \quad e_{23}A = \pm e_{41}, \quad e_{34}A = \pm e_{34}, \quad e_{45}A = \pm e_{63}, \\ &e_{56}A = \pm e_{56}, \quad e_{135}A = \pm e_{135}. \end{aligned} \tag{4.1}$$

Relative to A, $\mathfrak{E}_6$ decomposes into the direct sum of 24 two-dimensional invariant subspaces, 24 one-dimensional invariant subspaces, and $\mathfrak{H}$. $\mathfrak{H}$, in turn, splits into the direct sum of a four-dimensional subspace of fixed elements and a two-dimensional subspace of elements h such that $hA = -h$. From this fact regarding $\mathfrak{H}$ it follows that if any 4 of the signs in (4.1) are specified, all 6 are uniquely determined. Since A is completely determined by the table (4.1), we obtain all L.t.s. determined by outer automorphisms in $\mathfrak{E}_6$ by assigning all permissible arrangements of signs in (4.1) and calculating the effect of A on the remainder of the basis elements by using the multiplication table for $\mathfrak{E}_6$.

Precisely two nonisomorphic L.t.s., $\mathfrak{T}_1$, $\mathfrak{T}_2$, arise in this way. Dim $\mathfrak{T}_1 = 26$ and dim $\mathfrak{T}_2 = 42$. Thus $[\mathfrak{T}_1\mathfrak{T}_1]$ has 52 dimensions while $[\mathfrak{T}_2\mathfrak{T}_2]$ has 36 dimensions. It is possible to show that each $[\mathfrak{T}_i\mathfrak{T}_i]$ has a Cartan subalgebra of dimension 4. Using this fact together with Corollary 4.2 and the representation of $[\mathfrak{T}_i\mathfrak{T}_i]$ given by restricting Ad $[\mathfrak{T}_i\mathfrak{T}_i]$ to $\mathfrak{T}_i$, the representation theory for semi-simple Lie algebras allows us to conclude that $[\mathfrak{T}_1\mathfrak{T}_1] \cong \mathfrak{F}_4$([13]) and $[\mathfrak{T}_2\mathfrak{T}_2] \cong \mathfrak{C}_4$.

Next we provide an illustration of the results obtainable from Theorem 4.11 and complete the determination of L.t.s. of class $\mathfrak{E}_6$.

Suppose A is an inner automorphism of period 2 in $\mathfrak{E}_6$. One possibility is that $e_{12}A = -e_{12}$, $e_{i,i+1}A = e_{i,i+1}$ for $i = 2, 3, 4, 5$, and $e_{135}A = e_{135}$. Of course, $hA = h$ for all h in $\mathfrak{H}$. From the multiplication table we can compute that for 32 of the e_α, $e_\alpha A = -e_\alpha$. Thus the L.t.s. $\mathfrak{T}_3$ belonging to A is 32-dimensional and $[\mathfrak{T}_3\mathfrak{T}_3]$ has 46 dimensions. Knowing the general structure of $[\mathfrak{T}_3\mathfrak{T}_3]$, we obtain from an examination of its multiplication table the fact that $[\mathfrak{T}_3\mathfrak{T}_3] \cong \mathfrak{D}_5 \oplus (c)$ where (c) indicates the 1-dimensional center of $[\mathfrak{T}_3\mathfrak{T}_3]$. It turns out that only one other L.t.s. $\mathfrak{T}_4$ arises from the inner automorphisms

([13]) It follows from the theorem of [1] that $\mathfrak{T}_1 = \mathfrak{W}^{(1)}$, where $\mathfrak{W} = \mathfrak{W}^{(1)} \oplus (1)$ is the associator L.t.s. of the exceptional simple Jordan algebra.

of $\mathfrak{E}_6$. $\mathfrak{T}_4$ has dimension 40, $[\mathfrak{T}_4\mathfrak{T}_4]$ dimension 38, and $[\mathfrak{T}_4\mathfrak{T}_4]\cong\mathfrak{A}_5\oplus\mathfrak{A}_1$.

Although we have provided an explicit method for calculating all complex simple L.t.s., we have yet to show what can be done if the base field is arbitrary but algebraically closed. In this case we use the fact, noted before, that any simple Lie algebra has a basis for which the multiplication table is rational; that is, all basis products are rational combinations of the basis elements. Let * denote Kronecker products, Γ be the complex field, and P the rational field. The assertion amounts to the statement that there is a natural algebra $\mathfrak{L}$ over P for which ($\mathfrak{L}$ over P)$*\Gamma\cong\mathfrak{L}$. The algebra $\mathfrak{L}_\Phi=(\mathfrak{L}$ over $\mathrm{P})*\Phi$ is then a simple Lie algebra over Φ, the natural counterpart of $\mathfrak{L}$. This argument is symmetric and establishes a natural connection from which it is possible to obtain the simple Lie algebras over Φ from the complex simple algebras. It is clear that all that is needed to use the same type of argument for L.t.s. is a proof of the fact that every simple L.t.s. has a rational multiplication table.

LEMMA 4.12. *If A is an automorphism of period 2 in a simple Lie algebra $\mathfrak{L}$ over an algebraically closed field, then there is a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ for which $\mathfrak{H}A=\mathfrak{H}$.*

Proof. Let $\mathfrak{T}$ be the L.t.s. determined by A. It is clear that if $[\mathfrak{T}\mathfrak{T}]$ is not semi-simple, a Cartan subalgebra will consist of a Cartan subalgebra for $[\mathfrak{T}\mathfrak{T}]^{(1)}$ together with the 1-dimensional center. Let $\mathfrak{H}_1$ be a Cartan subalgebra of $[\mathfrak{T}\mathfrak{T}]$. $\mathfrak{H}_1$ is a commutative subalgebra of $\mathfrak{L}$, and the representation theory for semi-simple Lie algebras shows that Ad h has simple elementary divisors for each h in $\mathfrak{H}_1$. Let h_1 be a regular element of $\mathfrak{H}_1$. $\mathfrak{H}_1$ is then the intersection of $R^0_{h_1}$ with $[\mathfrak{T}\mathfrak{T}]$. The condition that each Ad h has simple elementary divisors implies that $\mathfrak{H}_1$ is contained in a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. Clearly $\mathfrak{H}\subseteq R^0_h$ for each h in $\mathfrak{H}_1$. Let h_r in $\mathfrak{H}$ be regular in $\mathfrak{L}$. Suppose for some x in $\mathfrak{H}^0_1=\bigcap R^0_h$, h in $\mathfrak{H}_1$, $[xh_r]\neq 0$. We may assume x is also in $\mathfrak{T}$. From this it follows that $[xh_r]$ is in $\mathfrak{H}_1$. $[[xh_r]h_r]=0$. But Ad h_r has simple elementary divisors. Thus $[xh_r]=0$, and $\mathfrak{H}^0_1=\mathfrak{H}$. Since dim $R^0_{h_r}=$ dim $R^0_{h_1}$, h_1 is regular in $\mathfrak{L}$, and $R^0_{h_1}$ is the required invariant Cartan subalgebra.

THEOREM 4.13. *Every simple L.t.s. has a rational multiplication table.*

Proof. Let A determine the simple L.t.s. $\mathfrak{T}$ in $\mathfrak{L}$. Choose a Cartan subalgebra $\mathfrak{H}$ for which $\mathfrak{H}\mathfrak{A}=\mathfrak{H}$. Since any automorphism maps roots into roots, $h_\alpha A=h_{\alpha^*}$ for some root α^*. From this it follows that A sends root vectors into root vectors, and, in fact, $e_\alpha A=\gamma e_{\alpha^*}$. Since A is of period 2, $\gamma=\pm 1$. Choose in $\mathfrak{L}$ a canonical rational basis with respect to $\mathfrak{H}$. Selecting a basis for $\mathfrak{H}$ from $\{h_\alpha\}$ preserves the rationality, for $[e_\alpha, h_\beta]=\alpha(h_\beta)e_\alpha$ and $\alpha(h_\beta)$ is a rational combination of $\alpha(h_1), \cdots, \alpha(h_r)$. $\mathfrak{T}$ will then have a basis of elements of the form h_α, $h_\alpha\pm h_{\alpha^*}$, e_α, and $e_\alpha\pm e_{\alpha^*}$, and consequently a rational multiplication table.

Bibliography

1. C. Chevalley and R. D. Schafer, *The exceptional simple Lie algebras F_4 and E_6*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 137.

2. N. Gantmacher, *Canonical representation of automorphisms of a semi-simple Lie group*, Rec. Math. (Mat. Sbornik) N.S. vol. 5 (1939) p. 101.

3. G. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math. vol. 64 (1942) p. 667.

4. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. vol. 69 (1950) p. 479.

5. N. Jacobson, *Completely reducible Lie algebras of linear transformations*, Proceedings of the American Mathematical Society vol. 2 (1951) p. 105.

6. ———, *Simple Lie algebras over a field of characteristic* 0, Duke Math. J. vol. 4 (1938) p. 694.

7. ———, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. vol. 70 (1951) p. 509.

8. E. V. Schenkman, *A theory of subinvariant Lie algebras*, Amer. J. Math. vol. 73 (1951) p. 453.

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