Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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CS-R9346 1993

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Abstract

The Structural Induction Theorem (Lehmann and Smyth, 1981; Plotkin, 1981) characterizes initial F-algebras of locally continuous functors F on the category of cpo's with strict and continuous maps. Here a dual of that theorem is presented, giving a number of equivalent characterizations of final coalgebras of such functors. In particular, final coalgebras are order strongly-extensional (sometimes called internal full abstractness): the order is the union of all (ordered) F-bisimulations. (Since the initial fixed point for locally continuous functors is also final, both theorems apply.) Further a similar co-induction theorem is given for a category of complete metric spaces and locally contracting functors.

AMS Subject Classification (1991): 68Q55
CR Subject Classification (1991): D.3.1, F.3.2
Keywords & Phrases: Co-induction, coalgebras, complete partial order, complete metric space, strong extensionality, F-bisimulation, ordered kernel pair.
Note: This is an improved version of Technical Report CS-R9346, CWI, Amsterdam, 1993. It will appear in the proceedings of the ninth conference on Mathematical Foundations of Programming Semantics, New Orleans, 1993.

1. Introduction

Consider a preorder $(P, \leq)$ and a monotone function $f : P \to P$. An element $q \in P$ is a post-fixed point of f (also called f-consistent) if $q \leq f(q)$. If the collection of post-fixed points of f has a largest element, then this is also the greatest fixed point of f. Defining p as the greatest post-fixed point of f is sometimes called *a co-inductive definition.* (A typical example is a complete lattice $(P, \subseteq)$ and a monotone function f, which by Tarski's fixed-point theorem has a greatest (post-)fixed point.) Being the greatest post-fixed point can also be used as a proof method: in order to establish $q \leq p$, for $q \in P$, it is sufficient to prove $q \leq f(q)$. This fact is sometimes called a *co-induction principle.*

A familiar example in computer science is the co-inductive definition of the bisimilarity relation on a labelled transition system. It is defined as the greatest fixed point of a monotone function on the lattice of relations on the states of this transition system (see [Mil89]). An example of the above co-induction proof principle can be found in [MT91], where it is used to prove the consistency of the static and the dynamic semantics of a simple functional programming language with recursive functions.

By generalizing preorders to categories $\mathcal{C}$ and monotone functions to functors $F : \mathcal{C} \to \mathcal{C}$, a co-induction principle can be obtained for recursive data types, which are often defined as fixed points. Post-fixed points of F are F-coalgebras (A, α), and consist of an object A in $\mathcal{C}$ together with an arrow $\alpha : A \to F(A)$ (generalizing $\leq$). These F-coalgebras form again a category, as the post-fixed points of a monotonic function form a preorder. Arrows between two F-coalgebras (A, α) and (B, β) are arrows $f : A \to B$ (in $\mathcal{C}$) such that $\beta \circ f = F(f) \circ \alpha$. A greatest post-fixed point for a functor F is a *final* F-coalgebra (A, α): for any other F-coalgebra (B, β) there exists a unique arrow $f : (B, \beta) \to (A, \alpha)$. If (A, α) is a final F-coalgebra then A is a fixed point of F (i.e., α is an isomorphism).

As will become apparent, the richer structure of categories allows for a number of different formulations of a co-induction principle for final coalgebras of functors. For instance, let (A, α) and (B, β) be F-coalgebras, and suppose that (A, α) is final. The following can be easily proved. For any $\pi : (A, \alpha) \to (B, \beta)$: if π is epi then π is an isomorphism (cf. [Smy92]). Note that this generalizes the fact that for an ordered set $(P, \leq)$ and a monotone function $f : P \to P$: if $p, q \in P$, with p the greatest post-fixed point of f and $q \geq p$, then $q \leq f(q)$ implies $p = q$—another formulation of the co-induction principle mentioned above.

In particular, *locally continuous* (endo-)functors on the category of complete partial orders will be investigated. These functors are well-known to have an initial F-algebra (see [SP82]), which is at the same time a final F-coalgebra. A structural co-induction theorem will be proved, giving a number of equivalent characterizations for such final F-coalgebras. Maybe the most surprising and interesting one is the equivalence between finality and so-called *order strong-extensionality*, stating that two elements are ordered if and only if they are related by a so-called *ordered bisimulation*. Order-bisimulations generalize the F-bisimulations of [AM89], which at their turn are categorical abstractions of the notion of bisimulation of [Par81, Mil89]. In the present paper, the definition of ordered bisimulation from [Fio93] is used, which generalizes the original definition from [RT93] by the use of lax-homomorphisms.

The co-induction theorem (Section 5) is presented as and named after a dualization of the *structural induction theorem* of [Plo81] (but see also [LS81]), which is repeated here in the Appendix. Part of this dualization is fairly straightforward; order strong-extensionality, however, does not arise as the dual of the structural induction principle for ω-inductive sets (clause (3) of the induction theorem), nor do the corresponding parts of the proof. Note that because initial algebras of locally continuous functors are also final, both the induction and the co-induction theorem apply to them.

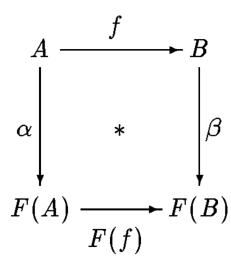
In Section 6, the co-induction theorem is used to extend the final semantics approach of [RT93] (initiated in [Acz88]) to the ordered case: the unique arrow from a coalgebra to a final coalgebra is shown to preserve and reflect the bisimulation order. The paper is concluded by proving, in Section 7, a slightly adapted version of the co-induction theorem for a category of *metric spaces* and locally contracting functors, in very much the same way. This last result is illustrated by the description of a metric hyperuniverse.

2. Preliminaries

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor from $\mathcal{C}$ to $\mathcal{C}$. An *F-coalgebra* is a pair (A, α), consisting of an object A and an arrow $\alpha : A \to F(A)$ in $\mathcal{C}$. It is dual to the notion of *F-algebra*: an *F-algebra* is a pair (A, α), consisting of an object A and an arrow $\alpha : F(A) \to A$ in $\mathcal{C}$.

For instance, any preorder $(P, \leq)$ is a category (with an arrow between two elements iff they are order related) and post-fixed points of monotone functions $f : P \to P$ are examples of f-coalgebras.

The collection of *F-coalgebras* constitutes a category by taking as arrows between coalgebras (A, α) and (B, β) those arrows $f : A \to B$ in $\mathcal{C}$ such that $\beta \circ f = F(f) \circ \alpha$; that is, the following diagram commutes:



Such an arrow f from (A, α) to (B, β) is called a homomorphism of F-coalgebras.

For example, a graph $(N, \to)$, consisting of a set N of nodes and a collection $\to$ of (directed) arcs between nodes can be regarded as a coalgebra of the (covariant) powerset functor $\mathcal{P}$ on the category *Set* of sets as follows: define $child : N \rightarrow \mathcal{P}(N)$ by, for all $n \in N$, $child(n) \equiv \{m \mid n \rightarrow m\}$. Arrows between graphs (as coalgebras) are those mappings between the sets of nodes that respect the child relation.

Definition 2.1 An object A in $\mathcal{C}$ is called *final* if for any other object B in $\mathcal{C}$ there exists a unique arrow from B to A. It is the dual notion of initial object (unique arrow *from* the object). Final and initial objects are unique up to isomorphism. □

The following is standard (see, e.g., [SP82]).

Proposition 2.2 *Every final F-coalgebra (A, α) is a fixed point of F (that is, α is an isomorphism).* □

3. Coalgebras in $CPO_\perp$

Let $CPO_\perp$ be the category with complete partial orders $(D, \sqsubseteq_D)$ as objects and strict and continuous functions as arrows. For any two cpo's D and E, the set $\hom(D, E)$ of arrows between D and E is itself a cpo, with the usual order: for all $f, g \in \hom(D, E)$,

$$f \leq g \equiv \forall x \in D,\ f(x) \sqsubseteq_E g(x).$$

Moreover composition of arrows is continuous with respect to this ordering. Therefore the category $CPO_\perp$ is called an order-enriched (or **O**-) category ([SP82]).

The structure on hom sets can be used to characterize a class of functors.

Definition 3.1 A functor $F : CPO_\perp \rightarrow CPO_\perp$ is *locally continuous* if, for any two objects $D, E \in CPO_\perp$, the mapping

$$F_{D,E} : \hom(D, E) \rightarrow \hom(F(D), F(E))$$

is continuous. Similarly, F is *locally monotonic* if $F_{D,E}$ is monotonic. □

Next we recall the definition of the subcategory CPO^E of $CPO_\perp$. If D and D' are cpo's and $\mu^e : D \rightarrow D'$ and $\mu^p : D' \rightarrow D$ are arrows in $CPO_\perp$ then $\langle \mu^e, \mu^p \rangle$ is called an *embedding-projection* pair from D to D' provided that

$$\mu^p \circ \mu^e = id_D \text{ and } \mu^e \circ \mu^p \leq_{\hom(D',D')} id_{D'}.$$

Note that the one half of such a projection pair determines the other. Let CPO^E denote the subcategory of $CPO_\perp$ that has cpo's as objects and embedding-projection pairs as arrows. Note that also CPO^E is an order-enriched category. The following theorem is standard (see [SP82]).

Theorem 3.2 *Every $F : CPO_\perp \rightarrow CPO_\perp$ that is locally continuous can be extended to a functor $F^E : CPO^E \rightarrow CPO^E$ that is ω-continuous (preserving colimits of ω-chains): on objects F^E is identical to F; and on arrows, F^E is given by*

$$F^E(\langle \mu^e, \mu^p \rangle) \equiv \langle F(\mu^e), F(\mu^p) \rangle.$$

A fixed point of F is obtained by constructing an initial F^E-algebra D in CPO^E as the colimit of the ω-chain $(D_n, \alpha_n)_n$, given by $D_0 \equiv \{\bot\}$, the trivial embedding $\alpha_0 : D_0 \to F(D_0)$, and for all $n \geq 0$, $D_{n+1} \equiv F(D_n)$, $\alpha_{n+1} \equiv F(\alpha_n)$. □

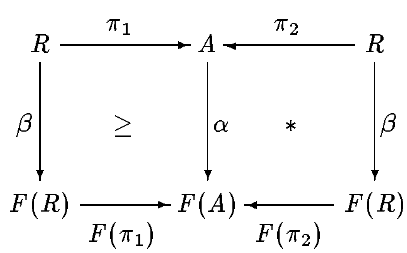
This fixed point D is an initial F^E-algebra (D, i^{-1}) in the category CPO^E. Moreover, it can also be seen to be an initial F-algebra in $CPO_\bot$: the fact that D is a colimit (of its defining chain) in CPO^E implies, by a little exercise (Exercise 4.17 from [Plo81]—to be precise), that it is a colimit in $CPO_\bot$ as well; then the 'Basic Lemma', from [SP82], immediately yields the result. By the so-called "limit-colimit coincidence" for **O**-categories, which is extensively discussed in [SP82], the dual of these facts also holds: Let CPO^P be defined as $(CPO^E)^{op}$, the opposite category of CPO^E. Thus arrows in CPO^P are mappings μ^p for which there exists a (unique) μ^e such that $\langle \mu^e, \mu^p \rangle$ is an embedding-projection pair. The fact that (D, i^{-1}) is an initial F^E-algebra (in CPO^E) implies that (D, i) is a final F^P-coalgebra in CPO^P. (Here F^P is defined analogously to F^E.) Again, (D, i) is a final F-coalgebra in $CPO_\bot$ as well, which can be shown by dualizing the little argument above. Summarizing, we have the following.

Theorem 3.3 *Let $F : CPO_\bot \to CPO_\bot$ be a locally continuous functor and let (D, i^{-1}) be the (in CPO^E) initial F^E-algebra as described above. Then (D, i) is a final F^P-coalgebra in CPO^P as well as a final F-coalgebra in $CPO_\bot$.* □

4. Ordered F-bisimulation

In [AM89], a categorical generalization of the notion of *bisimulation* of [Par81, Mil89] has been given in terms of coalgebras of functors on a category of classes. In [RT93], this definition is extended to functors F on arbitrary categories, yielding the notion of F-bisimulation. The order on hom sets in the category $CPO_\bot$ makes the following generalization of that definition possible. Let for the rest of this section $F : CPO_\bot \to CPO_\bot$ be a functor.

Definition 4.1 Let (A, α) be an F-coalgebra and R a relation on A with projections $\pi_1, \pi_2 : R \to A$. (That is, $R \subseteq A \times A$ is a cpo $(R, \sqsubseteq_R)$ such that the inclusion function $i : R \to A \times A$ is continuous.) Then R is called an *ordered F-bisimulation* on (A, α) if there exists an arrow $\beta : R \to F(R)$ such that



That is, π_2 is a homomorphism of coalgebras (satisfying $F(\pi_2) \circ \beta = \alpha \circ \pi_2$), and π_1 is a so-called *lax-homomorphism*: it satisfies $F(\pi_1) \circ \beta \geq \alpha \circ \pi_1$. □

The above definition is from [Fio93] and generalizes an earlier definition of ordered bisimulation given in [RT93], which required the existence of two coalgebra mappings $\beta_1, \beta_2 : R \to F(R)$ such that $\beta_1 \leq \beta_2$ and both π_1 and π_2 are coalgebra homomorphisms. The latter can be seen to be a special instance of the definition given above by taking $\beta \equiv \beta_2$. (Cf. the notion of simulation in [Pit92]; see also [Pit93], where proof principles that combine induction and co-induction are studied.)

The following definition generalizes the notion of *strong extensionality* used in [Acz88] (in the context of non-well-founded set theory). It is sometimes called *internal full abstractness* (cf. [Abr91]).

Definition 4.2 Let (A, α) be an F-coalgebra, and let $\sqsubseteq_A$ be the order on A. Let $\sqsubseteq^F \subseteq A \times A$ be defined by

$$\sqsubseteq^F \equiv \bigcup \{ R \subseteq A \times A \mid \; R \text{ is an ordered } F\text{-bisimulation on } (A, \alpha) \ \}.$$

Elements $a, b \in A$ with $a \sqsubseteq^F b$ are called (ordered) F-bisimilar. Now (A, α) is called *order strongly-extensional* if, for all $a, b \in A$,

$$a \sqsubseteq_A b \Leftrightarrow a \sqsubseteq^F b.$$

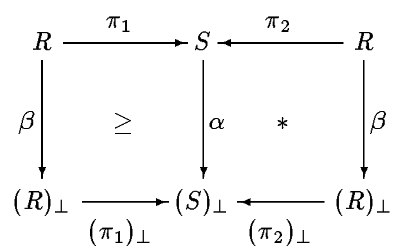
□

Example 4.3 A *deterministic partial transition system* is a pair $(S, \rightarrow)$ consisting of a set S of states and a transition relation $\rightarrow \subseteq S \times S$ that is a partial function. We assume that S contains a minimal element $\perp_S$ and is otherwise discretely ordered. Furthermore we assume that $\{ s \in S \mid \perp_S \rightarrow s \} = \emptyset$.

Such transition systems can be represented as coalgebras of the functor $(\cdot)_\perp : CPO_\perp \rightarrow CPO_\perp$, which maps a cpo D to its lifted version $(D)_\perp$ by extending D with a new minimal element $\perp_{\text{new}}$. For $(S, \rightarrow)$, define $\alpha : S \rightarrow (S)_\perp$, for $s \in S$, by

$$\alpha(s) = \begin{cases} s' & \text{if } s \rightarrow s' \\ \perp_{\text{new}} & \text{otherwise.} \end{cases}$$

An ordered $(\cdot)_\perp$-bisimulation (R, β) on (S, α),



satisfies for all $s, t \in S$ with $s \, R \, t$, and for all $s' \in S$,

$$\text{if } s \rightarrow s' \text{ then } \exists t' \in S, \ t \rightarrow t' \text{and } s' \, R \, t'.$$

Two states s and t in S are bisimilar whenever the number of subsequent transition steps that can be taken from t is at least as big as the number of steps that are possible starting from s. If β would be such that also π_1 is a coalgebra homomorphism, then two states are bisimilar if they can take the same number of steps. □

Example 4.4 A *nondeterministic transition system with divergence* is a triple

$$(S, \rightarrow, \uparrow)$$

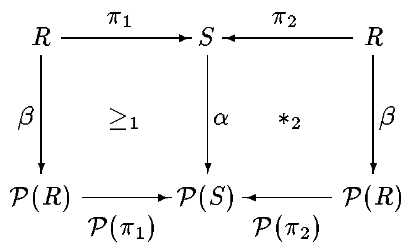
consisting of a set S of states, a transition relation $\rightarrow \subseteq S \times S$, and a divergence set $\uparrow \subseteq S$. (This is the—for simplicity—unlabelled version of the transition systems with divergence considered in [Abr91].) One should think of states s in $\uparrow$ (notation: $s \uparrow$) as having the possibility of divergence. Similarly $s \downarrow$ is used to indicate that s converges, that is, s not in $\uparrow$.

As above, we assume that S has a minimal element $\perp_S$, satisfying now $\{s \in S \mid \perp_S \rightarrow s\} = \emptyset = \{s \in S \mid s \rightarrow \perp_S\}$ (so $\perp_S$ is not involved in any transitions) and in addition $\perp_S\uparrow$. We shall only consider transition systems that are *finitely branching*, i.e., for all $s \in S$, the set $\{s' \in S \mid s \rightarrow s'\}$ is finite.

Transition systems with divergence can be represented as coalgebras of the functor $\mathcal{P} : CPO_\perp \rightarrow CPO_\perp$, which takes a cpo D to the Plotkin powerdomain of its lifted version $(D)_\perp$, extended (as in [Abr91]) with the empty set. In the ordering of $\mathcal{P}(D)$, the empty set is greater than the bottom element $\{\perp_{\mathrm{new}}\}$, and incomparable to all other elements; non-empty sets $X, Y \in \mathcal{P}(D)$ are ordered as usual by the Egli-Milner order. For $(S, \rightarrow, \uparrow)$ define $\alpha : S \rightarrow \mathcal{P}(S)$ by, for all $s \in S$,

$$\alpha(s) \equiv \{s' \in S \mid s \rightarrow s'\} \cup \{\perp_{\mathrm{new}} \in (S)_\perp \mid s \uparrow\}.$$

An ordered $\mathcal{P}$-bisimulation (R, β) on (S, α),



satisfies for all $s, t \in S$ with sRt, and for all $s', t' \in S$,

if $s \rightarrow s'$ then $\exists t' \in S,\ t \rightarrow t'$ and $s'Rt'$;

if $s\downarrow$ then ($t\downarrow$ and if $t \rightarrow t'$ then $\exists s' \in S,\ s \rightarrow s'$ and $s'Rt'$).

(Relations satisfying these two conditions are called partial bisimulations in [Abr91].) For suppose sRt and $s \rightarrow s'$. By the definition of α, $s' \in \alpha(s) = \alpha \circ \pi_1(s,t)$, and because of $\geq_1$, also $s' \in \mathcal{P}(\pi_1)(\beta((s,t))$. Thus there exists $t' \in S$ with $(s', t') \in \beta((s,t))$, satisfying $s'Rt'$; $*_2$ implies $t' \in \alpha(t)$ whence $t \rightarrow t'$.

Next suppose $s\downarrow$. Thus $\perp_{\mathrm{new}} \notin \alpha(s)$ and hence $\perp_{\mathrm{new}} \notin \mathcal{P}(\pi_1)(\beta((s,t))$, by $\geq_1$ and the definition of the Egli-Milner order. By the definition of $\mathcal{P}(\pi_1)$ it follows that $\perp_{\mathrm{new}} \notin \beta((s,t))$ (since for any $X \subseteq (S)_\perp$, $\mathcal{P}(\pi_1)(X)$ contains $\perp_{\mathrm{new}}$ iff X does). Thus by $*_2$, $\alpha(t)$ does not contain $\perp_{\mathrm{new}}$, that is, $t\downarrow$. Further suppose $t \rightarrow t'$. By $*_2$, there is $s' \in S$ with $(s', t') \in \beta((s,t))$. By $\geq_1$ and the fact that $\alpha(s)$ does not contain $\perp_{\mathrm{new}}$ (nor $\perp_S$), it follows that $s' \in \alpha(s)$, thus $s \rightarrow s'$.

Conversely, any relation $R \subseteq S \times S$ (not involving $\perp_S$) satisfying the two above conditions can be turned into a $\mathcal{P}$-coalgebra (T, β) by defining

$$T \equiv R \cup (\ \{\perp_S\} \times S\ )$$

and $\beta : T \rightarrow \mathcal{P}(T)$ by, for all sTt,

$$\begin{array}{rcl}\beta((s,t)) & \equiv & \{(s',t') \in T \mid s \rightarrow s' \text{ and } t \rightarrow t' \text{ and } s'Rt'\} \\ & \cup & \{(\perp_S, t') \in T \mid s\uparrow \text{ and } t \rightarrow t'\} \\ & \cup & \{\perp_{\mathrm{new}} \in (T)_\perp \mid s\uparrow \text{ and } t\uparrow\}\end{array}$$

It is left to the reader to verify that (T, β) is an ordered $\mathcal{P}$-bisimulation. □

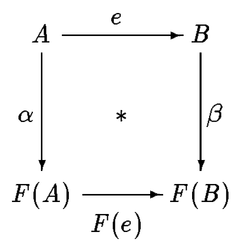
5. A STRUCTURAL CO-INDUCTION THEOREM

Next we formulate and prove the main theorem of this paper. (The definitions of some of the categorical and order-theoretic notions used here, can be found in the Appendix.)

Theorem 5.1 *Let $F : CPO_\perp \to CPO_\perp$ be a locally continuous functor. Let (A, α) be an F-coalgebra. Then of the following six statements, (1), (2), (2'), (4) and (5) are equivalent and all imply (3). If F moreover weakly preserves ordered kernel pairs then all statements are equivalent.*

1. *(A, α) is a final F-coalgebra.*

2. *α is epi; and for any F-coalgebra (B, β) and coalgebra homomorphism $e : (A, \alpha) \to (B, \beta)$: if e is epi then it is an isomorphism:*



2'. *As 2., but with* epi *replaced by* dense-epi, *twice.*

3. *α is dense-epi and (A, α) is order strongly-extensional; that is, if $\sqsubseteq_A$ is the order on A then*

$$\sqsubseteq_A = \bigcup\{R \subseteq A \times A \mid \; R \text{ is an ordered } F\text{-bisimulation on } (A, \alpha)\ \}.$$

4. *α is an isomorphism and $1_A = \mu h.\, \alpha^{-1} \circ F(h) \circ \alpha$ (the least fixed point).*

5. *(A, α) is maximally-final: it is a final F-coalgebra and for any F-coalgebra (B, β) the unique coalgebra homomorphism $e : (A, \alpha) \to (B, \beta)$ is maximal among the lax-homomorphisms between (A, α) and (B, β); that is, for any $f : B \to A$, if $\alpha \circ f \leq F(f) \circ \beta$ then $f \leq e$.*

Schematically:

$$1 \Leftrightarrow 2 \Leftrightarrow 2' \Leftrightarrow 4 \Leftrightarrow 5 \Rightarrow 3,$$

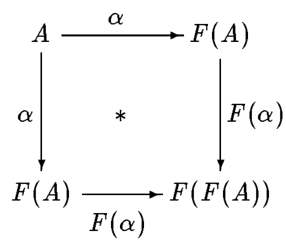
$$3 + F \text{ weakly preserves ordered kernel pairs } \Rightarrow 2'.$$

Proof:

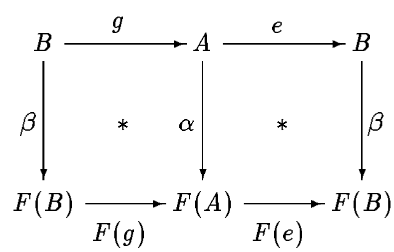
(1) $\Rightarrow$ **(2)**: By Proposition 2.2, α is an isomorphism and hence epi. Consider an epi $e : A \to B$ and suppose $e : (A, \alpha) \to (B, \beta)$ is a coalgebra homomorphism. Since (A, α) is final there exists a unique $h : (B, \beta) \to (A, \alpha)$. Thus both 1_A and $h \circ e$ are arrows from (A, α) to itself. By finality $h \circ e = 1_A$. From

$$\begin{aligned} (e \circ h) \circ e &= e \circ (h \circ e) \\ &= e \circ 1_A \\ &= 1_B \circ e \end{aligned}$$

and the fact that e is epi, it follows that $e \circ h = 1_B$.
(2) $\Rightarrow$ **(1)** : First we observe that α is an isomorphism, which follows from applying (2) to the following diagram (note that here the fact is used that α is epi):



Let (D, i^{-1}) be the initial F-algebra from Theorem 3.2. We saw (Theorem 3.3) that (D, i) is a final F^P-coalgebra (in CPO^P). Since α is an isomorphism it is also a projection, hence there exists a projection $e : A \to D$, which (by the construction of D) is also an arrow of coalgebras $e : (A, \alpha) \to (D, i)$. Now every projection is epi and by applying (2), e can be seen to be an isomorphism. Because (D, i) is a final F-coalgebra in $CPO_\perp$—again by Theorem 3.3—and (A, α) and (D, i) are isomorphic coalgebras, it follows that also (A, α) is a final F-coalgebra.
(1) $\Leftrightarrow$ **(2′)** : Inspection of the above two implications tells us that their proofs remain valid when epi is replaced by dense-epi.
(1) $\Rightarrow$ **(4)** : The finality of (A, α) implies that α is an isomorphism. Since F is locally continuous the function $\lambda h.\alpha^{-1} \circ F(h) \circ \alpha$ is continuous. Define $g \equiv \mu h.\ \alpha^{-1} \circ F(h) \circ \alpha$. It is immediate that $\alpha \circ g = F(g) \circ \alpha$, thus $g : (A, \alpha) \to (A, \alpha)$. By finality, $g = 1_A$.
(4) $\Rightarrow$ **(2)** : Since α is an isomorphism it is also epi. Consider an epi $e : (A, \alpha) \to (B, \beta)$. We prove that e is an isomorphism. Let $g \equiv \mu h.\ \alpha^{-1} \circ F(h) \circ \beta$. Then $\alpha \circ g = F(g) \circ \beta$, and we have the following diagram:



Next we show that $g \circ e = 1_A$ from which it follows—as in the proof of "(1) $\Rightarrow$ (2)"—that $e \circ g = 1_B$, using the fact that e is epi. First we prove $g \circ e \leq 1_A$, using the fixed-point definition of g:

- $(\lambda b \in B.\ \perp_A) \circ e = \lambda a \in A.\ \perp_A \leq 1_A$.
- Suppose $g \circ e \leq 1_A$, then

$$\begin{aligned} \alpha^{-1} \circ F(g) \circ \beta \circ e &= \alpha^{-1} \circ F(g) \circ F(e) \circ \alpha \\ &= \alpha^{-1} \circ F(g \circ e) \circ \alpha \\ &\leq 1_A \end{aligned}$$

  since, by assumption, $g \circ e \leq 1_A$, and the facts that α is an isomorphism and F is locally (continuous and hence) monotonic.

Next we shall use $1_A = \mu h.\ \alpha^{-1} \circ F(h) \circ \alpha$ from (4) to prove $1_A \leq g \circ e$:

- $\lambda a \in A.\ \perp_A \leq g \circ e$.

- Suppose $h \leq g \circ e$. Then

$$\begin{aligned} \alpha^{-1} \circ F(h) \circ \alpha &\leq \text{(since } F \text{ is locally monotonic)} \\ & \quad \alpha^{-1} \circ F(g \circ e) \circ \alpha \\ &= \alpha^{-1} \circ F(g) \circ F(e) \circ \alpha \\ &= \alpha^{-1} \circ F(g) \circ \beta \circ e \\ &= \alpha^{-1} \circ \alpha \circ g \circ e \\ &= g \circ e. \end{aligned}$$

(1) $\Rightarrow$ **(5)** : Let $f : (B, \beta) \rightarrow (A, \alpha)$ be a lax-homomorphism. By Proposition 2.2, α is an isomorphism. Define a sequence of functions from B to A inductively by

$$\begin{aligned} e_0 &\equiv f, \\ e_{n+1} &\equiv \alpha^{-1} \circ F(e_n) \circ \beta. \end{aligned}$$

Then $(e_n)_n$ is a chain ($f \leq \alpha^{-1} \circ F(f) \circ \beta$ because f is a lax-homomorphism) and its least upperbound e satisfies

$$\begin{aligned} e &= \bigsqcup e_n \\ &= \bigsqcup \alpha^{-1} \circ F(e_n) \circ \beta \\ &= \text{(by local continuity of } F\text{)} \\ & \quad \alpha^{-1} \circ F(\bigsqcup e_n) \circ \beta \\ &= \alpha^{-1} \circ F(e) \circ \beta. \end{aligned}$$

Hence e is the unique coalgebra homomorphism from (B, β) to (A, α). It follows from the definition of e that $f \leq e$.

(5) $\Rightarrow$ **(1)** : trivial.

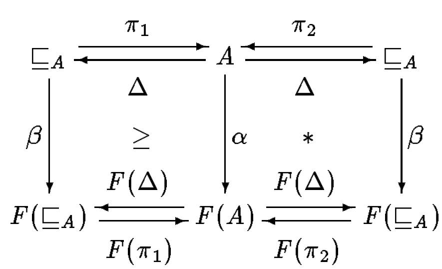
(4) $\Rightarrow$ **(3)** : The fact that α is an isomorphism implies that it is dense-epi. We have to show that

$$\sqsubseteq_A = \bigcup \{ R \subseteq A \times A \mid R \text{ is an ordered } F\text{-bisimulation on } (A, \alpha) \}.$$

The inclusion from left to right follows from the fact that $\sqsubseteq_A$ is an ordered F-bisimulation on (A, α): First observe that $\sqsubseteq_A$, with the inherited order from $A \times A$, is a cpo. Next define $\Delta : A \rightarrow \sqsubseteq_A$ by, for all $a \in A$, $\Delta(a) \equiv < a, a >$ and $\beta : \sqsubseteq_A \rightarrow F(\sqsubseteq_A)$ by

$$\beta \equiv F(\Delta) \circ \alpha \circ \pi_2.$$

Then $(\sqsubseteq_A, \beta)$ is an ordered F-bisimulation on (A, α):



since

$$\begin{array}{rcl} \alpha \circ \pi_1 & = & (\text{because } \pi_1 \circ \Delta = 1_A) \\ & & F(\pi_1 \circ \Delta) \circ \alpha \circ \pi_1 \\ & \leq & F(\pi_1) \circ F(\Delta) \circ \alpha \circ \pi_2 \\ & = & F(\pi_1) \circ \beta, \end{array}$$

and

$$\begin{array}{rcl} \alpha \circ \pi_2 & = & F(\pi_2 \circ \Delta) \circ \alpha \circ \pi_2 \\ & = & F(\pi_2) \circ \beta. \end{array}$$

Conversely, consider an ordered F-bisimulation (R, β) on (A, α):

$$\begin{array}{ccccc} R & \xrightarrow{\pi_1} & A & \xleftarrow{\pi_2} & R \\ \beta \downarrow & \geq & \downarrow \alpha & * & \downarrow \beta \\ F(R) & \xrightarrow[F(\pi_1)]{} & F(A) & \xleftarrow[F(\pi_2)]{} & F(R) \end{array}$$

We prove $R \subseteq \sqsubseteq_A$ or rather, equivalently, $\pi_1 \leq \pi_2$. We use fixed-point induction on 1_A (which by (4) is equal to $\mu h.\ \alpha^{-1} \circ F(h) \circ \alpha)$ to show $1_A \circ \pi_1 \leq \pi_2$:

- $(\lambda a \in A.\ \perp_A) \circ \pi_1 \leq \pi_2$.
- Suppose $h \circ \pi_1 \leq \pi_2$. Then

$$\begin{array}{rcl} \alpha^{-1} \circ F(h) \circ \alpha \circ \pi_1 & \leq & \alpha^{-1} \circ F(h) \circ F(\pi_1) \circ \beta \\ & = & \alpha^{-1} \circ F(h \circ \pi_1) \circ \beta \\ & \leq & (\text{because } h \circ \pi_1 \leq \pi_2 \text{ and } F \text{ is locally monotonic}) \\ & & \alpha^{-1} \circ F(\pi_2) \circ \beta \\ & = & \alpha^{-1} \circ \alpha \circ \pi_2 \\ & = & \pi_2 \end{array}$$

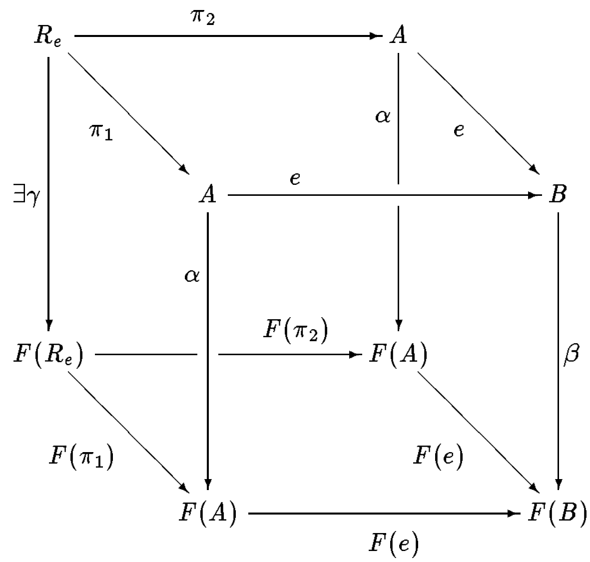
(3) $\Rightarrow$ **(2′)** : We prove this implication, from which the equivalence of (1) – (5) follows, under the assumption that F weakly preserves ordered kernel pairs.

By assumption α is dense-epi. Consider a homomorphism of coalgebras $e : (A, \alpha) \to (B, \beta)$ and suppose e is dense-epi. We shall prove that e is an isomorphism. Define

$$R_e \equiv \{(a, a') \in A \times A \mid e(a) \sqsubseteq e(a')\}.$$

The continuity and the strictness of e imply that R_e is a cpo. Below it is shown that it can be extended to an F-coalgebra (R_e, γ), such that (R_e, γ) is an ordered F-bisimulation on (A, α). Then from the order strong-extensionality of (A, α) it follows that $R_e \subseteq \sqsubseteq_A$. Hence e is a strict order-monic and since e is also dense-epi, it is an isomorphism (see the Appendix).

For the existence of an arrow $\gamma : R_e \to F(R_e)$ the assumption that F weakly preserves ordered kernel pairs will be used.



Since (R_e, π_1, π_2) is an ordered kernel pair for e, $(F(R_e), F(\pi_1), F(\pi_2))$ is by assumption a weak ordered kernel pair for $F(e)$. Now

$$\begin{aligned} F(e) \circ \alpha \circ \pi_1 &= \beta \circ e \circ \pi_1 \\ &\leq \beta \circ e \circ \pi_2 \\ &= F(e) \circ \alpha \circ \pi_2, \end{aligned}$$

from which the existence of an arrow $\gamma : R_e \to F(R_e)$, with $\alpha \circ \pi_1 \leq F(\pi_1) \circ \gamma$ and $\alpha \circ \pi_2 = F(\pi_2) \circ \gamma$ follows. Thus R_e is an ordered F-bisimulation. □

The fact that the final F-coalgebra (D, i) from Theorem 3.2 is order strongly-extensional was already proved in [RT93]. (The proof given there makes explicit use of the way D is constructed (as the projective limit of its defining ω-chain).) The equivalence of finality and maximal-finality ((1) and (5)) is due to [Plo91].

The main contribution of the above theorem is the proof of (3) $\Rightarrow$ (2), showing—for functors that weakly preserve ordered kernel pairs—that coalgebras are final if they are strongly extensional. Most functors (lifting, sum and so on) weakly preserve ordered kernel pairs.

Note that for locally continuous functors on $CPO_\perp$ there always exists an arrow from any F-coalgebra to an F-coalgebra (A, α) for which α is an isomorphism. For such functors, therefore, a final coalgebra is completely determined by the uniqueness part in the definition of finality. This explains why order strong-extensionality can be shown to be equivalent to finality.

Clearly, the clauses (1), (2) and (4) are fairly straightforward dualizations of the corresponding clauses in Plotkin's induction theorem (repeated here as Theorem 9.1 in the appendix). The proofs of the equivalence of (1) and (2), and of the implications (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) are immediate from the corresponding parts in the proof of the induction theorem. Clause (3) above cannot be seen as a dualization of any of the clauses of Theorem 9.1. For a further remark on this poin see Section 8.

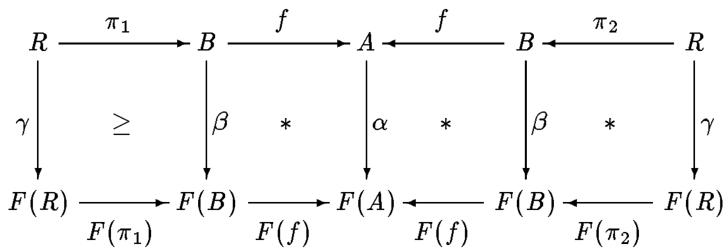
6. Ordered final semantics

Final coalgebras are furthermore characterized by the following theorem, which shows that they present a natural way of modelling bisimulation.

Theorem 6.1 *Let $F : CPO_\perp \to CPO_\perp$ be a locally continuous functor, and suppose that F weakly preserves ordered kernel pairs. Let (A, α) be a final F-coalgebra and let $f : (B, \beta) \to (A, \alpha)$ be a coalgebra homomorphism (which is unique by finality of (A, α)). For all $b, b' \in B$,*

$$b \sqsubseteq^F b' \Leftrightarrow f(b) \sqsubseteq_A f(b').$$

Proof:
From left to right: consider $b, b' \in B$ with $b \sqsubseteq^F b'$. Let (R, γ) be an ordered F-bisimulation on (B, β) with bRb'. From



it follows that $f \circ \pi_1$ is a lax-homomorphism from (R, γ) to (A, α) and that $f \circ \pi_2$ is the (by finality of (A, α)) unique coalgebra homomorphism from (R, γ) to (A, α). It follows from Theorem 5.1 (clause (5)) that $f \circ \pi_1 \leq f \circ \pi_2$. Thus $f(b) \sqsubseteq_A f(b')$.

As in the proof of $(3) \Rightarrow (2')$ in Theorem 5.1, it can be shown that the ordered kernel pair

$$R_f \equiv \{(b, b') \in B \times B \mid f(b) \sqsubseteq_A f(b')\}$$

of f can be extended to an ordered F-bisimulation (R_f, γ) on (B, β) (using the fact that F weakly preserves ordered kernel pairs), from which the implication from right to left follows. □

The unique arrow $f : (B, \beta) \to (A, \alpha)$ could be called (having in mind, e.g., a transition system represented by (B, β)) the *ordered final semantics* for (B, β). Cf. the final semantics of [Acz88, RT93], where symmetric F-bisimulations are used.

The above theorem can be seen as yet another characterization of final coalgebras, since its reverse also holds: if (A, α) is an F-coalgebra such that for all coalgebra homomorphisms $f : (B, \beta) \to (A, \alpha)$ and, for all $b, b' \in B$,

$$b \sqsubseteq^F b' \Leftrightarrow f(b) \sqsubseteq_A f(b'),$$

then (A, α) is a final F-coalgebra. Take (A, α) for (B, β) and 1_A for f to see that (A, α) is order strongly-extensional (using in addition the fact that $\sqsubseteq_A$ is itself an ordered F-bisimulation); by Theorem 5.1, (A, α) is final.

Example 4.3, continued. Let N be the set of natural numbers with the usual ordering and extended with a top element ω, and let $\phi : N \to (N)_\perp$ be the obvious isomorphism. Then (N, ϕ) is a final coalgebra of the functor $(\cdot)_\perp : CPO_\perp \to CPO_\perp$. For a deterministic partial transition system $(S, \to)$, represented as a $(\cdot)_\perp$-coalgebra (S, α), the final semantics $f : (S, \alpha) \to (N, \phi)$ maps a state $s \in S$ to the natural number (possibly ω) corresponding to the number of transition steps that can be taken starting in s. □

Example 4.4, continued. The functor $\mathcal{P} : CPO_\perp \to CPO_\perp$, which takes a cpo D to the Plotkin powerdomain (with empty set) of $(D)_\perp$ is locally continuous (see [Plo81]) and has by Theorem 3.3 a final coalgebra (P, ψ). By Theorem 5.1, we know that (P, ψ) is order strongly-extensional, thus finding back (an "unlabelled" version of) Proposition 3.10 from [Abr91]. Since $\mathcal{P}$ can be shown to preserve weakly ordered kernel pairs, Theorem 6.1 applies. Thus for the final semantics $f : (S, \alpha) \to (P, \psi)$ of a nondeterministic transition system $(S, \to, \uparrow)$, represented as the $\mathcal{P}$-coalgebra (S, α), we have for all $s, t \in S$,

$$s \sqsubseteq^{\mathcal{P}} t \Leftrightarrow f(s) \sqsubseteq_P f(t),$$

sometimes called the full abstractness of f. (Similar results are obtained in [Abr91] by means of Stone duality.) □

7. Metric spaces

In [AM89], bisimulations are defined as coalgebras (R, β) (in a category of classes) for which both projections π_1 and π_2 are coalgebra homomorphisms (not only π_2). For such symmetric bisimulations, the category of complete metric spaces offers a suitable framework as well. It has been studied in great detail in [RT93]. In this section, we shall point out that the preceding co-induction theorem also applies to metric spaces, and next use the resulting theorem to prove some properties of a metric hyperuniverse.

Let *CMS* be the category with (1-bounded) complete metric spaces (D, d_D) as objects and non-expansive (non-distance-increasing) functions as arrows. (For basic facts on metric spaces see, e.g., [Eng89].) Hom sets in *CMS* are themselves complete metric spaces, using as a metric on arrows the usual pointwise extension. A functor F on *CMS* is *locally contracting* if there exists ϵ with $0 \leq \epsilon < 1$ such that, for all D, E, the mapping $F_{D,E}$ is a contraction with factor ϵ. In [RT93], it is shown (extending earlier results of [AR89]) that every locally contracting functor F has a unique fixed point which is both an initial F-algebra and a final F-coalgebra.

A 'metric version' of Theorem 5.1 is obtained by dropping—both in the formulation of the theorem and in its proof—the word 'order(ed)' everywhere; considering in clause (3) only symmetric bisimulations; replacing in clause (4) the least fixed-point characterization of 1_A by the statement that it is the *unique* fixed point; and by dropping clause (5) (the notion of lax-homomorphism does not make sense in a metric setting). Note that the definitions of 'weakly preserving kernel pairs' and 'dense-epi' can be adapted straightforwardly. The proof can be almost literally copied: the proof of (4) $\Rightarrow$ (2) becomes somewhat simpler because of the uniqueness of 1_A; and in the proof of (3) $\Rightarrow$ (2$'$), the kernel pair of f should be taken rather than the ordered kernel pair.

Example 7.1 Let $\mathcal{P}_c : CMS \to CMS$ be defined by, for all $(D, d_D) \in CMS$,

$$\mathcal{P}_c(D) \equiv \{X \subseteq D \mid X \text{ is compact (w.r.t. } d_D)\ \}.$$

(The metric on $\mathcal{P}_c(D)$ is the so-called Hausdorff metric.) For every ϵ with $0 \leq \epsilon < 1$, the 'shrinking' functor id_ϵ is given by, for any (D, d_D),

$$id_\epsilon((D, d_D)) \equiv (D, \epsilon \cdot d_D).$$

Clearly id_ϵ is locally contracting. Taking the composition $\mathcal{P}_c \circ id_\epsilon$ (which we shall by abuse of notation again denote by $\mathcal{P}_c$) yields again a locally contractive functor. Thus there exists a fixed point

$$\gamma : H \cong \mathcal{P}_c(H),$$

and (H, γ) is a final $\mathcal{P}_c$-coalgebra. □

Because the metric space H is isomorphic to the collection of its compact subsets (note the presence of the 'metric shrinker' id_ϵ, though), it is an instance of a *hyperuniverse*. (See [FH92] for a general construction of hyperuniverses, and [FH83] and [Acz88] for a hyperuniverse based on a non-standard collection of axioms. Cf. [Abr88, MMO89, Rut91].) By putting, for $p, p' \in H$,

$$p' \in_H p \;\equiv\; p' \in \gamma(p),$$

H can be easily seen to contain all so-called hereditarily finite sets and their limits (with respect to the metric on H). Note that these limits need not be hereditarily finite themselves.

As pointed out in [Abr88], the standard axioms of set theory hold in H, with topological versions of separation, replacement and choice. By (the metric version of) Theorem 5.1, strong extensionality can be added to these axioms: two sets in H are equal if and only if they are $\mathcal{P}_c$-bisimilar. E.g., for $p, q \in H$ with (omitting the isomorphism γ)

$$p = \{p\}, \quad q = \{q\},$$

$p = q$ follows from the fact that $\{(p, q)\}$ is a $\mathcal{P}_c$-bisimulation on H.

8. Conclusion

As was observed above, the characterization of final coalgebras in terms of strong extensionality (clause (3) of Theorem 5.1) does not have a dual counterpart among the clauses of the structural induction theorem (Theorem 9.1 in the Appendix). However, the latter theorem can be extended with a fifth, equivalent clause that comes close to being the dual of clause (3) of Theorem 5.1, as follows. An *F-congruence* on an *F-algebra* (A, α) is an F-algebra (R, β) with R a relation on A such that the projections $\pi_1, \pi_2 : (R, \beta) \to (A, \alpha)$ are homomorphisms of F-algebras. This definition generalizes the standard notion of a congruence on Σ-algebras. Note that it is dual to the definition of symmetric bisimulation. Clauses (1) through (4) of Theorem 9.1 can be shown to be equivalent to the following statement: there exists $\beta : F(\Delta) \to \Delta$ (with $\Delta \equiv \{(a, a') \in A \times A \mid a = a'\}$) such that (Δ, β) is the smallest F-congruence on (A, α).

Acknowledgements

As always, we have benefitted from presentations of this work for the Amsterdam Concurrency Group, headed by Jaco de Bakker. We are in particular indebted to Daniele Turi: this paper builds directly on our previous joint work, and we have had some lively discussions on the contents of this paper. One of the anonymous referees is thanked for suggesting a useful reference.

References

[Abr88] S. Abramsky. A Cook's tour of the finitary non-well-founded sets. Department of Computing, Imperial College, London, 1988.

[Abr91] S. Abramsky. A domain equation for bisimulation. *Information and Computation*, 92:161–218, 1991.

[Acz88] P. Aczel. *Non-well-founded sets.* Number 14 in CSLI Lecture Notes. Stanford University, 1988.

[AM89] P. Aczel and N. Mendler. A final coalgebra theorem. In D.H. Pitt, D.E. Ryeheard, P. Dybjer, A.M. Pitts, and A. Poigné, editors, *Proceedings Category Theory and Computer Science*, volume 389 of *Lecture Notes in Computer Science*, pages 357–365, 1989.

[AR89] P. America and J.J.M.M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer and System Sciences*, 39(3):343–375, 1989.

[Eng89] R. Engelking. *General Topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, revised and completed edition, 1989.

[FH83] M. Forti and F. Honsell. Set theory with free construction principles. *Annali Scuola Normale Superiore, Pisa*, X(3):493–522, 1983.

[FH92] M. Forti and F. Honsell. A general construction of hyperuniverses. Technical Report 1992/9, Istituto di Matematiche Applicate 'U. Dini', Facoltà di Ingegneria, Università di Pisa, 1992.

[Fio93] M. Fiore. A coinduction principle for recursive data types based on bisimulation. In *Proceedings of the Eighth IEEE Symposium on Logic In Computer Science*, 1993.

[Lan71] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, 1971.

[LP82] D. Lehmann and A. Pasztor. Epis need not to be dense. *Theoretical Computer Science*, 17:151–161, 1982.

[LS81] D. Lehmann and M.B. Smyth. Algebraic specifications of data types: a synthetic approach. *Mathematical Systems Theory*, 14:97–139, 1981.

[Mil89] R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.

[MMO89] M.W. Mislove, L.S. Moss, and F.J. Oles. Non-well-founded sets obtained from ideal fixed points. In *Proc. of the Fourth IEEE Symposium on Logic in Computer Science*, pages 263–272, 1989. To appear in Information and Computation.

[MT91] R. Milner and M. Tofte. Co-induction in relational semantics. *Theoretical Computer Science*, 87:209–220, 1991.

[Par81] D.M.R. Park. Concurrency and automata on infinite sequences. In P. Deussen, editor, *Proceedings 5th GI Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer-Verlag, 1981.

[Pit92] A.M. Pitts. A co-induction principle for recursively defined domains. Technical Report 252, Computer Laboratory, University of Cambridge, 1992.

[Pit93] A.M. Pitts. Relational properties of recursively defined domains. In *Proceedings 8th Annual Symposium on Logic in Computer Science*, pages 86–97. IEEE Computer Society Press, 1993.

[Plo81] G.D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the "Pisa Notes"). Department of Computer Science, University of Edinburgh, 1981.

[Plo91] G.D. Plotkin. Some notes on recursive domain equations. Handwritten notes for the Domain Theory PG Course. University of Edinburgh, 1991.

[RT93] J.J.M.M. Rutten and D. Turi. On the foundations of final semantics: Non-standard sets, metric spaces, partial orders. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Proceedings of the REX workshop on Semantics: Foundations and Applications*, volume 666 of *Lecture Notes in Computer Science*, pages 477–530. Springer-Verlag, 1993.

[Rut91] J.J.M.M. Rutten. Hereditarily-finite sets and complete metric spaces. Technical Report CS-R9148, Centre for Mathematics and Computer Science, Amsterdam, 1991.

[Smy92] M.B. Smyth. I-categories and duality. In M.P. Fourman, P.T. Johnstone, and A.M. Pitts, editors, *Applications of categories in computer science*, volume 177 of *London Mathematical Society Lecture Note Series*, pages 270–287. Cambridge University Press, 1992.

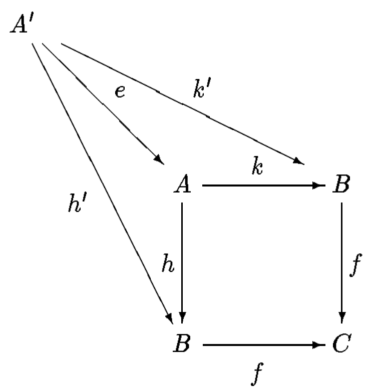
[SP82] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Comput.*, 11:761–783, 1982.

9. Appendix

Some categorical notions

Let $\mathcal{C}$ be a category. An arrow $m : A \to B$ is called *monic* if for any two arrows $f, g : D \to A$ the equality $m \circ f = m \circ g$ implies $f = g$. An arrow $e : A \to B$ is called *epi* if for any two arrows $f, g : B \to D$ the equality $f \circ e = g \circ e$ implies $f = g$.

A kernel pair (see [Lan71]) for an arrow $f : B \to C$ in $\mathcal{C}$ consists of an object A and a pair of arrows $h : A \to B$ and $k : A \to B$ such that $f \circ h = f \circ k$, and such that for any other object A' and arrows $h' : A' \to B$ and $k' : A' \to B$ with $f \circ h' = f \circ k'$, there exists a unique arrow $e : A' \to A$ satisfying $h' = h \circ e$ and $k' = k \circ e$:



Ordered kernel pairs

In $\mathcal{C} = CPO_\perp$, the above definition can be generalized as follows. An *ordered* kernel pair for a function $f : B \to C$ in $CPO_\perp$ consists of a cpo A and a pair of functions $h : A \to B$ and $k : A \to B$ such that $f \circ h \leq f \circ k$, and such that for any other cpo A' and functions $h' : A' \to B$ and $k' : A' \to B$ with $f \circ h' \leq f \circ k'$, there exists a unique arrow $e : A' \to A$ satisfying $h' = h \circ e$ and $k' = k \circ e$.

The cpo A with functions h and k is called a *weak ordered kernel pair* for f if for any other cpo A' and functions $h' : A' \to B$ and $k' : A' \to B$ with $f \circ h' \leq f \circ k'$, there exists an arrow $e : A' \to A$ (not necessarily unique) satisfying $h' \leq h \circ e$ (rather than $h' = h \circ e$) and $k' = k \circ e$.

A functor $F : CPO_\perp \to CPO_\perp$ *weakly preserves ordered kernel pairs* if it transforms ordered kernel pairs for functions f into weak ordered kernel pairs for $F(f)$.

Some further order-theoretic notions

Let D be a cpo and consider a continuous function $f : D \to D$. (That is, f preserves least upperbounds of ω-chains.) Then f has a least fixed point, which is denoted by $\mu x.\ f(x)$.

A subset $P \subseteq D$ is called ω-inductive if every chain $\langle x_n \rangle_n$ in P has its least upperbound in P.

The following is called the principle of fixed-point induction. Let $f : D \to D$ be continuous and let $P \subseteq D$ be ω-inductive. Then

$$(\perp \in P \ \wedge \ (\forall x \in D[x \in P \Rightarrow f(x) \in P]) \ \ \Rightarrow \ \ (\mu x.\ f(x)) \in P$$

A *strict order-monic* (see [Plo81]) is a strict continuous function (in $CPO_\perp$) $m : A \to B$ such that for any two arrows $f, g : D \to A$ the inequality $m \circ f \leq m \circ g$ implies $f \leq g$. It is easy to see that m is a strict order-monic if and only if, for all $a, a' \in A$,

$$a \sqsubseteq a' \Leftrightarrow m(a) \sqsubseteq m(a').$$

A strict continuous function $e : A \to B$ is *dense-epi* if it is epi and moreover satisfies $cl(e(A)) = B$, where $cl(e(A))$ is the least subset of B that contains $e(A)$ and that is closed under least upperbounds of ω-chains. (In fact the condition $cl(e(A)) = B$ can be shown, by transfinite induction, to imply the fact that e is epi. See [LP82] for an explanation why "Epis need not to be dense".)

If $m : A \rightarrow B$ is both a strict order-monic and dense-epi, then m is an isomorphism: $m(A) = cl(m(A))$ since e is a strict order-monic, and $cl(m(A)) = B$, since e is dense-epi. Thus e is a bijective order-embedding.

The structural induction theorem
In [Plo81] (Theorem 4 of Chapter 5), the following theorem is proved. (See also [LS81] for a similar result.)

Theorem 9.1 *Let $F : CPO_\perp \rightarrow CPO_\perp$ be a locally continuous functor which preserves inclusions. (That is, if $\iota : A \subseteq B$ then $F(\iota) : F(A) \subseteq F(B)$.) Let $\alpha : F(A) \rightarrow A$ be an F-algebra. Then the following four statements are equivalent:*

1. *(A, α) is an initial F-algebra.*
2. *α is a strict order-monic, and for every strict order-monic $m : B \rightarrow A$: if there exists $\beta : F(B) \rightarrow B$ such that $m : (B, \beta) \rightarrow (A, \alpha)$ is a homomorphism of algebras (i.e., $m \circ \beta = \alpha \circ F(m)$), then m is an isomorphism.*
3. *α is a strict order-monic, and for every ω-inductive $P \subseteq A$ the following principle of structural induction holds:*

$$(\perp \in P \ \wedge \ (\forall x \in F(A)[x \in F(P) \Rightarrow \alpha(x) \in P])) \ \ \Rightarrow \ \ P = A$$

4. *α is an isomorphism and $1_A = \mu h.\, \alpha \circ F(h) \circ \alpha^{-1}$.*

The assumption that F preserves inclusions is only used to prove the equivalence of (2) and (3). This property is satisfied by most covariant functors.