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A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions

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Abstract

In this paper, we introduce a new concept of coupled non-separated boundary conditions and solve a coupled system of fractional differential equations supplemented with these conditions. The existence results obtained in the given configuration are not only new but also yield some new special results corresponding to particular values of the parameters involved in the problem. For the illustration of the existence and uniqueness result, an example is constructed.

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1 Introduction

Fractional differential equations have gained considerable importance due to their varied applications in many problems of physics, chemistry, biology, applied sciences and engineering. The tools of fractional calculus are found to be of great support in developing a more realistic mathematical modeling of the applied problems in terms of fractional differential equations. Fractional-order models are regarded as better than the classical ones (based on differential equations) as fractional derivatives can take care of the hereditary properties of materials and processes involved in the problem at hand. For details and explanations, we refer the reader to the texts [1–4]. In particular, a great interest has been shown by many authors in the subject of fractional-order boundary value problems (BVPs), and a variety of results for BVPs equipped with different kinds of boundary conditions have been obtained, for instance, see [5–13] and the references cited therein.

Coupled systems of fractional-order differential equations constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [14], disease models [15–17], ecological models [18], synchronization of chaotic systems [19–21], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [22–27].

In this paper, we consider a new boundary value problem of coupled Caputo type fractional differential equations:

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x(t), y(t)), & t \in [0, T], 1 < \alpha \leq 2 \\ {}^c D^\beta y(t) = g(t, x(t), y(t)), & t \in [0, T], 1 < \beta \leq 2, \end{cases} \quad (1.1)$$

subject to the following non-separated coupled boundary conditions:

$$\begin{cases} x(0) = \lambda_1 y(T), & x'(0) = \lambda_2 y'(T), \\ y(0) = \mu_1 x(T), & y'(0) = \mu_2 x'(T), \end{cases} \tag{1.2}$$

where ${}^c D^\alpha, {}^c D^\beta$ denote the Caputo fractional derivatives of order α and β , respectively, $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are appropriately chosen functions, and $\lambda_i, \mu_i, i = 1, 2$, are real constants with $\lambda_i \mu_i \neq 1, i = 1, 2$.

Here we emphasize that our problem is new in the sense of non-separated coupled boundary conditions introduced here. To the best of our knowledge, fractional-order coupled system (1.1) has yet to be studied with the boundary conditions (1.2). In consequence, our findings of the present work will be a useful contribution to the existing literature on the topic. The existence and uniqueness results for the given problem are new, though they are proved by applying the well-known method based on Banach’s contraction principle and Leray-Schauder’s alternative.

The rest of the contents of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a pivotal role in obtaining the main results presented in Section 3. We also discuss an example for illustration of the existence-uniqueness result. The paper concludes with some interesting observations.

2 Preliminaries

First of all, we recall some basic definitions of fractional calculus.

Definition 2.1 The fractional integral of order r with the lower limit zero for a function f is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, r > 0,$$

provided the right-hand side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $r > 0, n - 1 < r < n, n \in \mathbb{N}$, is defined as

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-r-1} f(s) ds,$$

where the function $f(t)$ has an absolutely continuous derivative up to order $(n - 1)$.

Definition 2.3 The Caputo derivative of order r for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^c D^r f(t) = D_{0+}^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n - 1 < r < n.$$

Remark 2.4 If $f(t) \in C^n[0, \infty)$, then

$${}^c D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t), \quad t > 0, n-1 < r < n.$$

Now we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.5 *Let $\phi, h \in C([0, T], \mathbb{R})$ and $\lambda_i \mu_i \neq 1, i = 1, 2$. Then the solution of the linear fractional differential system*

$$\begin{cases} {}^c D^\alpha x(t) = \phi(t), & t \in [0, T], 1 < \alpha \leq 2, \\ {}^c D^\beta y(t) = h(t), & t \in [0, T], 1 < \beta \leq 2, \\ x(0) = \lambda_1 y(T), & x'(0) = \lambda_2 y'(T), \\ y(0) = \mu_1 x(T), & y'(0) = \mu_2 x'(T), \end{cases} \tag{2.1}$$

is equivalent to the system of integral equations

$$\begin{aligned} x(t) = & \frac{\mu_2}{1-\lambda_2\mu_2} \left(\frac{\lambda_1 T(\mu_1\lambda_2+1)}{1-\lambda_1\mu_1} + \lambda_2 t \right) B_2 + \frac{\lambda_2}{1-\lambda_2\mu_2} \left(\frac{T(\mu_1+\mu_2)\lambda_1}{1-\lambda_1\mu_1} + t \right) A_2 \\ & + \frac{\lambda_1}{1-\lambda_1\mu_1} (A_1 + \mu_1 B_1) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} y(t) = & \frac{\mu_2}{1-\lambda_2\mu_2} \left(\frac{T\mu_1(\lambda_1+\lambda_2)}{1-\lambda_1\mu_1} + t \right) B_2 + \frac{\lambda_2}{1-\lambda_2\mu_2} \left(\frac{T\mu_1(\lambda_1\mu_2+1)}{1-\lambda_1\mu_1} + \mu_2 t \right) A_2 \\ & + \frac{\mu_1}{1-\lambda_1\mu_1} (\lambda_1 A_1 + B_1) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} A_1 &= \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds, & B_1 &= \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds, \\ A_2 &= \int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} h(s) ds, & B_2 &= \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi(s) ds. \end{aligned}$$

Proof We know that the general solution of fractional differential equations in (2.1) can be written as

$$x(t) = a_0 + a_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds, \tag{2.4}$$

$$y(t) = b_0 + b_1 t + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds, \tag{2.5}$$

where $a_i, b_i, i = 0, 1$, are arbitrary real constants.

Using the boundary conditions in (2.4) and (2.5), we have

$$\begin{aligned} x(0) = \lambda_1 y(T) & \Rightarrow a_0 = \lambda_1 (b_0 + b_1 T + A_1), \\ y(0) = \mu_1 x(T) & \Rightarrow b_0 = \mu_1 (a_0 + a_1 T + B_1), \end{aligned}$$

$$x'(0) = \lambda_2 y'(T) \implies a_1 = \lambda_2 (b_1 + A_2),$$

$$y'(0) = \mu_2 x'(T) \implies b_1 = \mu_2 (a_1 + B_2).$$

From the last two relations we find

$$a_1 = \frac{\lambda_2}{1 - \lambda_2 \mu_2} (\mu_2 B_2 + A_2) \quad \text{and} \quad b_1 = \frac{\mu_2}{1 - \lambda_2 \mu_2} (B_2 + \lambda_2 A_2).$$

Substituting a_1 and b_1 in the first two relations, we find

$$\begin{aligned} a_0 &= \frac{\lambda_1}{1 - \lambda_1 \mu_1} [\mu_1 T a_1 + \mu_1 B_1 + T b_1 + A_1] \\ &= \frac{\lambda_1}{1 - \lambda_1 \mu_1} \left[\frac{\mu_1 T \lambda_2}{1 - \lambda_2 \mu_2} (\mu_2 B_2 + A_2) + \mu_1 B_{1f} + A_1 + \frac{T \mu_2}{1 - \lambda_2 \mu_2} (B_2 + \lambda_2 A_2) \right] \\ &= \frac{\lambda_1}{1 - \lambda_1 \mu_1} \left[\frac{T \mu_2 (\mu_1 \lambda_2 + 1)}{1 - \lambda_2 \mu_2} B_2 + \frac{T \lambda_2 (\mu_1 + \mu_2)}{1 - \lambda_2 \mu_2} A_2 + A_1 + \mu_1 B_1 \right], \end{aligned}$$

and

$$\begin{aligned} b_0 &= \mu_1 a_0 + \mu_1 a_1 T + \mu_1 B_1 \\ &= \frac{\lambda_1 \mu_1}{1 - \lambda_1 \mu_1} \left[\frac{T \mu_2 (\mu_1 \lambda_2 + 1)}{1 - \lambda_2 \mu_2} B_2 + \frac{T \lambda_2 (\mu_1 + \mu_2)}{1 - \lambda_2 \mu_2} A_2 + A_1 + \mu_1 B_1 \right] \\ &\quad + \frac{\mu_1 T \lambda_2}{1 - \lambda_2 \mu_2} [\mu_2 B_2 + A_2] + \mu_1 B_1 \\ &= \frac{T \mu_1 \mu_2}{1 - \lambda_2 \mu_2} \left(\frac{\lambda_1 + \lambda_2}{1 - \lambda_1 \mu_1} \right) B_2 + \frac{T \mu_1 \lambda_2}{1 - \lambda_2 \mu_2} \left(\frac{\lambda_1 \mu_2 + 1}{1 - \lambda_1 \mu_1} \right) A_2 \\ &\quad + \frac{\lambda_1 \mu_1}{1 - \lambda_1 \mu_1} A_1 + \frac{\mu_1}{1 - \lambda_1 \mu_1} B_1. \end{aligned}$$

Inserting the values of $a_i, b_i, i = 0, 1$, in (2.4) and (2.5), we get solutions (2.2) and (2.3). The converse follows by direct computation. This completes the proof. \square

3 Main results

Let us introduce the space $X = \{u(t) | u(t) \in C([0, T], \mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in [0, T]\}$. Obviously, $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(u, v)\|)$ is also a Banach space equipped with the norm $\|(u, v)\| = \|u\| + \|v\|$.

In view of Lemma 2.5, we define the operator $T : X \times X \rightarrow X \times X$ by

$$T(u, v)(t) = \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix},$$

where

$$\begin{aligned} T_1(u, v)(t) &= \frac{\mu_2}{1 - \lambda_2 \mu_2} \left(\frac{\lambda_1 T (\mu_1 \lambda_2 + 1)}{1 - \lambda_1 \mu_1} + \lambda_2 t \right) B_{2f} + \frac{\lambda_2}{1 - \lambda_2 \mu_2} \left(\frac{T (\mu_1 + \mu_2) \lambda_1}{1 - \lambda_1 \mu_1} + t \right) A_{2g} \\ &\quad + \frac{\lambda_1}{1 - \lambda_1 \mu_1} (A_{1g} + \mu_1 B_{1f}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), y(s)) ds, \end{aligned}$$

and

$$\begin{aligned}
 T_2(u, v)(t) &= \frac{\mu_2}{1 - \lambda_2\mu_2} \left(\frac{T\mu_1(\lambda_1 + \lambda_2)}{1 - \lambda_1\mu_1} + t \right) B_{2f} + \frac{\lambda_2}{1 - \lambda_2\mu_2} \left(\frac{T\mu_1(\lambda_1\mu_2 + 1)}{1 - \lambda_1\mu_1} + \mu_2 t \right) A_{2g} \\
 &\quad + \frac{\mu_1}{1 - \lambda_1\mu_1} (\lambda_1 A_{1g} + B_{1f}) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x(s), y(s)) \, ds, \\
 A_{1g} &= \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s, x(s), y(s)) \, ds, \quad B_{1f} = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), y(s)) \, ds, \\
 A_{2g} &= \int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(s, x(s), y(s)) \, ds, \quad B_{2f} = \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s), y(s)) \, ds.
 \end{aligned}$$

For convenience, we put

$$\begin{aligned}
 M_1 &= \frac{|\mu_2|}{|1 - \lambda_2\mu_2|} \left(\frac{|\lambda_1|(|\mu_1|\lambda_2 + 1)}{|1 - \lambda_1\mu_1|} + |\lambda_2| \right) \frac{T^\alpha}{\Gamma(\alpha)} \\
 &\quad + \left(\frac{|\lambda_1||\mu_1|}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\alpha}{\Gamma(\alpha + 1)}, \tag{3.1}
 \end{aligned}$$

$$M_2 = \frac{|\lambda_2|}{|1 - \lambda_2\mu_2|} \left(\frac{(|\mu_1| + |\mu_2|)|\lambda_1|}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\beta}{\Gamma(\beta)} + \frac{|\lambda_1|}{|1 - \lambda_1\mu_1|} \frac{T^\beta}{\Gamma(\beta + 1)}, \tag{3.2}$$

$$M_3 = \frac{|\mu_2|}{|1 - \lambda_2\mu_2|} \left(\frac{|\mu_1|(|\lambda_1| + |\lambda_2|)}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\alpha}{\Gamma(\alpha)} + \frac{|\mu_1|}{|1 - \lambda_1\mu_1|} \frac{T^\alpha}{\Gamma(\alpha + 1)}, \tag{3.3}$$

$$\begin{aligned}
 M_4 &= \frac{|\lambda_2|}{|1 - \lambda_2\mu_2|} \left(\frac{|\mu_1|(|\lambda_1|\mu_2 + 1)}{|1 - \lambda_1\mu_1|} + |\mu_2| \right) \frac{T^\beta}{\Gamma(\beta)} \\
 &\quad + \left(\frac{|\lambda_1||\mu_1|}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\beta}{\Gamma(\beta + 1)}. \tag{3.4}
 \end{aligned}$$

In the first result, we prove the existence and uniqueness of solutions of boundary value problem (1.1)-(1.2) via Banach’s contraction principle.

Theorem 3.1 *Assume that:*

(H₁) *f, g : [0, T] × ℝ × ℝ → ℝ are continuous functions and there exist positive constants ℓ₁ and ℓ₂ such that for all t ∈ [0, T] and x_i, y_i ∈ ℝ, i = 1, 2, we have*

$$\begin{aligned}
 |f(t, x_1, x_2) - f(t, y_1, y_2)| &\leq \ell_1 (|x_1 - y_1| + |x_2 - y_2|), \\
 |g(t, x_1, x_2) - g(t, y_1, y_2)| &\leq \ell_2 (|x_1 - y_1| + |x_2 - y_2|).
 \end{aligned}$$

If

$$(M_1 + M_3)\ell_1 + (M_2 + M_4)\ell_2 < 1,$$

where M_i, i = 1, 2, 3, 4, are given by (3.1)-(3.4), then system (1.1)-(1.2) has a unique solution.

Proof Define $\sup_{t \in [0, T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t \in [0, T]} g(t, 0, 0) = N_2 < \infty$ and $r > 0$ such that

$$r > \frac{(M_1 + M_3)N_1 + (M_2 + M_4)N_2}{1 - (M_1 + M_3)\ell_1 - (M_2 + M_4)\ell_2}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(u, v) \in X \times X : \|(u, v)\| \leq r\}$.

By assumption (H_1) , for $(u, v) \in B_r, t \in [0, T]$, we have

$$\begin{aligned} |f(t, u(t), v(t))| &\leq |f(t, u(t), v(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \ell_1(|u(t)| + |v(t)|) + N_1 \\ &\leq \ell_1(\|u\| + \|v\|) + N_1 \leq \ell_1 r + N_1, \end{aligned}$$

and

$$|g(t, u(t), v(t))| \leq \ell_2(\|u\| + \|v\|) + N_2 \leq \ell_2 r + N_2,$$

which lead to

$$\begin{aligned} |T_1(u, v)(t)| &\leq \frac{|\mu_2|}{|1 - \lambda_2 \mu_2|} \left(\frac{|\lambda_1|(|\mu_1| |\lambda_2| + 1)}{|1 - \lambda_1 \mu_1|} + |\lambda_2| \right) \frac{T^\alpha}{\Gamma(\alpha)} (\ell_1 r + N_1) \\ &\quad + \frac{|\lambda_2|}{|1 - \lambda_2 \mu_2|} \left(\frac{(|\mu_1| + |\mu_2|)|\lambda_1|}{|1 - \lambda_1 \mu_1|} + 1 \right) \frac{T^\beta}{\Gamma(\beta)} (\ell_2 r + N_2) \\ &\quad + \frac{|\lambda_1|}{|1 - \lambda_1 \mu_1|} \left(\frac{T^\beta}{\Gamma(\beta + 1)} (\ell_2 r + N_2) + |\mu_1| \frac{T^\alpha}{\Gamma(\alpha + 1)} (\ell_1 r + N_1) \right) \\ &\quad + \frac{T^\alpha}{\Gamma(\alpha + 1)} (\ell_1 r + N_1) \\ &= (M_1 \ell_1 + M_2 \ell_2)r + M_1 N_1 + M_2 N_2. \end{aligned}$$

Hence

$$\|T_1(u, v)\| \leq (M_1 \ell_1 + M_2 \ell_2)r + M_1 N_1 + M_2 N_2.$$

In the same way, we can obtain that

$$\|T_2(u, v)\| \leq (M_3 \ell_1 + M_4 \ell_2)r + M_3 N_1 + M_4 N_2.$$

Consequently,

$$\|T(u, v)\| \leq [(M_1 + M_3)\ell_1 + (M_2 + M_4)\ell_2]r + (M_1 + M_3)N_1 + (M_2 + M_4)N_2 \leq r.$$

Now, for $(u_2, v_2), (u_1, v_1) \in X \times X$ and for any $t \in [0, T]$, we get

$$\begin{aligned} &|T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| \\ &\leq \frac{|\mu_2|}{|1 - \lambda_2 \mu_2|} \left(\frac{|\lambda_1|(|\mu_1| |\lambda_2| + 1)}{|1 - \lambda_1 \mu_1|} + |\lambda_2| \right) \frac{T^\alpha}{\Gamma(\alpha)} \ell_1 (\|u_2 - u_1\| + \|v_2 - v_1\|) \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\lambda_2|}{|1 - \lambda_2\mu_2|} \left(\frac{(|\mu_1| + |\mu_2|)|\lambda_1|}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\beta}{\Gamma(\beta)} \ell_2 (\|u_2 - u_1\| + \|v_2 - v_1\|) \\
 & + \frac{|\lambda_1|}{|1 - \lambda_1\mu_1|} \frac{T^\beta}{\Gamma(\beta + 1)} \ell_2 (\|u_2 - u_1\| + \|v_2 - v_1\|) \\
 & + \frac{|\lambda_1||\mu_1|}{|1 - \lambda_1\mu_1|} \frac{T^\alpha}{\Gamma(\alpha + 1)} \ell_1 (\|u_2 - u_1\| + \|v_2 - v_1\|) \\
 & + \frac{T^\alpha}{\Gamma(\alpha + 1)} \ell_1 (\|u_2 - u_1\| + \|v_2 - v_1\|) \\
 & = (M_1\ell_1 + M_2\ell_2) (\|u_2 - u_1\| + \|v_2 - v_1\|),
 \end{aligned}$$

and consequently we obtain

$$\|T_1(u_2, v_2) - T_1(u_1, v_1)\| \leq (M_1\ell_1 + M_2\ell_2) (\|u_2 - u_1\| + \|v_2 - v_1\|). \tag{3.5}$$

Similarly,

$$\|T_2(u_2, v_2)(t) - T_2(u_1, v_1)\| \leq (M_3\ell_1 + M_4\ell_2) (\|u_2 - u_1\| + \|v_2 - v_1\|). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\|T(u_2, v_2) - T(u_1, v_1)\| \leq [(M_1 + M_3)\ell_1 + (M_2 + M_4)\ell_2] (\|u_2 - u_1\| + \|v_2 - v_1\|).$$

Since $(M_1 + M_3)\ell_1 + (M_2 + M_4)\ell_2 < 1$, therefore, T is a contraction operator. So, by Banach’s fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1)-(1.2). This completes the proof. \square

The second result is based on the Leray-Schauder alternative.

Lemma 3.2 (Leray-Schauder alternative [28], p.4) *Let $F : E \rightarrow E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded or F has at least one fixed point.

Theorem 3.3 *Assume that:*

(H₃) $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and there exist real constants $k_i, \gamma_i \geq 0$ ($i = 0, 1, 2$) and $k_0 > 0, \gamma_0 > 0$ such that $\forall x_i \in \mathbb{R}$ ($i = 1, 2$), we have

$$\begin{aligned}
 |f(t, x_1, x_2)| & \leq k_0 + k_1|x_1| + k_2|x_2|, \\
 |g(t, x_1, x_2)| & \leq \gamma_0 + \gamma_1|x_1| + \gamma_2|x_2|.
 \end{aligned}$$

If

$$(M_1 + M_3)k_1 + (M_2 + M_4)\gamma_1 < 1 \quad \text{and} \quad (M_1 + M_3)k_2 + (M_2 + M_4)\gamma_2 < 1,$$

where $M_i, i = 1, 2, 3, 4$, are given by (3.1)-(3.4), then system (1.1)-(1.2) has at least one solution.

Proof First we show that the operator $T : X \times X \rightarrow X \times X$ is completely continuous. By the continuity of functions f and g , the operator T is continuous.

Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \quad |h(t, u(t), v(t))| \leq L_2, \quad \forall (u, v) \in \Omega.$$

Then, for any $(u, v) \in \Omega$, we have

$$\begin{aligned} |T_1(u, v)(t)| &\leq \frac{|\mu_2|}{|1 - \lambda_2\mu_2|} \left(\frac{|\lambda_1|(|\mu_1|\lambda_2| + 1)}{|1 - \lambda_1\mu_1|} + |\lambda_2| \right) \frac{T^\alpha}{\Gamma(\alpha)} L_1 \\ &\quad + \frac{|\lambda_2|}{|1 - \lambda_2\mu_2|} \left(\frac{(|\mu_1| + |\mu_2|)|\lambda_1|}{|1 - \lambda_1\mu_1|} + 1 \right) \frac{T^\beta}{\Gamma(\beta)} L_2 \\ &\quad + \frac{|\lambda_1|}{|1 - \lambda_1\mu_1|} \left(\frac{T^\beta}{\Gamma(\beta + 1)} L_2 + |\mu_1| \frac{T^\alpha}{\Gamma(\alpha + 1)} L_1 \right) + \frac{T^\alpha}{\Gamma(\alpha + 1)} L_1 \\ &= M_1 L_1 + M_2 L_2, \end{aligned}$$

which implies that

$$\|T_1(u, v)\| \leq M_1 L_1 + M_2 L_2.$$

Similarly, we get

$$\|T_2(u, v)\| \leq M_3 L_1 + M_4 L_2.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded, since $\|T(u, v)\| \leq (M_1 + M_3)L_1 + (M_2 + M_4)L_2$.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned} &|T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))| \\ &\leq \frac{L_1|\lambda_2||\mu_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) + \frac{L_2|\lambda_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) \\ &\quad + L_1 \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \right| \\ &\leq \frac{L_1|\lambda_2||\mu_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) + \frac{L_2|\lambda_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) \\ &\quad + L_1 \left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right\} \\ &\leq \frac{L_1|\lambda_2||\mu_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) + \frac{L_2|\lambda_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) \\ &\quad + \frac{L_1}{\Gamma(\alpha + 1)} [2(t_2 - t_1)^\alpha + |t_2^\alpha - t_1^\alpha|]. \end{aligned}$$

Analogously, we can obtain

$$\begin{aligned} &|T_2(u(t_2), v(t_2)) - T_2(u(t_1), v(t_1))| \\ &\leq \frac{L_1|\mu_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) + \frac{L_2|\lambda_2||\mu_2|}{|1 - \lambda_2\mu_2|}(t_2 - t_1) + \frac{L_2}{\Gamma(\beta + 1)}[2(t_2 - t_1)^\beta + |t_2^\beta - t_1^\beta|]. \end{aligned}$$

Therefore, the operator $T(u, v)$ is equicontinuous, and thus the operator $T(u, v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(u, v) \in X \times X | (u, v) = \lambda T(u, v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u, v) \in \mathcal{E}$, with $(u, v) = \lambda T(u, v)$. For any $t \in [0, T]$, we have

$$u(t) = \lambda T_1(u, v)(t), \quad v(t) = \lambda T_2(u, v)(t).$$

Then

$$\begin{aligned} |u(t)| &\leq M_1(k_0 + k_1|u| + k_2|v|) + M_2(\gamma_0 + \gamma_1|u| + \gamma_2|v|) \\ &= M_1k_0 + M_2\gamma_0 + (M_1k_1 + M_2\gamma_1)|u| + (M_1k_2 + M_2\gamma_2)|v| \end{aligned}$$

and

$$\begin{aligned} |v(t)| &\leq M_3(k_0 + k_1|u| + k_2|v|) + M_4(\gamma_0 + \gamma_1|u| + \gamma_2|v|) \\ &= M_3k_0 + M_4\gamma_0 + (M_3k_1 + M_4\gamma_1)|u| + (M_3k_2 + M_4\gamma_2)|v|. \end{aligned}$$

Hence we have

$$\|u\| \leq M_1k_0 + M_2\gamma_0 + (M_1k_1 + M_2\gamma_1)\|u\| + (M_1k_2 + M_2\gamma_2)\|v\|$$

and

$$\|v\| \leq M_3k_0 + M_4\gamma_0 + (M_3k_1 + M_4\gamma_1)\|u\| + (M_3k_2 + M_4\gamma_2)\|v\|,$$

which imply that

$$\begin{aligned} \|u\| + \|v\| &\leq (M_1 + M_3)k_0 + (M_2 + M_4)\gamma_0 + [(M_1 + M_3)k_1 + (M_2 + M_4)\gamma_1]\|u\| \\ &\quad + [(M_1 + M_3)k_2 + (M_2 + M_4)\gamma_2]\|v\|. \end{aligned}$$

Consequently,

$$\|(u, v)\| \leq \frac{(M_1 + M_3)k_0 + (M_2 + M_4)\gamma_0}{M_0},$$

where $M_0 = \min\{1 - [(M_1 + M_3)k_1 + (M_2 + M_4)\gamma_1], 1 - [(M_1 + M_3)k_2 + (M_2 + M_4)\gamma_2]\}$, which proves that \mathcal{E} is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence boundary value problem (1.1)-(1.2) has at least one solution. The proof is complete. □

Example 3.4 Consider the following system of fractional boundary value problem:

$$\begin{cases} {}^c D^{3/2} x(t) = \frac{1}{4(t+2)^2} \frac{|x(t)|}{1+|x(t)|} + 1 + \frac{1}{32} \sin^2 y(t) + \frac{1}{\sqrt{t^2+1}}, & t \in [0, 1], \\ {}^c D^{3/2} y(t) = \frac{1}{32\pi} \sin(2\pi x(t)) + \frac{|y(t)|}{16(1+|y(t)|)} + \frac{1}{2}, & t \in [0, 1], \\ x(0) = \frac{1}{2}y(1), & x'(0) = \frac{1}{3}y'(1), \\ y(0) = \frac{4}{5}x(1), & y'(0) = \frac{3}{4}x'(1). \end{cases} \tag{3.7}$$

Here $\alpha = \beta = 3/2$, $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, $\mu_1 = 4/5$, $\mu_2 = 3/4$, $f(t, u, v) = \frac{1}{4(t+2)^2} \frac{|u|}{1+|u|} + 1 + \frac{1}{32} \sin^2 v$, and $h(t, u, v) = \frac{1}{32\pi} \sin(2\pi u) + \frac{|v|}{16(1+|v|)} + \frac{1}{2}$. With the given data, we find that $M_1 \approx 2.8216701$, $M_2 \approx 1.4892147$, $M_3 \approx 3.3860044$, $M_4 \approx 2.5499538$. Note that $|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \frac{1}{16} |u_1 - u_2| + \frac{1}{16} |v_1 - v_2|$, $|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \frac{1}{16} |u_1 - u_2| + \frac{1}{16} |v_1 - v_2|$, and $(M_1 + M_3)\ell_1 + (M_2 + M_4)\ell_2 \approx 0.6404276 < 1$. Thus all the conditions of Theorem 3.1 are satisfied, and consequently, its conclusion applies to problem (3.7).

4 Conclusions

In this work, we have established the existence and uniqueness results for a nonlinear coupled system of Caputo type fractional differential equations supplemented with non-separated coupled boundary conditions. Our results are not only new in the given setting but also lead to some new interesting situations for specific values of the parameters involved in the problem. For instance, if we choose $\lambda_1 = 1 = \lambda_2$ and $\mu_1 = -1 = \mu_2$ or vice versa, our results correspond to a boundary value problem of nonlinear coupled fractional differential equations subject to a combination of coupled periodic and anti-periodic boundary conditions of the form: $x(0) = y(T)$, $x'(0) = y'(T)$, $y(0) = -x(T)$, $y'(0) = -x'(T)$ or $x(0) = -y(T)$, $x'(0) = -y'(T)$, $y(0) = x(T)$, $y'(0) = x'(T)$. Further, by taking $\lambda_1 = 1 = -\lambda_2$ and $-\mu_1 = 1 = \mu_2$, we obtain the results for the coupled system with the boundary conditions: $x(0) = y(T)$, $x'(0) = -y'(T)$, $y(0) = -x(T)$, $y'(0) = x'(T)$, while the results for the boundary conditions $x(0) = -y(T)$, $x'(0) = y'(T)$, $y(0) = x(T)$, $y'(0) = -x'(T)$ can be obtained by letting $-\lambda_1 = 1 = \lambda_2$ and $\mu_1 = 1 = -\mu_2$. In the case of $x(0) = 0$, $y(0) = 0$ ($\lambda_1 = 0 = \mu_1$), $x'(0) = \lambda_2 y'(T)$, $y'(0) = \mu_2 x'(T)$, our results correspond to a problem with coupled flux type conditions. Thus, the work established in this article is of quite a general nature and covers a variety of special cases associated with particular values of the parameters involved in the problem.

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The research has been carried out within an appropriate ethical framework and each of the authors, HHA, SKN, RPA, BA and AA, participated in this research work.

Competing interests

The authors declare that they have no competing interests.

Consent for publication

Each of the authors, HHA, SKN, RPA, BA and AA, approved the final version of the manuscript for publication.

Authors' contributions

Each of the authors, HHA, SKN, RPA, BA and AA, contributed to each part of this work equally. All authors read and approved the final manuscript.

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