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# A study of modified holographic Ricci dark energy in the framework of $f(T)$ modified gravity and its statefinder hierarchy

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Inspired by the work of Bamba et al., *Phys. Rev. D* **85**, 104036 (2012) the present paper reports a study on the reconstruction of modified holographic Ricci dark energy (MHRDE) in the framework of modified gravity taken as  $f(T)$  gravity. A correspondence between modified Chaplygin gas (MCG) and MHRDE has also been considered and thereafter the  $f(T)$  gravity has been reconstructed via reconstruction of the Hubble parameter. The reconstructed equation of state (EoS) parameter obtained this way has been found to be able to cross the phantom boundary. In the next phase of the work a viable model of  $f(T)$  gravity has been considered and MHRDE has been discussed in this modified gravity frame. The EoS parameter due to the torsion contribution obtained this way has been found to behave like quintessence. The transition of the universe from the dark matter (DM) dominated to dark energy (DE) dominated phase is apparent from this model. Also, the model is exhibiting DE domination of the current universe. Finally, the statefinder hierarchy has been discussed through the statefinder and snap parameters. The model has been found to be able to attain the  $\Lambda$ CDM fixed point in the statefinder trajectory.

**Key words:** reconstruction; modified gravity; dark energy; statefinder parameters

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## I. INTRODUCTION

Accelerated expansion of the current universe has been established through cosmological observations obtained with Supernovae Ia (SNeIa), the Cosmic Microwave Background (CMB) radiation anisotropies, the Large Scale Structure (LSS) and X-ray experiments [1–4]. A missing energy component also known as Dark Energy (DE) characterized by negative pressure is widely considered as responsible for this accelerated expansion. It has been indicated by recent analysis of cosmological observations that two-thirds of the total energy content of the universe is occupied by the DE

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and the remaining part is occupied by dark matter (DM). The baryonic matter represents only a few percent of the total energy density of the universe [5]. The contribution of the radiation is practically negligible.

The exact nature of DE is yet to be revealed and different candidates have been proposed till date in order to describe it (e.g. [6–13]). The simplest candidate is a tiny positive cosmological constant, with a constant equation of state (EoS) parameter  $w = -1$ . However, the cosmological constant suffers from the fine-tuning and the cosmic coincidence problems. The first one asks why the vacuum energy density is so small and the latter says why vacuum energy and DM are nearly equal today [9]. To get rid off these difficulties dynamical scalar field models have been proposed by several authors and they have the potential to put some light into the evolution of the universe through time varying EoS parameters. These models include quintessence (reviewed in [14]), f-essence [15] and K-essence [16] among others. Suggestion of the so called holographic principle by Fischler and Susskind (1998) [17] is an important advancement in the studies of black hole theory and string theory. According to the holographic principle, the number of degrees of freedom and hence the total entropy of a physical system is finite and should scale with its bounding area rather than with its volume [18] and is constrained by an infrared cut-off. The Holographic DE (HDE), based on the holographic principle, is a well studied model of DE [19–22]. The HDE models have also been constrained by various astronomical observations [23]. The holographic Ricci dark energy is particular model from most general Nojiri-Odintsov HDEs introduced in [21], where, in an attempt to have a unifying approach to early-time and late-time universe based on phantom cosmology, the authors suggested a generalized holographic dark energy with infrared cutoff identified with combination of Hubble constant, particle and future horizons, cosmological constant and universe life-time (if finite). They [21] demonstrated, in this perspective, the possibility to solve the coincidence problem, crossing of phantom divide and unification of early-time inflationary and late-time accelerating phantom universe.

In recent years, the concept of modification of gravity has led to more accurate study of the cosmic acceleration as an alternative to DE [21, 24–26, 28–31]. Modified theories of gravity (reviewed in [32–34]) can be considered a new paradigm to get rid off the shortcomings of General Relativity at infrared and ultraviolet scales. Preserving the undoubtedly positive results of Einstein's Theory this approach aims to address the conceptual and experimental problems like Dark Energy, Dark Matter, Large Scale Structure etc. that have emerged in Astrophysics, Cosmology and High Energy Physics in recent years. One recent review on modified theories of gravity is by Nojiri et al. [35], where and extensive review on some standard issues and also the latest de-

velopments of modified gravity in cosmology, emphasizing on inflation, bouncing cosmology and late-time acceleration era has been presented. The modified theory of gravity gives a very natural gravitational alternative for exotic matter and can describe the phantom, non-phantom and quintom phases of the universe without the necessity of the introduction of a negative kinetic term in DE models. Another important feature of modified gravity is its capability of explaining both the scenarios of early inflation and late time acceleration. These features have been studied in Carroll et al. [36]. Some remarkable works to be noted include [28, 37–42]. One notable review of modified gravity is the Nojiri and Odintsov [43] that presented an elaborate discussion on various well-known models of modified gravity. Under the assumption of flat FRW cosmology they [43] investigated the unified scenario of the universe through modified gravity background evolution. The popular models of modified gravity include include:  $f(R)$  gravity,  $f(G)$  gravity,  $f(T)$  gravity and Horava-Lifshitz gravity. Myrzakulov [45] has presented a review on  $f(T)$  gravity. Considering modified gravity ( $f(R)$ ,  $f(G)$ ) non-minimally coupled with matter Lagrangian Nojiri et al. [26] described the early-time and late-time universe and revealed that such models of modified gravity in the absence of non-minimal coupling is viable theory which passes the local tests and reproduces the  $\Lambda$ CDM era. In a recent work, Chattopadhyay [27] studied the generalised second law of thermodynamics in reconstructed  $f(T)$  cosmology considering the universe as a closed bounded system with future event horizon as the cosmological boundary and discussed two different entropies with the cosmological horizons with a logarithmic correction term and a power-law correction term.

Applying the holographic principle to cosmology one can obtain the upper bound of the entropy contained in the universe. Following this line, Li [19] suggested a constraint on the energy density of the universe as  $\rho_\Lambda \leq 3\gamma M_p^2 L^{-2}$ , where  $\gamma$  is a numerical constant,  $L$  is the IR cut-off radius and  $M_p$  is the reduced Planck mass. If the the holographic bound is saturated then equality sign holds. In this work, we consider a recently proposed holographic cosmological model with IR cut-off given by the modified Ricci radius so that  $L^{-2}$  is a combination of  $H^2$  and  $\dot{H}$  (with  $H$  and  $\dot{H}$  being, respectively, the Hubble parameter and its first derivative with respect to the cosmic time  $t$ ) Chimento et al. [46, 47]. After that, the energy density  $\rho_\Lambda$  of the MHRDE model becomes  $\rho_\Lambda = \frac{2}{\alpha-\beta} \left( \dot{H} + \frac{3\alpha}{2} H^2 \right)$ , where  $\alpha$  and  $\beta$  are free constants. In the limiting case corresponding to  $(\alpha = 4/3, \beta = 1)$  we obtain that  $\rho_\Lambda$  becomes proportional to the Ricci scalar curvature  $R$  for a spatially flat FLRW space-time (corresponding to the curvature parameter  $k$  equal to 0). The use of the MHRDE is motivated by the holographic principle because one can relate the DE with an UV cut-off for the vacuum energy with an IR scale such as the one given by  $R$ . Alternatively, one could proceed in another way by considering  $R$  as a new kind of DE, for instance, a geometric DE

instead of evoking the holographic principle. Irrespective of the origin of the DE component, it modifies the Friedmann equation leading to a second order differential equation for the scale factor.

It is reflected in the title of the paper itself that we are working on a correspondence between MHRDE and modified Chaplygin gas (MCG). Also, the title reflects that the primary concern of the paper is the reconstruction of  $f(T)$  gravity. We have already discussed the issues related to MHRDE and in this paragraph we focus on MCG. In the subsequent section the  $f(T)$  gravity would be taken into account in the perspective of reconstruction. The Chaplygin gas (CG) is a candidate for DE and is characterized by an exotic equation of state (EoS)  $p = -A/\rho$  [50, 51], where  $A$  is a positive constant. This equation of state leads to a component which behaves as dust at early stage and as cosmological constant at later stage. Gorini et al. [51] generalized the EoS for CG to  $p = -A/\rho^\alpha$ , with  $0 \leq \alpha \leq 1$  and  $A$  as free parameter. This model is referred to as generalized Chaplygin gas model (GCG). Another modification to the EoS for CG leads to a class of EoS that interpolates between standard fluids at high energy densities and GCG at low energy densities and this class is dubbed modified Chaplygin gas (MCG) [52, 53], whose EoS is  $p = A\rho - \frac{B}{\rho^\alpha}$ , where  $A$ ,  $\alpha$  and  $B$  are parameters of the model. Usefulness of the Chaplygin-type models of DE in unification of the early and late time universe has been discussed in refs. [54–56]

Bamba et al. [48] demonstrated the appearance of finite-time future singularities in  $f(T)$  gravity with  $T$  as the torsion scalar and reconstruct a model of  $f(T)$  gravity that could realize the finite-time future singularities and explicitly showed that a power-law type correction term  $T^\beta$  ( $\beta > 1$ ) like  $T^2$  term could remove the finite-time future singularities in  $f(T)$  gravity. Purpose of the present work deviates from the earlier reconstruction approaches of modified gravity. In the first phase of the work we have considered a correspondence between MHRDE and MCG and thereby we have reconstructed the Hubble parameter. The reconstructed Hubble parameter has been used in modifying the EoS parameter for the  $f(T)$  gravity i.e. due to torsion component to examine the capability of the  $f(T)$  model reconstructed this way to cross the phantom boundary. In the next phase of the work we have tested a viable model of  $f(T)$  gravity through a reconstruction scheme for MHRDE and viewed the cosmological consequences under the assumption that background evolution of the universe is due to MHRDE. One notable work in this connection is the very recent work of Nojiri and Odintsov [40], where they have established that  $f(R)$  gravity can be rewritten in the holographic language at the level of background equivalence and obtained a realistic inflation or viable dark energy or a unified inflationary-dark energy universe in terms of covariant holographic dark energy.

## II. $f(T)$ RECONSTRUCTION SCHEME THROUGH CORRESPONDENCE BETWEEN MCG AND MHRDE

Like the generalization of Einstein-Hilbert action in the Ricci curvature scalar  $R$  to  $f(R)$  gravity, Linder [44] considered the extensions of teleparallel gravity to  $f(T)$  theories, where  $T$  is the torsion scalar. In the framework of  $f(T)$  theory, the action of modified teleparallel action is given by [44]:

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [f(T) + L_m], \quad (1)$$

where  $L_m$  is the Lagrangian density of the matter inside the universe,  $G$  is the gravitational constant and  $g$  is the determinant of the metric tensor  $g^{\mu\nu}$ . We consider a flat Friedmann-Robertson-Walker (FRW) universe filled with the pressureless matter. Choosing ( $8\pi G = 1$ ), the modified Friedmann equations in the framework of  $f(T)$  gravity are given by [44]:

$$H^2 = \frac{1}{3}(\rho + \rho_T), \quad (2)$$

$$2\dot{H} + 3H^2 = -(p + p_T), \quad (3)$$

where

$$\rho_T = \frac{1}{2}(2Tf_T - f - T), \quad (4)$$

$$p_T = -\frac{1}{2} \left[ -8\dot{H}Tf_{TT} + (2T - 4\dot{H})f_T - f + 4\dot{H} - T \right], \quad (5)$$

and

$$T = -6(H^2). \quad (6)$$

where,  $T$  is the torsion scalar. Various aspects of  $f(T)$  gravity have been discussed by Myrzakulov [45, 57]. Chattopadhyay and Pasqua [58] reconstructed  $f(T)$  gravity and Setare [59] reconstructed  $f(R)$  gravity from HDE.

Modified Chaplygin gas (MCG) is given by

$$p_c = A\rho_c - \frac{B}{\rho_c^n} \quad (7)$$

where,  $A$ ,  $B$  and  $n$  are constant parameters. Modified holographic Ricci dark energy (MHRDE) is given by

$$\rho_\Lambda = \frac{2}{\alpha - \beta} \left( \dot{H} + \frac{3\alpha}{2} H^2 \right) \quad (8)$$

where,  $\alpha$  and  $\beta$  are free constants. In this step we shall first find solution for MCG density  $\rho_c$  by inserting the Eq. (7) into the conservation equation for MCG given as

$$\dot{\rho}_c + 3H\rho_c(1 + w_c) = 0 \quad (9)$$

where  $\rho_c$  is the density of MCG and  $w_c = \frac{p_c}{\rho_c} = A - \frac{B}{\rho_c^{n+1}}$  is the equation of state parameter for MCG. The upper dot indicated the time derivative. Solving the differential equation (9) using  $w_c$  as stated above we get the energy density of MCG as a function of scale factor  $a$  as follows:

$$\rho_c = \left( \frac{B + a^{-3(1+A)(1+n)} e^{(1+A)(1+n)C_1}}{1 + A} \right)^{\frac{1}{1+n}} \quad (10)$$

It may be noted that we have used  $\dot{\rho} = aH \frac{d\rho(a)}{da}$  in (9) to get the solution for  $\rho_c$  in terms of  $a$ . After this, we consider a correspondence between  $\rho_c$  as obtained in Eq. (10) and  $\rho_\Lambda$  of MHRDE as shown in Eq. (8). This correspondence leads to a differential equation of  $H^2$  with  $a$  as the independent variable:

$$\frac{1}{\alpha - \beta} \left( a \frac{dH^2}{da} + 3\alpha H^2 \right) = \left( \frac{B + a^{-3(1+A)(1+n)} e^{(1+A)(1+n)C_1}}{1 + A} \right)^{\frac{1}{1+n}} \quad (11)$$

At this juncture let us discuss some motivation behind considering a correspondence between MCG and MHRDE. This kind of consideration is not new in the area of dark energy and modified gravity theories. By formulating independently the two cosmological scenarios, and by enforcing their simultaneous validity, Setare and Saridakis [61] have shown that the existence of a correspondence between the holographic dark energy scenario in flat universe and the phantom dark energy model in the framework of GaussBonnet theory with a potential. In another work, Jawad and Majeed [62] considered interacting pilgrim dark energy with cold dark matter in flat universe and developed the equation of state parameter in this scenario to analyze the behavior of scalar field and corresponding scalar potentials of various scalar field models. The current work is in line with the above mentioned work. As the Chaplygin gas represents a model of DE capable of unifying the early inflation and late time acceleration of the universe and the holographic models of DE are based on the holographic principle, reconstruction of various cosmological parameters based on their mutual correspondence is expected to be capable of getting hold of early as well as later phases of the universe.

Solving differential Eq. (11) we have the reconstructed Hubble parameter  $\tilde{H}$  as a function of

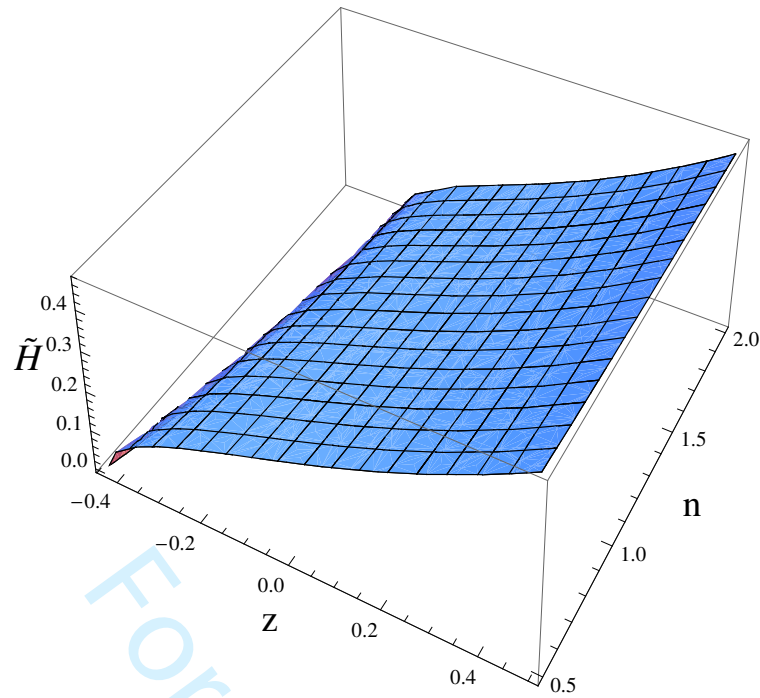


FIG. 1: Evolution of reconstructed Hubble parameter  $\tilde{H}$  with redshift  $z$  based on Eq. (12).

the scale factor  $a$ :

$$\begin{aligned} \tilde{H}^2(a) = & a^{-3\alpha} C_2 + \frac{1}{3\alpha} \left[ \left( \frac{B + a^{-3(1+A)(1+n)} e^{(1+A)(1+n)C_1}}{1+A} \right)^{\frac{1}{1+n}} \left( 1 + \frac{a^{-3(1+A)(1+n)} e^{(1+A)(1+n)C_1}}{B} \right)^{-\frac{1}{1+n}} \right. \\ & \left. \times (\alpha - \beta) {}_2F_1 \left[ -\frac{\alpha}{1+A+n+An}, -\frac{1}{1+n}, 1 - \frac{\alpha}{1+A+n+An}, -\frac{a^{-3(1+A)(1+n)} e^{(1+A)(1+n)C_1}}{B} \right] \right] \end{aligned} \quad (12)$$

The reconstructed Hubble Parameter  $\tilde{H}$  in Eq. (12) is plotted in Fig. 1 against red shift  $z$  for a range of values of  $n$ .

The figure shows that the reconstructed  $\tilde{H}$  exhibits a falling pattern from higher to lower red shifts, i.e., from early to late time universe. The rate of falling is found to be more, as we increase the value of  $n$ . The decreasing pattern of  $\tilde{H}$  with evolution of the universe is consistent with the accelerated expansion of the universe. This also proves the physical viability of considering a correspondence between MCG and MHRDE. In the next phase of our study, we shall use the reconstructed  $\tilde{H}$  to have a reconstructed  $T$  and thereafter we shall reconstruct the  $f(T)$  gravity accordingly.



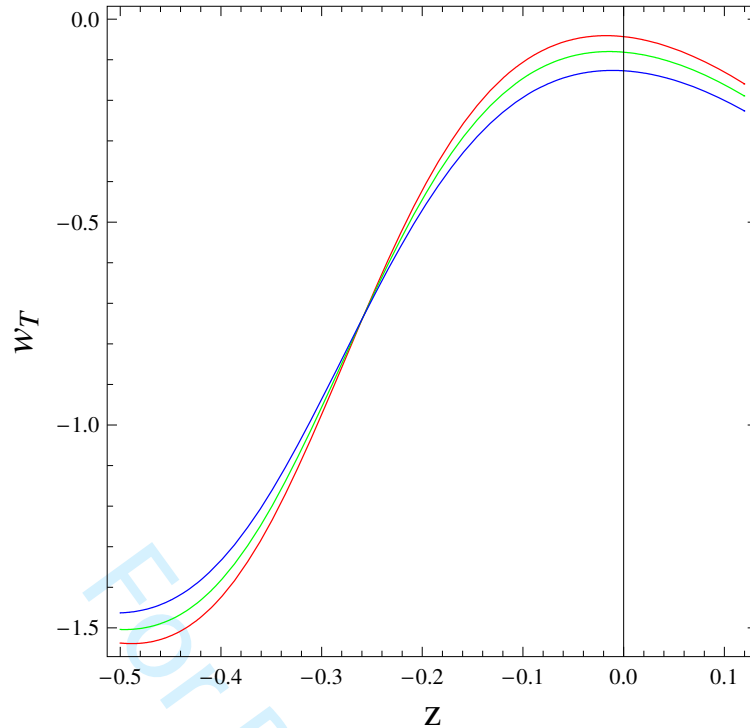


FIG. 2: Equation of state parameter due to torsion contribution  $w_T$  (see Eq. (14)) reconstructed through  $\tilde{H}$ . Red, green and blue lines correspond to  $n = 0.36, 0.40$  and  $0.46$  respectively.

From Eq. (12) we can have

$$\begin{aligned} \dot{\tilde{H}} = & \frac{1}{2} \left( -3a^{-3\alpha} C_2 \alpha + \left( \frac{B+a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{1+A} \right)^{\frac{1}{1+n}} \times \right. \\ & \left. \left( 1 + \frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right)^{-\frac{1}{1+n}} (\alpha - \beta) \left( \left( 1 + \frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right)^{\frac{1}{1+n}} - \right. \right. \quad (13) \\ & \left. \left. 2F1 \left[ -\frac{\alpha}{1+A+n+An}, -\frac{1}{1+n}, 1 - \frac{\alpha}{1+A+n+An}, -\frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right] \right) \right) \end{aligned}$$

In Eq. (13)  $2F1$  function has the series expansion  $2F1(a, b, c; z) = \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$

We consider a flat Friedmann-Robertson-Walker(FRW) universe filled with the pressureless matter and using Eqs. (2)- (6) we obtain the equation of state parameter  $w_T$  due to the torsion contribution as

$$\begin{aligned} w_T = & \left( 3a^{-3\alpha} C_2 (-1 + \alpha) + \left( \frac{B+a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{1+A} \right)^{\frac{1}{1+n}} \right. \\ & \left. (-\alpha + \beta) + \frac{1}{\alpha} \left( \frac{B+a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{1+A} \right)^{\frac{1}{1+n}} \left( 1 + \frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right)^{-\frac{1}{1+n}} (-1 + \alpha)(\alpha - \beta) \times \right. \\ & \left. 2F1 \left[ -\frac{\alpha}{1+A+n+An}, -\frac{1}{1+n}, 1 - \frac{\alpha}{1+A+n+An}, -\frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right] \times \right. \\ & \left. \left( 3a^{-3\alpha} C_2 - \frac{\rho_{m0}}{a^3} + \frac{1}{\alpha} \left( \frac{B+a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{1+A} \right)^{\frac{1}{1+n}} \left( 1 + \frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right)^{-\frac{1}{1+n}} (\alpha - \beta) \times \right. \right. \\ & \left. \left. 2F1 \left[ -\frac{\alpha}{1+A+n+An}, -\frac{1}{1+n}, 1 - \frac{\alpha}{1+A+n+An}, -\frac{a^{-3(1+A)(1+n)} e^{(1+A)C_1(1+n)}}{B} \right] \right)^{-1} \right) \quad (14) \end{aligned}$$

In Fig. 2 we plot Eq. (14) and we observe that the reconstructed  $w_T$  is crossing the phantom boundary of  $-1$  at  $z \approx -0.3$ . Thus, the reconstructed EoS parameter is capable of crossing the phantom boundary and hence it behaves like quintom.

### III. MHRDE IN A VIABLE MODEL OF $f(T)$ GRAVITY

In this section we consider the following viable model of  $f(T)$  gravity:

$$f(T) = \eta_0 T + 2\eta_1 \sqrt{-T} + \eta_2 \quad (15)$$

The model under consideration was proposed in [63]. In order to avoid analytic and computation problems the proposers worked on a suitable expression for  $f(T)$  containing a constant, linear and a non-linear form of torsion. This model is given by

$$f_T = \eta_0 - \frac{\eta_1}{\sqrt{-T}} \quad (16)$$

where,  $\eta_0$ ,  $\eta_1$  and  $\eta_2$  are arbitrary constants and the subscript  $T$  indicates the derivative of  $f$  with respect to  $T$ . The combination of the first and the third term of the model corresponds to the cosmological constant EoS in the background of  $f(T)$  gravity. This model has been further studied by [64] for its cosmic coincidence. In the present work using  $\rho_\Lambda = \rho_T$  we have the following differential equation

$$\frac{1}{\alpha - \beta} \left( 3\alpha H^2 + a \frac{dH^2}{da} \right) = -\eta_2 - 6(-1 + \eta_0)H^2 \quad (17)$$

solving which we obtain the reconstructed Hubble parameter as a function of  $a$ :

$$\tilde{H}(a) = \left[ a^{3\alpha - 6\beta + 6(-\alpha + \beta)\eta_0} C_2 + \frac{(-\alpha + \beta)\eta_2}{-3\alpha + 6\beta + 6(\alpha - \beta)\eta_0} \right]^{1/2} \quad (18)$$

and hence the time derivatives of the reconstructed Hubble parameter are:

$$\dot{\tilde{H}}(a) = \frac{1}{2} a^{3\alpha - 6\beta + 6(-\alpha + \beta)\eta_0} C_2 (3\alpha - 6\beta + 6(-\alpha + \beta)\eta_0) \quad (19)$$

$$\ddot{\tilde{H}}(a) = \frac{9}{2} a^{3\alpha - 6\beta + 6(-\alpha + \beta)\eta_0} C_2 (\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2 \times \sqrt{a^{3\alpha - 6\beta + 6(-\alpha + \beta)\eta_0} C_2 + \frac{(-\alpha + \beta)\eta_2}{-3\alpha + 6\beta + 6(\alpha - \beta)\eta_0}} \quad (20)$$

$$\ddot{\tilde{H}}(a) = -\frac{9}{4} a^{3(\alpha - 4\beta + 2(-2\alpha + \beta)\eta_0)} C_2 (\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2 \times \left( 9a^{3(\alpha + 2\beta)\eta_0} C_2 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + 2a^{6(\beta + \alpha\eta_0)} (-\alpha + \beta)\eta_2 \right) \quad (21)$$

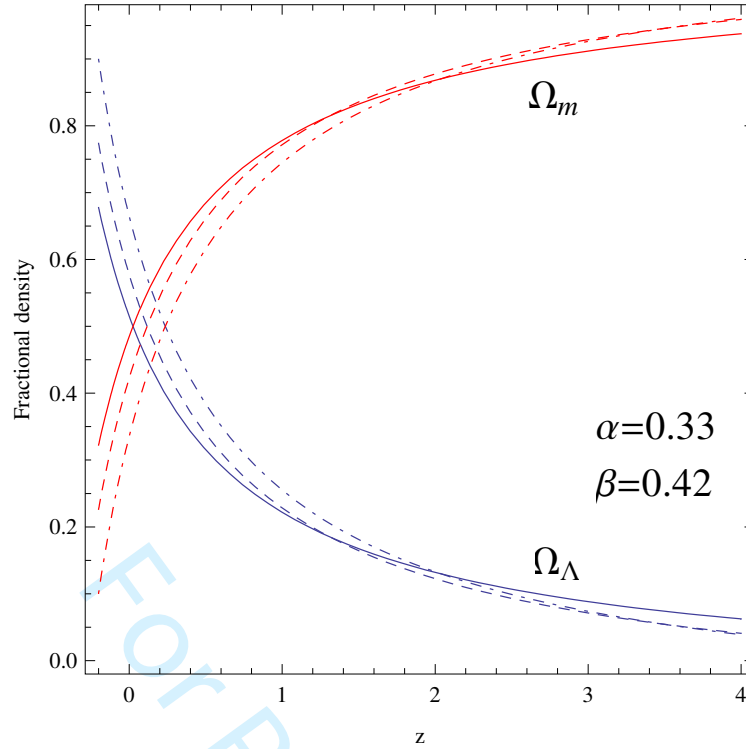


FIG. 3: Evolution of fractional densities  $\Omega_\Lambda$  and  $\Omega_m$  with  $z$  using Eqs. (23) and (24). The solid, dashed and dotdashed lines correspond to  $(\eta_0, \eta_2)$  pairs of (1.02, 1.3), (1.04, 1.5) and (1.05, 1.7) respectively. We have taken  $C_2 = -0.9$ .

from which we reconstruct the MHRDE as

$$\rho_\Lambda = -6a^{3\alpha-6\beta+6(-\alpha+\beta)\eta_0} C_2 (-1 + \eta_0) + \frac{\alpha\eta_2}{\alpha - 2\beta + 2(-\alpha + \beta)\eta_0} \quad (22)$$

Thus, the fractional densities are

$$\Omega_\Lambda = \frac{-6a^{3\alpha-6\beta+6(-\alpha+\beta)\eta_0} C_2 (-1 + \eta_0) + \frac{\alpha\eta_2}{\alpha - 2\beta + 2(-\alpha + \beta)\eta_0}}{3 \left( a^{3\alpha-6\beta+6(-\alpha+\beta)\eta_0} C_2 + \frac{(-\alpha+\beta)\eta_2}{-3\alpha+6\beta+6(\alpha-\beta)\eta_0} \right)} \quad (23)$$

and

$$\Omega_m = \frac{3a^{3(\alpha+2\beta\eta_0)} C_2 (-1 + 2\eta_0) (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + a^{6(\beta+\alpha\eta_0)} \beta\eta_2}{3a^{3(\alpha+2\beta\eta_0)} C_2 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + a^{6(\beta+\alpha\eta_0)} (-\alpha + \beta)\eta_2} \quad (24)$$

In Fig. 3 we have plotted the evolution of the fractional densities  $\Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}$  and  $\Omega_m = \frac{\rho_m}{3H^2}$  satisfying  $\Omega_\Lambda + \Omega_m = 1$ . It is observed that the fractional density of reconstructed  $\rho_\Lambda$  is increasing and fractional density of dark matter is decreasing with evolution of the universe. This is consistent with the evolution of the universe from matter dominated to dark energy dominated universe. The transition from matter dominated to energy dominated universe is occurring at  $z \approx 0.4$  and the current universe ( $z = 0$ ) is found to be dark energy dominated as consistent with observations.

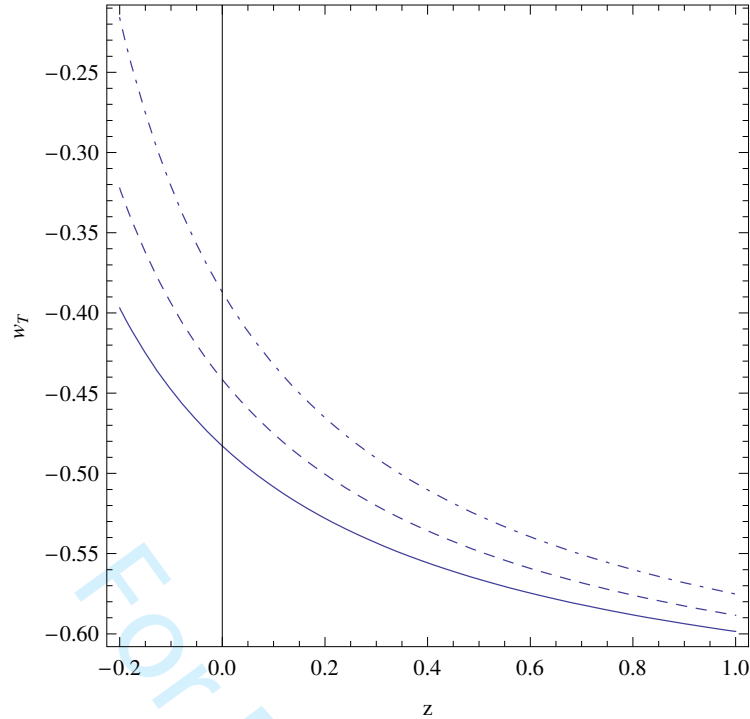


FIG. 4: Evolution of the EoS parameter  $w_T$  based on Eq. (26). The solid, dashed and dotdashed lines correspond to  $(\eta_0, \eta_2)$  pairs of (1.02, 1.3), (1.04, 1.5) and (1.05, 1.7) respectively. We have taken  $C_2 = -0.9$ .

Now we investigate the EoS parameter  $w_T = \frac{p_T}{\rho_T}$ , which is given by

$$w_T = -1 + 4\dot{H} \left( \frac{f_T + 2T f_{TT}}{-f + 2T f_T} \right) \quad (25)$$

In the present case Eq. (25) takes the form

$$w_T = \left( a^{-6\alpha\eta_0} \left( (\alpha - 2\beta) \left( 6a^{3(\alpha+2\beta\eta_0)} C_2 (1 + \alpha - 2\beta)\eta_0 + a^{6(\beta+\alpha\eta_0)} \eta_2 \right) + 2(\alpha - \beta) \left( 6a^{3(\alpha+2\beta\eta_0)} C_2 \eta_0 \eta_0 (-1 - 2\alpha + 4\beta + 2(\alpha - \beta)\eta_0) + a^{6(\beta+\alpha\eta_0)} (-\eta_2 \eta_0 + \eta_0 \eta_2) \right) \right) \right) \times \left( 6a^{3\alpha+6(-\alpha+\beta)\eta_0} C_2 \eta_0 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + a^{6\beta} ((-\alpha + 2\beta)\eta_2 + 2(\alpha - \beta)(\eta_2 \eta_0 - \eta_0 \eta_2)) \right)^{-1} \quad (26)$$

The EoS parameter plotted in Fig. 4 exhibits quintessence behaviour i.e.  $w_T > -1$ .

#### IV. STATEFINDER HIERARCHY FOR THE RECONSTRUCTED MHRDE

In this section we intend to work in the framework of spatially homogeneous and isotropic FRW universe. In this case, the scale factor  $a(t)$  is the only dynamical variable. As we are interested in the late time behavior of expansion of the universe, the Taylor expansion of the scale factor around

the present epoch  $t_0$  is considered as [65]:

$$(1+z)^{-1} = \frac{a(t)}{a_0} = 1 + \sum_{n=1}^{\infty} \frac{\alpha_n(t_0)}{n!} [H_0(t-t_0)]^n \quad (27)$$

where  $\alpha_n(t_0) = \frac{d^n a}{a H^n}$ .

At this juncture, let us have a small discussion on Eq. (27), where  $\alpha_n := \frac{a^{(n)}}{a H^n}$  with  $n \in N$  and  $a^{(n)}$  is the  $n^{\text{th}}$  derivative of the scale factor with respect to cosmic time  $t$ . Different letters of the alphabet are used to describe various derivatives of the scale factor.  $q = -\alpha_2$  is the deceleration parameter,  $\alpha_3$  is the statefinder  $r$  as well as jerk  $j$ ,  $\alpha_4$  is the snap,  $\alpha_5$  is the lerk  $l$ , etc. A detailed discussion in this connection is available in the work of [65]. It may be noted that the other statefinder parameter  $s$  can be obtained from  $r$  and  $q$  following [66].

The deceleration parameter is defined as:

$$q = -\frac{\ddot{a}}{a H^2} = -\frac{\dot{H}}{H^2} - 1 \quad (28)$$

The statefinder pair  $\{r, s\}$  and the Snap ( $\alpha_4$ ) are defined as [68]:

$$r = \alpha_3 = \frac{\ddot{a}}{a H^3} = \frac{\ddot{H}}{H^3} + 3 \frac{\dot{H}}{H^2} + 1 \quad (29)$$

$$s = \frac{r-1}{3(q-\frac{1}{2})} \quad (30)$$

$$\alpha_4 = \frac{\dddot{a}}{a H^4} = 1 + \frac{\ddot{H}}{H^4} + 4 \frac{\dot{H}}{H^3} + 3 \frac{\dot{H}^2}{H^4} + 6 \frac{\dot{H}}{H^2} \quad (31)$$

Using Eqs.(18)-(21) in Eqs. (28)-(31) we have

$$q = -1 + \frac{9(\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2}{2\left(-3\alpha + 6\beta + (\alpha - \beta)\left(6\eta_0 - \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)\right)} \quad (32)$$

$$r = 1 + \frac{27(-\alpha + 2\beta + 2(\alpha - \beta)\eta_0)^3}{2\left(-3\alpha + 6\beta + (\alpha - \beta)\left(6\eta_0 - \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)\right)} + \frac{27(\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2}{2\left(3(\alpha - 2\beta) + (\alpha - \beta)\left(-6\eta_0 + \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)\right)} \quad (33)$$

$$s = \left[ \frac{1}{-\alpha + 2\beta + 2(\alpha - \beta)\eta_0} + \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}(\alpha - \beta)\eta_2}{3C_2(-1 - \alpha + 2\beta + 2(\alpha - \beta)\eta_0)(\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2} \right]^{-1} \quad (34)$$

$$\begin{aligned}
\alpha_4 = 1 + & \frac{243a^{6(\alpha+2(-\alpha+\beta)\eta_0)}C_2^2(\alpha-2\beta+2(-\alpha+\beta)\eta_0)^4}{4(3a^{3\alpha+6(-\alpha+\beta)\eta_0}C_2(-\alpha+2\beta+2(\alpha-\beta)\eta_0)+a^{6\beta}(-\alpha+\beta)\eta_2)^2} - \\
& \frac{81a^{3\alpha+6(-2\alpha+\beta)\eta_0}C_2(\alpha-2\beta+2(-\alpha+\beta)\eta_0)^4 \times}{(9a^{3(\alpha+2\beta)\eta_0}C_2(-\alpha+2\beta+2(\alpha-\beta)\eta_0)+2a^{6(\beta+\alpha)\eta_0}(-\alpha+\beta)\eta_2)} + \\
& \frac{54(-\alpha+2\beta+2(\alpha-\beta)\eta_0)^3}{4(3a^{3\alpha+6(-\alpha+\beta)\eta_0}C_2(-\alpha+2\beta+2(\alpha-\beta)\eta_0)+a^{6\beta}(-\alpha+\beta)\eta_2)^2} + \\
& \frac{-3\alpha+6\beta+(\alpha-\beta)\left(6\eta_0-\frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)}{27(\alpha-2\beta+2(-\alpha+\beta)\eta_0)^2} + \\
& \frac{3(\alpha-2\beta)+(\alpha-\beta)\left(-6\eta_0+\frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)}{3(\alpha-2\beta)+(\alpha-\beta)\left(-6\eta_0+\frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)} \quad (35)
\end{aligned}$$

In the above equations all the parameters are same as they are stated after the equations where they appeared first.

Arabsalmani and Sahni [65] introduced the notion of the ‘Statefinder hierarchy’, which includes higher derivatives of the expansion factor  $d^n a/dt^n$ . The Statefinder hierarchy  $S_n$  is given by:

$$S_2 = \alpha_2 + \frac{3}{2}\Omega_m \quad (36)$$

$$S_3 = \alpha_3 \quad (37)$$

$$S_4 = \alpha_4 + \frac{3^2}{2}\Omega_m \quad (38)$$

where  $\Omega_m = \frac{\rho_m}{3H^2} = \frac{2}{3}(1+q)$ . Using this form of  $\Omega_m$  the alternate form of  $S_4$  is:

$$S_4^{(1)} = \alpha_4 + 3(1+q) \quad (39)$$

where  $q$  is the deceleration parameter as stated in Eq. (28). The second Statefinder corresponding to  $S_3^{(1)} = S_3$  is defined as follows:

$$S_3^{(2)} = \frac{S_3^{(1)} - 1}{3\left(q - \frac{1}{2}\right)} \quad (40)$$

Similarly the second statefinder is:

$$S_4^{(2)} = \frac{S_4^{(1)} - 1}{9\left(q - \frac{1}{2}\right)} \quad (41)$$

$$\begin{aligned}
S_4^{(1)} = 1 + & \frac{243a^{6(\alpha+2(-\alpha+\beta)\eta_0)} C_2^2 (\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^4}{4 (3a^{3\alpha+6(-\alpha+\beta)\eta_0} C_2 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + a^{6\beta}(-\alpha + \beta)\eta_2)^2} - 81a^{3\alpha+6(-2\alpha+\beta)\eta_0} \times \\
& \frac{C_2 (\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^4 (9a^{3(\alpha+2\beta\eta_0)} C_2 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + 2a^{6(\beta+\alpha\eta_0)}(-\alpha + \beta)\eta_2)}{4 (3a^{3\alpha+6(-\alpha+\beta)\eta_0} C_2 (-\alpha + 2\beta + 2(\alpha - \beta)\eta_0) + a^{6\beta}(-\alpha + \beta)\eta_2)^2} \\
& + \frac{54(-\alpha + 2\beta + 2(\alpha - \beta)\eta_0)^3}{-3\alpha + 6\beta + (\alpha - \beta) \left(6\eta_0 - \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)} \\
& + \frac{27(\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2}{2 \left(-3\alpha + 6\beta + (\alpha - \beta) \left(6\eta_0 - \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)\right)} \\
& + \frac{27(\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2}{3(\alpha - 2\beta) + (\alpha - \beta) \left(-6\eta_0 + \frac{a^{-3\alpha+6\beta+6(\alpha-\beta)\eta_0}\eta_2}{C_2}\right)}
\end{aligned} \tag{42}$$

$$\begin{aligned}
S_4^{(2)} = & - \left[ 2 \left( -3a^{3\alpha} C_2 (\alpha - 2\beta) + (\alpha - \beta) \left( 6a^{3\alpha} C_2 \eta_0 - a^{6(\beta+(\alpha-\beta)\eta_0)} \eta_2 \right) \right) \times \right. \\
& \left. \left( 3a^{3\alpha} C_2 (\alpha - 2\beta) (1 + \alpha - 2\beta) + (\alpha - \beta) \left( 6a^{3\alpha} C_2 \eta_0 (-1 - 2\alpha + 4\beta + 2(\alpha - \beta)\eta_0) + a^{6(\beta+(\alpha-\beta)\eta_0)} \eta_2 \right) \right) \right]^{-1} \times \\
& a^{3\alpha} C_2 (-1 - \alpha + 2\beta + 2(\alpha - \beta)\eta_0) (\alpha - 2\beta + 2(-\alpha + \beta)\eta_0)^2 \times \\
& (3a^{3\alpha} C_2 (2 + 9\alpha - 18\beta) (\alpha - 2\beta) + \\
& 2(\alpha - \beta) \left( 6a^{3\alpha} C_2 \eta_0 (-1 - 9\alpha + 18\beta + 9(\alpha - \beta)\eta_0) + a^{6(\beta+(\alpha-\beta)\eta_0)} (1 + 3\alpha - 6\beta + \right. \\
& \left. 6(-\alpha + \beta)\eta_0) \eta_2 \right)
\end{aligned} \tag{43}$$

All the parameters appeared in the above equations are as they are defined in the places where they appeared first.

#### A. Comparison of statefinder parameters with observational data

The statefinder parameters are discussed in the previous section. In this subsection, we provide the comparison of them with observational data. In this connection, we would like to mention the work of Kumar [67], where the author found the constraints on statefinder parameters from the latest  $H(z)$  and SNe Ia data. According to [67], the  $H(z)$  and SNe Ia data constrain the statefinders as  $r = -0.09_{-0.03}^{+0.04}$ ,  $s = 0.58_{-0.12}^{+0.04}$  and  $r = -0.09_{-0.02}^{+0.03}$ ,  $s = 0.41_{-0.03}^{+0.03}$  respectively. Kumar (2012) also found that joint test of  $H(z)$  and SNe Ia data puts the constraint of  $r = -0.11_{-0.01}^{+0.02}$  and  $s = 0.44_{-0.03}^{+0.03}$ . The errors in the study of [67] were at  $1\sigma$  level. Fig. 5 shows that the constraints are satisfied by the statefinder parameters up to the present universe i.e.  $z = 0$ . Also, the statefinder

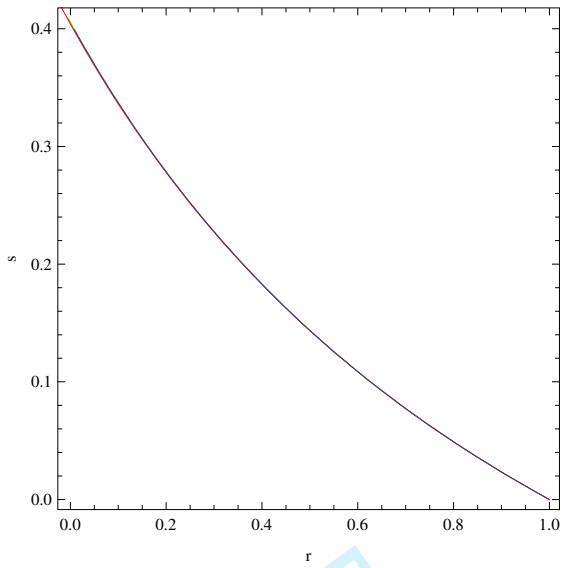


FIG. 5: Time evolution of the statefinder pair  $\{r, s\}$  for  $f(T)$ -reconstructed MHRDE.

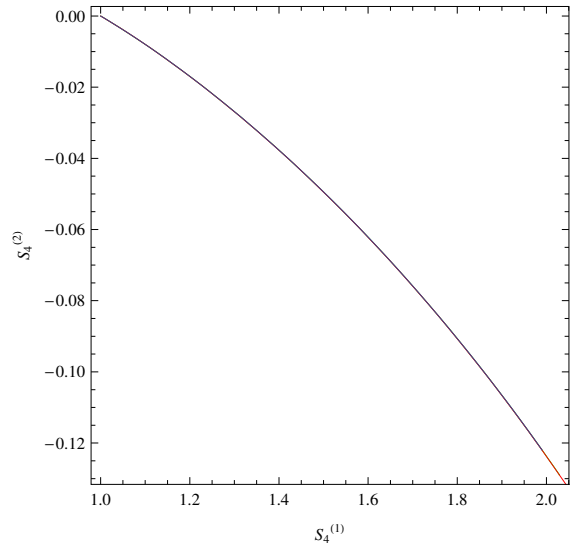


FIG. 6: Time evolution of the  $\{S_4^{(1)}, S_4^{(2)}\}$  for  $f(T)$ -reconstructed MHRDE.

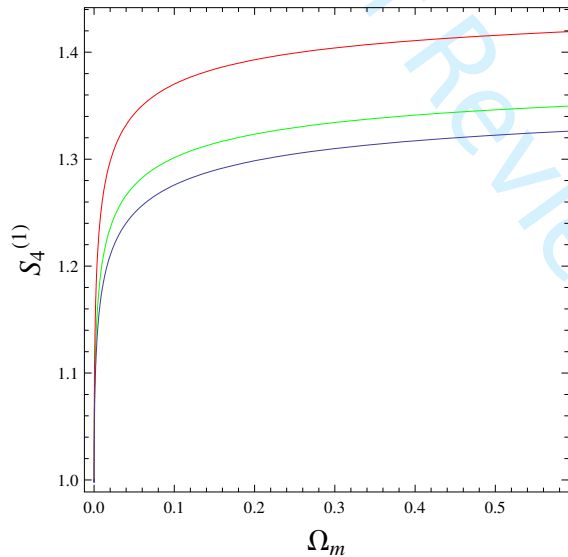


FIG. 7: Evolution of statefinder  $S_4^{(1)}$  plotted against  $\Omega_m$ . (Eq.(43))

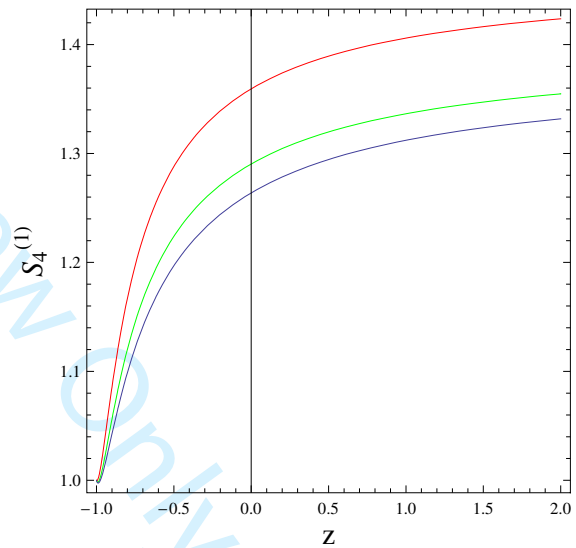


FIG. 8: Evolution of statefinder  $S_4^{(1)}$  plotted against  $z$  (Eq. (43)).

trajectories lie in the region of  $r < 1, s > 0$  for different choices of  $\eta_0$  and  $\eta_2$ . Furthermore, the trajectories of evolution, after crossing today's point at  $r \approx 0.47, s \approx 0.16$  approach towards  $\{r = 1, s = 0\}$  i.e.  $\Lambda$ CDM fixed point. Thus, the model tends to evolve like  $\Lambda$ CDM universe. This is in agreement with Cao et al. (2018).

With reference to Eq. (14), we furnish below the present values (i.e. for  $z = 0$ ) of  $w_{eff} = \frac{p_T}{\rho_m + \rho_T}$  taking pressureless dark matter:



TABLE I: Vales of  $w_{eff}$  using Eq.(14) for various combinations of  $(\alpha, \beta)$ , different values of  $n$ . We have taken  $\rho_{m0} = 0.23$ ,  $C_2 = 3$ ,  $C_1 = 1 \times 10^{-4}$ ,  $A = 0.6$ ,  $B = 1 \times 10^{-4}$ .

$n$	$\alpha$	$\beta$	$w_{eff}$
0.1	0.002	0.75	-0.945233
0.2	0.02	0.68	-0.931871
0.15	0.002	0.8	-0.940616

It is observed in Table I that for MCG- reconstructed MHRDE in the framework of  $f(T)$  gravity the effective EoS parameter becomes very close to  $-1$  for  $z = 0$  and hence it is compatible with the observational value of the EoS parameter. However, EoS due to torsion contribution crosses phantom boundary at a later stage.

## V. CONCLUDING REMARKS

In the present work we have reported a study on the reconstruction of modified holographic Ricci dark energy (MHRDE) in the framework of modified gravity taken as  $f(T)$  gravity. The basic objective is to discuss dynamically the actual embedding of modified HDE in terms of  $f(T)$  gravity. In fact, such effective equivalence at the level of fluid and scalar DE models was discussed in general terms in Bamba et al. [48, 49], and the work continues that line. In the first phase of the study we have discussed a correspondence between the MHRDE and modified Chaplygin gas (MCG). In this context we would like to draw the attention of the readers to the review by Bamba et al. [48] on how present one DE in terms of other DE candidates. Also, we would like to note the first work on unification of DE and inflation in modified gravity by Nojiri and Odintsov [28]. Inspired by these considerations we have reconstructed the Hubble parameter  $H$  (see Eq. (13)), whose evolution with  $z$  has been studied (see Eq. (12)). The decreasing pattern of the reconstructed Hubble parameter  $H$  has been observed and hence it has been concluded that the reconstructed Hubble parameter is consistent with the accelerated expansion of the current universe. This has also proved the physical viability of considering a correspondence between MCG and MHRDE. In the next phase of our study, we shall use the reconstructed  $\tilde{H}$  to have reconstructed  $T$  and thereafter we shall reconstruct the  $f(T)$  gravity accordingly. Subsequently we have used the reconstructed  $H$  to have a reconstructed  $T$  and thereafter we have reconstructed the  $f(T)$  gravity accordingly. Behaviour of the reconstructed  $f(T)$  gravity has been observed through

the reconstructed  $w_T$  i.e. the EoS parameter due to the torsion contribution. It has been observed (see Fig. 2) that the reconstructed  $w_T$  is crossing the phantom boundary of  $-1$  at  $z \approx -0.3$ . Thus, the reconstructed EoS parameter is capable of crossing the phantom boundary and hence it behaves like quintom. In the subsequent phase of the study we have considered a viable model of  $f(T)$  gravity proposed by [15]. Considering that the density contribution due to the torsion scalar  $\rho_T$  is equal to that of the density of MHRDE  $\rho_\Lambda$  we have reconstructed Hubble parameter and also obtained the different time derivatives of the Hubble parameter. As we have considered the non-interacting existence of dark energy and pressureless dark matter, we have computed the fractional densities of the reconstructed MHRDE through  $f(T)$ -reconstructed Hubble parameter and also of the dark matter (see Fig. 3). It has been observed that fractional density of dark energy is increasing and fractional density of dark matter is decreasing with evolution of the universe. Also, for  $z = 0$  we observe that the fractional density of  $f(T)$ -reconstructed dark energy is dominating the dark matter. Thus, the  $f(T)$ -reconstructed MHRDE model is consistent with the current dark energy dominated universe and also exhibits the transition of the universe from matter dominated phase to dark energy dominated universe. However, contrary to the previous model we observe that the  $w_T$  is staying at  $w > -1$  level. Thus, this EoS is behaving like quintessence. In the last phase of the study we have demonstrated the statefinder hierarchy due to Sahni et al. (2003). It is observed from the plots of statefinder parameters (see Figs. 5 and 6) that the model can reach  $\Lambda$ CDM fixed point. At this juncture it may be noted that Nojiri and Odintsov [21] formulated several versions of modified gravity compatible with Solar System tests generated the epochs of the universe in a sequence beginning with matter dominated phase (with or without usual matter), then transition from deceleration to acceleration, accelerating epoch consistent with recent WMAP data and  $\Lambda$ CDM cosmology without cosmological constant.

While concluding the work should be compared to other construction works done till date in this direction. Firstly, in comparison with Daouda et al. [70] it may be stated that contrary to what was observed in their work, here while considering MHRDE in the framework of  $f(T)$  gravity and discussing the EoS parameter thereby we observed that the reconstructed  $w_T$  is not crossing the phantom boundary i.e. the model is not leading to a phantom universe, in which future finite time singularity (Big rip or type III singularity) can occur. However, similar to Douda et al. [70] we have observed the crossing of phantom boundary by  $w_T$  if we consider a reconstructed  $H$  obtained through a correspondence between MCG and MHRDE. Thus, it may be interpreted that incorporation of MCG into the background evolution of the universe may lead to future finite time singularity. If the work is compared to Chattopadhyay and Pasqua [58],

in which a reconstruction of  $f(T)$  gravity was carried out holographically it may be stated that similar to them the universe can reach the  $\Lambda$ CDM fixed point  $\{r = 1, s = 0\}$  (see Figs. 5 and 6). Moreover, the statefinder hierarchy [68] has been used for higher derivatives of scale factor, which can easily distinguish different dark energy models. Myrzakulov and Shahalam [68] demonstrated the usefulness of statefinder  $S_2$  as an excellent discriminant of  $\Lambda$ CDM and modified gravity models. In the present case we have presented a statefinder hierarchy to view the influence of the model parameters of  $f(T) = \eta_0 T + 2\eta_1 \sqrt{-T} + \eta_2$ . In Figs. 7 and 8 we have observed that for none of the combinations of the values of the model parameters the  $S_4$  has any degeneracy i.e. is always staying in the positive level irrespective of whether the universe is matter dominated or DE dominated. However, it has also been noted that in the latter stage of the universe is prominence of the role of model parameters is gradually becoming indiscriminating. Observing Figs. 5-8 based on it is also apparent that higher order statefinders are more efficient in discriminating the role of the model parameters in the early stage of the universe. Another notable work to be mentioned here is by Setare [59], who reconstructed modified gravity through holographic DE. Compared to [59], in the present work, a further modification has been made through a correspondence between MHRDE and MCG and the reconstruction has been primarily schemed through reconstructed Hubble parameter. As a future study, the thermodynamics of the proposed reconstruction is proposed. In a recent study, Chattopadhyay [60] reported a reconstruction scheme for  $f(T)$  gravity and demonstrated the cosmological evolution of the primordial perturbations through scalar metric fluctuations and the reconstructed model was found consistent with the generic expansion of the universe. The current work, when compared to [60], shows that irrespective of the construction approach, the  $\Lambda$ CDM fixed point is being achieved.

At the end, we would like to mention it further that Ricci dark energy (RDE) is just a specific version of most general holographic dark energy (HDE) with Nojiri-Odintsov cut-off. We have limited the present study by this model only into consideration. Studies have been made by various authors attempting to embed  $f(T)$  gravity to various versions of HDE (e.g. [69, 70]). As future study, we propose to develop holographic reconstruction models of modified gravity theories constrained by observational data.

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