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A STUDY OF I-FUNCTIONS OF TWO VARIABLES

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In our present investigation we propose to study and develop the I-function of two variables analogous to the I-function of one variable introduced and studied by one of the authors [30]. The conditions for convergence, series representation, behaviour for small values, elementary properties, transformation formulas and some special cases for the I-function of two variables are also discussed.

1. Introduction

The well known H-function of one variable defined by Fox [13] was in fact first introduced by Pincherle in 1888 [12, section 1.19], but when Fox investigated and proved the H-function as a symmetric Fourier kernel to Meijers's G-function [12, p.206-222], the interest of most researchers, mathematicians and statisticians in this function has increased. By this fact, the H-function is often called Fox's H-function. Later on, in 1964, Braaksma [6] studied and developed this function in a resonable good manner by finding its asymptotic behaviour etc. Moreover, the H-function is recognized as a generalization of both Meijer's G-function and Wright's generalized hypergeometric functions [12, p.183]. Numerous research papers by the researchers, mathematicians and statisticians related to Fox's H-function, its properties and applications have been appeared in

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the literature. For more information about the results, we refer [21, 23] to the readers.

Recently good deal of progress has been done in the direction of generalizing the Fox's H-function and Meijer's G-function. Frankly speaking, the well known H-function of one variable, introduced by Fox [13] and studied by Braaksma [6] contains as particular cases most of the commonly used special functions of applied mathematics, but it does not contain some of the important functions such as the Riemann zeta functions, Polylogarithms etc. By demonstrating several examples of functions which are not included in the Fox's H-function, in 1997, Rathie [30] introduced a new function in the literature namely the "I-function" which is useful in Mathematics, Physics and other branches of Applied Mathematics. The newly defined function contains the polylogarithms, the exact partition of Gaussian free energy model from statistical mechanics, Feynmann integrals and functions useful in testing hypothesis from statistics as special cases.

Several authors like Chaurasia [10], Devra [11], Rathie [30], Nambisan and Roshina [32], Vyas and Rathie [38, 39], Agarwal and Shilpi Jain [28] have obtained several interesting results involving I-function of one variable and related functions ($\overline{\text{H}}$-function [20], H-function [13], G-function [24] etc.).

Very recently, the I-function introduced by Rathie [30] has found useful applications in *wireless communications*. It is not out of place to mention here that, in two of papers, Ansari, et. al. [2, 3] have successfully developed an efficient Mathematica$^{@}$ implementation of the I-function, in order to give numerical results of their research. In one of their useful and very interesting papers, Ansari, et.al. [3] derived novel closed form expressions for the PDF and CDF of the sum of independently but not identically distributed (i.n.i.d.) gamma or equivalently squared Nakagami- m random variables in the case of both integer-order as well as non integer-order fading figure parameters. They have expressed their results in terms of Fox's H - function and Rathie's I-function.

The vast popularity and immense usefulness of generalized hypergeometric functions in one variable, inspired and stimulated a number of research workers to the study of hypergeometric functions involving two variables. Serious and significant study of the functions of two variables began with the introduction of F_1 , F_2 , F_3 and F_4 by Appell [4] and their confluent form by Humbert [19]. These functions were further generalized by Kampé de Fériet by means of a function popularly known as the Kampé de Fériet function.

Appell and Kampé de Fériet [4] studied the functions of two variables in detail and recorded all the useful and important results concerning these functions in their famous work. Later on Srivastava and Daoust [35] studied the Kampé de Fériet function in a more general form. The hypergeometric functions of two variables have attracted attention of eminent mathematicians. Thus W. N. Bailey [5], J. L. Burchhal and T. W. Chaundy [8, 9], A. Erdelyi [12] etc. have contributed a lot of their developement and progress.

Further generalization of these functions of two variables were introduced almost simultaneously by Sharma [33, 34] and Agarwal [1]. These two functions, infact are the same except in their notational representations. Agarwal has given a method of estimating this function for large values of the variable and has also obtained a formal pair of unsymmetrical Fourier kernals for it. While Sharma [33, 34] has discussed various solutions of the partial differential equations satisfied by his function. He has also evaluated a number of interesting integrals and established several properties of the function.

Since then a number of mathematicians, notably Pathak [27], Bora and Kalla [7], Verma [37] and several others have studied functions of two variables which are more general than the functions studied earlier by Agarwal and Sharma.

Finally in 1972, Mittal and Gupta [25] defined a generalized function of two variables popularly known as the H-function of two variables, which generalizes almost all the hypergeometric functions involving two variables mentioned above.

A detailed study of further generalization of the H-function of two variable due to Mittal and Gupta [25] were given in the standard book of Hai and Yakubovich [18]. It is also interesting to refer a book on Asymptotics and Mellin-Barnes integrals by Paris and Kaminiski [26], which describes the theory of Mellin-Barnes integrals and illustrates their power and usefulness in asymptotic analysis. The asymptotic behaviours of some classical special functions are also illustrated in this book.

Very recently, the hypergeometric function of two variables introduced by Agarwal [1] and Sharma [33, 34] have found interesting applications in *wireless communications*. For this in a paper by Xia, et.al [40] by considering dual-hop channel state information (CSI)-assisted amplify-and-forward (AF) relaying over Nakagami-m fading channels, the cumulative distribution function (CDF) of the end-to-end signal-to-noise ratio(SNR) is derived. In particular, when the fading shape factors m_1 and m_2 at consecutive hops take non-integer values, the bivariate H-function and G-function are exploited to obtain an exact analytical expression for the CDF.

The remainder of the paper will be organised as follows. In section 2, we shall define the generalization of I-function namely, I-function of two variables. In section 3, we give the notations and results used throughout this paper. In section 4, convergence conditions for this function have been derived. In section 5, we will obtain the series representations and behaviour of the function for small values of the variables. In section 6, we list special cases of our function giving relations with other functions available in the literature, including H-functions of two variables and G-functions of two variables. In section 7, we mention few important properties.

2. The I-function of two variables

The double Mellin Barnes type contour integral occuring in this paper will be referred to as the I-function of two variables throughout our present study and will be defined and represented in the following manner.

$$\begin{aligned} &\mathrm{I}[\mathrm{z}_1,\mathrm{z}_2] \\ &= \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle| \begin{array}{l} (a_j;\alpha_j,A_j;\xi_j)_{1,p_1} : (c_j,C_j;U_j)_{1,p_2};\,(e_j,E_j;P_j)_{1,p_3} \\ (b_j;\beta_j,B_j;\eta_j)_{1,q_1} : (d_j,D_j;V_j)_{1,q_2};\,(f_j,F_j;Q_j)_{1,q_3} \end{array}\right] \\ &= \frac{1}{(2\pi i)^2}\int_{\mathcal{L}_s}\int_{\mathcal{L}_t}\phi(s,t)\,\theta_1(s)\,\theta_2(t)\,z_1^s\,z_2^t\,ds\,dt \end{aligned} \tag{1}$$

where $\phi(s,t)$, $\theta_1(s)$, $\theta_2(t)$ are given by

$$\phi(s,t) = \frac{\prod_{j=1}^{n_1}\Gamma^{\xi_j}(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma^{\xi_j}(a_j-\alpha_j s-A_j t)\prod_{j=1}^{q_1}\Gamma^{\eta_j}(1-b_j+\beta_j s+B_j t)} \tag{2}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{n_2}\Gamma^{U_j}(1-c_j+C_j s)\prod_{j=1}^{m_2}\Gamma^{V_j}(d_j-D_j s)}{\prod_{j=n_2+1}^{p_2}\Gamma^{U_j}(c_j-C_j s)\prod_{j=m_2+1}^{q_2}\Gamma^{V_j}(1-d_j+D_j s)} \tag{3}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{n_3}\Gamma^{P_j}(1-e_j+E_j t)\prod_{j=1}^{m_3}\Gamma^{Q_j}(f_j-F_j t)}{\prod_{j=n_3+1}^{p_3}\Gamma^{P_j}(e_j-E_j t)\prod_{j=m_3+1}^{q_3}\Gamma^{Q_j}(1-f_j+F_j t)} \tag{4}$$

Also :

- $z_1 \neq 0$, $z_2 \neq 0$;
- $i = \sqrt{-1}$;
- an empty product is interpreted as unity ;

- the parameters $n_j, p_j, q_j (j = 1, 2, 3)$, $m_j (j = 2, 3)$ are nonnegative integers such that $0 \leq n_j \leq p_j (j = 1, 2, 3)$, $q_1 \geq 0$, $0 \leq m_j \leq q_j (j = 2, 3)$ (not all zero simulataneously);

- $\alpha_j, A_j (j = 1, \ldots, p_1)$, $\beta_j, B_j (j = 1, \ldots, q_1)$, $C_j (j = 1, \ldots, p_2)$, $D_j (j = 1, \ldots, q_2)$, $E_j (j = 1, \ldots, p_3)$, $F_j (j = 1, \ldots, q_3)$ are assumed to be positive quantities for standardisation purpose.

  However, the definition of I-function of two variables will have a meaning even if some of the quantities are zero or negative numbers. For these we may obtain corresponding transformation formulas which will be given in later section.

- $a_j (j = 1, \ldots, p_1)$, $b_j (j = 1, \ldots, q_1)$, $c_j (j = 1, \ldots, p_2)$, d_j $(j = 1, \ldots, q_2)$, $e_j (j = 1, \ldots, p_3)$ and $f_j (j = 1, \ldots, q_3)$ are complex numbers;

- The exponents $\xi_j (j = 1, \ldots, p)$, η_j $(j = 1, \ldots, q)$, U_j $(j = 1, \ldots, p_2)$, $V_j (j = 1, \ldots, q_2)$, $P_j (j = 1, \ldots, p_3)$, $Q_j (j = 1, \ldots, q_3)$ of various gamma functions involved in (2), (3) and (4) may take non-integer values.

- $\mathcal{L}_s$ and $\mathcal{L}_t$ are suitable contours of Mellin - Barnes type. Moreover, the contour $\mathcal{L}_s$ is in the complex s-plane and runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$, (σ_1 real) so that all the singularities of $\Gamma^{V_j} (d_j - D_j s)$ $(j = 1, \ldots, m_2)$ lie to the right of $\mathcal{L}_s$ and all the singularities of $\Gamma^{U_j} (1 - c_j + C_j s)$ $(j = 1, \ldots, n_2)$, $\Gamma^{\xi_j} (1 - a_j + \alpha_j s + A_j t) (j = 1, \ldots, n_1)$ lie to the left of $\mathcal{L}_s$;

- The contour $\mathcal{L}_t$ is in the complex t-plane and runs from $\sigma_2 - i\infty$ to $\sigma_2 + i\infty$, (σ_2 real) so that all the singularities of $\Gamma^{Q_j} (f_j - F_j t)$ $(j = 1, \ldots, m_3)$ lie to the right of $\mathcal{L}_t$, and all the singularities of $\Gamma^{P_j} (1 - e_j + E_j t)$ $(j = 1, \ldots, n_3)$, $\Gamma^{\xi_j} (1 - a_j + \alpha_j s + A_j t)$ $(j = 1, \ldots, n_1)$ lie to the left of $\mathcal{L}_t$.

3. Notations and Results used

All the assumptions made above will be retained throughout this paper. For the sake of brevity, the function defined in (1) will be simply denoted by either $\mathrm{I}[z_1, z_2]$ or $\mathrm{I}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

Throughout the present work we shall use the following notations.

- $(a_j; \alpha_j, A_j; \xi_j)_{1,p}$ stands for $(a_1; \alpha_1, A_1; \xi_1), (a_2; \alpha_2, A_2; \xi_2), \ldots, (a_p; \alpha_p, A_p; \xi_p)$.
- $(c_j, C_j; U_j)_{1,p_2}$ stands for $(c_1, C_1; U_1), (c_2, C_2; U_2), \ldots, (c_{p_2}, C_{p_2}; U_{p_2})$.
- $(a_j; \alpha_j, A_j; 1)_{1,p}$ stands for $(a_1; \alpha_1, A_1; 1), (a_2; \alpha_2, A_2; 1), \ldots, (a_p; \alpha_p, A_p; 1)$.
- $(a_j; \alpha_j, A_j)_{1,p}$ stands for $(a_1; \alpha_1, A_1), (a_2; \alpha_2, A_2), \ldots, (a_p; \alpha_p, A_p)$.

- $(a_j;\alpha_j,1)_{1,p}$ stands for $(a_1;\alpha_1,1),(a_2;\alpha_2,1),\ldots,(a_p;\alpha_p,1)$.
- $(a_j;\alpha_j)_{1,p}$ stands for $(a_1;\alpha_1),(a_2;\alpha_2),\ldots,(a_p;\alpha_p)$.
- $(a_j;1)_{1,p}$ stands for $(a_1;1),(a_2;1),\ldots,(a_p;1)$.
- $(a_p)=(a_j)_{1,p}$ stands for $(a_1),(a_2),\ldots,(a_p)$.

4. Convergence Conditions

Following the results of Braaksma [6, p.278] and Rathie [30], it can easily be shown that the function defined by (1) is an analytic function of z_1 and z_2 if $R<0$ and $S<0$ where

$$R=\sum_{j=1}^{p_1}\xi_j\alpha_j+\sum_{j=1}^{p_2}U_jC_j-\sum_{j=1}^{q_1}\eta_j\beta_j-\sum_{j=1}^{q_2}V_jD_j \tag{5}$$

$$S=\sum_{j=1}^{p_1}\xi_jA_j+\sum_{j=1}^{p_3}P_jE_j-\sum_{j=1}^{q_1}\eta_jB_j-\sum_{j=1}^{q_3}Q_jF_j \tag{6}$$

The sufficient conditions for the convergence of (1) are given in the following theorem.

Theorem 4.1. *The integral* (1) *is convergent if* $\Delta_1>0$, $\Delta_2>0$, $|arg(z_1)|<\frac{1}{2}\Delta_1\pi$, *and* $|arg(z_2)|<\frac{1}{2}\Delta_2\pi$ *where*

$$\Delta_1=\left[\sum_{j=1}^{n_1}\xi_j\alpha_j-\sum_{j=n_1+1}^{p_1}\xi_j\alpha_j-\sum_{j=1}^{q_1}\eta_j\beta_j+\sum_{j=1}^{n_2}U_jC_j\right.$$
$$\left.-\sum_{j=n_2+1}^{p_2}U_jC_j+\sum_{j=1}^{m_2}V_jD_j-\sum_{j=m_2+1}^{q_2}V_jD_j\right] \tag{7}$$

$$\Delta_2=\left[\sum_{j=1}^{n_1}\xi_jA_j-\sum_{j=n_1+1}^{p_1}\xi_jA_j-\sum_{j=1}^{q_1}\eta_jB_j+\sum_{j=1}^{n_3}P_jE_j\right.$$
$$\left.-\sum_{j=n_3+1}^{p_3}P_jE_j+\sum_{j=1}^{m_3}Q_jF_j-\sum_{j=m_3+1}^{q_3}Q_jF_j\right] \tag{8}$$

and if $|arg(z_1)|=\frac{1}{2}\Delta_1\pi$, $|arg(z_2)|=\frac{1}{2}\Delta_2\pi$ *and* $\Delta_1,\Delta_2\geq 0$, *then integral* (1) *converges absolutely under the following conditions.*
(i) $\mu_1=\mu_2=0$, $\Omega_1>1$ *and* $\Omega_2>1$, *where*

$$\mu_1=\sum_{j=1}^{q_1}\eta_j\beta_j-\sum_{j=1}^{p_1}\xi_j\alpha_j+\sum_{j=1}^{q_2}V_jD_j-\sum_{j=1}^{p_2}U_jC_j \tag{9}$$

$$\mu_2 = \sum_{j=1}^{q_1} \eta_j B_j - \sum_{j=1}^{p_1} \xi_j A_j + \sum_{j=1}^{q_3} Q_j F_j - \sum_{j=1}^{p_3} P_j E_j \tag{10}$$

$$\Omega_1 = \left[\sum_{j=1}^{p_1} \xi_j \left(\Re(a_j) - \frac{1}{2} \right) - \sum_{j=1}^{q_1} \eta_j \left(\Re(b_j) - \frac{1}{2} \right) + \sum_{j=1}^{p_2} U_j \left(\Re(c_j) - \frac{1}{2} \right) - \sum_{j=1}^{q_2} V_j \left(\Re(d_j) - \frac{1}{2} \right) \right] \tag{11}$$

$$\Omega_2 = \left[\sum_{j=1}^{p_1} \xi_j \left(\Re(a_j) - \frac{1}{2} \right) - \sum_{j=1}^{q_1} \eta_j \left(\Re(b_j) - \frac{1}{2} \right) + \sum_{j=1}^{p_3} P_j \left(\Re(e_j) - \frac{1}{2} \right) - \sum_{j=1}^{q_3} Q_j \left(\Re(f_j) - \frac{1}{2} \right) \right] \tag{12}$$

(ii) $\mu_1 \neq 0, \mu_2 \neq 0$, *if with* $s = \sigma_1 + it_1$, $t = \sigma_2 + it_2$ (σ_1, σ_2 , t_1 , t_2 *are real),* σ_1 *and* σ_2 *are chosen so that for* $|t_1| \to \infty$, *we have* $(\Omega_1 + \sigma_1\mu_1) > 1$ *and for* $|t_2| \to \infty$, *we have* $(\Omega_2 + \sigma_2\mu_2) > 1$.

Proof. The existence of $I[z_1, z_2]$ may be recognised by the convergence of the integral (1) which depends on the asymptotic behaviour of the functions $\phi(s,t)$, $\theta_1(s)$ and $\theta_2(t)$ defined by (2), (3) and (4) respectively. Such an asymptotic is based on the following relation for gamma function [22]

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\, |y|^{x-\frac{1}{2}}\, exp\left(-\frac{1}{2}\pi|y| \right), \quad (|y| \to \infty) \tag{13}$$

where x and y are real numbers.
Along the contour $\mathcal{L}_s$, if we put $s = \sigma_1 + it_1$ and take limit as $|t_1| \to \infty$ we obtain by virtue of (13),

$$\left| \Gamma^{\xi} \left(1 - a_j + \alpha_j s + A_j t\right) \right| \leq (2\pi)^{\frac{1}{2}\xi_j} \, (\alpha_j|t_1|)^{\xi_j\left(\frac{1}{2} - \Re(a_j) + \alpha_j\sigma_1\right)} \times exp\left(-\frac{\pi}{2}\xi_j\left(\alpha_j|t_1| + |\Im(a_j)|\right) \right) \tag{14}$$

$$\prod_{j=1}^{n_1} \left| \Gamma^{\xi_j} \left(1 - a_j + \alpha_j s + A_j t\right) \right| \leq (2\pi)^{\frac{1}{2}\sum_{j=1}^{n_1}\xi_j} \prod_{j=1}^{n_1} (\alpha_j|t_1|)^{\xi_j\left(\frac{1}{2} - \Re(a_j) + \alpha_j\sigma_1\right)} \times exp\left(-\frac{\pi}{2} \sum_{j=1}^{n_1} \xi_j\left(\alpha_j|t_1| + |\Im(a_j)|\right) \right) \tag{15}$$

Similarly, we have

$$\prod_{j=n_1+1}^{p_1} \left|\Gamma^{\xi_j}\left(a_j-\alpha_j s-A_j t\right)\right| \geq (2\pi)^{\frac{1}{2}\sum_{j=n_1+1}^{p_1}\xi_j} \prod_{j=n_1+1}^{p_1} (\alpha_j|t_1|)^{\xi_j\left(\Re(a_j)-\alpha_j\sigma_1-\frac{1}{2}\right)} \times exp\left(-\frac{\pi}{2}\sum_{j=n_1+1}^{p_1}\xi_j\left(\alpha_j|t_1|+|\Im(a_j)|\right)\right) \quad (16)$$

$$\prod_{j=1}^{q_1}\left|\Gamma^{\eta_j}\left(1-b_j+\beta_j s+B_j t\right)\right| \geq (2\pi)^{\frac{1}{2}\sum_{j=1}^{q_1}\eta_j} \prod_{j=1}^{q_1}(\beta_j|t_1|)^{\eta_j\left(\frac{1}{2}-\Re(b_j)+\beta_j\sigma_1\right)} \times exp\left(-\frac{\pi}{2}\sum_{j=1}^{q_1}\eta_j\left(\beta_j|t_1|+|\Im(b_j)|\right)\right) \quad (17)$$

$$\prod_{j=1}^{m_2}\left|\Gamma^{V_j}\left(d_j-D_j s\right)\right| \leq (2\pi)^{\frac{1}{2}\sum_{j=1}^{m_2}V_j} \prod_{j=1}^{m_2}(D_j|t_1|)^{V_j\left[\Re(d_j)-\sigma_1 D_j-\frac{1}{2}\right]} \times exp\left(-\frac{\pi}{2}\sum_{j=1}^{m_2}V_j\left(D_j|t_1|+|\Im(d_j)|\right)\right) \quad (18)$$

$$\prod_{j=m_2+1}^{q_2}\left|\Gamma^{V_j}(1-d_j+D_j s)\right| \geq (2\pi)^{\frac{1}{2}\sum_{j=m_2+1}^{q_2}V_j} \prod_{j=m_2+1}^{q_2}(D_j|t_1|)^{V_j\left[\frac{1}{2}-\Re(d_j)+\sigma_1 D_j\right]} \times exp\left(-\frac{\pi}{2}\sum_{j=m_2+1}^{q_2}V_j\left(D_j|t_1|+|\Im(d_j)|\right)\right) \quad (19)$$

$$\prod_{j=1}^{n_2}\left|\Gamma^{U_j}\left(1-c_j+C_j s\right)\right| \leq (2\pi)^{\frac{1}{2}\sum_{j=1}^{n_2}U_j} \prod_{j=1}^{n_2}(C_j|t_1|)^{U_j\left[\frac{1}{2}-\Re(c_j)+\sigma_1 C_j\right]} \times exp\left(-\frac{\pi}{2}\sum_{j=1}^{n_2}U_j\left(C_j|t_1|+|\Im(c_j)|\right)\right) \quad (20)$$

$$\prod_{j=n_2+1}^{p_2}\left|\Gamma^{U_j}\left(c_j-C_j s\right)\right| \geq (2\pi)^{\frac{1}{2}\sum_{j=n_2+1}^{p_2}U_j} \prod_{j=n_2+1}^{p_2}(C_j|t_1|)^{U_j\left[\Re(c_j)-\sigma_1 C_j-\frac{1}{2}\right]} \times exp\left(-\frac{\pi}{2}\sum_{j=n_2+1}^{p_2}U_j\left(C_j|t_1|+|\Im(c_j)|\right)\right) \quad (21)$$

Also

$$\begin{aligned} |z_1^s| &= |exp\,[s\,(log|z_1| + iarg(z_1))]\,| \\ &= |exp\,[(\sigma_1 + it_1)\,(log|z_1| + iarg(z_1))] \\ &= |z_1|^{\sigma_1} exp\,[-t_1 arg(z_1)] \end{aligned} \tag{22}$$

Hence substituting (15)-(17) in (2), (18)-(21) in (3) and using (22) we have, after **much** simplification,

$$|\phi(s,t)\,\theta_1(s)\,z_1^s| \sim k_1\,|t_1|^{-\Omega_1-\sigma_1\mu_1}\,exp\left(-t_1 arg(z_1) - \frac{\pi}{2}|t_1|\Delta_1\right) \tag{23}$$

where Δ_1, Ω_1 and μ_1 are given by (7), (11) and (9) respectively.

Similarly along the contour $\mathcal{L}_t$ by putting $t = \sigma_2 + it_2$ in various Gamma functions of (2), (4) and taking limit as $|t_2| \to \infty$, with the help of (13) we have, after **much** simplification,

$$|\phi(s,t)\,\theta_2(t)\,z_2^t| \sim k_2\,|t_2|^{-\Omega_2-\sigma_2\mu_2}\,exp\left(-t_2 arg(z_2) - \frac{\pi}{2}|t_2|\Delta_2\right) \tag{24}$$

where Δ_2, Ω_2 and μ_2 are given by (8), (12) and (10) respectively, and k_1 and k_2 are independent of t_1 and t_2. Hence the result follows. □

Remark 4.2. If $V_j = 1 (j = 1,\ldots,m_2)$ and $Q_j = 1 (j = 1,\ldots,m_3)$ in (1), then the function will be denoted by

$$\begin{aligned} \overline{\mathrm{I}}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1} : (c_j,C_j;U_j)_{1,n_2}, \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1} : (d_j,D_j;1)_{1,m_2}, \end{array}\right. \\ &\qquad \left.\begin{array}{l} (c_j,C_j;U_j)_{n_2+1,p_2};(e_j,E_j;P_j)_{1,n_3},(e_j,E_j;P_j)_{n_3+1,p_3} \\ (d_j,D_j;V_j)_{m_2+1,q_2};(f_j,F_j;1)_{1,m_3},(f_j,F_j;Q_j)_{m_3+1,q_3} \end{array}\right] \end{aligned} \tag{25}$$

$$= \frac{1}{(2\pi i)^2}\int_{\mathcal{L}_s}\int_{\mathcal{L}_t}\phi(s,t)\,\overline{\theta}_1(s)\,\overline{\theta}_2(t)\,z_1^s\,z_2^t\,ds\,dt \tag{26}$$

where $\phi(s,t)$ is given by (2) and $\overline{\theta}_1(s)$, $\overline{\theta}_2(t)$ are given by

$$\overline{\theta}_1(s) = \frac{\prod_{j=1}^{n_2}\Gamma^{U_j}(1-c_j+C_js)\,\prod_{j=1}^{m_2}\Gamma(d_j-D_js)}{\prod_{j=n_2+1}^{p_2}\Gamma^{U_j}(c_j-C_js)\,\prod_{j=m_2+1}^{q_2}\Gamma^{V_j}(1-d_j+D_js)} \tag{27}$$

$$\overline{\theta}_2(t) = \frac{\prod_{j=1}^{n_3} \Gamma^{P_j}(1-e_j+E_jt) \prod_{j=1}^{m_3} \Gamma(f_j-F_jt)}{\prod_{j=n_3+1}^{p_3} \Gamma^{P_j}(e_j-E_jt) \prod_{j=m_3+1}^{q_3} \Gamma^{Q_j}(1-f_j+F_jt)} \tag{28}$$

Remark 4.3. If $V_j = 1(j = 1,\ldots,m_2)$, $Q_j = 1(j = 1,\ldots,m_3)$, $U_j = 1(j = 1,\ldots,n_2)$ and $P_j = 1(j = 1,\ldots,n_3)$ and $n_1 = 0$ in (1), then the corresponding function will be denoted by

$$\overline{\mathrm{I}}_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \mathrm{I}^{0,0:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j;\alpha_j,A_j;\xi_j)_{1,p_1} : (c_j,C_j;1)_{1,n_2}, \\ (b_j;\beta_j,B_j;\eta_j)_{1,q_1} : (d_j,D_j;1)_{1,m_2}, \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j,C_j;U_j)_{n_2+1,p_2};(e_j,E_j;1)_{1,n_3},(e_j,E_j;P_j)_{n_3+1,p_3} \\ (d_j,D_j;V_j)_{m_2+1,q_2};(f_j,F_j;1)_{1,m_3},(f_j,F_j;Q_j)_{m_3+1,q_3} \end{matrix} \right] \tag{29}$$

$$= \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_s} \int_{\mathcal{L}_t} \phi_1(s,t)\, \overline{\theta}_3(s)\, \overline{\theta}_4(t)\, z_1^s\, z_2^t\, ds\, dt \tag{30}$$

where $\phi_1(s,t)$, $\overline{\theta}_3(s)$, $\overline{\theta}_4(t)$ are given by

$$\phi_1(s,t) = \frac{1}{\prod_{j=1}^{p_1} \Gamma^{\xi_j}(a_j-\alpha_j s-A_j t) \prod_{j=1}^{q_1} \Gamma^{\eta_j}(1-b_j+\beta_j s+B_j t)} \tag{31}$$

$$\overline{\theta}_3(s) = \frac{\prod_{j=1}^{n_2} \Gamma(1-c_j+C_js) \prod_{j=1}^{m_2} \Gamma(d_j-D_js)}{\prod_{j=n_2+1}^{p_2} \Gamma^{U_j}(c_j-C_js) \prod_{j=m_2+1}^{q_2} \Gamma^{V_j}(1-d_j+D_js)} \tag{32}$$

$$\overline{\theta}_4(t) = \frac{\prod_{j=1}^{n_3} \Gamma(1-e_j+E_jt) \prod_{j=1}^{m_3} \Gamma(f_j-F_jt)}{\prod_{j=n_3+1}^{p_3} \Gamma^{P_j}(e_j-E_jt) \prod_{j=m_3+1}^{q_3} \Gamma^{Q_j}(1-f_j+F_jt)} \tag{33}$$

5. Series Representation for $\overline{\mathrm{I}}[z_1,z_2]$

Following the lines of Rathie [30], we can obtain the series representation and behaviour for small values for the function $\overline{\mathrm{I}}[z_1,z_2]$ defined and represented by (26). The series representation may be given as follows :

If
(i) $D_h(d_j+r) \neq D_j(d_h+\mu)$, for $j \neq h;\ j,h = 1,\ldots,m_2;\ r,\mu = 0,1,2,\ldots$
(ii) $F_l(f_j+k) \neq F_j(f_l+\nu)$, for $j \neq l;\ j,l = 1,\ldots,m_3;\ k,\nu = 0,1,2,\ldots$
(iii) $z_1 \neq 0$, $z_2 \neq 0$, $R < 0$, $S < 0$ (where R and S are defined by (5) and (6) respectively),
(iv) and if all the poles of $\Gamma(d_j - D_j s)$, $\Gamma(f_j - F_j t)$ in (26) are simple,
then the integral (26) can be evaluated with the help of the Residue theorem to give

$$\overline{\mathrm{I}}[z_1,z_2] = \sum_{h=1}^{m_2}\sum_{l=1}^{m_3}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{(-1)^{r+k}}{D_hF_lr!k!}\phi\left(\frac{d_h+r}{D_h},\frac{f_l+k}{F_l}\right)\right. \\ \left.\times\ \overline{\theta}_5\left(\frac{d_h+r}{D_h}\right)\overline{\theta}_6\left(\frac{f_l+k}{F_l}\right)z_1^{\frac{d_h+r}{D_h}}\ z_2^{\frac{f_l+k}{F_l}}\right\} \tag{34}$$

where $\phi\left(\frac{d_h+r}{D_h},\frac{f_l+k}{F_l}\right)$ is defined analogous to $\phi(s,t)$ given by (2) and $\overline{\theta}_5\left(\frac{d_h+r}{D_h}\right)$, $\overline{\theta}_6\left(\frac{f_l+k}{F_l}\right)$ are respectively given by

$$\overline{\theta}_5\left(\frac{d_h+r}{D_h}\right) \\ = \frac{\prod_{j=1}^{n_2}\Gamma^{U_j}\left(1-c_j+C_j\left(\frac{d_h+r}{D_h}\right)\right)\prod_{\substack{j=1\\(j\neq h)}}^{m_2}\Gamma\left(d_j-D_j\left(\frac{d_h+r}{D_h}\right)\right)}{\prod_{j=n_2+1}^{p_2}\Gamma^{U_j}\left(c_j-C_j\left(\frac{d_h+r}{D_h}\right)\right)\prod_{j=m_2+1}^{q_2}\Gamma^{V_j}\left(1-d_j+D_j\left(\frac{d_h+r}{D_h}\right)\right)} \tag{35}$$

$$\overline{\theta}_6\left(\frac{f_l+k}{F_l}\right) \\ = \frac{\prod_{j=1}^{n_3}\Gamma^{P_j}\left(1-e_j+E_j\left(\frac{f_l+k}{F_l}\right)\right)\prod_{\substack{j=1\\(j\neq l)}}^{m_3}\Gamma\left(f_j-F_j\left(\frac{f_l+k}{F_l}\right)\right)}{\prod_{j=n_3+1}^{p_3}\Gamma^{P_j}\left(e_j-E_j\left(\frac{f_l+k}{F_l}\right)\right)\prod_{j=m_3+1}^{q_3}\Gamma^{Q_j}\left(1-f_j+F_j\left(\frac{f_l+k}{F_l}\right)\right)} \tag{36}$$

for $|z_1| < 1$, $|z_2| < 1$.
From (34) it follows that

$$\overline{\mathrm{I}}[\mathrm{z}_1,\mathrm{z}_2] = O\left(|z_1|^{\alpha}\ |z_2|^{\beta}\right), \qquad max\{|z_1|,\ |z_2|\} \to 0 \tag{37}$$

where

$$\alpha = \min_{1 \le j \le m_2} \left[\Re\left(\frac{d_j}{D_j} \right) \right] \tag{38}$$

$$\beta = \min_{1 \le j \le m_3} \left[\Re\left(\frac{f_j}{F_j} \right) \right] \tag{39}$$

for small values of z_1 and z_2.
Similarly, when all the poles of $\bar{\mathrm{I}}_1[z_1, z_2]$ are simple , it can be shown that

$$\bar{\mathrm{I}}_1[z_1, z_2] = \begin{cases} O\left(|z_1|^{\alpha}\ |z_2|^{\beta}\right), & max\{|z_1|,\ |z_2|\} \to 0 \\ O\left(|z_1|^{\alpha'}\ |z_2|^{\beta'}\right), & min\{|z_1|,\ |z_2|\} \to \infty \end{cases} \tag{40}$$

where α and β are given by (38) and (39) and α' and β' are given by

$$\alpha' = \max_{1 \le j \le n_2} \left[\Re\left(\frac{c_j - 1}{C_j} \right) \right] \tag{41}$$

$$\beta' = \max_{1 \le j \le n_3} \left[\Re\left(\frac{e_j - 1}{E_j} \right) \right] \tag{42}$$

for large values of z_1 and z_2.

6. Special Cases

In this section, we mention some interesting and useful special cases of the I-function of two variables.
(i) If all the exponents $\xi_j(j = 1,\dots,p_1)$, $\eta_j(j = 1,\dots,q_1)$, $U_j(j = 1,\dots,p_2)$, $V_j(j = 1,\dots,q_2)$, $P_j(j = 1,\dots,p_3)$ and $Q_j(j = 1,\dots,q_3)$ are equal to unity, then (1) reduces to the H-function of two variables defined by Mittal and Gupta [25].
(ii) If we take $p_1 = q_1 = n_1 = 0$ in (1) then it degenerates into the product of two I- functions of one variable introduced by Rathie [30] as

$$\mathrm{I}^{0,0:\, m_2, n_2;\, m_3, n_3}_{0,0:\, p_2, q_2;\, p_3, q_3} \left[\begin{array}{c} z_1 \\ z_2 \end{array} \middle| \begin{array}{l} ---: (c_j, C_j; U_j)_{1,p_2};\ (e_j, E_j; P_j)_{1,p_3} \\ ---: (d_j, D_j; V_j)_{1,q_2};\ (f_j, F_j; Q_j)_{1,q_3} \end{array} \right]$$
$$= \mathrm{I}^{m_2, n_2}_{p_2, q_2} \left[z_1 \middle| \begin{array}{c} (c_j, C_j; U_j)_{1,p_2} \\ (d_j, D_j; V_j)_{1,q_2} \end{array} \right] \times \mathrm{I}^{m_3, n_3}_{p_3, q_3} \left[z_2 \middle| \begin{array}{c} (e_j, E_j; P_j)_{1,p_3} \\ (f_j, F_j; Q_j)_{1,q_3} \end{array} \right] \tag{43}$$

(iii) If we take $m_3 = 1$, $n_3 = p_3$, $f_1 = 0$, $(F_j)_{1,q_3} = 1$, $(A_j)_{1,p_1}$ = $(B_j)_{1,q_1}$ = $(E_j)_{1,p_3}$ = $(F_j)_{1,q_3} = 1$, equate the exponents $P_j(j = 1,\dots,p_3)$, $Q_j(j = 1,\dots,q_3)$ to unity, replace q_3 by q_3+1 and let $z_2 \to 0$ in (1), we get the following relation by virtue of (1) and known results [12, p.208],

$$\lim_{z_2 \to 0} \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,1;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,1;\eta_j)_{1,q_1}: \end{array}\right. \left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,1;1)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(0,1;1),(f_j,1;1)_{2,q_3+1} \end{array}\right]$$

$$= \frac{\prod_{j=1}^{p_3}\Gamma(1-e_j)}{\prod_{j=1}^{q_3}\Gamma(1-f_j)} \times \mathrm{I}^{m_2,n_1+n_2}_{p_1+p_2,\,q_1+q_2}\left[\begin{array}{c|l} z_1 & (a_j,\alpha_j;\xi_j)_{1,n_1},(c_j,C_j;U_j)_{1,p_2},(a_j,\alpha_j;\xi_j)_{n_1+1,p_1} \\ & (d_j,D_j;V_j)_{1,q_2},(b_j,\beta_j;\eta_j)_{1,q_1} \end{array}\right] \tag{44}$$

where $p_1+p_3 < q_1+q_3+1$
(iv) A relationship between the I-function of two variables and the G-function of two variables [36, p.7, eq.1.2.3] is,

$$\mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;1,1;1)_{1,p_1}:(c_j,1;1)_{1,p_2};\ (e_j,1;1)_{1,p_3} \\ z_2 & (b_j;1,1;1)_{1,q_1}:(d_j,1;1)_{1,q_2};\ (f_j,1;1)_{1,q_3} \end{array}\right]$$

$$= \mathrm{G}(z_1,z_2) = \mathrm{G}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_{p_1}):(c_{p_2});\ (e_{p_3}) \\ z_2 & (b_{q_1}):(d_{q_2});\ (f_{q_3}) \end{array}\right] \tag{45}$$

(v) Another specialization of parameters in the I-function of two variables yields the interesting relationship :

$$\mathrm{I}^{0,p_1:\,1,p_2;\,1,p_3}_{p_1,q_1:\,p_2,q_2+1;\,p_3,q_3+1}\left[\begin{array}{c|l} -z_1 & (1-a_j;\alpha_j,A_j;1)_{1,p_1}: \\ -z_2 & (1-b_j;\beta_j,B_j;1)_{1,q_1}: \end{array}\right. \left.\begin{array}{l} (1-c_j,C_j;1)_{1,p_2};(1-e_j,E_j;1)_{1,p_3} \\ (0,1;1),(1-d_j,D_j;1)_{1,q_2};(0,1;1),(1-f_j,F_j;1)_{1,q_3} \end{array}\right]$$

$$= \mathrm{S}^{p_1;p_2;p_3}_{q_1;q_2;q_3}\left[\begin{array}{l} (a_j;\alpha_j,A_j)_{1,p_1}:(c_j,C_j)_{1,p_2};(e_j,E_j)_{1,p_3}\ ; \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,D_j)_{1,q_2};(f_j,F_j)_{1,q_3}\ ; \end{array} z_1,z_2\right] \tag{46}$$

where $\mathrm{S}(\mathrm{z}_1,\mathrm{z}_2)$ is generalized Kampe de Feriet function defined by Srivastava and Daoust [35, p.6,eq.1.2.2].

(vi) $$\mathrm{I}^{0,n_1:\,1,p_2;\,1,p_3}_{p_1,q_1:\,p_2,q_2+1;\,p_3,q_3+1}\left[\begin{array}{c|l} z_1 & (1-a_j;1,1;1)_{1,p_1}: \\ z_2 & (1-b_j;1,1;1)_{1,q_1}: \end{array}\right. \left.\begin{array}{l} (1-c_j,1;1)_{1,p_2};(1-e_j,1;1)_{1,p_3} \\ (0,1;1),(1-d_j,1;1)_{1,q_2};(0,1;1),(1-f_j,1;1)_{1,q_3} \end{array}\right]$$

$$= \frac{\prod_{j=1}^{p_1}\Gamma(a_j)\,\prod_{j=1}^{p_2}\Gamma(c_j)\,\prod_{j=1}^{p_3}\Gamma(e_j)}{\prod_{j=1}^{q_1}\Gamma(b_j)\,\prod_{j=1}^{q_2}\Gamma(d_j)\,\prod_{j=1}^{q_3}\Gamma(f_j)}$$

$$\times\, \mathrm{F}^{p_1:\,p_2;\,p_3}_{q_1:\,q_2;\,q_3}\left[\begin{array}{ll}(a_j)_{1,p_1}:(c_j)_{1,p_2},(e_j)_{1,p_3} & ;z_1\\ (b_j)_{1,q_1}:(d_j)_{1,q_2};(f_j)_{1,q_3} & ;z_2\end{array}\right] \qquad (47)$$

where the function $\mathrm{F}[z_1,z_2]$ is the Kamp$\acute{e}$ de F$\acute{e}$riet function [36, p.5, eq.1.2.1].

(vii)
$$\mathrm{I}^{0,\,0:\,1,\,p_2;\,1,\,p_3}_{0,\,0:\,p_2,\,q_2+1;\,p_3,\,q_3+1}\left[\begin{array}{c}-z_1\\ -z_2\end{array}\middle|\begin{array}{l}---:\\ ---:\end{array}\right.$$
$$\left.\begin{array}{l}(1-c_j,C_j;1)_{1,p_2};(1-e_j,E_j;1)_{1,p_3}\\ (0,1;1),(1-d_j,D_j;1)_{1,q_2};(0,1;1),(1-f_j,F_j;1)_{1,q_3}\end{array}\right]$$

$$= {}_{p_2}\Psi_{q_2}\left[\begin{array}{ll}(c_j,C_j)_{1,p_2} & ;\,z_1\\ (d_j,D_j)_{1,q_2} & \end{array}\right] \times {}_{p_3}\Psi_{q_3}\left[\begin{array}{ll}(e_j,E_j)_{1,p_3} & ;\,z_2\\ (f_j,F_j)_{1,q_3} & \end{array}\right] \qquad (48)$$

where the function ${}_{p_2}\Psi_{q_2}$ and ${}_{p_3}\Psi_{q_3}$ are Wright's generalized hypergeometric functions [36, p. 19,eq.2.6.11].

(viii)
$$\mathrm{I}^{0,\,0:\,1,0;\,1,0}_{0,\,0:\,0,2;\,0,2}\left[\begin{array}{c}z_1\\ z_2\end{array}\middle|\begin{array}{lllll}--- & : & --- & ; & ---\\ --- & : & (0,1;1),(-\mu,\alpha;1);(0,1;1),(-\nu,\beta;1) & & \end{array}\right]$$
$$= J^{\alpha}_{\mu}(z_1) \times J^{\beta}_{\nu}(z_2) \qquad (49)$$

where the functions $J^{\alpha}_{\mu}(z_1)$ and $J^{\beta}_{\nu}(z_2)$ are Wright's generalized Bessel functions [36, p.19, eq.2.6.10].

(ix)
$$\mathrm{I}^{0,\,0:\,1,2;\,1,2}_{0,\,0:\,2,2;\,2,2}\left[\begin{array}{c}-z_1\\ -z_2\end{array}\middle|\begin{array}{l}---:(1,1;1),(1-\alpha,1;p);(1,1;1),(1-\beta,1;q)\\ ---:(0,1;1),(-\alpha,1;p);(0,1;1),(-\beta,1;q)\end{array}\right]$$
$$= \Phi(z_1,p,\alpha) \times \Phi(z_2,q,\beta) \qquad (50)$$

where $\Phi(z_1,p,\alpha)$ and $\Phi(z_2,q,\beta)$ are the generalised Riemann-zeta functions [12, p.27, 1.11, eq(1)], which are generalizations of Hurwitz zeta functions and Riemann zeta functions [12, p.24, 1.10, eq(1) and 1.12, eq(1)]

(x)
$$\mathrm{I}^{0,0:\,1,2;\,1,2}_{0,0:\,2,2;\,2,2}\left[\begin{array}{c}-z_1\\ -z_2\end{array}\middle|\begin{array}{l}\text{—}:(1,1;1),(1,1;p);(1,1;1),(1,1;q)\\ \text{—}:(0,1;1),(0,1;p);(0,1;1),(0,1;q)\end{array}\right]$$
$$= F(z_1,p) \times F(z_2,q) \qquad (51)$$

where $F(z_1,p)$ and $F(z_2,q)$ are the polylogarithms of order p and q respectively. For $p=2$ and $q=2$, the R.H.S. of (51) equation reduces to product of Eulers's dilogarithm [12, p.31, 1.11.1, eq(2)].

(xi) If we take $p_1=q_1=n_1=0$, $U_j=1(j=n_2+1,\ldots,p_2)$ and $P_j=1(j=n_3+1,\ldots,p_3)$ in $\overline{\mathrm{I}}\left[\begin{array}{c}z_1\\ z_2\end{array}\right]$ defined by (26) then it degenerates into the product of two $\overline{H}$ - functions introduced by Inayat Hussain [20].

7. Elementary properties and Transformation formulas.

The properties given below are immediate consequence of the definition (1) and hence they are given here without proof:

i) $$I^{0,0:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}:(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}:(d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$
$$= I^{0,0:n_2,m_2;n_3,m_3}_{q_1,p_1:q_2,p_2;q_3,p_3}\left[\begin{array}{c|l} z_1^{-1} & (1-b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \\ z_2^{-1} & (1-a_j;\alpha_j,A_j;\xi_j)_{1,p_1}:\end{array}\right.$$
$$\left.\begin{array}{l}(1-d_j,D_j;V_j)_{1,q_2};(1-f_j,F_j;Q_j)_{1,q_3} \\ (1-c_j,C_j;U_j)_{1,p_2};(1-e_j,E_j;P_j)_{1,p_3}\end{array}\right] \quad (52)$$

ii) $z_1^{k_1}\, z_2^{k_2}\, \mathrm{I}[z_1,z_2]$
$$= I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j+k_1\alpha_j+k_2A_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j+k_1\beta_j+k_2B_j;\beta_j,B_j;\eta_j)_{1,q_1}:\end{array}\right.$$
$$\left.\begin{array}{l}(c_j+k_1C_j,C_j;U_j)_{1,p_2};(e_j+k_2E_j,E_j;P_j)_{1,p_3} \\ (d_j+k_1D_j,D_j;V_j)_{1,q_2};(f_j+k_2F_j,F_j;Q_j)_{1,q_3}\end{array}\right] \quad (53)$$

for $k_1>0,\ k_2>0$.

iii) $$\frac{1}{k_1}\,\frac{1}{k_2}\,\mathrm{I}[z_1,z_2] = I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1^{k_1} & (a_j;k_1\alpha_j,k_2A_j;\xi_j)_{1,p_1}: \\ z_2^{k_2} & (b_j;k_1\beta_j,k_2B_j;\eta_j)_{1,q_1}:\end{array}\right.$$
$$\left.\begin{array}{l}(c_j,k_1C_j;U_j)_{1,p_2};(e_j,k_2E_j;P_j)_{1,p_3} \\ (d_j,k_1D_j;V_j)_{1,q_2};(f_j,k_2F_j;Q_j)_{1,q_3}\end{array}\right] \quad (54)$$

where $k_1>0,\ k_2>0$.

iv) $$I^{0,n_1:m_2,n_2,m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a;\alpha,0;\xi),(a_j;\alpha_j,A_j;\xi_j)_{2,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}:\end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2},(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$

$$= I^{0,n_1-1:m_2,n_2+1;m_3,n_3}_{p_1-1,q_1:p_2+1,q_2;p_3\,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{2,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}:\end{array}\right.$$
$$\left.\begin{array}{l}(a,\alpha;\xi),(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right] \quad (55)$$

where $p_1 \geq n_1 \geq 1$.

$$\text{v) } \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1},(b;\beta,0;\eta): \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{array}\right]$$

$$= \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1-1:\,p_2,q_2+1;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1}: \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2},(b,\beta;\eta);(f_j,F_j;Q_j)_{1,q_3} \end{array}\right] \quad (56)$$

where $q_1 - 1 \geq 0$

$$\text{vi) } \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a;0,0;\xi),(a_j;\alpha_j,A_j;\xi_j)_{2,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{array}\right]$$

$$= \Gamma^{\xi}(1-a)\ \mathrm{I}^{0,n_1-1:\,m_2,n_2;\,m_3,n_3}_{p_1-1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{2,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{array}\right] \quad (57)$$

where $\geq n_1 \geq 1$, $\Re(1-a) > 0$.

$$\text{vii) } \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1},(b;0,0;\eta): \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{array}\right]$$

$$= \frac{1}{\Gamma^{\eta}(1-b)}\ \mathrm{I}^{0,n_1:\,m_2,n_2;\,m_3,n_3}_{p_1,q_1-1:\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1}: \end{array}\right.$$
$$\left.\begin{array}{l} (c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{array}\right] \quad (58)$$

where $q_1 - 1 \geq 0$, $\Re(1-b) > 0$.

viii) $$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l}(c,0;U),(c_j,C_j;U_j)_{2,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$

$$= \Gamma^U(1-c)\, I^{0,n_1:m_2,n_2-1;m_3,n_3}_{p_1,q_1:p_2-1,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{2,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right] \tag{59}$$

where $p_2 \geq n_2 \geq 1$, $\Re(1-c) > 0$.

ix) $$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d,0;V),(d_j,D_j;V_j)_{2,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$

$$= \Gamma^V(d)\, I^{0,n_1:m_2-1,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2-1;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}: \end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{2,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right] \tag{60}$$

where $q_2 \geq n_2 \geq 1$, $\Re(d) > 0$.

x) $$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1},(a_1;\alpha_1,A_1;\xi_1): \end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ (d_j,D_j;V_j)_{1,q_2-1},(c_1,C_1;U_1);(f_j,F_j;Q_j)_{1,q_3-1},(e_1,E_1;P_1)\end{array}\right]$$

$$= I^{0,n_1-1:m_2,n_2-1;m_3,n_3-1}_{p_1-1,q_1-1:p_2-1,q_2-1;p_3-1,q_3-1}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{2,p_1}: \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1-1}: \end{array}\right.$$
$$\left.\begin{array}{l}(c_j,C_j;U_j)_{2,p_2};(e_j,E_j;P_j)_{2,p_3} \\ (d_j,D_j;V_j)_{1,q_2-1};(f_j,F_j;Q_j)_{1,q_3-1}\end{array}\right] \tag{61}$$

provided that $p_1 \geq n_1 \geq 1$, $p_2 \geq n_2 \geq 1$, $p_3 \geq n_3 \geq 1$, $q_1 \geq 1$, $q_2 \geq m_2+1$, $q_3 \geq m_3+1$.

7.1. Special Cases :

When all the exponents $\xi_j (j = 1,\ldots,p_1)$, $\eta_j (j = 1,\ldots,q_1)$, $U_j (j = 1,\ldots,p_2)$, $V_j (j = 1,\ldots,q_2)$, $P_j (j = 1,\ldots,p_3)$, $Q_j (j = 1,\ldots,q_3)$ are equal to unity, the I-function of two variables reduce to the H-function of two variables and therefore we obtain the corresponding results in H- function of two variables recorded in [36].

Concluding Remark

In this research paper we have introduced the natural generalization of the H-function of two variables introduced by Mittal and Gupta [25]. In addition to this we have also obtained convergence conditions, series representation, elementary properties , transformation formulas and special cases.

As we have seen that the generalized function of two variables introduced by Agarwal [1] and Sharma [33, 34] have found interesting applications in *wireless communication*, therefore the function which have been introduced in this paper may be potentially useful.

We conclude this paper by remarking that double integrals and summation formulas for this newly defined I-functions of two variables are under investigations and the same will form a subsequent paper in this direction.

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