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RESEARCH ARTICLE

A Study of Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifold

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Abstract

We study conformal η-Einstein solitons on the framework of trans-Sasakian manifold in dimension three. Existence of conformal η-Einstein solitons on trans-Sasakian manifold is discussed. Then we find some results on trans-Sasakian manifold which are conformal η-Einstein solitons where the Ricci tensor is cyclic parallel and Codazzi type. We also consider some curvature conditions with addition to conformal η-Einstein solitons on trans-Sasakian manifold. We also use torse-forming vector fields in addition to conformal η-Einstein solitons on trans-Sasakian manifold. Finally, an example of conformal η-Einstein solitons on trans-Sasakian manifold is constructed.

Keywords Trans-Sasakian manifold · Einstein soliton · Conformal η-Einstein soliton · Codazzi type Ricci tensor · $\mathcal{C}$-Bochner curvature tensor · $\mathcal{W}_2$ curvature tensor · $\mathcal{M}$-projective curvature tensor

1 Introduction

The Ricci flow on a smooth manifold M with Riemannian metric $g(t)$ is given by

$$\frac{\partial}{\partial t}g(t) = -2Ric,$$

where Ric is the Ricci tensor of the metric $g(t)$. A Ricci soliton is a solution of Ricci flow (see details [24, 25, 57]), defined on a pseudo-Riemannian manifold (M, g) by

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$$\frac{1}{2}\pounds_V g + Ric = \lambda g,$$

where $\pounds_V$ denotes the Lie-derivative with respect to $V \in \chi(M)$, Ric is the Ricci tensor of g and λ is a constant. The Ricci soliton is shrinking, steady, and expanding depending on $\lambda < 0, \lambda = 0, \lambda > 0$ respectively. Otherwise, it will be called indefinite.

Cho and Kimura [13], generalized the notion of Ricci soliton, by introducing the notion of η-Ricci soliton. Later Calin and Crasmareanu [9] studies it on Hopf hypersurfaces in complex space forms. An η-Ricci soliton equation is given by:

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0$$

where S is the Ricci tensor and for λ and μ are constants.

The η-Einstein soliton [4] on a Riemannian manifold (M, g) is given by,

$$\pounds_\zeta g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1}$$

where r is the scalar curvature of the metric g and λ and μ are constants. For $\mu = 0$, the data (g, ζ, λ) is called Einstein soliton [11].

In [49], Roy, Dey and Bhattacharyya considered conformal Einstein soliton, defined by on an n-dimensional manifold:

$$\pounds_V g + 2S + \left(2\lambda - r + \left(p + \frac{2}{n}\right)\right)g = 0,$$

where Λ is real constant, p is a scalar non-dynamical field.

Moreover, an n-dimensional Riemannian manifold (M, g) is said to admit conformal η-Einstein soliton if

$$\pounds_\zeta g + 2S + \left(2\lambda - r + \left(p + \frac{2}{n}\right)\right)g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

as shown in [11]. Note that, the conformal η-Einstein soliton becomes the Einstein soliton (g, ζ, λ).

Ricci solitons and Einstein solitons are considered by many authors in different contexts for instant: on K*ä*hler manifolds [14], on contact and Lorentzian manifolds [1, 2], on K-contact manifolds [55] etc. We also refer to similar studies in [10] and [43]. In 2017, Yaning Wang [59] proved that if cosymplectic manifold M^3 admits a Ricci soliton, then either M^3 is locally flat or the potential vector field is an infinitesimal contact transformation. Also, in [44], authors have provided some insight on trans-Sasakian manifolds. Dey et al. [50] also have set up some new results on conformal η-Einstein soliton. Very recently, $*$-Ricci soliton and Yamabe soliton and their generalizations and related research have been studied by many authors ([15–18, 20, 22, 23, 26–40, 48–54, 58, 61]).

Motivated by the above study, we discuss here conformal η-Einstein soliton on 3-dimensional trans-Sasakian manifold. Our paper is organized as follows: after a brief introduction, in Sect. 2, we recall some basic knowledge on trans-Sasakian manifolds. Section 3 deals with (0, 2)-tensor field L which is parallel(i.e., ∇L=0) as conformal η-Einstein soliton on 3-dimensional trans-Sasakian manifold. In the next section, we study the characteristics of the scalar curvature of the manifold and obtain the nature of the soliton. In Sect. 5, we have evolved Codazzi type and cyclic parallel Ricci tensor admitting conformal η-Einstein soliton on trans-Sasakian 3-manifold. Sections 6, 7, 8 deals with some curvature conditions Einstein semi-symmetric, $R(\zeta, X) \cdot S = 0$, $\mathcal{W}_2(\zeta, X) \cdot S = 0$ and $\mathcal{B}(\zeta, X) \cdot S = 0$, where $\mathcal{W}_2(X, Y)Z$ and $\mathcal{B}(X, Y)Z$ are $\mathcal{W}_2$ curvature tensor and $\mathcal{C}$-Bochner curvature tensor respectively. In Sect. 9, we have studied the nature of conformal η-Einstein solitons on trans-Sasakian 3-manifold whose vector field is torse-forming. Section 10, we have contrived the curvature condition $M(\zeta, X) \cdot S = 0$ and $S(\zeta, X) \cdot \mathcal{M} = 0$, where $\mathcal{M}$ is a $\mathcal{M}$-projective curvature tensor. In last section, we have set up an example to illustrate the existence of conformal η-Einstein soliton on 3-dimensional trans-Sasakian manifold.

2 Preliminaries

A Riemannian manifold (M, g), dim$M = 2n + 1$ is said to be an almost contact metric manifold [6] if there is (1, 1) tensor field $\mathcal{F}$, a vector field ζ, a 1-form η satisfying the following:

$$\mathcal{F}^2(V_1) = -I + \eta(V_1)\zeta \tag{3}$$

$$\eta(\zeta) = 1 \tag{4}$$

$$\eta(V_1) = g(V_1, \zeta) \tag{5}$$

$$g(\mathcal{F}V_1, \mathcal{F}V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2) \tag{6}$$

$$g(V_1, \mathcal{F}V_2) + g(\mathcal{F}V_1, V_2) = 0 \tag{7}$$

for all $V_1, V_2 \in TM$, where TM is the tangent bundle of the manifold M. Also it can be easily seen that $\mathcal{F}(\zeta) = 0$,$\eta(\mathcal{F}V_1) = 0$ and rank of $\mathcal{F}$ is $(n - 1)$.

An almost contact metric manifold $M\ (\mathcal{F}, \zeta, \eta, g)$ is said to be trans-Sasakian manifold if $(M \times \mathbb{R}, J, G)$ belong to the class ω_4 of the Hermitian manifold, where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(V_3, f\frac{d}{dt}\right) = \left(\mathcal{F}V_3 - f\zeta, \eta(V_3)\frac{d}{dt}\right) \tag{8}$$

for any vector field V_3 on M and smooth funtion f on $M \times \mathbb{R}$.

An almost contact metric manifold is said to be trans-Sasakian if there are smooth functions α and β on M satisfying,

$$(\nabla_{V_1}\mathcal{F})V_2 = \alpha\{g(V_1, V_2)\zeta - \eta(V_2)V_1\} + \beta\{g(\mathcal{F}V_1, V_2)\zeta - \eta(V_2)\mathcal{F}V_1\} \tag{9}$$

$$\nabla_{V_1}\zeta = -\alpha\mathcal{F}V_1 + \beta\{V_1 - \eta(V_1)\zeta\} \tag{10}$$

$$(\nabla_{V_1}\eta)V_2 = -\alpha g(\mathcal{F}V_1, V_2) + \beta g(\mathcal{F}V_1, \mathcal{F}V_2). \tag{11}$$

The manifold $(M, \mathcal{F}, \zeta, \eta, g, \alpha, \beta)$ is said to be a trans-Sasakian manifold of type (α, β). A trans-Sasakian manifold of type $(0, 0), (\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifold respectively. Sasakian manifold appear as examples of α-Sasakian manifold with $\alpha = 1$ and $\beta = 0$. And Kenmotsu manifold appear when $\alpha = 0$ and $\beta = 1$, Marrero [41] has shown that a trans-Sasakian manifold of dimension > 5 is either cosymplectic or α-Sasakian or β-Kenmotsu.

Again in a trans-Sasakian 3-manifold (M, g) the Ricci tensor is given by

$$\begin{aligned} S(V_1, V_2) =& \left[\frac{r}{2} + \zeta\beta - (\alpha^2 - \beta^2)\right] g(V_1, V_2) - \left[\frac{r}{2} + \zeta\beta - 3(\alpha^2 - \beta^2)\right] \eta(V_1) \\ & \eta(V_2) - [V_2\beta + \mathcal{F}(V_2)\alpha]\eta(V_1) - [V_1\beta + \mathcal{F}(V_1)\alpha]\eta(V_2) \end{aligned} \tag{12}$$

$$R(V_1, V_2)\zeta = (\alpha^2 - \beta^2)[\eta(V_2)V_1 - \eta(V_1)V_2] \tag{13}$$

$$R(\zeta, V_1)V_2 = (\alpha^2 - \beta^2)[g(V_1, V_2)\zeta - \eta(V_2)V_1] \tag{14}$$

$$R(\zeta, V_1)\zeta = (\alpha^2 - \beta^2)[\eta(V_1)\zeta - V_1] \tag{15}$$

$$S(V_1, V_2) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] g(V_1, V_2) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(V_1)\eta(V_2) \tag{16}$$

$$S(V_1, \zeta) = 2(\alpha^2 - \beta^2)\eta(V_1). \tag{17}$$

3 Conformal η-Einstein Soliton on Trans-Sasakian Manifolds with L Parallel

In order to study the existence conditions of conformal η-Einstein solitons on trans-Sasakian manifolds, first we consider a symmetric tensor field L that is parallel $(\nabla L = 0)$. As an outcome we see that

$$L(R(V_1, V_2)V_3, V_4) + L(V_3, R(V_1, V_2)V_4) = 0 \tag{18}$$

for an arbitrary vector field V_1, V_2, V_3, V_4 on M. Using $V_3 = V_4 = \zeta$, we obtain

$$L(R(V_1, V_2)\zeta, \zeta) = 0 \tag{19}$$

for any $V_1, V_2 \in TM$. Using (13) and replacing V_1 by ζ, we get

$$L(V_2, \zeta) = g(V_2, \zeta)L(\zeta, \zeta) \tag{20}$$

for any $V_2 \in TM$. Taking covariant derivative in equation (20) in the direction of the vector field $V_1 \in TM$, we acquire

$$L(\nabla_{V_1}V_2,\zeta)+L(V_2,\nabla_{V_1}\zeta)=g(\nabla_{V_1}V_2,\zeta)L(\zeta,\zeta)+g(V_2,\nabla_{V_1}\zeta)L(\zeta,\zeta). \tag{21}$$

Using the Eq. (10), we have

$$\beta L(V_1,V_2)-\alpha L(\mathcal{F}V_1,V_2)=-\alpha g(\mathcal{F}V_1,V_2)L(\zeta,\zeta)+\beta L(\zeta,\zeta)g(V_1,V_2). \tag{22}$$

Now, we interchange V_1 by V_2 in above equation to yield

$$\beta L(V_1,V_2)-\alpha L(V_1,\mathcal{F}V_2)=-\alpha g(V_1,\mathcal{F}V_2)L(\zeta,\zeta)+\beta L(\zeta,\zeta)g(V_1,V_2). \tag{23}$$

Then we add the above two Eqs. (22) and (23) to achieve

$$\beta L(V_1,V_2)-\frac{\alpha}{2}[L(\mathcal{F}V_1,V_2)+L(V_1,\mathcal{F}V_2)]=\beta L(\zeta,\zeta)g(V_1,V_2). \tag{24}$$

We see that $\beta L(V_1,V_2)-\frac{\alpha}{2}[L(\mathcal{F}V_1,V_2)+L(V_1,\mathcal{F}V_2)]$ is a symmetric tensor of type (0, 2).

Let

$$\begin{aligned}&\beta L(V_1,V_2)-\frac{\alpha}{2}[L(\mathcal{F}V_1,V_2)+L(V_1,\mathcal{F}V_2)]\\&=\mathcal{L}_\zeta g(V_1,V_2)+2S(V_1,V_2)+2\mu\eta(V_1)\eta(V_2)\\&-rg(V_1,V_2)+\{2\lambda+(p+\frac{2}{n})\}g(V_1,V_2)\end{aligned}.$$

Then, we compute

$$\beta L(\zeta,\zeta)g(V_1,V_2)=\mathcal{L}_\zeta g(V_1,V_2)+2S(V_1,V_2)+2\mu\eta(V_1)\eta(V_2)-rg(V_1,V_2)+\{2\lambda+(p+\tfrac{2}{n})\}g(V_1,Y).$$

As L is parallel so, $L(\zeta,\zeta)$ is constant. Hence, we can write $L(\zeta,\zeta)=-\frac{1}{\beta}(2\lambda+(p+\frac{2}{n}))$ where β is constant and $\beta\neq 0$. Therefore $\mathcal{L}_\zeta g(V_1,V_2)+2S(V_1,V_2)+2\mu\eta(V_1)\eta(V_2) \quad -rg(V_1,V_2)=-(2\lambda+p+\frac{2}{n})g(V_1,V_2)$

and so (g,ζ,λ,μ) becomes a conformal η-Einstein soliton. Hence, we have the following theorem:

Theorem 1 *Let $(M,g,\mathcal{F},\eta,\zeta,\alpha,\beta)$ be a trans-Sasakian manifold, $dimM=3$ with α, β constant $(\beta\neq 0)$. If the symmetric $(0,2)$ tensor field L satisfying the condition* $\beta L(V_1,V_2)-\frac{\alpha}{2}[L(\mathcal{F}V_1,V_2)+L(V_1,\mathcal{F}V_2)]=\mathcal{L}_\zeta g(V_1,V_2)+2S(V_1,V_2)+2\mu\eta(V_1)\eta(V_2)-rg(V_1,V_2)+\{2\lambda+(p+\frac{2}{n})\}g(V_1,V_2)$ *is parallel with respect to the Levi-Civita connection associated to g. Then (g,ζ,λ,μ) becomes a conformal η-Einstein soliton.*

Corollary 2 *Let $(M,g,\mathcal{F},\eta,\zeta,\alpha,\beta)$ be a trans-Sasakian manifold, $dimM=3$ with α, β constant $(\beta\neq 0)$. If the symmetric $(0,2)$ tensor field L satisfying the condition* $\beta L(V_1,V_2)-\frac{\alpha}{2}[L(\mathcal{F}V_1,V_2)L(V_1,\mathcal{F}V_2)]=\mathcal{L}_\zeta g(V_1,V_2)+2S(V_1,V_2)+2\mu\eta(V_1)\eta(V_2)$

is parallel with respect to the Levi-Civita connection associated to g. Then (g, ζ, μ) *becomes an* η*-Ricci soliton.*

Next we obtain some results on 3-dimensional trans-Sasakian manifold satisfying a conformal η-Einstein soliton when the manifold is Ricci-symmetric has η-recurrent Ricci curvature tensor.

Theorem 3 *Let* (M, g) *be a trans-Sasakian manifold,* $dimM = 3$ *with* α, β *constant* $(\beta \neq 0)$ *satisfying conformal* η*-Einstein soliton.*

(i) *If the manifold* (M, g) *is Ricci symmetric* (*i.e.*, $\nabla S = 0$) *then* $\mu = \beta$.

(ii) *If the Ricci tensor is* η*-recurrent* (*i.e.*, $\nabla S = \eta \otimes S$) *then* $\mu = 2\beta - \frac{\alpha^2}{\beta}$.

Proof From the Eq. (2), we get

$$2S(V_1, V_2) = -g(\nabla_{V_1}\zeta, V_2) - g(V_1, \nabla_{V_2}\zeta) - [2\lambda - r + (p + \frac{2}{n})]g(V_1, V_2) - 2\mu\eta(V_1)\eta(V_2). \tag{25}$$

Now, we use the Eq. (10) into the identity (25) to yield

$$S(V_1, V_2) = \left[\frac{r}{2} - \lambda - \beta - \left(\frac{p}{2} + \frac{1}{n}\right)\right]g(V_1, V_2) + (\beta - \mu)\eta(V_1)\eta(V_2) \tag{26}$$

and

$$S(V_1, \zeta) = \left\{\frac{r}{2} - \lambda - \left(\frac{p}{2} + \frac{1}{n}\right) - \mu\right\}\eta(V_1). \tag{27}$$

Also employing the identity (17) to (27), we obtain

$$\frac{r}{2} - \lambda - \left(\frac{p}{2} + \frac{1}{n}\right) - \mu = 2(\alpha^2 - \beta^2). \tag{28}$$

The Ricci operator Q is defined by $g(QV_1, V_2) = S(V_1, V_2)$. Then, we get

$$QV_1 = \{\mu - \beta + 2(\alpha^2 - \beta^2)\}V_1 + (\beta - \mu)\eta(V_1)\zeta \tag{29}$$

(i) We consider that the manifold (M, g) is Ricci symmetric i.e.,

$$\nabla S = 0. \tag{30}$$

Now, we have

$$\nabla_{V_1}S(V_2, V_3) = V_1 S(V_2, V_3) - S(\nabla_{V_1}V_2, V_3) - S(\nabla_{V_1}V_3, V_2)$$

Using the Eqs. (26) and (30), we obtain

$$(\beta - \mu)[-\alpha\{g(\mathcal{F}V_1, V_2) + g(\mathcal{F}V_1, V_3)\} + \beta\{g(V_1, V_2)\eta(V_3) - g(V_1, V_3)\eta(V_2)\} - 2\beta\eta(V_1)\eta(V_2)\eta(V_3)] = 0.$$

By putting $V_2 = V_3 = \zeta$, the above equation becomes $\mu = \beta$.

(ii) We assume that the manifold (M, g) is η-recurrent, i.e.,

$$\nabla S = \eta \otimes S. \tag{31}$$

Now, we have

$$\nabla_{V_1} S(V_2, V_3) = \eta(V_1) S(V_2, V_3) \tag{32}$$

for all vector fields V_1, V_2, V_3. Using the Eqs. (26) and (32), we obtain $\mu = 2\beta - \frac{\alpha^2}{\beta}$. Hence, we complete the proof. □

4 3-Dimensional Trans-Sasakian Manifold Admitting Conformal η-Einstein Soliton

Let us consider a trans-Sasakian 3-manifold (M, g) admitting a conformal η-Einstein soliton (g, ζ, λ, μ) then from the Eq. (2), we can write

$$(\mathcal{L}_\zeta g)(V_1, V_2) + 2S(V_1, V_2) + \left[2\lambda - r + \left(p + \frac{2}{n}\right)\right] g(V_1, V_2) + 2\mu\eta(V_1)\eta(V_2) = 0 \tag{33}$$

for all $V_1, V_2 \in TM$.

Using $(\mathcal{L}_\zeta g)(V_1, V_2) = g(\nabla_{V_1}\zeta, V_2) + g(\nabla_{V_2}\zeta, V_1)$ and Eq. (10), we get

$$(\mathcal{L}_\zeta g)(V_1, V_2) = 2\beta[g(V_1, V_2) - \eta(V_1)\eta(V_2)]. \tag{34}$$

Using equations (33) and (34), we achieve

$$S(V_1, V_2) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right] g(V_1, V_2) + (\beta - \mu)\eta(V_1)\eta(V_2). \tag{35}$$

Consequently, (M, g) is an η-Einstein manifold.

Also, we plug $V_2 = \zeta$ into (35) to find

$$S(V_1, \zeta) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \mu\right] \eta(V_1). \tag{36}$$

Comparing the above Eq. (36) with the identity (17), we obtain

$$r = 4(\alpha^2 - \beta^2) + 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\lambda + 2\mu. \tag{37}$$

Taking an orthonormal basis $\{e_1, e_2, e_3\}$ of (M, g) and then setting $V_1 = V_2 = e_i$ in the Eq. (35) and summation over i we obtain

$$r = 6\left(\frac{p}{2} + \frac{1}{n}\right) + 6\lambda + 4\beta + 2\mu. \tag{38}$$

Finally combining Eqs. (37) and (38), we arrive at

$$\lambda = (\alpha^2 - \beta^2) - \beta - \left(\frac{p}{2} + \frac{1}{n}\right). \tag{39}$$

Thus the above discussion leads to the following:

Theorem 4 *If a trans-Sasakian* 3*-manifold* (M, g) *admits a conformal* η*-Einstein soliton* (g, ζ, λ, μ) *then the manifold* (M, g) *becomes a* η*-Einstein manifold of constant scalar curvature* $r = 6(\frac{p}{2} + \frac{1}{n}) + 6\lambda + 4\beta + 2\mu$. *Furthermore, the soliton is shrinking, steady or expanding according as* $\alpha^2 < \beta(\beta + 1) + (\frac{p}{2} + \frac{1}{n})$, $\alpha^2 = \beta(\beta + 1) + (\frac{p}{2} + \frac{1}{n})$, $\alpha^2 > \beta(\beta + 1) + (\frac{p}{2} + \frac{1}{n})$ *respectively.*

Next, considering a trans-sasakian 3-manifold (M, g) that admits a conformal η-Einstein soliton $(g, \mathcal{V}, \lambda, \mu)$ such that $\mathcal{V}$ is parallel to ζ, i.e. $\mathcal{V} = b\zeta$, for some function b, and using Eq. (2) it follows that

$$\begin{aligned} &bg(\nabla_{V_1}\zeta, V_2) + (V_1 b)\eta(V_2) + bg(\nabla_{V_2}\zeta, V_1) + (V_2 b)\eta(V_1) \\ &+2S(V_1, V_2) + \left[2\lambda - r + 2\left(\frac{p}{2} + \frac{1}{n}\right)\right]g(V_1, V_2) + 2\mu\eta(V_1)\eta(V_2) = 0. \end{aligned} \tag{40}$$

Then we utilize the identity (10) in the above Eq. (40) to get

$$\begin{aligned} &\left[2b\beta + 2\lambda - r + 2\left(\frac{p}{2} + \frac{1}{n}\right)\right]g(V_1, V_2) + (V_1 b)\eta(V_2) \\ &+ (V_2 b)\eta(V_1) + 2S(V_1, V_2) + 2(\mu - b\beta)\eta(V_1)\eta(V_2) = 0. \end{aligned} \tag{41}$$

Now, we insert $V_2 = \zeta$ into the identity (41) to yield

$$\left[2\lambda - r + 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\mu\right]\eta(V_1) + (V_1 b) + (\zeta b)\eta(V_1) + 2S(V_1, \zeta) = 0. \tag{42}$$

Again taking $V_1 = \zeta$ in the above Eq. (42) and by virtue of (17), we acquire

$$2(\zeta b) = \left(r - 2\lambda - 2\left(\frac{p}{2} + \frac{1}{n}\right) - 2\mu\right) - 4(\alpha^2 - \beta^2). \tag{43}$$

Using the value from (43) in the Eq. (42) and recalling (17), we can write

$$db = \left[\frac{r}{2} - \lambda - \left(\frac{p}{2} + \frac{1}{n}\right) - \mu - 2(\alpha^2 - \beta^2)\right]\eta. \tag{44}$$

Now, taking exterior derivative on both sides of (44) we obtain

$$r = 2\lambda + 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\mu + 4(\alpha^2 - \beta^2). \tag{45}$$

In view of the above identity (45), the Eq. (44) gives $db = 0$ i.e., the function b is constant. Then the Eq. (41) reduces to

$$S(V_1, V_2) = \left[\frac{r}{2} - \lambda - b\beta - \left(\frac{p}{2} + \frac{1}{n}\right)\right] g(V_1, V_2) + (b\beta - \mu)\eta(V_1)\eta(V_2) \tag{46}$$

for all $V_1, V_2 \in TM$. Hence we can state the following:

Theorem 5 *If a trans-Sasakian 3-manifold (M, g) admits a conformal η-Einstein soliton $(g, \mathcal{V}, \lambda, \mu)$ such that $\mathcal{V}$ is pointwise collinear with ζ, then $\mathcal{V}$ is constant multiple of ζ and the manifold (M, g) becomes an η-Einstein manifold of constant scalar curvature.* $r = 2\lambda + 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\mu + 4(\alpha^2 - \beta^2)$.

5 Conformal η-Einstein Soliton on Trans-Sasakian 3-Manifold with Cyclic Parallel Ricci Tensor

In this section we study conformal η-Einstein solitons on trans-Sasakian 3-manifolds having certain special types of Ricci tensor.

Definition 5.1 [21] A trans-Sasakian 3-manifold is said to have Codazzi type Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$(\nabla_{V_1} S)(V_2, V_3) = (\nabla_{V_2} S)(V_1, V_3), \quad \forall V_1, V_2, V_3 \in TM \tag{47}$$

We consider a trans-Sasakian 3-manifold that has Codazzi type Ricci tensor and admits a conformal η-Einstein soliton (g, ζ, λ, μ) then Eq. (35) holds. Taking covariant derivative in Eq. (35) and using (11), we conclude

$$\begin{aligned}(\nabla_{V_1} S)(V_2, V_3) = (\beta - \mu)[&\eta(V_3)(-\alpha g(\mathcal{F}V_1, V_2) + \beta g(\mathcal{F}V_1, \mathcal{F}V_2)) \\ + \ &\eta(V_2)(-\alpha g(\mathcal{F}V_1, V_3) + \beta g(\mathcal{F}V_1, \mathcal{F}V_3))].\end{aligned} \tag{48}$$

Also, we have

$$\begin{aligned}(\nabla_{V_2} S)(V_1, V_3) = (\beta - \mu)[&\eta(V_3)(-\alpha g(\mathcal{F}V_2, V_1) + \beta g(\mathcal{F}V_2, \mathcal{F}V_1)) \\ + \ &\eta(V_1)(-\alpha g(\mathcal{F}V_2, V_3) + \beta g(\mathcal{F}V_2, \mathcal{F}V_3))].\end{aligned} \tag{49}$$

As the Ricci tensor is of Codazzi type on using (48) and (49) in the Eq. (47) and then recalling (67), we arrive at

$$\begin{aligned}&(\beta-\mu)[\eta(V_2)(-\alpha g(\mathcal{F}V_1, V_3) + \beta g(V_1, V_3)) - \eta(V_1)(-\alpha g(\mathcal{F}V_2, V_3) \\ &+\beta g(V_2, V_3)) - 2\alpha\eta(V_3)g(\mathcal{F}V_1, V_2)] = 0.\end{aligned} \tag{50}$$

Now, we put $V_3 = \zeta$ in the above Eq. (50) and view of (45) to obtain

$$2\alpha(\beta - \mu)g(\mathcal{F}V_1, V_2) = 0 \tag{51}$$

for all $V_1, V_2 \in TM$. Therefore from (50) we can conclude that either $\alpha = 0$ or $\beta = \mu$.
Thus, we have:

Theorem 6 *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η-Einstein soliton (g, ζ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes a β-Kenmotsu manifold provided $\mu \neq \beta$.*

We use $\alpha = 0$ in the Eq. (39), we get $\lambda = -\{\beta(\beta+1) + (\frac{p}{2} + \frac{1}{n})\}$. Thus, we conclude the following:

Corollary 7 *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η-Einstein soliton (g, ζ, λ, μ) with $\beta \neq \mu$. If the Ricci tensor of the manifold is of Codazzi type then the soliton is shrinking if $\beta(\beta+1) + (\frac{p}{2} + \frac{1}{n}) > 0$, steady if $\beta(\beta+1) + (\frac{p}{2} + \frac{1}{n}) = 0$ and expanding if $\beta(\beta+1) + (\frac{p}{2} + \frac{1}{n}) < 0$ respectively.*

Again from the Eq. (50), we can write that $\mu = \beta$ if $\alpha \neq 0$, then from the Eq. (35) we find

$$S(V_1, V_2) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right] g(V_1, V_2) \tag{52}$$

for all $V_1, V_2 \in TM$. Then contracting the Eq. (51) we get $r = 6(\frac{p}{2} + \frac{1}{n}) + 6\lambda + 6\beta$. Hence in view of this identity and Eq. (51), we have the following theorem:

Theorem 8 *Let (M,g) be a trans-Sasakian 3-manifold admitting a conformal η-Einstein soliton (g, ζ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6(\frac{p}{2} + \frac{1}{n}) + 6\lambda + 6\beta$ provided $\alpha \neq 0$.*

Definition 5.2 [21] A trans-Sasakian 3-manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$(\nabla_{V_1} S)(V_2, V_3) + (\nabla_{V_2} S)(V_1, V_3) + (\nabla_{V_3} S)(V_1, V_2) = 0, \forall V_1, V_2, V_3 \in TM. \tag{53}$$

On considering a trans-Sasakian 3-manifold, that has cyclic parallel Ricci tensor, that also admits a conformal η-Einstein soliton (g, ζ, λ, μ), then Eq. (35) holds. On taking covariant derivative in Eq. (35) and using Eq. (11) we obtain relations (48) and (49). Similarly we have

$$\begin{aligned}(\nabla_{V_3} S)(V_1, V_2) = (\beta - \mu)[&\eta(V_1)(-\alpha g(\mathcal{F}V_3, V_2) + \beta g(\mathcal{F}V_3, \mathcal{F}V_2)) \\ &+ \eta(V_2)(-\alpha g(\mathcal{F}V_3, V_1) + \beta g(\mathcal{F}V_3, \mathcal{F}V_1))].\end{aligned} \tag{54}$$

As the Ricci tensor is cyclic parallel on using Eqs. (48), (49) and (54) in the Eq. (53) and then making use of (67), we conclude

$$2\beta(\beta - \mu)[\eta(V_1) g(\mathcal{F}V_2, \mathcal{F}V_3) + \eta(V_2) g(\mathcal{F}V_3, \mathcal{F}V_1) + \eta(V_3) g(\mathcal{F}V_1, \mathcal{F}V_2)] = 0. \tag{55}$$

Taking $V_3 = \zeta$ in Eq. (55), we obtain

$$2\beta(\beta - \mu)g(\mathcal{F}V_1, \mathcal{F}V_2) = 0 \tag{56}$$

for all $V_1, V_2 \in TM$. Since $g(\mathcal{F}V_1, \mathcal{F}V_2) \neq 0$, Eq. (56) implies that either $\beta = 0$ or $\mu = \beta$. Thus we have

Theorem 9 *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η-Einstein soliton (g, ζ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an α-Sasakian manifold provided $\mu \neq \beta$.*

On using $\beta = 0$ in Eq. (39), we obtain $\lambda = \alpha^2 - (\frac{p}{2} + \frac{1}{n})$. Therefore, we have

Corollary 10 *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η -Einstein soliton (g, ζ, λ, μ) with $\beta \neq \mu$. If the manifold has cyclic parallel Ricci tensor then the soliton is shrinking if $\alpha^2 < (\frac{p}{2} + \frac{1}{n})$ steady if $\alpha^2 = (\frac{p}{2} + \frac{1}{n})$ and expanding if $\alpha^2 > (\frac{p}{2} + \frac{1}{n})$*

Also if $\beta \neq 0$ then using (56)we have $\mu = \beta$. Thus, with a similar calculation like Eq. (52) we can state

Theorem 11 *Let (M, g) be trans-Sasakian 3-manifold admitting a conformal η-Einstein soliton (g, ζ, λ, μ). If the manifold has cyclic parallel Ricci tensor then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6(\frac{p}{2} + \frac{1}{n}) + 6\lambda + 6\beta$ provided $\beta \neq 0$.*

6 Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifolds Satisfying $R(\zeta, V_1) \cdot S = 0$ and $\mathcal{W}_2(\zeta, V_1) \cdot S = 0$

In this section, first we consider a trans-Sasakian 3-manifold that admits a conformal η-Einstein soliton (g, ζ, λ, μ) and the manifold satisfies the curvature condition $R(\zeta, V_1) \cdot S = 0$ then, we can write

$$S(R(\zeta, V_1)V_2, V_3) + S(V_2, R(\zeta, V_1)V_3) = 0. \tag{57}$$

Now using the Eq. (35) into (57), we get

$$\begin{aligned}
&\left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right) g(R(\zeta, V_1)V_2, V_3) + (\beta - \mu)\eta(R(\zeta, V_1)V_2)\eta(V_3) \\
&\quad + \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right) g(V_2, R(\zeta, V_1)V_3) \\
&\quad + (\beta - \mu)\eta(R(\zeta, V_1)V_3)\eta(V_2) = 0.
\end{aligned} \tag{58}$$

Using (14) in the previous equation, we obtain

$$(\alpha^2-\beta^2)(\beta-\mu)[g(V_1,V_2)\eta(V_3)+g(V_1,V_3)\eta(V_2)-2\eta(V_1)\eta(V_2)\eta(V_3)]=0. \tag{59}$$

By taking $V_3=\zeta$ in Eq. (59) and using (67), we get

$$(\alpha^2-\beta^2)(\beta-\mu)g(\mathcal{F}V_1,\mathcal{F}V_2)=0 \tag{60}$$

for all $V_1, V_2 \in TM$. As $g(\mathcal{F}V_1,\mathcal{F}V_2)\neq 0$ and for non-trivial case $\alpha^2\neq\beta^2$, we can conclude from the Eq. (60) that $\mu=\beta$. Thus, using Eq. (35) we have

$$S(V_1,V_2)=\left[\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\beta\right]g(V_1,V_2) \tag{61}$$

for all $V_1, V_2 \in TM$. On contracting Eq. (61), we have $r=6(\frac{p}{2}+\frac{1}{n})+6\lambda+6\beta$. Plugging this information together with Eq. (61) we have the following:

Theorem 12 *Let* (M, g) *be trans-Sasakian* 3*-manifold admitting a conformal* η *-Einstein soliton* (g,ζ,λ,μ)*. If the manifold satisfies the curvature condition* $R(\zeta,V_1)\cdot S=0$*, then the manifold becomes an Einstein manifold of constant scalar curvature* $r=6(\frac{p}{2}+\frac{1}{n})+6\lambda+6\beta$*.*

Our next result of this section uses $\mathcal{W}_2$-Curvature tensor that is introduced in 1970 by Pokhariyal and Mishra in [46].

Definition 6.1 The $\mathcal{W}_2$-curvature tensor in a trans-Sasakian 3-manifold (M, g) is defined as

$$\mathcal{W}_2(V_1,V_2)V_3=R(V_1,V_2)V_3+\frac{1}{2}\{g(V_1,V_3)QV_2-g(V_2,V_13)QV_1\} \tag{62}$$

Next, we consider that (M, g) is a trans-Sasakian 3-manifold that admits a conformal η-Einstein soliton (g,ζ,λ,μ) and that the manifold satisfies the curvature condition $\mathcal{W}_2(\zeta,V_1)\cdot S=0$. Then we can write

$$S(\mathcal{W}_2(\zeta,V_1)V_2,V_3)+S(V_2,\mathcal{W}_2(\zeta,V_1)V_3)=0, \quad \forall V_1,V_2\in TM. \tag{63}$$

Using (35) the Eq. (63) takes the form

$$\begin{aligned}&\left(\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\beta\right)[g(\mathcal{W}_2(\zeta,V_1)V_2,V_3)+g(\mathcal{W}_2(\zeta,V_1)V_3,V_2)]\\&\quad+(\beta-\mu)[\eta(\mathcal{W}_2(\zeta,V_1)V_2)\eta(V_3)+\eta(\mathcal{W}_2(\zeta,V_1)V_3)\eta(V_2)]=0.\end{aligned} \tag{64}$$

Also, Eq. (35) implies

$$QV_1=\left(\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\beta\right)V_1+(\beta-\mu)\eta(V_1)\zeta, \tag{65}$$

that is,

$$Q\zeta = \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \mu\right)\zeta. \tag{66}$$

Now, we replace $V_1 = \zeta$ into (62) and then using Eqs. (14), (65) and (66) to obtain

$$\mathcal{W}_2(\zeta, V_2)V_3 = N' g(V_2, V_3)\zeta - M'\eta(V_3)V_2 + (M' - N')\eta(V_2)\eta(V_3), \tag{67}$$

where $M' = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - (\frac{p}{2} + \frac{1}{n}) - \lambda - \beta)$ and $N' = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - (\frac{p}{2} + \frac{1}{n}) - \lambda - \mu)$. On taking the inner product in Eq. (67) with the vector field ζ yields

$$\eta(\mathcal{W}_2(\zeta, V_2)V_3) = N'[g(V_2, V_3) - \eta(V_2)\eta(V_3)]. \tag{68}$$

Using (67) and (68) in the equation (64) and then taking $V_3 = \zeta$, we arrive at

$$(N' - M')\left[2N' - \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right)\right]\left[g(V_1, V_2) - \eta(V_1)\eta(V_2)\right] = 0,$$

which in view of (67) implies

$$(N' - M')\left[2N' - \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right)\right]g(\mathcal{F}V_1, \mathcal{F}V_2) = 0 \tag{69}$$

for all $V_1, V_2 \in TM$. As $g(\mathcal{F}V_1, \mathcal{F}V_2) \neq 0$, we can conclude from the Eq. (69) that either $M' = N'$ or $2N' = \frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta$. Thus recalling the values of M' and N' it implies that either $\mu = \beta$ or

$$r - 2\left(\frac{p}{2} + \frac{1}{n}\right) - 2\lambda - \mu - \beta = 2(\alpha^2 - \beta^2). \tag{70}$$

Now for $\beta = \mu$, in a similar way as in Eq. (61) we can easily say that the manifold becomes an Einstein manifold. On using Eq. (70) with (37) we have

$$r = 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\lambda + 2\beta. \tag{71}$$

Therefore we can state the following:

Theorem 13 *Let* (M, g) *be a trans-Sasakian* 3*-manifold admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ)*. If the manifold satisfies the curvature condition* $\mathcal{W}_2(\zeta, V_1) \cdot S = 0$ *then either the manifold becomes an Einstein manifold or it is a manifold of constant scalar curvature* $r = 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\lambda + 2\beta$.

Again in view of (38), the Eq. (71) implies $\lambda = -\frac{1}{2}\{2(\frac{p}{2} + \frac{1}{n}) + \mu + \beta\}$. Hence we have

Corollary 14 *Let* (M, g) *be a trans-Sasakian* 3*-manifold admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ) *with* $\beta \neq \mu$*. If the manifold satisfies the curvature* *condition* $\mathcal{W}_2(\zeta, V_1) \cdot S = 0$, *then the soliton is shrinking if* $\mu > -\{2(\frac{p}{2} + \frac{1}{n}) + \beta\}$, *steady if* $\mu = -\{2(\frac{p}{2} + \frac{1}{n}) + \beta\}$ *and expanding if* $\mu < -\{2(\frac{p}{2} + \frac{1}{n}) + \beta\}$.

7 Einstein Semi-Symmetric Trans-Sasakian 3-Manifolds Admitting Conformal η-Einstein Solitons

Definition 7.1 A trans-Sasakian 3-manifold (M, g) is called Einstein semi-symmetric [56] if $R \cdot E = 0$ where E is the Einstein tensor given by

$$E(V_1, V_2) = S(V_1, V_2) - \frac{r}{3} g(V_1, V_2) \tag{72}$$

for all vector fields $V_1, V_2 \in TM$ and r is the scalar curvature of the manifold.

Now consider a trans-Sasakian 3-manifold is Einstein semi-symmetric i.e. the manifold satisfies the curvature condition $R \cdot E = 0$ Then for all the vector fields $V_1, V_2, V_3, V_4 \in TM$ we can write

$$E(R(V_1, V_2)V_3, V_4) + E(V_3, R(V_1, V_2)V_4) = 0. \tag{73}$$

In view of (72), the Eq. (73) becomes

$$S(R(V_1, V_2)V_3, V_4) + S(V_3, R(V_1, V_2)V_4) = \frac{r}{3}[g(R(V_1, V_2)V_3, V_4) + g(V_3, R(V_1, V_2)V_4)]. \tag{74}$$

Replacing $V_1 = V_3 = \zeta$ in the above Eq. (74) and then using (14), (15), we arrive at

$$(\alpha^2 - \beta^2)S(V_2, V_4) = (\alpha^2 - \beta^2)[\eta(V_2)S(\zeta, V_4) + \eta(V_4)S(\zeta, V_2) - g(V_2, V_4)S(\zeta, \zeta)]. \tag{75}$$

So, now in view of (17) the above Eq. (75) finally yields

$$S(V_2, V_4) = -2(\alpha^2 - \beta^2)g(V_2, V_4) + 4(\alpha^2 - \beta^2)\eta(V_2)\eta(V_4) \tag{76}$$

for all $V_2, V_4 \in TM$. This implies that the manifold is an η-Einstein manifold. Hence we have the following:

Lemma 15 *An Einstein semi-symmetric trans-Sasakian* 3*-manifold is an* η*-Einstein manifold.*

Now, let us assume that the Einstein semi-symmetric trans-Sasakian 3-manifold (M, g) admits a conformal η-Einstein soliton (g, ζ, λ, μ). Then Eq. (35) holds and combining (35) with the above Eq. (76), we get

$$r = 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\lambda + \mu + \beta. \tag{77}$$

Again recalling the equation (38) in the above (77), we achieve

$$\lambda = -\frac{1}{4}\left\{\left(\frac{p}{2} + \frac{1}{n}\right) + \mu + 3\beta\right\}. \tag{78}$$

Therefore, we can state the following:

Theorem 16 *Let* (M, g) *be a trans-Sasakian* 3*-manifold admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ). *If the manifold is Einstein semi-symmetric, then the manifold becomes an* η*-Einstein manifold of constant scalar curvature* $r = 2\left(\frac{p}{2} + \frac{1}{n}\right) + 2\lambda + \mu + \beta$ *and the soliton is shrinking, steady or expanding as* $\left(\frac{p}{2} + \frac{1}{n}\right) + \mu + 3\beta > 0$, $\left(\frac{p}{2} + \frac{1}{n}\right) + \mu + 3\beta = 0$ *or,* $\left(\frac{p}{2} + \frac{1}{n}\right) + \mu + 3\beta < 0$ *respectively.*

8 Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifolds Satisfying $\mathcal{B}(\zeta, V_1) \cdot S = 0$

Recall that Bochner curvature tensor [7] was introduced as complex analogue of conformal curvature tensor. However, its geometric significance was later revealed by the work in [5] through the Boothby-Wang fibration. The notion of C-Bochner curvature tensor in a Sasakian manifold was introduced by Matsumoto and Chuman [42] in 1969. The $\mathcal{C}$-Bochner curvature tensor in trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned}\mathcal{B}(V_1, V_2)V_3 = {}& R(V_1, V_2)V_3 + \frac{1}{6}[g(V_1, V_3)QV_2 - S(V_2, V_3)V_1 - g(V_2, V_3)QV_1 \\ & + S(V_1, V_3)V_2 + g(\mathcal{F}V_1, V_3)Q\mathcal{F}V_2 - S(\mathcal{F}V_2, V_3)\mathcal{F}V_1 \\ & - g(\mathcal{F}V_2, V_3)Q\mathcal{F}V_1 + S(\mathcal{F}V_1, V_3)\mathcal{F}V_2 + 2S(\mathcal{F}V_1, V_2)\mathcal{F}V_3 \\ & + 2g(\mathcal{F}V_1, V_2)Q\mathcal{F}V_3 + \eta(V_2)\eta(V_3)QV_1 - \eta(V_2)S(V_1, V_3)\zeta \\ & + \eta(V_1)S(V_2, V_3)\zeta - \eta(V_1)\eta(V_3)QV_2] - \frac{D+2}{6}[g(\mathcal{F}V_1, V_3)\mathcal{F}V_2 \\ & - g(\mathcal{F}V_2, V_3)\mathcal{F}V_1 + 2g(\mathcal{F}V_1, V_2)\mathcal{F}V_3] + \frac{D}{6}[\eta(V_2)g(V_1, V_3)\zeta \\ & - \eta(V_2)\eta(V_3)V_1 + \eta(V_1)\eta(V_3)V_2 - \eta(V_1)g(V_2, V_3)\zeta] \\ & - \frac{D-4}{6}[g(V_1, V_3)V_2 - g(V_2, V_3)V_1],\end{aligned} \tag{79}$$

where $D = \frac{r+2}{4}$.

Let us consider a trans-Sasakian 3-manifold (M, g) which admits a conformal η-Einstein soliton (g, ζ, λ, μ) and also the manifold satisfies the curvature condition $B(\zeta, V_1) \cdot S = 0$. Then $\forall V_1, V_2, V_3 \in TM$ we can write

$$S(\mathcal{B}(\zeta, V_1)V_2, V_3) + S(V_2, \mathcal{B}(\zeta, V_1)V_3) = 0. \tag{80}$$

Now using (35) in (80), we get

$$\left\{\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\beta\right\}[g(\mathcal{B}(\zeta,V_1)V_2,V_3)+g(\mathcal{B}(\zeta,V_1)V_3,V_2)] \\ +(\beta-\mu)[\eta(\mathcal{B}(\zeta,V_1)V_2)\eta(V_3)+\eta(\mathcal{B}(\zeta,V_1)V_3)\eta(V_2)]=0. \tag{81}$$

Moreover, using Eq. (35), we obtain

$$QV_1=\left[\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\beta\right]V_1+(\beta-\mu)\eta(V_1)\zeta, \tag{82}$$

that is,

$$Q\zeta=\left[\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\mu\right]\zeta. \tag{83}$$

Thus, on taking $V_1=\zeta$ in (79) we get

$$\mathcal{B}(\zeta,V_2)V_3=R(\zeta,V_2)V_3+\frac{1}{6}[S(\zeta,V_3)V_2-g(V_2,V_3)Q\zeta+\eta(V_2)\eta(V_3)Q\zeta \\ -\eta(V_2)S(\zeta,V_3)\zeta]+\frac{4}{6}[\eta(V_3)V_2-g(V_2,V_3)\zeta]. \tag{84}$$

We insert the Eqs. (14), (36) and (83) in the Eq. (84) to yield

$$\mathcal{B}(\zeta,V_2)V_3=\left[(\alpha^2-\beta^2)-\frac{1}{6}\left\{\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\mu\right\}-\frac{4}{6}\right][g(V_2,V_3)\zeta-\eta(V_3)V_2]. \tag{85}$$

In view of (85) the Eq. (81) becomes

$$\left[\left(\alpha^2-\beta^2\right)-\frac{1}{6}\left\{\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\mu\right\}-\frac{4}{6}\right](\beta-\mu)[g(V_1,V_2)\eta(V_3) \\ +g(V_1,V_3)\eta(V_2)-2\eta(V_1)\eta(V_2)\eta(V_3)]=0. \tag{86}$$

Now, we plug $V_3=\zeta$ in the above Eq. (86) and recalling (67) to arrive

$$\left[(\alpha^2-\beta^2)-\frac{1}{6}\left\{\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\mu\right\}-\frac{4}{6}\right](\beta-\mu)g(\mathcal{F}V_1,\mathcal{F}V_2)=0 \tag{87}$$

for all vector fields $V_1, V_2 \in TM$ and $g(\mathcal{F}V_1,\mathcal{F}V_2)\neq 0$, hence from (87) we can conclude that either

$$\left[(\alpha^2-\beta^2)-\frac{1}{6}\left\{\frac{r}{2}-\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\mu\right\}-\frac{4}{6}\right]=0 \tag{88}$$

or, $\beta=\mu$. Also for $\beta=\mu$ with a similar way as in Eq. (52) it can be easily shown that the manifold is an Einstein manifold. If $\beta\neq\mu$, then on using Eq. (39) in the Eq. (88) we have

$$r=14\lambda+14\left(\frac{p}{2}+\frac{1}{n}\right)+12\beta+2\mu-8, \tag{89}$$

that is, the scalar curvature is a constant.

Theorem 17 *Let* (M, g) *be a trans-Sasakian* 3*-manifold admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ)*. If the manifold satisfies the curvature condition* $\mathcal{B}(\zeta, V_1) \cdot S = 0$*, then either the manifold is an Einstein manifold or it is a manifold of constant scalar curvature* $r = 14\lambda + 14(\frac{p}{2} + \frac{1}{n}) + 12\beta + 2\mu - 8$.

Now for the case $\beta \neq \mu$, using the Eq. (38) in (89) we obtain $\lambda = 1 - \left(\frac{p}{2} + \frac{1}{n} + \beta\right)$. Hence we have

Corollary 18 *Let* (M, g) *be a trans-Sasakian* 3*-manifold admitting a conformal* η *-Einstein soliton* (g, ζ, λ, μ) *with* $\mu \neq \beta$*. If the manifold satisfies the curvature condition* $\mathcal{B}(\zeta, V_1) \cdot S = 0$*, then the soliton is shrinking, steady or expanding according as* $\frac{p}{2} + \frac{1}{n} + \beta > 1$, $\frac{p}{2} + \frac{1}{n} + \beta = 1$ *or* $\frac{p}{2} + \frac{1}{n} + \beta < 1$ *respectively.*

9 Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifolds with Torse-Forming Vector Field

Here we study the nature of conformal η-Einstein solitons on trans-Sasakian 3-manifold with torse-forming vector field.

Definition 9.1 A vector field $\mathcal{V}$ on a trans-Sasakian 3-manifold is a torse-forming vector field [62] if

$$\nabla_{\mathcal{V}} = fV_1 + \gamma(V_1)\mathcal{V}, \tag{90}$$

where f is a smooth function and γ is a 1-form.

Now let (g, ζ, λ, μ) be a conformal η-Einstein soliton on a trans-Sasakian 3-manifold (M, g) and assume that the Reeb vector field ζ of the manifold is a torse-forming vector field. Then ζ being a torse-forming vector field, by definition (90) we have

$$\nabla_{V_1}\zeta = fV_1 + \gamma(V_1)\zeta, \tag{91}$$

$\forall V_1 \in TM$.

Taking the inner product in Eq. (10) with ζ we can write

$$g(\nabla_{V_1}\zeta, \zeta) = (\beta - 1)\eta(V_1). \tag{92}$$

Taking the inner product in Eq. (91), with ζ we obtain

$$g(\nabla_{V_1}\zeta, \zeta) = f\eta(V_1) + \gamma(V_1). \tag{93}$$

We combine (92) and (93) to get, $\gamma = (\beta - 1 - f)\eta$. Thus from (91) it implies that, for torse forming vector field ζ in a trans-Sasakian 3-manifold, we have

$$\nabla_{V_1}\zeta = f(V_1 - \eta(V_1)\zeta) + (\beta - 1)\eta(V_1)\zeta. \tag{94}$$

Now from the formula of Lie differentiation and using (94) yields

$$\begin{aligned}(\mathcal{L}_\zeta g)(V_1, V_2) =& g(\nabla_{V_1}\zeta, V_2) + g(\nabla_{V_2}\zeta, V_1)\\ =& 2f[g(V_1, V_2) - \eta(V_1)\eta(V_2)] + 2(\beta - 1)\eta(V_1)\eta(V_2).\end{aligned} \tag{95}$$

Since (g, ζ, λ, μ) is a conformal η-Einstein soliton, the Eq. (2) holds. So in view of (95), the Eq. (2) reduces to

$$S(V_1, V_2) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - f\right]g(V_1, V_2) + (f - \mu - \beta + 1)\eta(V_1)\eta(V_2). \tag{96}$$

Thus, the manifold is an η-Einstein manifold. On letting $V_2 = \zeta$ in (96) we get

$$S(V_1, \zeta) = \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \mu - \beta + 1\right)\eta(V_1). \tag{97}$$

Combining (97) with the Eq. (17) implies

$$2(\alpha^2 - \beta^2) = \left(\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \mu - \beta + 1\right). \tag{98}$$

On taking trace in Eq. (96) we obtain

$$r = 6\lambda + 6\left(\frac{p}{2} + \frac{1}{n}\right) + 2\mu + 4f + 2\beta - 2. \tag{99}$$

Using Eq. (99) in (98), we get $\lambda = (\alpha^2 - \beta^2) - f - \left(\frac{p}{2} + \frac{1}{n}\right)$, and we have the following:

Theorem 19 *Let* (M, g) *be a conformal* η*-Einstein soliton on a trans-Sasakian* 3*-manifold* (M, g) *with torse-forming vector field* ζ, *then the manifold becomes an* η *-Einstein manifold and the soliton is shrinking, steady or expanding according as* $f > (\alpha^2 - \beta^2) - \left(\frac{p}{2} + \frac{1}{n}\right)$, $f = (\alpha^2 - \beta^2) - \left(\frac{p}{2} + \frac{1}{n}\right)$ *or* $f < (\alpha^2 - \beta^2) - \left(\frac{p}{2} + \frac{1}{n}\right)$.

10 Conformal η-Einstein Solitons on a Trans-Sasakian 3-Manifold Satisfying $\mathcal{M}(\zeta, V_1) \cdot S = 0$ and $S(\zeta, V_1) \cdot \mathcal{M} = 0$

Definition 10.1 Let (M, g) be a trans-Sasakian 3-manifold. Then $\mathcal{M}$-projective curvature tensor of M is defined by [47]

$$\begin{aligned}\mathcal{M}(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{4}[&S(V_2, V_3)V_1 - S(V_1, V_3)V_2 + g(V_2, V_3)QV_1\\ &- g(V_1, V_3)QV_2].\end{aligned} \tag{100}$$

We assume 3-dimensional trans-sasakian manifold with conformal η-Einstein solitons (g, ζ, λ, μ) satisfying the condition

$$\mathcal{M}(\zeta, V_1) \cdot S = 0.$$

Then, we have

$$(\mathcal{M}(\zeta, V_1)V_2, V_3) + S(V_2, \mathcal{M}(\zeta, V_1)V_3) = 0 \tag{101}$$

for any $V_1, V_2, V_3 \in TM$.

Now from (17) and (36), we have

$$2(\beta^2 - \alpha^2) = \left[\lambda + \mu + \left(\frac{p}{2} + \frac{1}{n}\right) - \frac{r}{2}\right]. \tag{102}$$

Now, we use the Eqs. (14), (35), (36), (102) and (100) into the identity (101) to get

$$\begin{aligned}
&\Big[\frac{1}{2}(\alpha^2 - \beta^2)\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \mu\Big\} + \frac{1}{4}\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \beta\Big\}\\
&+(\alpha^2 - \beta^2)\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\} + \frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\Big\{\lambda + \mu + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\\
&-\frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\{\beta - \mu + 2(\alpha^2 - \beta^2)\}\Big]\big[g(V_1, V_2V_1)\eta(V_3) + g(V_1, V_3)\eta(V_2)\big]\\
&+\Big[-(\alpha^2 - \beta^2)(\beta - \mu) + \frac{(\beta - \mu)}{4}\Big\{2(\alpha^2 - \beta^2) + \frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \beta\Big\}\Big]\eta(V_1)\eta(V_2)\eta(V_3) = 0.
\end{aligned}$$

By putting $V_3 = \zeta$ in the above equation, we obtain

$$\begin{aligned}
&\Big[\frac{1}{2}(\alpha^2 - \beta^2)\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \mu\Big\} + \frac{1}{4}\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \beta\Big\}\\
&+(\alpha^2 - \beta^2)\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\} + \frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\Big\{\lambda + \mu + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\\
&-\frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\{\beta - \mu + 2(\alpha^2 - \beta^2)\}\Big]\big[g(V_1, V_2) + \eta(V_1)\eta(V_2)\big]\\
&+\Big[-(\alpha^2 - \beta^2)(\beta - \mu) + \frac{(\beta - \mu)}{4}\Big\{2(\alpha^2 - \beta^2) + \frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \beta\Big\}\Big]\eta(V_1)\eta(V_2) = 0.
\end{aligned}$$

Now, we set $V_1 = \mathcal{F}V_1$ and $V_2 = \mathcal{F}V_2$ in the previous equation to arrive

$$\begin{aligned}
&\Big[\frac{1}{2}(\alpha^2 - \beta^2)\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \mu\Big\} + \frac{1}{4}\Big\{\frac{r}{2} - \Big(\frac{p}{2} + \frac{1}{n}\Big) - \lambda - \beta\Big\} + (\alpha^2 - \beta^2)\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\\
&+\frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\Big\{\lambda + \mu + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\\
&-\frac{1}{4}\Big\{\lambda + \beta + \Big(\frac{p}{2} + \frac{1}{n}\Big) - \frac{r}{2}\Big\}\{\beta - \mu + 2(\alpha^2 - \beta^2)\}\Big]g(\mathcal{F}V_1, \mathcal{F}V_2) = 0.
\end{aligned} \tag{103}$$

Again using the Eq. (102), we have

$$\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta - \left(\frac{p}{2} + \frac{1}{n}\right) + \frac{r}{2}.$$

So, we have the theorem:

Theorem 20 *If a* 3*-dimensional trans-Sasakian manifold* $(M, g, \mathcal{F}, \eta, \zeta, \alpha, \beta)$ *with* α, β *constants admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ) *satisfies the condition* $\mathcal{M}(\zeta, V_1) \cdot S = 0$ *then* $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta - (\frac{p}{2} + \frac{1}{n}) + \frac{r}{2}$.

Corollary 21 *A* 3*-dimensional trans-Sasakian manifold* (M, g) *with* α, β *constants satisfies the condition* $\mathcal{M}(\zeta, V_1) \cdot S = 0$, *there is no Ricci soliton with the potential vector field* ζ.

Suppose that 3-dimensional trans-Sasakian manifolds with conformal η-Einstein solitons satisfying the conditions

$$S(\zeta, V_1) \cdot \mathcal{M} = 0.$$

So, we have

$$\begin{aligned} &S(V_1, \mathcal{M}(V_2, V_3)\mathcal{V})\zeta - S(\zeta, \mathcal{M}(V_2, V_3)\mathcal{V})V_1 + S(V_1, V_2)\mathcal{M}(\zeta, V_3)\mathcal{V} \\ &- S(\zeta, V_2)\mathcal{M}(V_1, V_3)\mathcal{V} + S(V_1, V_3)\mathcal{M}(V_2, \zeta)\mathcal{V} - S(\zeta, V_3)\mathcal{M}(V_2, V_1)\mathcal{V} \\ &+ S(V_1, \mathcal{V})\mathcal{M}(V_2, V_3)\zeta - S(\zeta, \mathcal{V})\mathcal{M}(V_2, V_3)V_1 = 0. \end{aligned}$$

Taking inner product with ζ then the above equation becomes

$$\begin{aligned} &S(V_1, \mathcal{M}(V_2, V_3)\mathcal{V}) - S(\zeta, \mathcal{M}(V_2, V_3)\mathcal{V})\eta(V_1) + S(V_1, V_2)\eta(\mathcal{M}(\zeta, V_3)\mathcal{V}) \\ &- S(\zeta, V_2)\eta(\mathcal{M}(V_1, V_3)\mathcal{V}) + S(V_1, V_3)\eta(\mathcal{M}(V_2, \zeta)) - S(\zeta, V_3)\eta(\mathcal{M}(V_2, V_1)\mathcal{V}) \\ &+ S(V_1, \mathcal{V})\eta(\mathcal{M}(V_2, V_3)\zeta) - S(\zeta, \mathcal{V})\eta(\mathcal{M}(V_2, V_3)V_1) = 0. \end{aligned} \tag{104}$$

By letting $\mathcal{V} = \zeta$ and using the Eqs. (10), (14), (35), (36), (100) and (102) the Eq. (104) becomes

$$\begin{aligned} &[(2\lambda + p + \frac{2}{n} + \mu + \beta)(\alpha^2 - \beta^2) + \frac{1}{4}(2\lambda + p + \frac{2}{n} + \mu + \beta)^2 \\ &+ (2\lambda + p + \frac{2}{n} + \mu + \beta)\{(\alpha^2 - \beta^2) + \frac{1}{4}(2\lambda + p + \frac{2}{n} + \mu + \beta)\}] \\ &\{g(V_1, V_3)\eta(V_2) - g(V_1, V_2)\eta(V_3)\} = 0. \end{aligned} \tag{105}$$

Now as $g(QV_1, V_2) = S(V_1, V_2)$, then from (35) we have

$$QV_1 = \left\{\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda - \beta\right\}V_1 + (\beta - \mu)\eta(V_1)\zeta. \tag{106}$$

Using the Eq. (106) we have

$$\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta - \left(\frac{p}{2} + \frac{1}{n}\right)$$

or

$$\lambda = 2(\alpha^2 - \beta^2) - \left(\frac{p}{2} + \frac{1}{n}\right) - \beta, \quad \mu = -4(\alpha^2 - \beta^2) + \beta.$$

So, we have the following theorem:

Theorem 22 *If let a* 3*-dimensional trans-Sasakian manifold* $(M, g, \mathcal{F}, \eta, \zeta, \alpha, \beta)$ *with* α, β *constants admitting a conformal* η*-Einstein soliton* (g, ζ, λ, μ) *satisfies the condition* $S(\zeta, V_1) \cdot \mathcal{M} = 0$ *then*

$$\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta - \left(\frac{p}{2} + \frac{1}{n}\right)$$

or $\lambda = 2(\alpha^2 - \beta^2) - \left(\frac{p}{2} + \frac{1}{n}\right) - \beta$, $\mu = -4(\alpha^2 - \beta^2) + \beta$.

Corollary 23 *A* 3*-dimensional trans-Sasakian manifold* (M, g) *with* α, β *constants satisfies the condition* $S(\zeta, V_1) \cdot \mathcal{M} = 0$, *there is no Ricci soliton with the potential vector field* ζ.

11 Example of Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifold

In this section we construct an example that supports our outcomes.

Example 1 We consider $M = \{(V_1, V_2, V_3) \in \mathbb{R}^3 : V_2 \neq 0\}$, where (V_1, V_2, V_3) are the standard coordinates on the Euclidean space. Then M being an open subset is a 3-dimensional smooth manifold. The vector fields as defined bellow:

$$e_1 = e^{2V_3} \frac{\partial}{\partial V_1}, \quad e_2 = e^{2V_3} \frac{\partial}{\partial V_2}, \quad e_3 = \frac{\partial}{\partial V_3}$$

are orthogonal with respect to the Riemannian metric g is defined by:

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } i,j \in \{1,2,3\} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\zeta = e_3$. Then the 1-form η is defined by $\eta(V_3) = g(V_3, e_3)$, for arbitrary $V_3 \in \chi(M)$, then we have the following relations:

$$\eta(e_1) = \eta(e_2) = 0, \quad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field $\mathcal{F}$ as

$$\mathcal{F}e_1 = e_2, \quad \mathcal{F}e_2 = -e_1, \quad \mathcal{F}e_3 = 0,$$

then it satisfies,

$$\mathcal{F}^2(V_3) = -V_3 + \eta(V_3)e_3,$$
$$g(\mathcal{F}V_3, \mathcal{F}V_4) = g(V_3, V_4) - \eta(V_3)\eta(V_4)$$

for arbitrary $V_3, V_4 \in \chi(M)$.

Thus $(\mathcal{F}, \zeta, \eta, g)$ defines an almost contact metric structure on M. We can now easily conclude:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -2e_2, \quad [e_1, e_3] = -2e_1.$$

Let ∇ be the Levi-Civita connection of g. Then from *Koszul′sformula* for arbitrary $V_1, V_2, V_3 \in \chi(M)$ given by:

$$2g(\nabla_{V_1} V_2, V_3) = V_1 g(V_2, V_3) + V_2 g(V_3, V_1) - V_3 g(V_1, V_2) - g(V_1, [V_2, V_3]) \\ -g(V_2, [V_1, V_3]) + g(V_3, [V_1, V_2]),$$

we can have:

$$\nabla_{e_1} e_1 = 2e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -2e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = 2e_3, \quad \nabla_{e_2} e_3 = -2e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From here we can easily verify that the relations (10) and (11) are satisfied. Hence the considered manifold is trans-Sasakian manifold of type $(0, -2)$. The components of Riemannian curvature tensor are given by

$$R(e_1, e_2)e_1 = -4e_3, \quad R(e_1, e_2)e_2 = -4e_1, \quad R(e_1, e_2)e_3 = 0,$$
$$R(e_1, e_3)e_1 = 4e_2, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = -4e_1,$$
$$R(e_2, e_3)e_1 = 0, \quad R(e_2, e_3)e_2 = -4e_2, \quad R(e_2, e_3)e_3 = -4e_2.$$

And the components of Ricci tensor and $*$-Ricci tensor are given by:

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = -8.$$

From here, we can easily deduce that the scalar curvature of the manifold $r = \sum_{i=1}^{3} S(e_i, e_i) = -8$. Let us define a vector field by, $V = \zeta$. Then we can obtain:

$$(\mathcal{L}_V g)(e_1, e_1) = -4, \quad (\mathcal{L}_V g)(e_2, e_2) = -4, \quad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (2) and using the result $r = -8$, we deduce $\lambda = -\frac{(3p+2)}{6} - \frac{\mu}{3}$. Now using the identity (28), we get $\mu = 6$. Then $\lambda = -\left(\frac{p}{2} + \frac{7}{3}\right)$. The value of λ and μ satisfies the relation (38) and (39). So, g defines a conformal η-Einstein solitons on trans-Sasakian 3-manifold for $\lambda = -\left(\frac{p}{2} + \frac{7}{3}\right)$ and $\mu = 6$. Conformal η-Einstein soliton is shrinking i. e., $\lambda > 0$ if $\left(\frac{p}{2} + \frac{7}{3}\right) < 0$, steady i. e., $\lambda = 0$ if $\left(\frac{p}{2} + \frac{7}{3}\right) = 0$ and expanding i. e., $\lambda < 0$ if $\left(\frac{p}{2} + \frac{7}{3}\right) > 0$.

12 Geometrical and Physical Motivations

The study of conformal η-Einstein solitons on Riemannian manifolds and pseudo-Riemannian manifolds are the substantial momentousness in the area of differential geometry, especially in Riemannian geometry and in special relativistic physics as well. As an application to relativity there are some physical models of perfect fluids conformal η-Einstein soliton space times which generates a curvature inheritance symmetry. Here, we can find some physical and geometrical models of perfect conformal η-Einstein solitons space time and that will give the physical significance, the concept of conformal η-Einstein solitons.

The mathematical notion of an almost conformal η-Einstein soliton should not be confused with the notion of soliton solutions, which arise in several areas of mathematical or theoretical physics and its applications. It expresses a geometric and physical applications with relativistic viscous fluid spacetime admitting heat flux and stress, dark and dust fluid general relativistic spacetime, radiation era in general relativistic spacetime. Conformal Einstein solitons and conformal η-Einstein solitons have the applications in the renormalization group (RG) flow of a nonlinear sigma model [60]. We can review the concept of a conformal η-Einstein solitons to discuss the RG flow of mass in 2-dimensions. General relativistic spacetime models are of considerable interest in several areas of astrophysics [8, 12], plasma physics [3], string theory and nuclear physics [45].

As an application to cosmology and general relativity by investigating the kinetic and potential nature of relativistic space time, we present a physical models of 3-class namely, shrinking, steady and expanding of perfect and dust fluid solutions of conformal η-Einstein solitons space time. The first case shrinking ($\lambda < 0$) which exists on a minimal time interval $-1 < t < b$ where $b < 1$, steady ($\lambda = 0$) that exists for all time or expanding ($\lambda > 0$) which exists on maximal time interval $a < t < 1, a > -1$. These three classes give an example of ancient, eternal and immortal solutions, respectively.

13 Conclusion

In this article, we have used the methods of local Riemannian or semi-Riemannian geometry to interpretation solutions of (2) and impregnate Einstein metrics in a large class of metrics of conformal η-Einstein solitons on contact geometry, specially on trans-Sasakian manifolds. Besides, our results creates a requisite and persuasion mantle in the field of differential geometry. Also, conformal η-Einstein solitons has a significant and motivational contribution in the area of mathematical physics, general relativity and quantum cosmology, further research of complex geometry. The physical characteristics and motivational contribution of conformal η-Einstein solitons can be thought from references [19, 60]. There are some questions arise from our article to study further research.

1. Is the Theorem 5 true without assuming the vector field V is collinear with ξ?
2. Is the Theorem 16 true if we consider the manifold is not Einstein semi-symmetric?
3. If we consider the vector field is not torse-forming, then whether Theorem 19 is true?
4. Whether the results of the this paper are also true for nearly Kenmotsu, paracontact manifolds and f-cosymplectic manifolds?

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