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A Study Of Multi-Echelon Inventory Systems with Stochastic Capacity and Intermediate Product Demand

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A STUDY OF MULTI-ECHELON INVENTORY SYSTEMS WITH STOCHASTIC CAPACITY AND INTERMEDIATE PRODUCT DEMAND

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I HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER MY SUPERVISION BY Suman Niranjan ENTITLED A Study Of Multi-Echelon Inventory Systems With Stochastic Capacity And Intermediate Product Demand BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Doctor of Philosophy.

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ABSTRACT

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The research in this dissertation involves the study of several multi-echelon inventory systems with stochastic capacity and intermediate product demand. Specifically we analyze the behavior of the system which consists of several intermediate product demands. The analysis is primarily three fold i) developed update (relational) equations for all the multi-echelon inventory systems under several inventory allocation policies, ii) develop two simulation optimization approaches 1) *OptQuest* framework, and 2) IPA (Infinitesimal Perturbation Analysis) framework, used to minimize the total cost of the inventory systems that satisfy the desired customer service level, iii) obtain numerical results for all the multi-echelon inventory systems under several scenarios and instances, and an extensive analysis and implications of the results.

The research done in this dissertation differ from earlier works, since it considers a complex (combination of serial and assembly systems) multi-period multi-echelon inventory system with several sources of demand (specifically intermediate product demands). We obtain the best found base-stock levels for each node in the system that satisfies the required customer service level. A SIO (Simulation based Inventory Optimization) approach is used to obtain the best found base-stock level for the system under several inventory allocation policies. We consider a system which is closer to the actual world and can be used to solve contemporary issues like, 1) manufacturing firm

that produces finished products as well as spare parts, 2) manufacturer – warehouse – distribution center – retail outlets etc. I am not aware of any work that studies the impact of inventory allocation polices for multi-period in a multi-echelon inventory system, and obtains best found base stock level for each node using an IPA framework. Moreover the best found base-stock level for each node is obtained under realistic conditions like stochastic demand, stochastic capacity, and lead time.

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Finally I owe my education and stay in United States to my parents whom I would like to thank from the depth of my heart.

Dedicated to my Father "Chittaranjan Niranjan",
Mother "Niranjan Priyamvatha",
Advisor "Dr. Frank W Ciarallo",
Family and Friends

EXECUTIVE SUMMARY

This research involves the study of multi-echelon inventory systems with intermediate, external product demand in one or more upper echelons. This type of problem is of general interest in inventory theory and of particular importance in supply chain systems with both end-product demand and spare parts demand. The multi-echelon inventory system considered in the research consists of a combination of assembly and serial stages with demand directly from more than one node. The demand and capacity are considered as stochastic in nature. A fixed supply and manufacturing lead time is used between the stages. This research develops a mathematical model for capacitated multi-echelon systems (three-, four-, five-, and *m*-echelons) with demand for intermediate and final products. The equations describing the multi-echelon inventory dynamics including all of these factors allows simulation of the system. The multi-echelon inventory system is shown to be Lipschitz continuous. The update equations for the multi-echelon system are based on the following variables for each node: the outstanding orders, net inventory, and on-hand inventory.

This research utilizes two frameworks to carry out simulation-based optimization of the inventory parameters in the mathematical model: 1) an *OptQuest* framework, and 2) an Infinitesimal Perturbation Analysis (IPA) framework. The primary goal is to determine best found base stock levels for the components, intermediate product and final product based on a required customer service level at each stage. The *OptQuest* framework consists of a combination of *ARENA* (a simulation package) and *OptQuest* (black-box optimization engine in *ARENA*). *OptQuest* in *ARENA* implements a simulation based inventory optimization to determine the best found base-stock levels.

The status equations are updated every period with the help of *ARENA*. The objective function, service level constraints, and bounds for the decision variables are represented in *OptQuest*. The initial points, i.e. base-stock levels, for each node for this simulation optimization are provided by *OptQuest* to *ARENA*. Based on the service level obtained for each node from *ARENA* at the end of the simulation run (consisting of a sufficient number of periods), a new set of points are determined. The process iterates until a best solution (a non- improving base-stock value) for all nodes that satisfy the service level constraints is obtained.

The IPA framework utilizes *ARENA*, VBA (Visual Basic for Applications a tool in *ARENA*), and Xpress (optimization package). A Simulation-based Inventory Optimization (SIO) algorithm is developed to obtain the best found base-stock levels with the help of nonlinear programming methods. *ARENA* in combination with VBA and Xpress are used to develop the SIO algorithm, and solve the nonlinear problem. A Lagrange function that incorporates all the constraints in a single equation is developed. The first order derivative estimation equations for all the status equations, service level equations, and Lagrange equations are used in the SIO. Infinitesimal Perturbation Analysis (IPA) is used for gradient estimation. The objective function is shown to be Lipschitz continuous. A modified Zoutendijk's feasible direction algorithm is used to obtain the best found base-stock levels. Xpress is used to solve a linear program (LP) within this procedure. A golden section algorithm is use to solve the line search (a part of the feasible direction algorithm). *ARENA* is used to update (simulate) the first order equations, and status equations. The feasible direction algorithm, the line search algorithm, and termination conditions are implemented in VB. Initial points, (i.e. base-stock values, and Lagrange multipliers for each node) are specified in *ARENA*. The information is exchanged between *ARENA* and VBA to guide the search to an best found base-stock level using the IPA-based gradients. VBA is also used to invoke Xpress to solve the LP when required. The search continues until the termination condition is satisfied, which is based on the magnitudes of the gradients and the required service level for each node.

A number of scenarios are solved using both the frameworks, revealing some interesting interactions among inventory, cost, and service levels. A comparative analysis for few specific cases between the two frameworks show that the IPA-based approach more often provides a better solution much quicker than *OptQuest* framework. One interesting aspect of the system is how scarce inventory is allocated between different sources of demand. Four allocation policies have been used for computing the current results for three-echelon and five-echelon assembly systems. In order to show that the multi-echelon inventory model developed in this research is applicable on a wide range of multi-echelon network models, results for two large network models are computed. The two network models focus on the contemporary issues and further show the robustness, and practicality of the multi-echelon inventory system studied. Two heuristic approaches were constructed, 1) rule based approach, 2) decomposition approach, which are used as initial points in the IPA-based search.

1. INTRODUCTION

1.1 Background

Operations and production managers constantly face the challenge of determining the correct level of inventories, and service levels at each stage of the supply chain. Over the last two decades considerable emphasis is put on the coordination of all operations of the material supply chain. Effective management of inventories at every stage of the supply chain offers a great prospective for increasing system efficiency, customer service level and further helps in diminution of total system costs. Typically, the inventory position maintained in each stage should be based on a pre-specified customer service level while not violating budgeting constraints. In order to accomplish these goals mathematical models must be brought closer to the complexity of systems seen in practice.

One of the main difficulties of cost-efficient and effective supply-chain management is to determine the internal service-levels, so that the pre-specified external service levels are met at minimum cost. Unpredictable variability in demand is a factor in many systems. Increased complexity in modern manufacturing has led to significant levels of uncertainty in production as well, making it even more difficult to achieve the target service level. These uncertainties are due to the occurrence of machine failures, maintenance, service, rework, and other unpredictable events. These events cause difficulties in achieving the desired production quantity and service level in view of demand. The rapid development of manufacturing processes based on state-of-the-art technologies can outpace efforts to maintain quality, and can lead to uncertainties in many production processes. These uncertainties in production output can come from the use of relatively new technology in day-to-day manufacturing. Because of competition and shifting consumer preferences, predicting demand for a product is difficult, and it is even more difficult if there are multiple demands. Production needs to be carried out in an economic manner, despite the intrinsic uncertainty of the production processes and demands. In these situations an accurate production planning model would consider both uncertainties in production capacity as well as demand for all the stages in the model. There are relatively few studies that have dealt with this model. An appropriate model for planning the production quantity at each stage should include consideration of these production or supply uncertainties.

Determining appropriate production and procurement quantities and their timings is a critical issue. Management of material flow from suppliers through various stages of manufacturers and finally to the end customers is also crucial, especially because of the large investments involved. The control of the net investment made in raw materials, work in progress, and finished good's is important. Several factors, such as the review period, costs, the nature of demand and supply, and system structure play a crucial role in defining an optimal production policy. In order to effectively control the inventory, all the aforementioned mentioned factors should be considered carefully.

Inventory review and decisions can occur on two different time bases, periodic review, and continuous review. In the case of periodic review time is split into discrete periods, and ordering decisions are made only at the *beginning* of the period, but not *during* the period. In the continuous review case ordering decisions are made at any instance of time. The type of review considered affects how accounting of costs occurs in a model of the decision. A cost function can be developed based on the setup cost, inventory holding cost, shortage or backorder cost, per unit cost. Per unit costs could occur directly from manufacturing or could be as a result of procuring from a supplier.

The nature of both demand and supply affects the difficulty of the decision problem. Demand can be categorized into four different categories: deterministic & stationary, random & stationary, deterministic & time varying, and random & time varying. In the case of deterministic & stationary, the demand is a known quantity and is constant over all periods. In the case of random & stationary, the demand is uncertain and its distribution is unchanging over all the periods. In case of a deterministic & time varying, the demand is a known quantity and it could vary over all periods. Finally, in a random & time varying case the demand is an unknown quantity and its distribution could vary over all periods.

The nature of supply can be categorized based on three different issues:

- Infinite production capacity (supply)/ limited production capacity (supply).
- Perfect yield/ imperfect yield.
- Deterministic production (supply)/ random production (supply).

When supply is infinite the ordered quantity is obtained irrespective of the order size. When supply is capacitated, we may not obtain the entire order quantity i.e., the actual quantity received is the minimum of the quantity ordered and the available capacity. Supply process imperfections lead to defects in items being manufactured. In the face of such process imperfections the planned quantity may not be obtained. In the case of a perfect yield, the entire planned quantity is manufactured without any defective items. In the case of imperfect yield, some of the completed items are defective and not usable. In case of a deterministic supply, the supply is known in advance, whereas in case of a random supply it is an unknown quantity.

Uncertainty in production could be broadly classified into external uncertainty (supply uncertainty) and internal uncertainty (yield, capacity uncertainty). External uncertainty is caused due to the supply uncertainty, usually due to the delay in the delivery of materials on time, or due to shipment delays. Decisions in these cases require consideration of the distribution of supply in order to specify the optimal inventory stocking policy. Internal uncertainties are fundamentally classified into two types: yield models and capacity models. Yield models focus on output loss due to process imperfections, while capacity models deal with production loss due to resource unavailability. In random yield models, the defective units are identified after processing the entire input quantity and incurring the production cost. In uncertain capacity models, one may not be able to process the entire input material due to resource constraints, so that unused input incurs no production cost. In the case of random yield models safety stock is typically used to control the uncertainty involved. In case of capacity uncertainty usually a single critical number policy or dual critical number policy is used to control the uncertainty involved (Ciarallo et al., 1994; Bashyam and Fu, 1998). In recent years production uncertainty has received more attention in the academic literature.

1.2 Research Preview

For over 50 years researchers have been addressing a variety of problems in multi-echelon inventory systems. Multi-echelon inventory systems are broadly categorized as serial or assembly systems (as discussed in a subsequent sub-section). The current research is unique and distinct from previous research because it combines all of the following features:

- A multi-echelon inventory system which is a combination of serial and assembly components
- A multi-echelon system with multiple product demands, where some of the demands are in the form of intermediate products, as well as finished goods (final product)
- The presence of intermediate product demand gives rise to an interesting stock allocation problem at each echelon in the system, i.e. how to assign limited stock to multiple customers (through priorities assigned to intermediate demand and downstream final product demand)
- An Infinitesimal Perturbation Analysis (IPA) framework is used to obtain the best found base-stock levels at each node. The IPA estimates are used in a simulation-based, non-linear optimization search.
- Each node in the multi-echelon problem is subject to a random capacity, and the demands on the system are also random.
- A deterministic lead time for supply exists between nodes.

No existing research has considered all of these factors simultaneously, and provided an optimization based solution to stocking decisions. Each of the points discussed above has been discussed separately in previous research, but no research has addressed all the issues. In particular, a combination of serial, and assembly components with intermediate product demand and capacity allocation has not been widely explored. This makes the problems considered here closer to a generic multi-echelon system and most widely applicable to actual inventory and supply networks. The following section explores how the features included in this research apply to systems encountered in practice.

1.2.1 Practical Applicability

The multi-echelon model considered in this research is sufficiently generic so that it can be used to solve many practical problems. This sub-section describes some contemporary scenarios that can be tackled using the proposed framework. The two network structures discussed below are presented in detail in Chapter 10.

Manufacturer supporting a major finished-product and spare-parts

Many manufacturing companies produce a combination of a finished product, and supporting spare-parts (consisting of a major sub-assembly or component part). For example car manufacturers produce finished vehicles as well as spare vehicle body components for crash repairs. There is often a decision on how many of the component parts will be sold as spares and how many will be used for satisfying the final product demand. The spare-parts market is often very lucrative because of significant price markups and less competition, leading to a high profit margin on the spare-parts. In general, final products have a high allocation priority in order to maintain a high service level for the final products, and not to lose the good will of the customer in the market for the final product. Higher sales of the final-product will also result in higher sales of spare parts, since they are directly related to one another. Moreover there is usually more competition in the market for final products compared to spare parts, so the profit margin for the final product is relatively lower. So there is an important trade-off that needs to be considered, which determines the right amount of inventory to support spare-parts and final-products. The models and procedures described in this research address this type of tradeoff.

Manufacturer-Warehouse-Distribution Center-Retail Industry

The multi-echelon problems considered in this research can also be used to address problems in a supply chain consisting of a manufacturer with a warehouse, which supplies customer distribution centers (DC) that replenish retail locations. This type of supply chain is common in industries such as groceries and electronics, among others. The manufacturer (e.g. Sony) performs some fabrication or assembly for a product, which it stores at a warehouse. Demand on the manufacturer's warehouse comes from multiple customers (e.g. Best Buy, Circuit City) who take the product from the warehouse to their DC. The manufacturer must make a decision on how to allocate scarce stock to the demand from the various DC's. Further downstream a decision needs to be made at the DC's on the allocation of scarce stock to each of several retail outlets (e.g. individual Best Buy stores). The multi-echelon problem considered in this research addresses the inventory stocking and allocation decisions in this scenario.

1.3 Multi-Echelon System Elements

In planning and control of supply chains there are several elements that define, the characteristics of the problem: the network structure, ordering policies, system control, service measures, and formulation. The first sub-section discusses the ordering policy, followed by the network structures, system control, service measures, and formulation.

1.3.1 Ordering Policy

An inventory order policy can generally be broken down into two distinct parts:

- Determining the order quantity, or the amount of inventory that will be purchased or produced with replenishment.
- Determining the reorder point, or the inventory level at which a replenishment will be triggered.

System structures vary and can be classified as: single or a multiple item, single stage or multiple stage, and having assembly or distribution characteristics. The extensions of the single item models lead to larger systems in two specific directions, single-stage systems producing multiple items, or a multi-stage system producing single or multiple items. A single-stage/multi-item system could consist of a type of equipment which is used for processing/producing multiple items. In the case of single-stage, multiple item systems the number of items produced/processed is usually constrained by the production capacity, budget issues, or demand, in turn linking decisions between the items. Decisions in the case of multi-stage systems depend on effective coordination between different stages in the network.

Several standard policies are used to control inventory. These policies include *EOQ*, *base stock*, (*Q, r*), *order-up-to,* and (*S, s*). EOQ works in a deterministic, continuous review frame work, and typically the order is placed when the inventory reaches zero. In case of a continuous review *base stock policy,* orders are placed each time a demand occurs, with a goal of returning inventory to the base-stock level. In a continuous review (*Q, r*), an order is placed when the inventory drops to *r*, with an order quantity of *Q*. For the periodic review case, the order-up-to policies (which are similar to base-stock) place an order each period that will return inventory back to the *order-up-to level*. The periodic review *(S, s)* policy is similar to the continuous review (*Q, r*) policy. The parameters of these policies are based on the costs, nature of demand & supply, and system structure (Hopp and Spearman, 2000). The standard models discussed in the literature are known as: The classic news vendor model, economic order quantity (EOQ) or economic manufacturing quantity (EMQ) model, and time varying and deterministic model (Wagner-Whitin). Figure 1.1 summarizes the preceding discussion.

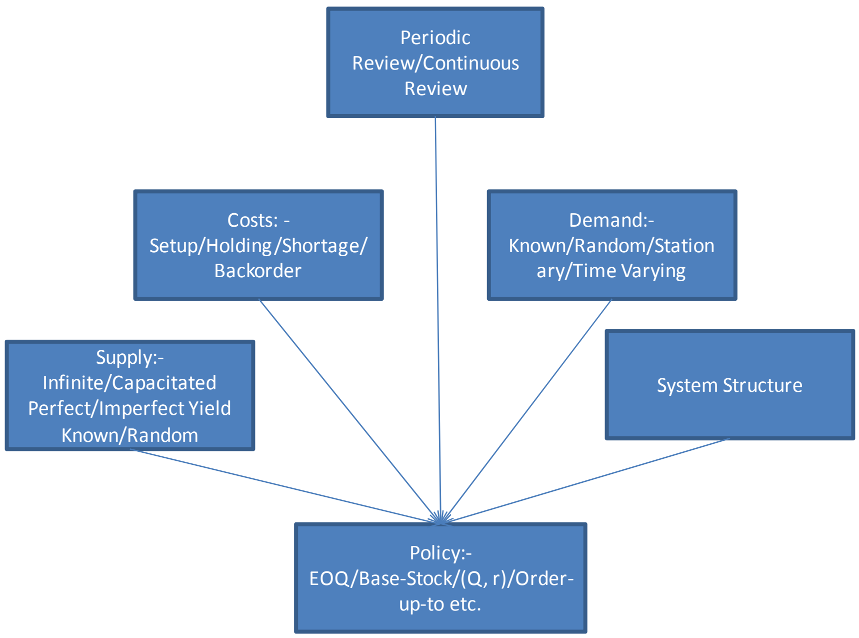

Figure 1.1: Factors involved in making a decision on type of policy

EOQ/EMQ is one of the earliest models that dealt with setting of manufacturing lot sizes. The classical EOQ/EMQ model is used when demand or production is deterministic in nature. The classical news vendor model is used when a scenario similar to the typical newspaper vendor ordering decision arises. In this case there is a single period demand with uncertainty. The news vendor model could also be extended beyond a single period (multiple-periods). In the classical news vendor model the demand occurs only once, since it is a single period problem, and only single replenishment can be done. The issue is to determine an appropriate order quantity in face of uncertain demand.

The classical Wagner-Whitin model represents a situation with deterministic but dynamic demand in every period. In case of the optimal Wagner-Whitin solution the inventory carried over from period t to $t+1$ will be zero or the quantity produced in period $t+1$ will be zero. This simply means that, if inventory is carried over consecutive periods, production will not take place, and similarly production for a period could take place if the inventory carried over from the past period is zero and has a positive demand (D_t). These characteristics of optimal solutions are tightly coupled to the assumption of no uncertainty in this model.

Variations in each classical model have been extensively studied and optimal policies derived in the academic literature. In traditional inventory management studies the production process is usually deterministic in nature with known lead times and no setup costs. But more recently, the stochastic component in demand, supply, yield, and capacity has been studied with fixed setup cost and variable lead times. When one or more uncertainties are involved in a model, deriving optimal policies tends to become increasingly complex.

In a base stock policy, inventory is replenished one unit at a time as the demand occurs. Because of this, the policy consists of a single critical number and specifying the policy requires determining this "base stock level". The base stock level is the target inventory level set for the system. Because of the structure of ordering, the base stock level is the total amount of inventory in the system at any time (in stock and on order). The extension of news vendor and base stock models in the literature include uncertainty in supply/capacity apart from uncertainty in demand.

The (Q, r) policy structure that guides the management of a system can be as simple as determining the re-order quantity using standard EOQ model to a relatively complex re-order point, and re-order quantity computation. The re-order quantity is defined as the amount of inventory that will be produced when a replenishment order is placed, whereas the re-order point is defined as the inventory position at which the production will be triggered. In models with uncertainty, determining the optimal re-order point is difficult. In these models safety stock is employed to reduce the risk of running out of stock. In practice, to some extent, the insights from the classical models, and some modified models are encapsulated in packages like ERP, MRP, and MRP II. The decisions driven by these packages help in smoothing production and controlling inventory.

The remaining portion of this chapter reviews some of the important concepts of supply chain management with respect to colloquial research on multi-echelon inventory optimization with multiple intermediate product demands. Concepts like multi-echelon system elements, information sharing, allocation policies, and simulation based optimization are discussed in the subsequent sub-sections followed by my contributions to research. This chapter also summarizes the main points of the research done. Chapter 2 discusses the literature review, Chapter 3 discusses the multi-echelon model, Chapter 4 discusses the multi-echelon simulation based optimization model and initial results using *ARENA* and *OptQuest* as tools, Chapter 5 presents the gradient estimation results, Chapter 6 discusses in detail the algorithm used for the simulation based optimization using IPA, Chapter 7 includes details on the numerical analysis for three-echelon assembly system., Chapter 8 describes the 1) four different inventory allocation polices used, 2) the change in the update equations for the three-echelon and five-echelon assembly systems, 3) numerical results and inferences based of several cases, 4) statements based of hypothesis testing. Chapter 9 discusses, 1) the contemporary issues the multi-echelon inventory models can address, 2) update equations for the two large multi-echelon network models, 3) numerical results based of several cases considered. Chapter 10 describes the two heuristic approaches used for generating good starting points for the IPA based search, 1) rule based approach, 2) decomposition approach. Chapter 11 provides the conclusion of this research, future research directions, and additional results are provided in the appendix.

1.3.2 Network Structure

The defining feature of a multi-echelon supply chain is that the downstream stages are supplied by the upstream stages. However within this framework there are several possible variations, and if transshipments between locations at same level are allowed, the very definition of a stage becomes unclear. For this reason only two types of network structure are discussed:

- General Arborescent systems (Graves and Schwarz , 1977)
    - Serial State Arborescent System
    - One Manufacturer - One Warehouse - Four Retailer Arborescent System
    - A Seven Stage General Arborescent system
- Convergent Systems

In *arborescent systems* each inventory location is supplied by a single source. A few examples of a general arborescent system are described: the serial system, one warehouse- three retailer system, and a seven stage system. In a serial state arborescent system there are a series of stages in which each stage supplies only a single destination, and the supply is received from a single source. Figure 1.2 shows the schematic representation of a serial state arborescent system.

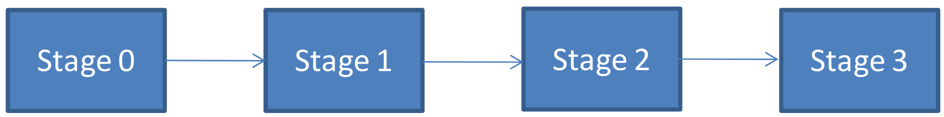

Figure 1.2: Schematic Representation of Serial State Arborescent System

A one manufacturer - one warehouse- four retailer arborescent system is characterized by a node that is supplied by exactly one other node, and supplies four nodes. The schematic representation of a divergent system is shown in Figure 1.3.

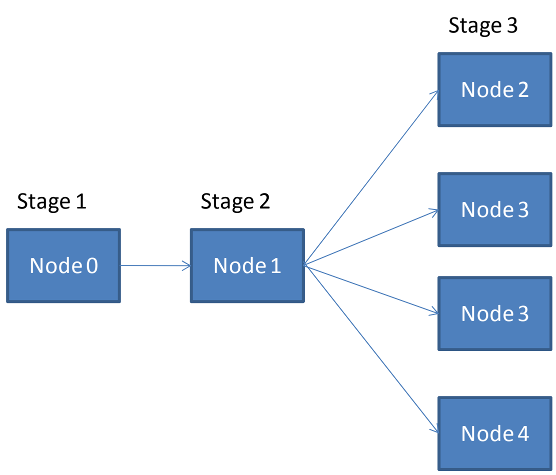

Figure 1.3: Schematic representation One Manufacturer - One Warehouse - Four Retailer Arborescent System

The schematic representation of a seven stage general arborescent system is shown in Figure 1.4. The system shown below is a combination of the earlier two systems, with more than one node diverging.

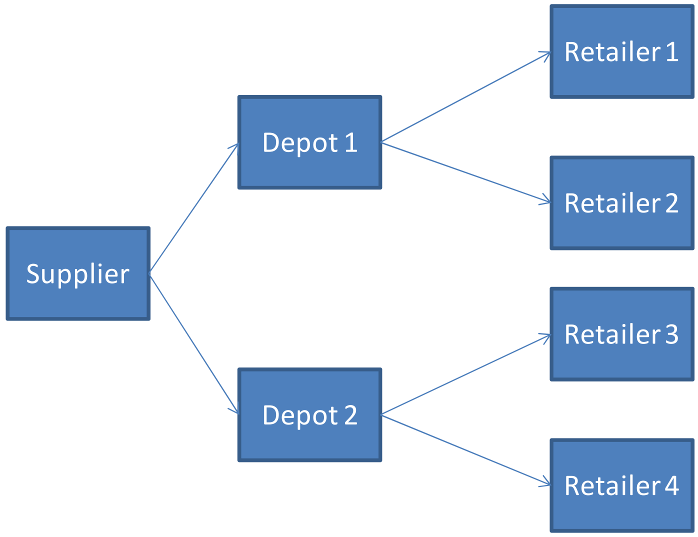

Figure 1.4: Schematic Representation of a Seven Stage General Arborescent system

A *convergent multi-echelon system* is characterized by the property that a node is supplied by more than one other stage, and supplies exactly one stage. The schematic representation of a convergent system is shown in Figure 1.5. Usually convergent systems represent an assembly process for a subassembly or a final finished product, where the upstream nodes represent the components and the downstream node represents a final or intermediate product.

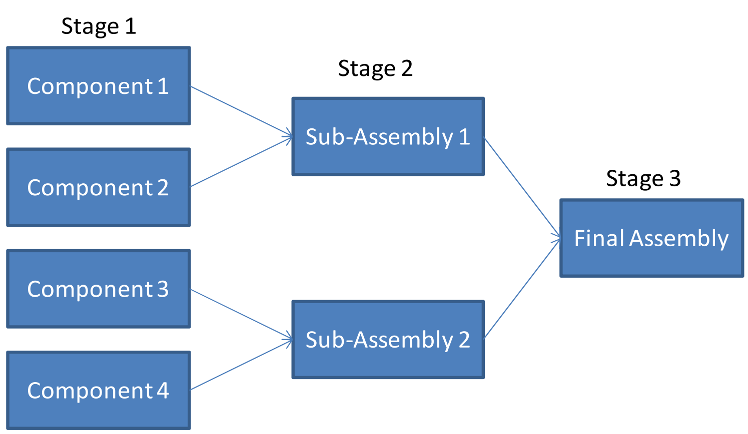

Figure 1.5: Schematic Representation of a Convergent System

1.3.3 System Control

The control of multi-echelon systems is broadly classified into two categories: decentralized, and centralized. Multi-echelon systems most often are decentralized in the sense that the ordering decisions are solely based on the installation stock, i.e., the inventory position at a particular stock point (node). The inventory position is defined as the outstanding orders at the stock point plus physical inventory minus backorders. An advantage of an installation stock based policy is that it does not require any information about the inventory at other stages/nodes in the system. Due to the lack of information availability about the entire system the cost effectiveness of these policies could be limited (Diks, 1996). One way of taking control of the entire system is to use the echelon based stock policy, i.e. the echelon inventory position is defined as the sum of all outstanding orders at the current stock point plus its physical inventory plus pipeline inventory minus eventual backorders at earliest stock point downstream which has an external demand. Since in the echelon based stocking policy the ordering decisions are made with complete knowledge of the downstream stages it is essential that the information of other stages/nodes are known.

An important difference between installation stock and echelon stock is stated by Chen and Zheng (1994). In the former policies the inventory position of a stock point includes all outstanding orders, i.e., in transit to this stock point or backorder at the supplier, whereas in the case of an echelon stock policy the echelon inventory position only includes the in transit orders to the stock point. As a result a policy based on installation inventory can always raise a node's inventory position to the desired inventory level, and if some part of the order is not satisfied it can be immediately backordered. So every stocking point can be modeled as a single location inventory system with random lead time. In echelon stock policies this lead time exactly equals the transportation time. However it is more difficult in echelon-based inventory models to determine the inventory position due to the fact that the stocking point cannot be considered as a single location inventory system (Diks, 1996).

1.3.4 Service Measures

In order to determine the cost minimizing parameters of the ordering policy inventory holding costs, order costs and cost of stockout need to be determined. A stockout cost is ascribed to a lost sale or a rush delivery. Often the stockout cost is due to a loss of customer goodwill. It is difficult to assign a precise cost to these factors. Because of this, in practice, these factors are determined indirectly by assigning service levels. There are several ways to measure the service level, but we limit our discussion to three different service measures which are widely used (Hopp and Spearman, 2000).

- The non-stock out probability (α/Type-I): the probability that the net stock (the stock on-hand minus backorder) is non-negative at the end of an arbitrary replenishment cycle.
- The fill rate (β/Type-II): the fraction of the demand that is satisfied directly from the stock on hand.
- The modified fill rate (γ): one minus the ratio of the average shortage immediately before arrival of a replenishment order and the average demand during an arbitrary replenishment cycle.

1.3.5 Formulation

There are two types of optimization formulations for inventory management that are widely used in the literature: a service level constrained model, and a backorder cost model. The succeeding paragraphs describe each of these.

The cost function for a traditional service level constrained model uses an inventory, holding cost, and decision variables like the base-stock level (target stock level in case of the base-stock policy). The holding cost creates an incentive in the formulation to minimize inventory in order to minimize cost. The service level constraint creates an incentive to hold sufficient inventory to ensure the required service is maintained at each stage, (i.e. probability of demand or collection of demands is met). The service level could either be a type-I or a type-II service level. As discussed in earlier, Type-I service level (also denoted as α) is defined as the proportion of cycles (periods) in which no stock out occurs, whereas type-II (also denoted as β , and known as fill rate) service level measures the proportion of demand that is met directly from the stock without delay (Nahmias, 2001). Existence of customer lead time would be usually based on the environment. For instance, in a make-to-stock environment the customer lead-time is usually zero, whereas in a make-to-order environment the customer lead time would be the amount of time the customer allows the firm to produce and deliver the item. Typically this kind of formulation is associated with make-to-order environment, and usually involves type-I service level constraint, but however type-II service level constraint can also be used.

Equation 1 represents a mathematical formulation for a single-stage system with a type-I service-level constraint (Bollapragada et al., 2004).

$$
\begin{aligned}
&C(s) = \min_{s \geq 0} \; cs \\
&\text{subject to } P\left[I \geq D\right] \geq \alpha
\end{aligned}
\tag{1.1}
$$

where c is the unit cost, s represents the base-stock level, I represent the on-hand inventory level, and D represents the demand. The objective function in equation (1.1) is to minimize the total cost subject to the type-I service level constraint. The cost function assumes that holding cost is proportional to the unit cost of the item. Equation (1.2) represents another example of single-stage service level constrained formulation, where the cost function involves the use of holding cost, and the service constraint is of type-II (Boyaci and Gallego, 2001). In 1.2 h is the holding cost, and $\beta(s)$ is the fill rate.

$$
\begin{aligned}
&C(s) = \min_{s \geq 0} \; hE\left[I\left(s\right)\right] \\
&\text{subject to } P(D < s) \geq \beta(s)
\end{aligned}
\tag{1.2}
$$

As an alternative to the service level constrained formulations, the traditional backorder cost formulation considers shortage penalty costs (backorder costs), holding costs, and set-up costs (if any) in developing the cost function. The backorder cost replaces the service level constraint and acts as the balance to the inventory minimizing influence of the holding cost. The service level in this kind of formulation is obtained as an output, i.e. the choice of backorder penalty cost induces a service level based on the solution of the problem (Boyaci and Gallego, 2001). Either type of service levels could be used to evaluate the resulting solution of the inventory optimization, but usually type-II service level is used. In this formulation customers experience some of the lead time when the system is out of stock, i.e. the time between the order placement and order arrival.

Equation (1.3) represents a simple mathematical formulation of the backorder plus holding cost function in a period for a single-stage inventory system. The inventory level *I* will depend on the base-stock level chosen (Glasserman et al., 1995).

$$C(s) = [I]^{+} b + [I]^{-} h \tag{1.3}$$

where b is the backorder cost per unit, and h is the holding cost per unit, $[I]^{+} = max(0, I)$, and $[I]^{-} = min(0, I)$.

To summarize the preceding discussion, the service level constrained formulation and the backorder constraint formulation are different ways of achieving a balance between too much and too little inventory. The objective of both models is similar, i.e. they are cost minimization formulations. The service level constraint model minimizes the cost with respect to the bounds placed on the service, whereas the backorder cost model minimizes the cost with respect to the tradeoff between shortage costs and holding costs.

1.4 Inventory Allocation Policies

Inventory allocation becomes important when a supplier has a collection of orders that are due immediately whose total requirement exceeds the current inventory level. In this case, there is an inventory shortage at the supplier end. The supplier checks the received orders with his available inventory and if the orders sum up to be more than the available inventory, the inventory is allocated among the retailers using an inventory allocation policy. A list of possible allocation policies is discussed below (Cachon and Lariviere, 1999).

Guaranteed fraction allocation: The supplier promises in advance that a guaranteed fraction of the retailers order will be filled.

Maximum allocation: A supplier promises in advance the maximum amount allocated to each retailer

Fixed allocation: When there is no information regarding the past sales data, the supplier may use a fixed allocation. The retailer receives at most the fraction of the scarce stock agreed upon.

Uniform shortage allocation: In this mechanism the supplier indexes the retailers in the decreasing order of their order size and supplies each retailer with his order minus some common deduction.

Proportional allocation: Each retailer receives an equal proportion of his order.

Pareto allocation: This kind of allocation policy is used when a set of allocation policies are currently in practice. The aim of this allocation policy is maximizing the sum of retailer's profits by improving the profit of at least one retailer without changing the profit of all the other retailers, assuming all the retailers submit their optimal orders truthfully. It can be interpreted as maximizing supply chain profits subject to no retailer ever receives more than his true need.

Lexicographic allocation: In this allocation policy the retailers are ranked in some manner independent of their order size. Orders from retailers are filled completely in the order specified in this ordering, until all stock is exhausted. As the rank of the retailer decreases the chance of having orders completely filled also decreases.

Uniform allocation: All retailers receive the same quantity, regardless of order size. Retailers with smaller orders usually benefit most from this allocation policy.

Inventory allocation plays an important role in determining levels of service in a supply network, especially when there is more than one demand on a particular node/stage. It is crucial that an appropriate allocation policy is chosen so as to optimize the performance of the entire supply chain.

1.5 Research Summary

This sub-section describes in brief the research that has been done for the dissertation, and summarizes some of the important results which are discussed in detail in subsequent chapters. The notation that is used throughout the research is provided below.

1.5.1 Notation

The lists of notations used are described below:

ξ_n^j : Product j demand in period n

l^i : Resupply/processing lead time for component i

η_n^i : Realized capacity at stage i in period n

s^i : Base-stock level for item i

c^i : Cost per unit of item i

α^i : Required type-I service level for retailer i (external source of demand)

Y_n^i : Outstanding orders of item i in period n that have not been delivered due to limited capacity (shortfall of item i)

I_n^i : On-hand inventory level of item i in period n before the demand is realized

IP_n^i : It is the on-hand inventory plus the pipeline inventory for component i in period n

NI_n^i : The net inventory for component i in period n

DS_n^1 : It is the downstream shortage at node 1 in period n

1.5.2 Analytical Review

A mathematical model for a capacitated multi-echelon systems (three-, four-, five-, and *m*-echelons) with demand for intermediate and final products has been developed. Figure 1.6 shows a three-echelon assembly system. As seen in the figure 1.6 components 1 and 2 are procured from the suppliers 1 and 2 by nodes 3 and 2. The procured components are assembled at node 1, where node 1 satisfies an intermediate product demand as well as a downstream demand. Node 0 procures the intermediate product from node 1 and it is further processed to be sold as a final product.

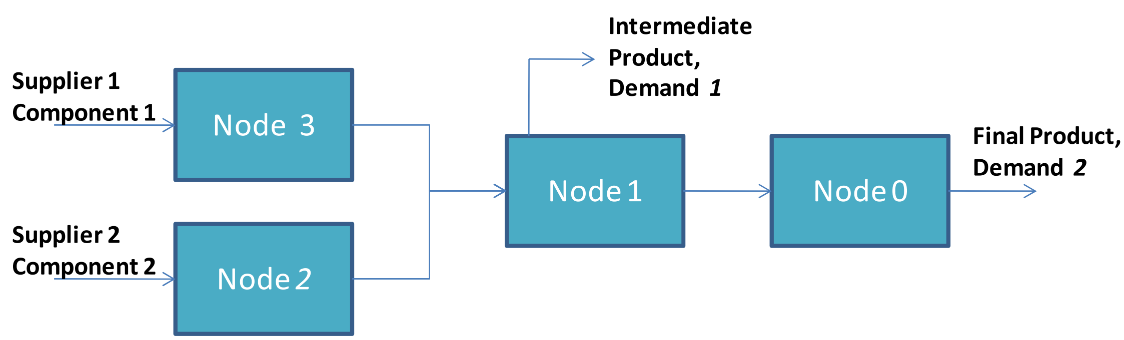

Figure 1.6: Three-echelon assembly system

All the events occur at the beginning of each period in the following sequence:

i) The outstanding orders are updated, i.e. items that have not been delivered in the previous period due to limited capacity

ii) The on-hand inventory is updated, i.e. the physical inventory

iii) Demand is realized

iv) Capacity is realized.

An order is placed with the supplier in response to demand ξ_n^j. The order can be received either in full or could be constrained by the node's capacity. Thus the amount received is $\min\left(Y_n^i + \xi_n^j, \eta_n^i\right)$, where η_n^i the realized capacity, and ξ_n^j is the demand facing that stage.

Since a base-stock policy is followed for each component, at the beginning of period n we have

$$I_n^i + Y_n^i - Y_n^0 = s^i, \text{for all } i \tag{1.1a}$$

Equation 1.2a represents the outstanding order (shortages) for stage 0, from the equation we can observe that the outstanding orders are zero or constrained by the node's capacity, η_n^i .

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^2 - \eta_n^i\right) where\, i \in \{0\} \tag{1.2a}$$

Let us assume that ξ_n^2, is the demand for the final product. The net inventory depends on demand during the lead time, and the outstanding orders. The on-hand inventory (physical inventory) is an extension of the net inventory, but on-hand inventory must always positive. The update of these variables between periods $n-1$ and n is described in (1.3) and (1.4) below:

$$NI_n^i = s^i - \xi_{n-1}^2 - \ldots\ldots - \xi_{n-l^i}^2 - Y_{n-l^i}^i \text{ where } i \in 0 \quad \textit{(Node 0)} \tag{1.3}$$

$$I_n^i = \max\left[0, s^i - Y_{n-2}^i - \xi_{n-1}^2 - \xi_{n-2}^2\right], \text{where } i \in 0 \; \textit{(Node 0)} \tag{1.4}$$

The first order derivative equation of Y_{n+1}^0, I_n^0 is as shown in equation (1.5) and (1.6) respectively. As you can observe the first-order derivative for Y_{n+1}^0 has two possible states, it is equal to zero (first state) when there are no shortages, and equal to dY_n^0 / ds^0 if shortages exist. Similarly we can see that the first order derivative of I_n^0 shown in equation (1.6) also has two states.

$$\frac{dY_{n+1}^0}{ds^0} = \begin{cases} 0 & if\ Y_{n+1}^0 = 0 \\ \dfrac{dY_n^0}{ds^0} & if\ Y_{n+1}^0 = Y_{n+1}^0 + \xi_n^2 - \eta_n^0 \end{cases} \tag{1.5}$$

$$dI_n^0 / ds^0 = \begin{cases} 0 & \text{if } NI_n^0 \le 0 \\ 1 - Y_{n-2}^0 / ds^0 & \text{otherwise} \end{cases} \tag{1.6}$$

More complex and detailed interactions are discussed in chapters 3 and 5. First order derivative estimates for each node in the supply chain networks are derived using the IPA approach in the succeeding chapters, and are used during a simulation to update derivatives of these variables with respect to the inventory parameters.

1.5.3 Simulation Based Optimization Using OptQuest

A simulation-based inventory optimization is carried out using the *OptQuest* framework. The aim of the *OptQuest* framework is to obtain the best base-stock levels for each node in the multi-echelon inventory system. Figure 1.7 shows a flowchart of how the framework is designed and carried out. The base-stock values are obtained from the *OptQuest*, and initial values are assigned in *ARENA* and the equations are updated (simulated) for a pre specified period of time (periods). At the end of the simulation run the service level values for each node are sent to *OptQuest* from *ARENA*. *OptQuest* then uses this information to check if all the constraints are satisfied and the process iterates until there is no further improvement in the solution or an iteration limit is reached.

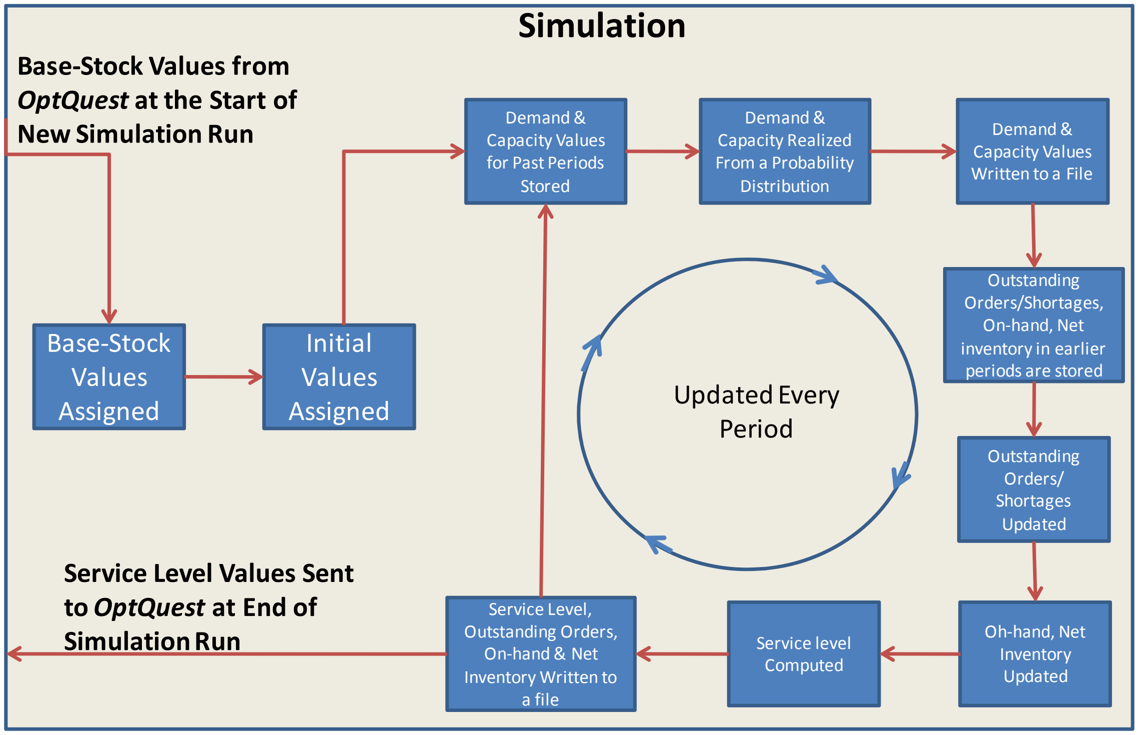

Figure 1.7: Simulation in *ARENA* within *OptQuest* Framework

1.5.4 Simulation Based Optimization Using IPA

A simulation based inventory optimization is carried out using an IPA (Infinitesimal Perturbation Analysis) framework. The aim of the IPA framework is to obtain best found base-stock levels for each node in the multi-echelon inventory system with the help of nonlinear optimization methods. The IPA framework uses a combination of *ARENA*, Visual Basic (VB), and Xpress. *ARENA* is used to update (simulate) the equations (on-hand inventory, outstanding orders etc.), first-order equations, and service level equations. A feasible direction algorithm and line search is implemented in VB. A modified Zoutendijk's feasible direction algorithm is the basis of the optimization, with a golden section algorithm used for a lower-level line search. Figure 1.8 shows a flowchart for simulation based Inventory optimization (SIO) using IPA. A detailed description of the feasible direction algorithm and line search is provided in chapter 6.

Figure 1.9 shows the block diagram of various activities that are performed every period in an IPA framework. The base-stock and the Lagrange multiplier values are initially obtained from VB and *ARENA*. The Initial values that are used in the update equations, first order equations, and service level equations are assigned. Once the initial values are assigned, the simulation runs for a pre-defined number of periods. The gradient estimates from the simulation are used to compute the Lagrange derivatives, which determine the direction of then nonlinear programming search. A line search uses this direction vector to determine the next point in the search. The process continues until the termination condition is satisfied. More details for this process are provided in chapter 6.

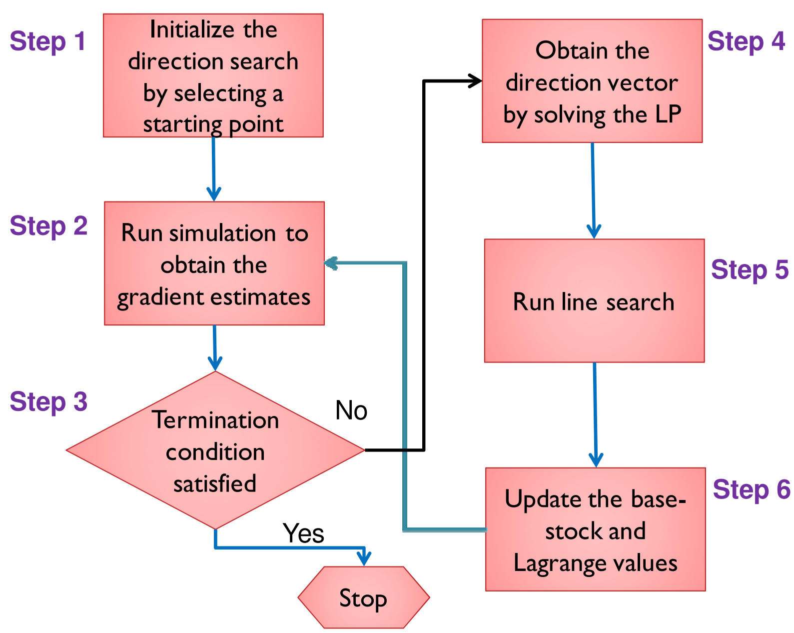

Figure 1.8 Flowchart of SIO for IPA

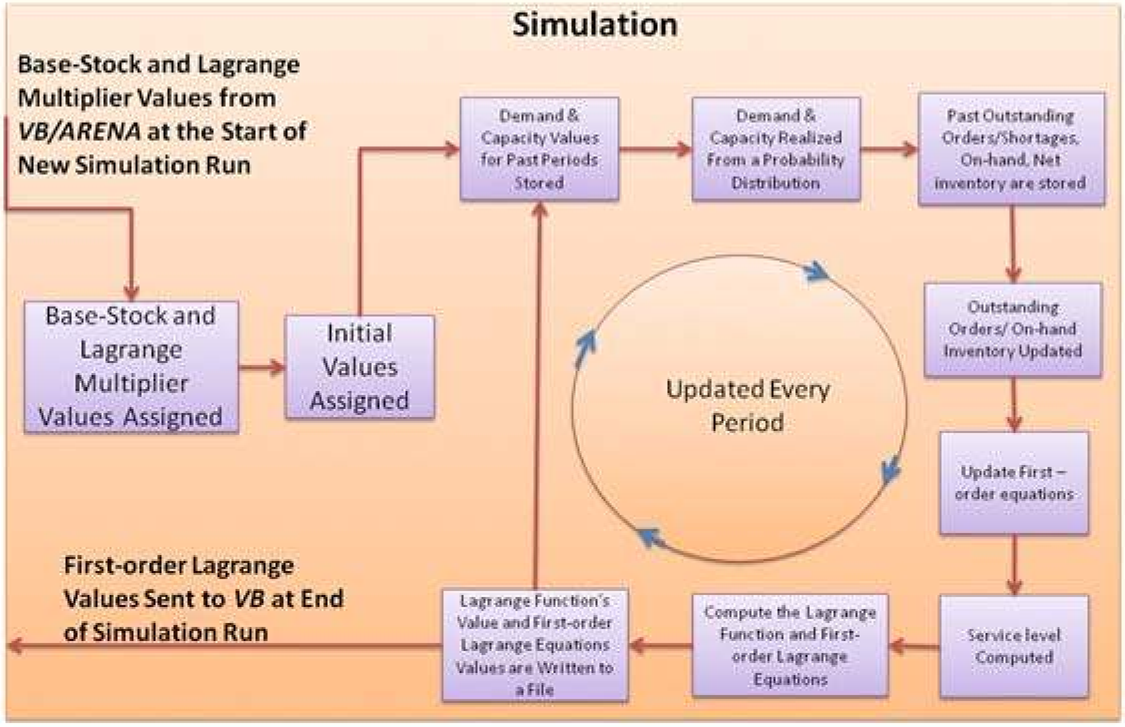

Figure 1.9 Block Diagram of Simulation in *ARENA* within IPA framework

1.5.5 Justification of Derivatives

The first order equations for the on-hand inventory, outstanding orders, and net inventory with respect the all the base-stock levels are shown as valid and that the sample-path derivatives they generate are unbiased estimators of derivative of expectations. To accomplish this we show that the conditions in proposition 1 and proposition 2 that ensure the following properties are satisfied:

- The outstanding orders, on-hand inventory, and the net inventory are differentiable with probability one, with respect to all base-stock levels
- The expectation and derivatives are interchangeable for outstanding orders, on-hand inventory, and net-inventory
- Show that the Lagrange function (discussed in chapter 5) is Lipchitz continuous

In order to show that the Lagrange function is Lipchitz continuous: - we initially prove that all the components of the Lagrange function are continuous. This includes the demand, capacity, outstanding orders, on-hand inventory, and net inventory. Since the demand and capacity are derived from the probability distributions, it is known that they are continuous. The outstanding orders, on-hand inventory and the net-inventory are also relations which are based on the demand and capacity for period 0. The important step in verifying that the derivative estimates based on the Lagrange function are unbiased is showing that the on-hand inventory, outstanding orders, and net inventory, with probability one are Lipchitz functions and have integrable moduli. This requires according to generalized mean value theorem, and dominated convergence theorem to show that, expectation and derivatives are interchangeable. The two propositions are stated below, the proofs of the propositions are derived in chapter 5.

Proposition 1

If $\left\{\xi_n^j, n=1,2,...N; j=1,2,...,m-2\right\}$, $\left\{\eta_n^i, n=1,2,...N; i=1,2,...,m\right\}$ are independent, and each ξ_n^j, η_n^i has a density on $(0,\infty)$, and then the following hold:

i) For $i=1,....,m$ and $n=1,2,...N$ each Y_n^i, I_n^i, NI_n^i are differentiable, with probability one, at s^i with respect to s^i, $i=1,2,...,m$

ii) If $\left[\sum_{j=1}^{m-2}\xi_n^j\right]<\infty$, and $\eta_n^i<\infty$ for all n, then $E\left[Y_n^i\right]'$, $E\left[NI_n^i\right]'$, and $E\left[I_n^i\right]'$ exist and equal $E\left[\left(Y_n^i\right)'\right]$, $E\left[\left(NI_n^i\right)'\right]$, and $E\left[\left(I_n^i\right)'\right]$

Proposition 2

Let $\{L(\vec{s},\vec{u}), \vec{s} \in S, \vec{u} \in U\}$ be a random function with $S, U \subseteq \Box$. If $E[L(\vec{s},\vec{u})] < \infty$ for all $\vec{s} \in S, \vec{u} \in U$. Assume that L is differentiable at $s^i \in S$ and $u^i \in U$ with probability one, and that L defined on set S, and U is almost surely Lipschitz continuous with modulus M_L satisfying $E[M_L] < \infty$. Then $E[L(\vec{s},\vec{u})]'$ exists and equals $E[L'(\vec{s},\vec{u})]$.

1.5.6 Inferences from Three-echelon Computational Results

In the experiments to date, the following are some of the important factors which were varied to study their effect on the best found base-stock levels, best found safety-stock, and minimum total cost. The effects shown in table 1.1 are based on the computational results from the three-echelon assembly model using IPA framework. Detailed computational results for *OptQuest* and IPA frameworks are discussed in chapter 7.

Table 1.1: Factors and Effects Based on Computational Results

No.	Factors	Effects
1	Increase in the value of the demand for each node	Increase in base-stock levels, and total cost
2	Increase in the Capacity Utilization for each node	Increase in base-stock levels, and total cost
3	Increase in the Capacity Utilization for each node	Increase in Safety Stock Costs
4	Increase in the Service Level for each node	Increase in base-stock levels, and total cost
4	Increase in demand CV (coefficient of variation) for each node X increase in service level for each node	Increase in the base-stock level, safety stock cost, and total cost increases
5	Increase in capacity CV for each node X increase in service level for each node	Increase in base-stock level, safety stock cost, and total cost increases
6	Increase in lead time for each node X constant demand CV for each node X constant Service Level for each node	Increase in the base-stock level, safety stock cost, and total cost increases
7	Increase in lead time for each node X increase Demand CV for each node X constant service level for each node	Increase in the base-stock level, safety stock cost, and total cost increases
8	Increase in lead time for each node X increase demand CV for each node X increase service level for each node	Increase in the base-stock level, safety stock cost, and total cost increases
9	Increase in number of components for each node X constant demand CV for each node	Increase in the base-stock level, and total cost increases

1.5.7 Inventory Allocation Policies

The four inventory allocation policies are used in the three-echelon assembly system, five-echelon assembly system, and the larger networks. The detailed description of the allocation policy and the numerical results can be found in chapter 8, this sub-section provides he names of the allocation policies and its inferences based of several numerical cases.

Lexicographic Allocation (Priority to Intermediate Product Demand)(LAPI): The Lexicographic allocation policy ranks the retailers in some manner independent of their order size, and based on the ranking the retailer receives the amount of supplier capacity. By assigning priority to intermediate products the following are observed:

- Decrease in the total cost when the upstream nodes have a greater proportion of demand when compared to the downstream nodes.
- Decrease in total safety-stock cost when the upstream nodes have a greater proportion of demand when compared to the downstream nodes.

Lexicographic Allocation (Priority to Downstream Demand) (LAPD): By assigning priority to downstream products the following are observed:

- Decrease in the total cost when the downstream nodes have a greater proportion of demand when compared to the upstream nodes.
- Decrease in total safety-stock cost when the downstream nodes have a greater proportion of demand when compared to the upstream nodes.
- Decrease in the total cost and safety-stock cost when there is a high demand CV for all the demands (intermediate and final products) and high capacity CV.

Predetermined Proportional Allocation (PPA): In a predetermined proportional allocation if the supplier's capacity is less than the sum of all the demand, then the supplier could use a predetermined proportional allocation mechanism. Each retailer receives not more than the pre-determined ratio of the available inventory on-hand. By using a PPA inventory policy with a fixed ratio of 0.5 (50%) we observe the following:

- In most cases using this inventory allocation policy leads to higher total cost compared to when other (LAPI, LAPD, and PA) inventory allocation policies are used.

Proportional Allocation (PA): In a proportional allocation, when upstream capacity is insufficient, each retailer receives an equal proportion of his current order. By using proportional inventory allocation we observe:

- Decrease in the total cost and safety-stock cost when there is a high demand CV for all the demands (intermediate and final products).

The update equations and numerical analysis of three and five-echelon assembly system based of all four inventory allocation policies are discussed in chapter 8.

1.5.8 Hypothesis Testing

Statistical analysis and hypothesis testing is done to derive a few implications based on the results from a five-echelon assembly model. In order to determine which inventory allocation policy would help in minimizing the total cost and safety-stock cost across the entire multi-echelon network, four instances, each consisting of several scenarios is solved to optimality. The instances are described in table 8.7, which shows the name of the instance and coefficient of variance for the demand and capacity. The capacity is denoted as “average” capacity when the mean capacity utilization is between 65% and 75%, and the capacity is defined as “tight” capacity when the mean capacity utilization is between 85% and 95%. The demand is categorized as “high” and “low” demand. If the value of the mean demand is between 8 and 10 it is referred as high demand, whereas low demand is defined between 3.5 and 5. The variance for the demand and capacity is defined as high and low if the coefficient of variance is 0.3 and 0.1 respectively.

Table 1.2: Instances for Five-echelon Assembly System

Instance #	Name of Instance	CV for Capacity	CV for Demand
1	Average Capacity (AC)	0.1	0.1
2	Tight Capacity (TC)	0.1	0.1
3	High Demand Variance with Average Capacity (HDVAC)	0.3	0.3
4	High Demand Variance with Tight Capacity (HDVTC)	0.3	0.3

The formal hypothesis statements and the statistical tests are described in chapter 8. The important implications based of the five-echelon assembly system are described below:

- *Statement 1.* If only one type of inventory allocation policy is used across the entire supply chain (five-echelon assembly system), using LAPI inventory allocation policy results in the lowest safety-stock cost compared to models which use other inventory allocation polices under average capacity instance (instance 1)
- *Statement 2.* If a combination of inventory allocation policies are used across the entire supply chain (five-echelon assembly system), using LAPI & LAPD inventory allocation policy in combination will results in the lowest safety-stock cost, compared to models which use other combinations of inventory allocation polices under average capacity instance (instance 1)
- *Statement 3.* LAPD inventory allocation policy results in the lowest safety-stock cost across the supply chain (five-echelon assembly system) under tight capacity instance (instance 2).
- *Statement 4.* If only one type of inventory allocation policy is used under high demand average capacity instance (instance 3) across the entire supply chain

(five-echelon assembly system), using PA inventory allocation policy results in the lowest safety-stock cost compared to models which use only one type of inventory allocation policy

- *Statement 5.* If a combination of inventory allocation policies are used across the entire supply chain (five-echelon assembly system), using LAPD & PA inventory allocation policy in combination will results in the lowest safety-stock cost, compared to models which use other combinations of inventory allocation polices under high demand variance average capacity instance (instance 3)
- *Statement 6.* LAPD inventory allocation policy results in the lowest safety-stock cost across the supply chain (five-echelon assembly system) under high demand variance tight capacity instance (instance 4).

1.5.9 Multi-echelon Networks

Two contemporary network models were evaluated in order to show that the multi-echelon inventory model developed in this research is applicable on a wide range of multi-echelon network models. The two networks studied were:

- *Network 1:* Multiple suppliers – representing a manufacturing industry with demand for spare parts (figure 1.10)
- *Network 2:* Representing a system with multiple manufacturers/suppliers – warehouse - distribution center – retailer interactions (figure 1.11)

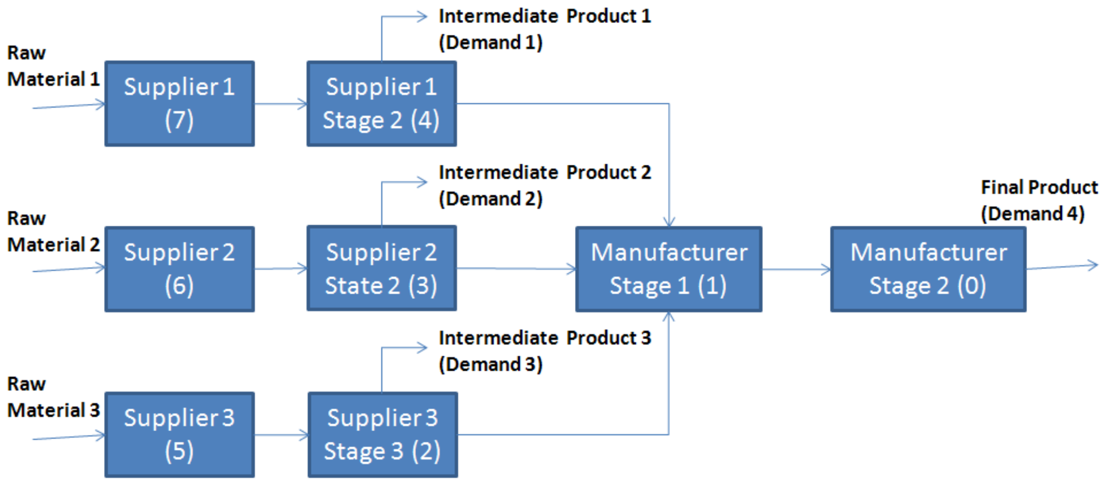

Figure 1.10: Multiple Suppliers-Manufacturing Industry Setup

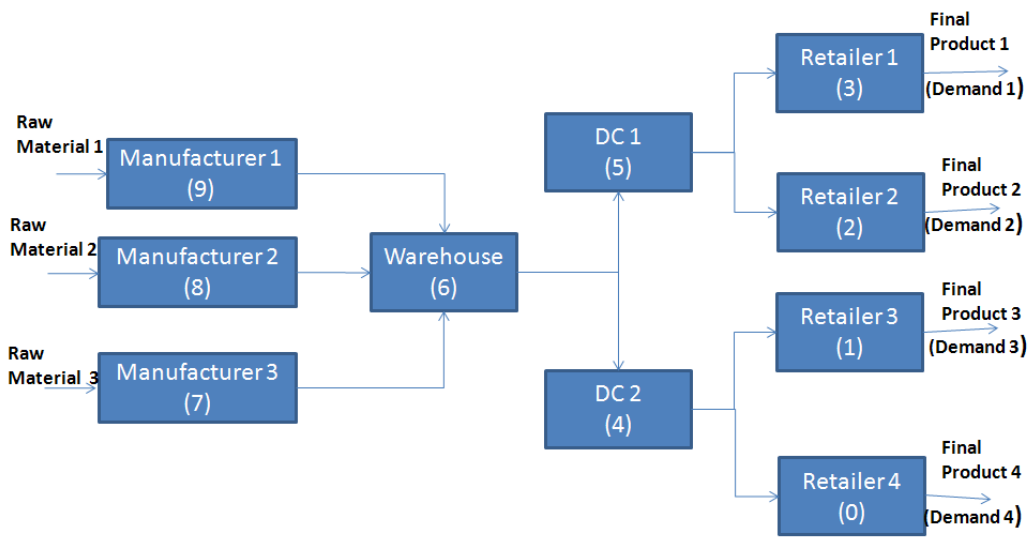

Figure 1.11: Manufacturers – Warehouse - Distribution Center – Retailer

Chapter 9 discusses the update equations, and detailed numerical results and analysis of the two large multi-echelon networks.

1.5.10 Heuristic Starting Points

Two heuristic approaches that determine the starting points for an IPA based search are developed. The following are the two approaches:

- *Rule based approach* : Determining good initial points/starting points for the search based on a set of rules

- *Decomposition approach* : Determine the initial base-stock levels for each node based on a decomposition approach

Chapter 10 describes the two heuristic approaches in much detail. The two heuristic approaches are applied to three-echelon, five-echelon and larger multi-echelon network models. The numerical results from the two approaches are compared with traditional approaches on a time and Percentage Relative Error (PRE). A few insights are provided on the bases of the analysis.

1.7 Summary of Contributions

This dissertation describes a mathematical model for a three-echelon, and m-echelon assembly systems with random capacity and random, intermediate demand. The model features include assembly operations in the upstream nodes, product demands in one or more of the upstream nodes, and a final product demand. The model incorporates both assembly and serial systems. Based on the mathematical model of inventory dynamics a simulation based inventory optimization is carried out in two frameworks, 1) *OptQuest*, and 2) Infinitesimal Perturbation Analysis (IPA). The frameworks are used to obtain the best found base-stock levels for the nodes in the network.

The *OptQuest framework uses OptQuest* as a black box optimization engine and *ARENA* as a simulation tool. The IPA framework consists of a combination of *ARENA*, Visual Basic, and Xpress. Visual basic is used for developing the details of the optimization algorithm, whereas Xpress is used to solve an ancillary linear program. The results for the best found base stock levels for different scenarios show how variation in capacity and demand parameters effects the distribution of inventory in the network. An inventory optimization approach would help planners in effectively managing the total cost of inventory and the level of customer service, especially when demand and capacity are varying. Four allocation policies have been used in the multi-echelon inventory systems, which are used at the appropriate nodes to assign priorities between intermediate product demand (local demands) and the downstream demand (final product). Initially the update equations for the three-echelon and m-echelon assembly systems are based on the inventory allocation policy that assigns priority to intermediate product demand, and later to all the inventory allocation policies. Two simple heuristic approaches, 1) based on a set of rules, 2) based on a decomposition approach are used to obtain better starting points that reduces the search time significantly.

The rest of the document is organized as follows: chapter 2 reviews the relevant literature, chapter 3 presents the completed multi-echelon inventory model, chapter 4 discusses the completed simulation optimization framework using *OptQuest*, chapter 5 discusses the completed gradient estimation analysis for the IPA framework and proves Lipschitz continuity for the multi-echelon inventory system, chapter 6 discusses the completed simulation optimization framework using IPA, chapter 7 discusses the completed computational results from *OptQuest* and IPA, Chapter 8 describes the 1) four different inventory allocation polices used, 2) the change in the update equations for the three-echelon and five-echelon assembly systems, 3) numerical results and inferences based of several cases, 4) statements based of hypothesis testing. Chapter 9 discusses, 1) the contemporary issues the multi-echelon inventory models addresses, 2) update equations for the two large multi-echelon network models, 3) analysis of numerical results based of several cases considered. Chapter 10 describes the two heuristic approaches used for generating good starting points for the IPA based search, using a 1) rule based approach, and a 2) decomposition approach. Chapter 11 provides the conclusion of this research, future research directions, and additional results are provided in the appendix.

2. LITERATURE REVIEW

Some of the prominent works involving single-stage and multi-stage inventory on lot sizing decisions that include supply uncertainty and/or capacity restrictions have been summarized in this section. The review is broadly categorized into six sub-sections: single-stage models with deterministic/random demand, multi-echelon problems with single/multiple products, multi-echelon problems with multiple-products, multi-echelon problems with spare products, information sharing, and simulation based optimization.

2.1 Single-Stage Models

Consider a single stage system as illustrated in figure 2.1, where a circle denotes a production facility, η denotes capacity, and the square box denotes a warehouse/storage location. The line between the circle and the square box indicates lead time. The demand, capacity, and lead time could be deterministic / random in nature. The inventory control theory has its origin in very simple systems, like the single-stage systems with holding costs for positive inventory and backorder costs for negative inventory. The earliest inventory model dates back to 1913 by Harris, known popularly as the economic order quantity model (EOQ). It was made popular by Wilson in 1934 (Moreira, 1999). This policy considers a single stage of production of a single product with a deterministic and constant demand.

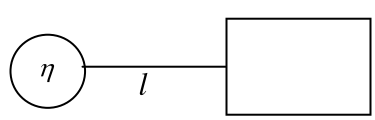

Figure 2.1: Single Stage System

The models like news vendor, base-stock, *(Q, r)* etc. have also been commonly used in practice. Based on the underlying assumptions the appropriate model is chosen. Similar to the EOQ model, the base-stock model also constitutes one of the the basic building blocks of inventory theory. Arrow *et al.* in 1951 show that the optimal policy is of base-stock type for a multi-stage inventory system (Moreira, 1999). More complex models of inventory control developed after the advent of dynamic programming by Bellman in 1957. One of the earliest papers in a single-stage setting based on EOQ with deterministic demand and supply uncertainty was written by Silver (1976). In this model, randomness in the number of non-defective units is a source of uncertainty affecting the quantity received. This model allows the quantity received to be larger than the quantity ordered. Two specific cases are considered; in either case the optimal lot size is found to be a slight modification of EOQ.

Groenevelt, Pintelon, and Siedmann (1992) studied the problem of selecting an economic lot size for an unreliable manufacturing facility with constant failure rate and randomly distributed repair times. Safety stocks were introduced in order to maintain the required customer service level (represented as fraction of lost sales). A policy was developed using an analogy between the safety stock dynamics and the renewal process of workload for a special single server queuing system. This analogy is used to derive exact and approximate expressions for the safety stock and holding costs. A clear tradeoff is shown to exist between the overall investment in increasing the maintenance level and the resulting savings in safety stocks and repair costs. A subsequent paper by Groenevelt *et al.* (1992a) focused on similar effects of machine failure and repair on optimal sizing decisions, but assuming exponential distributed time between failures and instantaneous time to repair of the machine. Groenevelt showed that the optimal lot size was always larger than that given by the classical EMQ model without machine failures and that optimal lot size increases with failure rate.

Chang, Yushin, and Soo-Young (1997) developed a model on similar lines, but the model was more general so that Groenevelt et al. (1992a) model was a special case. The model developed by Chang, Yushin, and Soo-Young (1997) derived an average cost function for arbitrarily distributed time between failures and constant time to repair. For exponentially distributed time to failure an optimal lot size was derived. Three different inventory paths were described when the machine failure occurs, and based on these paths the cost function was derived. A unique assumption was made which states that if on-hand inventory exists after the completion of repair, a new production cycle is not started until the entire on-hand inventory is depleted. Some special properties were derived for this model, several numerical experiments and sensitivity analysis were carried out to examine the effects of machine parameters as well as cost parameters on optimal lot size.

Many deterministic and stochastic inventory models assume lead time as a given parameter, but in fact most practical situations lead time is an unknown or uncertain parameter. Many modern production management efforts involve significant effort controlling lead time. Among these one of the most successful is JIT (Just-In-Time). One of the reasons for the success of JIT is the efforts in handling the lead time. A mixed inventory policy is developed by Kun-Shan Wu (2001) for variable lead time when the supplier capacity is assumed to be random. The optimal operating policies for two kinds of lead times are studied. Initially, a normally distributed lead time is studied and a distribution free lead time is studied. They derive an optimal bound (bound on the policy) for order quantity, reorder point and lead-time that minimizes the total cost. Optimal bounds were developed for a distribution free lead time model, and normally distributed lead time. The cost function derived for the distribution free model is unimodal and quasi-convex in nature. Irrespective of the distribution function, an optimal solution for the lead time was shown to exist.

There has been relatively little research done on the continuous time models with random demand and random yield, or random demand in combination with random capacity in a single-stage setting. For the case of random demand with backlogging, the model is tractable if there exists at most one replenishment order outstanding all times, and when the backlog costs are a function of the units short. Noori and Keller (1986) consider the standard reorder point (Q, r) model, where Q represents the order quantity, and r represents the reorder level. The approach used was similar to the one used by Silver. The optimal r and Q values are obtained using an iterative approach. Gerchack (1992) analyzes the same model, and a variant with lost sales. He shows that in the case of stochastically proportional yields, only Q is affected by the yield variance, whereas r is affected only by the mean yield rate. In the case of binomial yields, Gerchack show that the value of Q is affected only by the mean yield rate, and the value of r is not affected by the yield parameters.

There has been extensive research done in periodic-review inventory models with deterministic capacity and random demand, Federgruen and Zipkin (1986) consider a single-item, periodic-review inventory model with uncertain demands. A finite production capacity is assumed in each period. With stationary data, a convex one period cost function and a continuous demand distribution, Federgruen and Zipkin show that a modified base-stock policy is optimal under a set of discounted criterion for both finite and infinite planning horizons. Karlin (1960a, b) considers an infinite-horizon inventory problem with stochastic demands where the data vary periodically. The data is assumed to be nonstationary but they repeat for every “cycle” of *K* periods. The discounted version of this problem is considered, assuming costs are stationary but allowing demands to vary. Karlin shows that a periodic critical number policy is optimal. There may be different critical numbers each period but the critical numbers repeat for every cycle. Zipkin (1989) further extends the problem for the average cost case, where the demand as well as costs is cyclic. Zipkin shows that the best policy is optimal in much stronger sense than that considered in Karlin (1960a, b).

Gerchack, Vickson and Parlar (1988) investigate a model with variable yield and uncertain demand. A complete analysis of a general profit maximizing single period model with variable yield and uncertain demand is provided. A continuous demand, yield, and cost proportional to net yield is assumed. The expected profit function is proved to be concave in initial stock and lot size. When the initial stock is above a critical level, an order is not placed, and this level remains unchanged even for the certain yield case. However, when the initial stock is below the critical level, the expected yield corresponding to the amount ordered will in general not simply equal to the difference, i.e. the policy is not order-up-to type, as in case of most of the models with certain yield. The problem is further investigated for a two period case, and it is shown that the critical level for the first period is higher than for the last period.

There has been recent interest shown in periodic-review inventory models with random capacity and random demand. Obtaining a tractable solution is difficult for such models. With development of increasingly sophisticated products and complex environments theses models seem to be more applicable. Ciarallo et al. (1994) were first to present inventory models for a single product with random demand and random capacity in a single-stage setting. Different cases of the problem with single and multiple periods in finite and infinite horizon settings were explored. They showed that randomized capacity does not affect the order policy for a single period case. In other words, the derived optimal policy is identical to the one of the newsvendor model. Moreover, the random capacity resulted in unimodal, nonconvex cost function in both the single- and multi- period models with finite horizon. They further showed that the order-up-to policy structure is optimal for the infinite horizon model. Jain and Silver (1995) extended the single period model of Ciarallo et al. (1994) by adding the option of reserving ahead of time a dedicated capacity level. This option ensures that any order size up to the dedicated capacity will be delivered after paying a premium charge to the supplier. They found that the optimal order size was independent of the random capacity and showed that the optimal dedicated capacity, which was difficult to obtain for general demand and capacity distribution, is bounded by zero and the optimal ordering quantity.

It is known that near myopic policies are close to optimal in their performance for inventory problems without production capacity, where the term 'near myopic policies' refer to policies derived by considering information for only the short term ahead. However, it has not been known whether near myopic policies are still close to optimality for problems with a production capacity. Tetsuo (2002) shows that a similar kind of result holds for the problems with production capacity, i.e. information from a few periods is sufficient to obtain a policy with results close to optimality. Tetsuo considers a single stage nonstationary production inventory model with uncertain production capacity and uncertain demand. The objective of the model is to minimize the total discounted expected costs which include production, inventory holding costs, and penalty costs. The production, inventory holding and penalty costs are assumed to be linear. Upper and lower bounds on the optimal policy for the infinite horizon problems were derived by considering some finite horizon problems. Further, Tetsuo show that, lower and upper bounds converge as the length of the planning horizon considered becomes longer, and under mild conditions differences between the upper and lower bounds converge to zero.

All the single-stage models discussed consider the lead time as known, but it is not always possible to know the exact lead time in reality. Random lead time has been studied extensively in literature. Arrow, Karlin and Scarf were one of the first to consider lead time as a stochastic process which is similar to a queuing problem (Kaplan, 1970). Kaplan (1970) illustrate that if there were no crossover of orders, and if the probability of delivery of outstanding orders are independent of the number and size of the outstanding orders, the sequential multi-dimensional minimization problem can be reduced to a sequence of single-stage minimizations. Random lead time for inventory systems in the past has been handled under the key assumptions that, the orders are received in the same sequence as they are placed (Bashyam, 1998). Bashyam et al. (1998) allowed the orders to cross in time, and moreover considered a constrained optimization problem unlike the unconstrained optimization that was previously considered in the literature. The optimal results were found using perturbation analysis, and a feasible directions procedure using simulation. In many standard models, the order lead time is a fixed constant or a random variable (Song, 1996). In many situations managers try to obtain dynamic information on their sources of supply. The model considered by Song and Zipkin (1996) generalized the stochastic lead time model developed by Kaplan (1970), and includes specific information related to lead time. Song and Zipkin developed an inventory-control model which involves a Markovian model of supply system, where the optimal policy has the same structure but its parameters change dynamically to reflect supply conditions.

2.2 Multi-Echelon Models with Single Final Product

Over the last three decades there has been extensive progress on developing inventory theory for multi-echelon systems (Graves, 1996). Multi-echelon inventory models are one of the many significantly investigated fields in the mathematical inventory theory (Kochel, 2005), but solving an analytical multi-echelon model is a difficult task. Most analytically solved multi-echelon models suffer from various restrictive assumptions. Multi-echelon problems are studied under assumptions which are generally a combination of the following: deterministic or random demand or/and capacity, deterministic or random lead time, constant or varying customer service level in each stage, constant or varying purchase/manufacturing/procurement costs in each stage, and component demand in upper echelons. This sub-section will review papers that are related to multi-echelon problems with a single final product.

A requirement for successful coordination of the supply chain is found to be the measurement of operational performance in terms of due date reliability, stock availability and other customer service measures (Diks, 1996). One of the major difficulties of cost-efficient and effective supply chain management is to determine the internal target service levels so that the desired external service level is achieved with minimum cost (Diks, 1996). There are five common types of control policies for multi-echelon inventory systems, installation stock (decentralized), echelon stock (centralized), kanban, order-up-to-*S*, and MRP (Axsater, 1994). The *installation stock* is the on-hand inventory plus the outstanding orders minus the backlog. Installation stock is also often referred to as the *inventory position*. The *echelon stock* is obtained from the inventory positions by adding the installation inventory positions at the current installation and also at all the downstream installations (Axsater, 1993). Decentralized inventory policies, i.e. installation stock have obvious advantage that they do not require any information from other installations. The disadvantage of installation stock is the limited cost effectiveness due to the lack of information about the entire system. The echelon stock concept was first introduced by Clark and Scarf (1960). And later also discussed by Federgruen and Zipkin (1984), and Bodt and Graves (1985) and many others. Badinelli (1992), Lee and Moinzadeh (1987), Svoronos and Zipkin (1988) and Axsater (1993) suggest using installation stock based policies in controlling inventory. Sherbrooke (1968), Axsater (1990) and Graves (1985) considered installation and echelon stock (Axsater, 1993).

A kanban policy indicates that the replenishment at installation n is in lots of Q_n units, where each lot is put into a container to which a kanban (ID card or tag) is used as a production order for new lot (Axsater, 1994). In an order-up-to-S policy, an order is released when the inventory position drops below S. The size of the order is chosen in a way that the inventory position reaches S after the order is placed. In Material Requirement Planning (MRP) a production plan for a total of n periods ahead is produced periodically but only implemented for the first period.

Clark and Scarf (1960) was one of the first to study multi-echelon problems with stochastic demand. Dynamic programming was used as the primary analytical tool to solve the multi-echelon problem. The base-stock policy structure is shown to be optimal in terms of multi-echelon inventory. That is, the multi-echelon inventory of a given node is defined as the sum of inventory from that node to the last node at the end of the system (end product inventory). An order-up to level is determined for each of these echelon inventories. The nodes considered do not have any capacity constraints. An optimal purchasing quantity for a finite horizon two-installation model is determined, and it could be very well used for the multi-installation model. The results from Clark and Scarf (1960) were generalized to assembly systems (Rosling, 1989). The model developed by Rosling (1989) demonstrated that assembly systems can be modeled equivalently as a series system. Optimal policies for general assembly systems under a restriction on the initial stock level are developed. Under the assumed conditions the assembly system is interpreted as a series system and solved using the Clark and Scarf's (1960) approach.

Bodt and Graves (1985) presents an approximate cost model for a continuous review inventory control policy. Their model is an extension to Clark and Scarf's (1960) serial system with periodic review. The approximation provided by Bodt and Graves (1985) is comparable to that of single item, continuous review inventory model that assumes a reorder point, reorder quantity policy. An approximate model for a two-stage system is developed and the analysis is further extended to an *M-stage* serial system. One of the key assumptions is that each stage has a reorder point that corresponds to the stage's echelon inventory, moreover only nested policies are considered, i.e. whenever a stage reorders all the downstream stages also reorder. For the policy to be stationary the order quantity has to be an integral multiple of order quantity at the immediate downstream stage. In case of a two-stage system the order quantity and reorder points are found easily by finding the best choice of order multiplier using a line search. Whereas in the case of an *M-stage* system finding an optimal solution becomes difficult, this opens the scope for using heuristic procedures. Two heuristic procedures are used to find the appropriate order quantity multiplier. The heuristic is tested over 5000 test problems and found to provide good results. Badinelli (1992) considered a model of steady-state values of on-hand inventory and backorders for each facility of a serial inventory system, where each facility follows a *(Q, r)* policy based on installation stock. The lead time considered in this model is stochastic in nature. The stochastic lead time is a result of stockout delays. The inventory model by Badinelli (1992) models stockout delays exactly. Once the stockout occurs the lead time is the nominal lead time plus the delay until the next replenishment arrives at the supplier facility. Optimal lot sizes and reorder points at each facility are obtained by taking into account the holding costs, setup costs, and stockout costs.

Lee and Yano (1988) consider a single-period problem for a serial system with stochastic yield at each stage, deterministic demand and no set-up costs considered. Under specific conditions on shortage costs, holding costs, and processing it is shown that the single critical number policy is optimal representing the optimal target input at each stage. Bassok and Akella (1991) consider a two-stage production and inventory problem, for a single product, with uncertain demand, and a supply constraint in the form of service level. The production and component ordering decisions are jointly optimized using an integrated model. Gurnani et al. (1996) considers an assembly problem where two critical components are required for the assembly into a final product, the demand for the final product is stochastic. The supply process of the components is a case of "all-or-nothing", i.e. with a probability of β the supplier provides 100% of the order and with a probability of $(1 - \beta)$ the supplier provides nothing. The optimal policy structure of the assembly problem for a single-period is shown to be a single critical number (base-stock policy), and under certain conditions for cost and delivery parameters the optimal policy for a multi-period is shown to be a base-stock policy. In a variant of the above paper Gurnani et al. (2000) consider the same assembly system, where the components that are procured from the suppliers are random due to the production yield losses. An exact cost function is formulated, which is analytically complex in nature. So a modified cost function is introduced, using combined ordering and assembly decisions. Gurnani et al. (2000) discusses both a single-period and a multi-period case, and show that the order-up-to structure is the optimal ordering structure.

The base-stock/single critical number policy is shown to be an optimal policy for a single-item, periodic-review problems with no fixed ordering/production costs and a capacity constraint (Federgruen, 1986). The base-stock policy is also shown to be optimal policy for multi-echelon systems in which the fixed costs are ignored (Erkip, 1990). Glasserman and Tayur (1995) obtain the optimal base-stock levels for a capacitated multi-echelon serial system. Simulation based optimization methods are used by Glasserman and Tayur for estimating sensitivities of inventory costs with respect to the policy parameters. Simulation-based derivative estimates help steer the search to an improved policy and at the same time allowing for complex features that are usually difficult to handle by analytical models (Glasserman, 1995). A feasible direction search algorithm is used to obtain the optimal base-stock levels, which is based on the derivative estimates computed from the simulation. In a capacitated multi-echelon production-inventory system it is shown by Glasserman and Tayur (1995) that for various cost and performance measures derivatives with respect to base-stock levels can be consistently estimated from simulation or real data.

Bollapragada et al. (2004a) considers a system similar to Glasserman et al. (1995). A two-echelon serial system with demand and supply uncertainty is considered. A non-zero lead time for component and end product assembly exists. Two-supply models are considered, one allowing unlimited backordering and the other allowing backordering only for one period. Assuming an installation base-stock policy with quasi-concave demand distribution, Bollapragada et al. (2004a) determine the optimal base-stock levels which minimizes the total inventory investment subject to the specific service level. Under both demand and supply uncertainties Bollapragada et al. (2004a) show the optimal component stock level is a convex-decreasing function of the stock level of finished products. A simple illustration of the optimal internal-service level used to decompose and coordinate component end product replenishment is established. In a different paper Bollapragada et al. (2004) considers a multi-echelon assembly system that utilizes simulation-based inventory optimization to obtain the optimal base-stock levels. The assembly system considered by Bollapragada et al. (2004) employs installation base-stock policies with random capacity and random end product demand. Infinitesimal Perturbation Analysis (IPA) is used to estimate the gradient and further obtain the optimal stocking levels. A decomposition heuristic approach that uses an internal service level to independently determine near-optimal stock levels for each component is proposed. The multi-echelon system is converted into an equivalent two-echelon assembly system analytically. Several important managerial insights are illustrated from the numerical analysis.

The most frequently used approach (both in the literature and in practice) has been decomposition, which treats each stage as an independent separate entity with certain pre-assigned parameters and constraints. (Bollapragada, 2004; Inderfurth, 1998; Shang, 2006). The decomposition approach clearly provides an advantage computationally. Instead of solving a multi-dimensional problem only a one-dimensional problem needs to be solved. Moreover only local information is required for a manager to make a decision rather than information from the entire system. The literature addresses two seemingly contradictory views for setting internal fill rates; some researchers and practitioners assume upstream locations should achieve higher fill rates to guarantee a desired system fill rate, whereas some other show that it is not necessarily accurate (Shang, 2006). Most of these contradictory views are based on empirical observations. There are only a few studies that address this problem theoretically, due to the complexity of the problem. Boyaci and Gallego (2001), Sobel (2004), Shang and Song (2006) address the problem theoretically.

Boyaci and Gallego (2001) develop a service level constrained formulation to minimize the total cost of a two stage serial system. Boyaci and Gallego present exact and approximate algorithms for minimizing costs subject to service level constraints. A base-stock policy is used to minimize the total inventory costs. They develop a unique method of determining the optimum bounds for the base-stock level. Initially the optimal base-stock level of downstream stage is found by making the base-stock level of the upstream a constant. Later Boyaci and Gallego determine the bounds for the downstream base-stock variable followed by the upstream base-stock variable. With the help of the bounds they confine/limit their search space to find the optimal base-stock level. The downstream bounds are further refined by including cost considerations. Due to the connection between the stages, a limit exists on number of units that can be sent downstream. Utilizing this factor Boyaci and Gallego find that, the lower bound of the downstream base-stock variable is used as an upper bound for the upstream base-stock variable. A bottom-up approach is used to obtain the optimal base-stock levels. Once the bounds are found a simple search can be used to find the optimal base-stock values.

Sobel (2004) presents exact and approximate formulas for the fill rate of periodic review supply systems that use base-stock policy. In the first part fill rate formula is presented for a single-stage system and with general distributions of demand. When demand is normally distributed with only standard normal distribution and density functions an exact expression is derived. In the second part, the probability distribution of finished goods inventory level in a serial inventory system with buffer inventory between stages is derived. This distribution leads to formulas for the fill rate. Based on the numerical computation, Sobel (2004) states that shorter supply chains have higher fill rates. Shang and Song (2006) develop analytical guidelines for managing service-constrained systems, with attention on the linkage between stages. A serial base-stock inventory model with Poisson demand and a fill rate constraint is considered. Closed form approximations for the optimal base-stock levels are developed. Two key steps are considered in the process of developing closed form approximations: 1) convert the service-constrained model to a backorder cost model by assigning an appropriate backorder cost rate, and 2) use a logistical distribution to approximate the lead time demand distribution in the single-stage approximation obtained earlier. These closed form expressions are used to conduct sensitivity analysis and establish qualitative properties on the system design issues, total optimal system cost, stock positioning, and internal fill rates. Shang and Song (2006) find that all the internal fill rates are lower than the target system fill rates as long as the latter is sufficiently high, high internal fill rates can lead to significant overstocking. Shang and Song (2006) have a few interesting observations: 1) moving a high value-adding stage to a downstream location may increase the optimal system stock but reduce the optimal system cost, 2) the optimal system stock is larger when the upstream stage has a longer lead time, and 3) the optimal base-stock level decreases as we move upstream.

There are only a few approaches where multi-echelon safety stock optimization along with service level constraints and inventory holding costs which would allow for varied holding costs at different echelons as it is typical in production environments (Inderfurth, 1998). These few approach mentioned by Inderfurth, 1998 has a serious shortcoming in the sense that it is assumed that any safety stock insufficiency would result in order delay downstream and moreover would propagate over the entire multi-echelon system. In manufacturing systems this does not always describe what exactly happens when there is an internal shortage. The demand uncertainties and fluctuations that occur internally are up to certain extent reduced by using the slack capability (rescheduling priorities, internal order expediting, initiating express deliveries from outside) of the system (Inderfurth, 1998). Under this supposed no-delay assumption, safety stock determination is decomposed into a multi-echelon buffer allocation and single-stage buffer sizing problem at all echelons, and the optimal safety-stock is found by Simpson (1958). Inderfurth (1993) extended Simpson's approach to divergent systems. Inderfurth et al. (1998) provided a further extension of this approach by considering the fill-rate related measures to size and duration of stockout, using this model it is found that service levels representing duration and size of stockout need less protection. A considerably simplified solution property is shown by Inderfurth et al. (1998) by solving a non-linear optimization problem of allocating safety stocks in multi-echelon inventory problems (serial, divergent, and convergent).

Optimal base-stock at each store (the stocking location for each part or end product) for a supply network model which considers the bill of materials, the nominal lead times, the demand and cost data, and the required customer service levels is obtained by Ettle et al. (2000). The base-stock levels minimize the overall inventory capital throughout the network and provide the required customer service level. The base-stock policy assumes a one-for-one replenishment mechanism. The essential ingredient of the model is an approximate analysis of the actual lead times at each store and the associated demand over such lead times, along with the classification of the operation at each store via an inventory-queue model. A constrained nonlinear optimization problem which minimizes the total average dollar value of inventory in the network is formulated. Unlike in Glasserman and Tayur (1995) where simulation-based inventory optimization (Perturbation analysis) is developed to obtain gradient estimation, Ettle et al. (2000) derive approximate gradient formulas that are analytically computed. The approximate gradient obtained is used to carry out the search for the optimal answer.

Allocation policies are important in a divergent multi-echelon system, especially in a warehouse-retail environment when a decision has to be made on distribution of available inventory to several retail outlets from a warehouse. McGavin et al. (1993) discuss three allocation policies that have been extensively used in the literature: a “ship-all” policy in which the central warehouse does not hold any inventory but transships to all the retail sites upon receipt of a bulk shipment from the supplier; an equal-interval policy in which the central warehouse ships equally the available inventory to all its retail outlets; a two-interval policy in which the warehouse makes an initial shipment to its retail sites upon receipt of its shipment from the supplier, and makes a final shipment just prior to the next replenishment. McGavin et al. (1993) review the major findings, and show the relative effectiveness of allocating quantities when a two-interval policy is used. Graves (1996) introduces a new allocation scheme for a warehouse-retail multi-echelon problem, which is known as virtual allocation. Under this new allocation policy each site in the supply chain commits or reserves a unit of its upstream inventory (if available), i.e. warehouse, to replenish its downstream stages (retail outlets). However the actual shipment of this reserved inventory does not occur until next order occasion. Graves provides an analogy of a waiting truck which is gradually filled, but will depart only after the entire truck is loaded. Consequently, the central warehouse has a high probability of stock out. Moreover, Graves (1996) show that the virtual allocation policy is near optimal for a set of test problems.

An assemble-to-order (ATO) system is an important business model, since it is used in a wide-ranging class of supply chains (Song, 2002). Managing the component inventory in ATO systems is of critical importance to the business. a stockout of any component will delay order fulfillment, and on the other hand having excess inventory can severely reduce the firms profit margin. Song and Yao (2002) study a single-product assembly system in which the final product is assembled to order, whereas the components are built to stock. Optimal trade-off between inventory and service in an ATO with several components and one final product is addressed by Song and Yao (2002). The replenishment lead times for components are random in nature. An independent base-stock policy is used to control each component's inventory. Two optimization problems are studied to determine the base-stock levels of the components. i) minimize the expected number of backorders subject to the total upper limit on the investment for component inventory, ii) minimize the average component inventory subject to the fill rate requirement for the customer. Song and Yao (2002) show it is desirable to keep higher base-stock levels for components with longer mean lead times (and lower unit costs). In order to overcome the computational difficulty evaluating performance measures such as back-order and the fill rates Song and Yao (2002) develop upper and lower bounds. These bounds are used as surrogates in the optimization problems, which are used to developed effective solution. Gallien and Wein (2001) consider a problem similar to Song and Yao (2002), however they assume synchronization, i.e. replenishment of all components triggered by the same customer demand are later assembled into same product. So there is only one lead time, which corresponds to the longest component lead time. A simple approximate solution to the problem is derived.

Different customers often have a different willingness to pay for the speed with which their orders are fulfilled (order expediting). A build-to-order company could improve its profits by responding to customers who are willing to pay higher prices for shorter lead times (Hariharian, 1995). There is a growing consensus that the manufacturers can benefit from having a portfolio of customers with different lead times (Gallego, G. 2001). Customers with positive demand lead times usually place their orders in advance of their requirements, resulting in what is known as advance demand information. Gallego and Ozer (2001) analyze a discrete-time, single-item, single-stage, periodic review inventory problem under advance demand information. They show that an *(s, S)* policy is optimal under a positive set-up cost scenario for an infinite-horizon case, and a base-stock policy is optimal under a zero setup cost. In a follow up paper Gallego and Ozer (2003) develop optimal replenishment policies for a multi-echelon inventory problem (serial system) under advance demand information. Gallego and Ozer (2003) prove optimality of state-dependent, echelon base-stock policies for both finite and infinite horizon problems with advance demand information, to show this the problem is decomposed into single location periodic review problems. Gallego and Ozer (2003) also show that myopic base-stock policies are optimal for finite and infinite horizon problems when the demand and cost parameters are stationary.

Tree/network structure is a class of supply chains which has been studied in detail. Two distinct models arise when tree/network structure is considered: stochastic-service model and the guaranteed-service model. In the stochastic service model, each stage in the supply chain has orders which are subject to stochastic delays. These delays are due to stockout at upstream stages. In the case of the guaranteed-service model, it is assumed that each stage has an external source, so that if a stockout occurs at any stage then the additional demand is fulfilled by the external source. In the guaranteed-service model the service time is always guaranteed. Lee and Billington (1993) developed a multi-stage inventory model for the Hewlett-Packard DeskJet printer supply chain under the stochastic service assumption. The objective was to provide managers with tools to evaluate various stock positioning strategies. Each stage is assumed to control its inventory by an installation periodic review base-stock policy. Approximations were developed by Lee and Billington for replenishment lead times at all stages. Ettl et al. (2000) analyzed supply chins under the same assumption, i.e. stochastic service model, in which each stage controls its inventory with a continuous-time base-stock policy. Ettl et al. (2000) differentiated the nominal lead time from actual lead time at each stage. The actual lead time will exceed the nominal lead time when there is a supplier stockout. The authors analyzed the assembly system by assuming that at most one supplier can stockout at any given time. Approximations and bounds were derived on the expected backorder delays to downstream customers by modeling the replenishment process at each stage as $M/G/\infty$ queue. Ettl et al. (2000) optimized the total inventory investment, i.e. pipeline inventory and finished goods inventory, in the supply chain subject to meeting the required service level of the external customers. Safety factors were used as the decision variables to develop analytical expressions for gradients, and further were used to solve the constrained nonlinear optimization problem. Graves and Willems (2000) applied the guaranteed-service model approach to tree-structure supply chains. Instead of using the base-stock levels or service levels, planned lead time at all stages were used as the decision variable. The planned lead time is assumed to be 100% guaranteed to the downstream stages, thus the lead time between two stages is deterministic. A fast algorithm based on dynamic programming is developed by Graves and Willems (2000) to optimize the safety stock placement. Simchi-Levi and Zhao (2005) study the safety stock positioning problem in single-product multi-stage supply chains with a tree network under the stochastic service model assumption. An independent Poisson process is used to model external demands, and unsatisfied demand is fully backordered. An installation, continuous-time, base-stock policy is used to control inventory at each stage. The lead time is considered to be stochastic, sequential, and exogenously determined. Based on these assumptions the recursive equations are derived by Simchi-Levi and Zhao (2005) are used to develop insights into the impact of safety stock, approximations and algorithms to coordinate the base-stock levels in theses supply chains, so as to minimize system-wide inventory cost subject to service level requirements.

2.3 Multi-Echelon Problems with Multiple-Products

Competitive pressures in today's emerging marketplace are forcing companies to offer quicker response to customer needs. This is increasingly difficult for firms when they produce more than one product. As a result, managers need to pay close attention to various performance measures that reflect the system responsiveness. For example, manufacturer and distributors may manage huge stocks of several items. A customer order typically consists of several different items of varied order size, and it is the firm's challenge to satisfy the orders placed by customers in a stipulated period of time. The order fill rate, probability of filling an entire customer order from the available stock, or in general a pre-specified order time is an important measure of service in models with multi-echelons and multi-products. There are two important approaches that need to be discussed before models of multi-echelon, multi-product systems are considered: 1) order-based approach, and 2) item-based approach. In an order-based approach connection between the items are considered, whereas an item-based approach considers the demand between the items as independent. This issue has been crucial to many big firms like AT&T, IBM and Phillips which practice "assemble-to-order", since the component procurement lead times are often longer than the products shelf-life (Song, J. 1998). In order avoid the costly pile of unsold products and to compete in the fast-changing market, new products are usually designed around the interchangeable modules (Song, 1998). Thus companies can assemble products to order while producing modules to stock.

Srinivasan et al. (1992) study a multi-period, multi-component requirement planning problem in which each component can be common to several products whose demand is uncertain. A stochastic programming formulation of the problem is developed along with heuristic solution approaches. The component procurement lead times is not considered in the formulation. Hausman et al. (1998) study the evaluation and optimization of the order fill rate in a discrete-time, multi-item, base-stock system. Customer orders arrive in each period, and the total demand for that item is the sum of all the orders that arrive for that item in that period. Demands are correlated over the items in a period but they are independent across the periods. Since a discrete-time formulation is used in which the demands of a period are aggregated, it is impossible to identify the individual customer orders, so only bounds could be computed on the order fill rate. The computation also involves evaluation of multivariate normal distributions, which is computationally complex. Anupindi and Tayur (1998) consider both item-based and order-based performance measures in a multi-product cyclic production system. A simulation procedure is used to obtain good base-stock policies for each kind of performance measure. The numerical results in Anupindi and Tayur (1998) indicate that the item fill rates are not good indicators for order fill rates. Song (1998) considered a multi-item inventory system consisting of several items in different amounts. A base-stock system is considered in which the demand process forms a compound multivariate Poisson process and the replenishment lead times are constant. Song (1998) show that the order fill rate can be computed through a series of convolutions of one-dimensional compound Poisson distributions. Simpler bounds were formulated to estimate the order fill rate.

Erkip et al. (1990) considers a single-depot multi-warehouse system, and show that the *S*-type policy or the base-stock policy was optimal. The demands were considered to be correlated for different items in a given time period. Erkip et al. (1990) computed the optimal stocking policies as explicit functions of the correlation coefficients. Langenhoff and Zijm (1990) developed an analytical framework for a multi-echelon production system to determine optimal control policies for such systems under an average cost criterion. Exact decomposition of a complex multi-dimensional (serial and assembly) system to a series of one-dimensional problems is achieved. Later Houtum and Zijm (1991) developed numerical procedures for the analytical decomposition in Langenhoff and Zijm (1990). The numerical procedures enabled obtaining the optimal order-up-to levels for all the stages.

Numerous researchers have proposed power-of-two lot-sizing rules for stationary, continuous-time, infinite-horizon multi-stage production/inventory systems (Roundy, 1986). A power-of-two policy is a sequence of positive numbers $T = \left(T_n : n \in N\right)$ with the following three properties. First, orders for product n are placed once every $T_n > 0$ units of time beginning at zero. Second, $T_n = 2^{k_n}\beta$ for all products n and for $1 \le \beta \le 2$, where K_n is an integer. Finally, the zero-inventory property, i.e. an order is placed only when the inventory of that product is zero. Roundy (1986) study a multi-product, multi-stage production inventory system in a continuous time. External demand occurs for any or all products at a constant, product-dependent rate. Power-of-two policy policies are used in which each product employs a stationary interval of time between successive orders, and the ratio of order intervals of any two products is an integer of power of two. Roundy (1986) show that there is always a policy in this class that is within 2% of optimal answer.

Capacity or inventory allocation plays an important role when there is a shortage of capacity or inventory at the supplier end. Hausman et al. (1998) consider a multi-product assembly model where the product structure and demand processes are random and correlated within any period but stationary across the review periods. Replenishment lead times (from external suppliers) for orders of components are considered to be deterministic. A base-stock policy is used to manage the component stockpiles. Within a given period, demands are assumed to occur in some unknown sequence and the allocation of stock is done in the sequence in which these orders are received. Since the sequence of orders is unknown Hausman et al. (1998) focus only on the service level provided to the last order within a given period. Hausman et al. (1998) obtain optimal base-stock levels for the components and the final products under the assumptions discussed above. Zhang (1998) considers a similar problem as Hausman et al. (1998), the only major difference is the allocation policy; a "periodic" allocation policy is used. Moreover it is assumed that allocation of components is done according to a predetermined list. Thus, the finished products with higher priority always receive components before the products with lower priority. Agrawal and Cohen (2001) also consider a problem similar to Hausman et al. (1998) where the allocation policy is based on “fare-shares / proportional allocation”, in which the quantity of component allocated to a particular product is determined by the ratio of the realized demand due to the product order to total demand based on all orders. Under these assumptions, Agrawal and Cohen (2001) derive an expression that links component inventory levels to inventory cost and measures for finished product delivery service, such as order completion rates and response times to customer orders. These results are then later used to determine the optimal base-stock levels that use the fare-shares allocation policy. In another multi-product setting, Bish et al. (2005) consider a two-plant two-product capacitated manufacturing problem, and show that the performance of a system heavily depends on the allocation mechanism used to assign products to available capacity.

Similar to capacity allocation, optimal procurement policy in a multi-component assembly system plays an important role. Kim et al. (2006) consider an assembly system where a firm produces a single-product which is assembled using two types of components (component 1 and component 2). The components are provided by two individual suppliers (supplier 1 and supplier 2). Kim et al. (2006) assume that the firm makes different procurement contracts with supplier 1 and supplier 2. To supplier 1 the firm specifies a maximum inventory level of component 1 and makes a commitment to purchase the component as long as inventory is below this target level. Whereas to supplier 2, the firm has an option of purchasing or rejecting component 2 at each instant supplier 2 provides it. A Markov decision problem is formulated, leading to a component

2 purchasing policy which maximizes the firms profit subject to costs of rejecting component 1, holding component 2, and purchasing component 2. The optimal purchase policy is obtained only under certain states, and these states are used to investigate how the change in sale price and cost parameters affects the optimal purchasing policy.

Order fulfillment is increasingly important for companies if they need to adapt quickly to market and technology changes and thus move toward assemble-to-order as opposed to traditional make-to-stock (Song, J. 1999). Adapting assemble-to-order raises the following two questions: For a given safety-stock level of each item, what is the probability of a demand being immediately satisfied (order fill rate)? What is the probability a customer order can be met within a time window (order response time)? Song et al. (1999) study a multi-component, multi-product production inventory system in which the individual items are made to stock but the final item is assembled to customer's orders. An exact analysis on a wide range of performance measures in the assemble-to-order systems with sequential, capacitated stochastic production process is carried out. The demand is modeled as a multivariate Poisson process. That is, the overall demand arrives according to a Poisson process, but there is a fixed probability that the demand requests a particular kit of different items. Demands for items that cannot be filled are backordered in a capacitated queue (capacity varies from zero to infinite). For a given base-stock policy and backlog queue capacity, a procedure to evaluate the item-based, order-based, and system-based performance measures, such as fill-rate, service level (probability that an order will be backlogged and eventually filled) is found. The procedure relies on the computation of steady-state joint distribution of the outstanding orders. In a similar setup Song (2002) shows how to evaluate the order-based backorders in a continuous review, base-stock inventory system with constant lead times and multiple type orders. Instead of using the joint-distribution of item-backorder, a new approach to the exact analysis that leads to the closed form expression was employed. The approach consists of two steps, the first step deals with the equal lead times across the items. The important aspect here was to relate the customer waiting time distribution with immediate order fill rate. The second step of the analysis deals with the general case of unequal lead times. Lu et al. (2005) also considered a multi-product assemble-to-order system, but the lead time for component replenishment is stochastic in nature. The components are built to stock with inventory controlled by base-stock rules, but the final products are assembled to order. The customer orders follow a Poisson process. An optimal allocation of the given budget among component inventories so as to minimize the total backorders over the product type is studied. Bounds and approximations for the expected number of backorders are found to formulate surrogate optimization problems.

Rao et al. (2004) consider a single period multi-product inventory problem with stochastic demand, setup cost for production, and one-way product substitution in downward direction. Rao et al. (2004) present some important properties and an effective solution methodology that exploits the problem structure and utilizes a combination of optimization techniques that include network flows, dynamic programming and infinitesimal perturbation analysis. The problem is also extended to a multi-period problem in which product set selection can be made one at the beginning of the period and cannot be revised later. New and efficient solutions are obtained with the help of heuristic methods which combine dynamic programming and simulation based optimization while exploring the network flow structure of the allocation problem.

Numerical study indicates that the heuristic method employed is very effective in terms of accuracy.

2.4 Multi-Echelon Problems with Spare Parts

Spare parts are members of what is also known as MRO (maintenance, repair, and purchasing), which represent an important portion of all purchasing activities and have been the recent object of an explosion of e-market sites to reduce the purchasing costs (Diaz, 2003). Spare parts optimization requires an integrated approach that starts with the removal of factors that create noise: bad coding, resulting in parts proliferation; lack of classification; poor network practices (uncoupled warehouses, poor relations with key suppliers); and poor data integrity (non-centralized, non-real time data) (Diaz, 2003).

An important decision-making problem associated with multi-echelon inventory systems is determining stocking policies at various stock sites in the system (Moinzadeh, 1986). A key component in the design of multi-echelon systems for recoverable items is the determination of appropriate stock levels of spare inventory at each echelon (Graves, 1985). METRIC (A Multi-Echelon Technique for Recoverable Item Control) is a mathematical model of base-depot supply system developed by Sherbrooke (1967). The demand considered is a compound Poisson with a mean value estimated through a Bayesian procedure. When a unit fails at the base level there is a probability associated with it to either repair it at the base or send it to a depot. In the later case the base sends in a request for a resupply, and if no lateral resupply exits then Sherbrooke has shown that an $(s-1, s)$ policy is appropriate. This policy would be valid only if the items in consideration are high-cost and low-demand. The problem is shown to have a simple analytical solution which is function of repair times, minimizing expected backorders for any system investment. A common approach to solve a multi-echelon inventory system with recoverable items has two components (Graves, 1985). The first component is to characterize the service performance of multi-echelon system for a given inventory stock levels. This can be done either using an exact model or an approximate model. The second component is to search methodically to find the appropriate inventory stock that satisfies both inventory costs and service performance. Along the similar lines as METRIC, Graves (1985) determines the inventory stock level for a multi-echelon inventory problem with recoverable items. The approximation used in the Graves (1985) model dominates the results of the approximate results of METRIC

The repairable products of high value with infrequent failures have been shown in past studies to require a one-for-one, *(s-1, s,)* inventory ordering policy. For multi-echelon reparable inventory systems with high set-up costs for order and high demand rate, the batch ordering was shown to more cost effective than *(s-1, s)* policy (Moinzadeh, 1986). Moinzadeh and Lee (1986) show that when the set-up cost for ordering a shipment is high relative to the holding cost of the product the *(s-1, s)* policy might not be optimal. Moinzadeh and Lee (1986) determine the optimal batch size by minimizing the total expected costs of ordering, holding, inventory, and backorder costs. The optimal batch size is found using a power approximation scheme; given a batch size an upper bound on stocking level is found. With the help of the upper bound the depot stocking level is obtained using a "one-pass" search.

The METRIC model with minor variations has been studied by several authors. Svoronos and Zipkin (1988) studied a similar two-echelon problem with the same purpose of estimating the performance measures, the long-run average backorders at retailers, and average inventory at each location. The approach used is a decomposition technique, where each facility is approximated as a single location inventory system. However the parameters describing these single-locations are dependent on the policies and performance of other locations. The important aspect of the problem is that the model uses variance information in the approximations. A variant of the METRIC model was developed by Axsater (1990) which consists of one warehouse and *N* retailers. A one-for-one replenishment policy is employed and shown to be optimal, due to the low ordering cost involved. The approach employed by Axsater (1990) uses the inventory cost function that reflects costs incurred on an average unit. The approach is found to be more efficient and direct at finding the optimal inventory policy compared to METRIC. Later Axsater (1993) extended his previous results to derive general policies when both retailers and the warehouse order in batches compared to one-for-one replenishment. One of the major assumptions of the problem is that of identical retailers. The same problem considered by Svoronos and Zipkin (1988) are solved by different approximate methods and the results are compared and found to outperform Svoronos and Zipkin (1988) for large problems.

Most of the multi-echelon inventory models assume the demand that is not satisfied immediately can be backordered (Andersson, 2001), which may not be a realistic assumption is some cases. Andersson and Melchiors (2001) consider an inventory model which consists of warehouse and several retailers, where unsatisfied demand faced by retailer is replenished from an outside supplier. The METRIC- approximation framework which was discussed earlier is used to develop a heuristic for finding cost-effective base-stock policies. Approximate warehouse and retailer costs are used to obtain the overall solution, and the solution obtained using the heuristic on average performed 0.40% better than the $(s-1, s)$ policy.

Most of the inventory models dealing with spare parts considered until now assume independent part failures and hence one-at-a-time failure. In, practice usually more than one part is required when a failure occurs. Cheung and Hausman (1995) consider a situation of multiple failures, where the multiple failures are represented in a continuous, infinite horizon, order-for-order spares replenishment inventory model. Exact expressions for the distribution function and the expectations of the number of backlogged jobs are presented. These results are used to optimize the spares inventories, and to evaluate the effects of correlated failures, and to model the impact of parts commonality.

2.5 Information Sharing

Information sharing plays a key role in cost reduction and increasing the target customer service level. Information sharing can be defined as sharing of demand and inventory data between the key supply chain players, such as suppliers and retailers. It plays an important role in evaluating supply chain efficiency. Greater information sharing about actual demand between stages of the supply chain is an intuitive step towards reduction of the bullwhip effect. Gavirneni et al. (1999) described the role of information sharing in a two stage supply chain with a capacitated supplier and a single retailer. The three types of information sharing listed below are most often discussed in the literature:

- Classical information sharing / no information sharing

- Partial information sharing
- Full information sharing

The basic difference between these three types of information sharing is the extent of information that the supplier is willing to share with retailer and vice-versa.

Classical information sharing / no information sharing: In the classical information sharing the only information a supplier is aware of are the retailer's orders. In these situations if the supplier is short of material to deliver to his retail locations, the supplier uses the past history to determine the quantity that each retailer would receive.

Partial information sharing: In case of partial information sharing the supplier has information about the retailer's specific order, as well as the policy used and the end item demand distribution. Since the end item demand distribution is known the supplier can estimate the incoming order in advance. The literature on this type of information sharing shows that there is a high possibility of gaming behavior in these situations.

Full information sharing: With full information sharing the supplier has access to retailer's order, day-to-day inventory levels, and end product demand. This improves the efficiency with which the supplier allocates the available stock to retailers in response to actual customer demand and also improves the ordering decisions.

Sharing information, i.e. demand information is often not enough to mitigate the problems caused with supply chain coordination (e.g. bullwhip effect) (Cachon and Terwiesch, 2006). Demand can also be influenced by retailer actions on pricing, merchandise, promotion, advertisement etc. The process of sharing all the information that can directly/indirectly affect the demand is known as *Collaborative Planning, Forecasting, and Replenishment*, or CPFR for short.

2.6 Simulation Based Optimization

The rapid developments in the optimization and computation field allow optimization packages to easily solve models that involve thousands of variables. In parallel, simulation modeling and analysis of complex real systems has been used increasingly as both a descriptive tool and as a decision support tool. The developments in the field of simulation optimization have also moved forward significantly in recent years, primarily due the advances in the optimization field, and in the use of heuristic search methods.

Simulation based optimization (SBO) is the process of finding the best input variable values from among all possibilities without explicitly evaluating each possibility and using simulation as the evaluation mechanism (Carson and Maria, 1997). Finding an optimum solution traditionally requires identifying all possible alternatives, evaluating each possible alternative accurately, comparing each alternative fairly, and coming up with a best solution (Kunter, 1996). This process is the best way to find an accurate answer when the search space is relatively small and the problem is fairly complex. When the problem gets more complex and the search space is multi-modal in nature, it is very difficult to proceed by the traditional methods, primarily due to time constraints, and resource constraints. Using the SBO approach the amount of time taken to solve a relatively complex problem is reduced. Depending on the effectiveness of the SBO an optimal solution is not guaranteed, but a good solution is obtained most of the time. This establishes a trade-off between the quality of the final solution required and the computational time expended in finding that solution. Figure 1.6 represents a typical SBO framework, whereas figure 1.7 represents a commercial version of the SBO. Commercial software that implements general purpose SBO has been available for several years.

Many of these commercial products use some type of heuristic search technique as a part of their optimization subroutine.

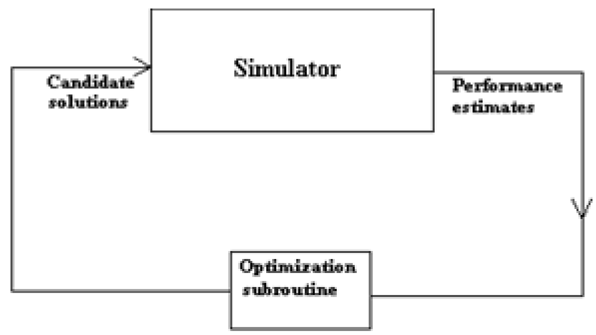

Figure 2.2: Optimization for Simulation (Fu, 1994)

The basic optimization problem for an SBO is shown in 2.1, where the goal is to maximize or minimize an objective function H(X), as a function of decision variables X. $H(X) = E[L(X,\varepsilon)]$ is the performance measure for the system (Eylem and Ihsan, 2004).

$$Maximize \text{ (or } Minimize\text{) } H(X) \quad X \in \Theta \tag{2.1}$$

The quantity $L(X,\varepsilon)$ is the sample performance, ε represents the stochastic effect in the system, X is a p-vector of controllable factors, and Θ is the constraint set on X. H(X) could be a single objective function or a multi-objective function (Eylem and Ihsan, 2004). Simulation optimization problems can be broadly classified into two types, local optimization and global optimization. Most of the traditional procedures fall into the local optimization category, whereas most new heuristic procedures fall into the global optimization category. Global optimization procedures provide a means of escaping from a local optimum). The classification is summarized in Figure 2.3. Local optimization is further classified into discrete decision spaces and continuous decision spaces. Under each of these categories there are many approaches as shown in Figure 2.3 (Eylem and Ihsan, 2004).

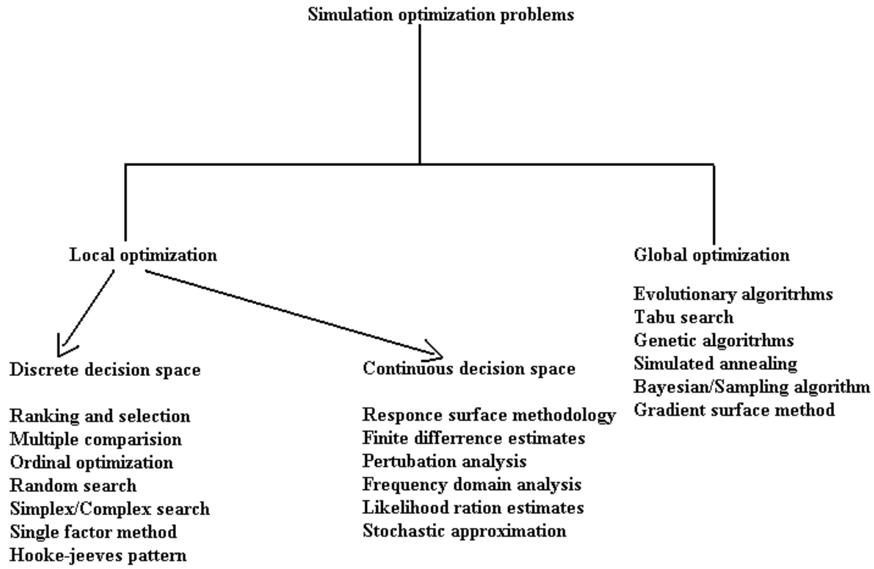

Figure 2.3: Classification of simulation optimization (Fu, 1994)

2.6.1 Meta-modeling methods

A meta-model provides a relationship between performance measure and parameters of interest (Fu, 1994). The meta-model approach to SBO divides the optimization task into two sub-problems, estimation and optimization. Simulation is used to fit a global response curve which is also referred to as the meta-model (Fu, 1994). Some type of search procedure is then used to explore that response curve. The meta-modeling methods can be broadly classified as:

- Gradient based.
- Stochastic approximation (StA)
- Response surface methodology (RSM)
- Statistical methods.

The **gradient based approach** could be further subdivided as follows:

- Finite difference estimates (FDE)

- Perturbation analysis (PA)
- Frequency Domain analysis (FDA)
- Likelihood ratio estimates (LRE)

In this sub-section we limit our discussion to Perturbation Analysis.

Perturbation analysis (PA): Perturbation analysis has two types, the finite perturbation analysis (FPA) and the infinitesimal perturbation analysis (IPA). FPA is especially used when the problem parameters are discrete in nature. "FPA is a heuristic which approximates the difference in a performance measure when a discrete parameter is changed by one unit" (Eylem and Ihsan, 2004). The primary aim of any PA would be avoiding additional simulation runs to evaluate the performance measure at the perturbed value. When PA is applied properly the estimates of the gradient can be obtained in just a single run when certain conditions are satisfied. IPA is used to obtain the derivatives of continuous parameters by estimating the partial derivative during a single simulation run, in turn keeping all the related statistics of certain events (Eylem and Ihsan, 2004). Estimates in IPA result from causing small changes in the input parameters that do not cause any event order changes in the simulation of the subject system (Fu, 1994).

$$\frac{\partial L}{\partial X} = \sum_{i,k} \frac{\partial L}{\partial T_k} \ \frac{\partial T_k}{\partial T_i} \ \frac{\partial T_i}{\partial X} \tag{2.2}$$

Here $\partial T_i / \partial X$ represents the change in the value of the system parameter X as the timing of the events change, $\partial T_k / \partial T_i$ represents the change in timing of event T_i as the other event T_k change, $\partial L / \partial T_k$ represents the how the timing of event T_k change with respect the system performance. The difficulty with the application of PA is that the modeler must have deep knowledge of the simulation model and the system it represents, and in some situations must build the model right from start (Eylem and Ihsan, 2004, Carson and Maria, 1997). PA helps reduce the number of simulations runs required to get derivative estimates, but it is not feasible for all systems and problems. When the system is sufficiently complex the analysis is too difficult and the PA approach is not possible.

Stochastic approximation (StA)

Stochastic approximation is a procedure for finding the minimum or maximum of a function. It was first introduced by Robbins and Monro in 1951 and Kiefer and Wolfowitz in 1952 (Eylem and Ihsan, 2004). The assumption underlying the problem is given as, minimization of H(X) can be solved using $\nabla H(X) = 0$. This method uses a recursive formula which is shown in (2.3).

$$X_{n+1} = \prod\nolimits_{\Theta} \left(X_n - a_n \hat{\nabla} H_n \right) \tag{2.3}$$

where a_n is a series of real valued step sizes that must satisfy $\sum a_n < \infty, \sum a_n^2 < \infty$. The quantity X_n is the estimated value of the minimum at the start of the iteration n, $\hat{\nabla} H_n$ is an estimate of the gradient $\nabla H(X_n)$ from iteration n, and $\prod\nolimits_{\Theta}$ is projection onto Θ (Eylem and Ihsan, 2004). As the number of iterations approaches infinity, X_n approaches a value such that the theoretical regression function of the stochastic response is minimized.

Response surface methodology (RSM)

RSM is a class of procedures that fits regression models to the output responses of a simulation model evaluated at several points, and optimizing the fit of the resulting response function (Eylem and Ihsan, 2004). The process could also be accomplished using neural networks in place of the regression. Sequential RSM is one of the popular forms of simulation optimization found in the research literature. The goal of RSM is to obtain an approximate functional relationship between the input variables and the output response function (Fu, 2002) (Fu, 2001). The basic algorithm is divided into two phases: In phase I, the first-order model is fit to the response surface. Then the steepest decent is computed and the process is repeated until there is no further improvements. In phase II, a quadratic response surface is fitted using a second-order experimental designs, and then the optimum is derived from the fit (Eylem and Ihsan, 2004).

"RSM provides a general methodology for optimization via simulation" (Eylem and Ihsan, 2004). The greatest advantage of this method is that it employs well-known statistical tools. In general RSM requires fewer simulation experiments compared to other gradient type methods (Carson and Maria, 1997).

Statistical Methods

The statistical methods could be classified into ranking and selection, multiple comparisons, and importance sampling. In case of ranking and selection two different approaches are employed. The first approach is called the indifference-zone, where the performance measure value selected differs from the optimum solution by a small value with at least certain a probability. The second approach is known as the subset selection where the optimum solution is most likely to be present in a set of best performance measures with a certain probability.

In case of multiple comparison procedures, a number of replications are run and the conclusion on performance measure is made by constructing confidence interval (Eylem and Ihsan, 2004).

The basic idea of the importance sampling is to simulate the system with different underlying probability measures so as to increase the probability of typical sample paths of interest. For each observation the estimated value is multiplied by a correction factor in order to obtain the unbiased estimate of the measure of the original system (Eylem and Ihsan, 2004).

All the above methods discussed above act as a means of providing the local optimum. The probability of convergence is higher when these methods are used, though the convergence time is long. The global optimization methods do not assure of best optimal solution, but in many cases they provide good solutions. They converge relatively faster compared to the traditional methods. The meta-modal methods were used in manufacturing problems, queuing networks etc. and are still being used in solving many problems. IPA is used in obtaining optimum inventory levels, intelligent traffic signaling system at the intersections, and ramp meter control. These meta-models still serve as a backbone for many real-world problems because they assure optimal answers for appropriately structured problems.

2.6.2 Nested Partitions for Global Optimization

There are many popular and effective methods to perform global optimization, some of them have been listed in the figure 2.3. The earliest and the most robust method involve the pure random search (PRS) (Shi and Olafsson, 2000). Under this method points in feasible search space are selected at random but uniformly, and their cardinal values are compared. The PRS is slow since it does not have a learning curve. Since then there have been many methods like simulated annealing algorithm, pure adaptive search (PAS), tabu search, multi-start algorithm etc.

Shi and Olafsson (2000) proposed a new randomized method for solving the global optimization problems. The new randomized search known as the nested partitions (NP) method is similar to that of the clustering methods. Unlike in clustering methods where an attempt is made to cluster together sample points that are "close", in NP method the feasible region is systematically partitioned into several sub-regions, assessment of a sub-region is performed and the entire computational effort is concentrated on one particular region. Let us consider the problems of the following form (Shi and Olafsson, 2000):

$$\theta^* \in arg \min_{\theta \in \Theta} f\left(\theta\right) \tag{2.4}$$

Where Θ is the finite solution set and $f : \Theta \to \mathrm{R}$ is the performance function to be optimized. In each iteration of the algorithm the *most promising region* is assumed, which would be a subset of the region Θ, and is further partitioned into *M* sub-regions and aggregate the entire surrounding region into one region. So there are *M+1* regions. At each iteration *M+1* disjoint sub-sets of the feasible region are analyzed. Using a random sampling method the performance function values are selected, and a promising index (PI) of each region is calculated. The regions for the next iteration are based on the PI of the region. The sub-regions found to be best is partitioned and the surrounding regions are aggregated into one region. If the surrounding region is later found as the best, the algorithm backtracks to the larger region that contains the older most promising region. The new most promising region is similarly partitioned and sampled as performed earlier.

2.6.3 Simulation Based Inventory Optimization

Min and Zhou (2002) categorized supply chain models that involve inventory and simulation as hybrids, since they involve components that are deterministic and stochastic in nature. The table 2.1 and 2.2 below shows various papers which use simulation and simulation optimization for inventory based models. Table 2.1 provides a short summary of the research conducted in studying the performance of supply chain using simulation. Unfortunately it is often not enough to only study the performance of the supply chain, but rather a need exists to optimize the supply chain parameters. So, table 2.2 show the research conducted in order to optimize the performance of supply chains using simulation-based approaches.

Table 2.1 Performance of Supply Chain Using Simulation*

No #	Authors, Year	Short Summary
1	Towill et al. (1992)	Simulation is used as a means to study the impact of various inventory policies
2	Lee and Billington (1993)	Developed a supply chain model for HP to manage their 1) material flows in their supply chains 2) assess inventory investment 3) evaulate alternatives
3	Strader et al. (2000)	A multi-agent simulation as a tool to study the impact of information sharing on order fullfilment in divergent assembly supply-chains
4	Souza et al. (2000)	The impact of causal factors on the dynamic performance of the supply chain
5	Beamon and Chen (2001)	The performance of conjoined supply chains is studied through experimental design and simulation
6	Holweg and Bicheno (2002)	Described how a participative simulation model is used to demonstrate the supply chain dynamics and provde improvements to the entire supply chain

(Daniel and Rajendran, 2005)

Table 2.2 Optimized Performance of Supply Chain Using Simulation

No #	Authors, Year	Short Summary
1	Glasserman and Tayur (1995)	Investigated multi-echelon systems working under a base stock policy with capacity constraints, and found the optimal solution using an (IPA) framework
2	Disney et al. (1997)	Demonstrated the use of a model with decision support system along with simulation facility and genetic algorithm (GA) optimization procedure
3	Bashyam and Fu (1998)	A constrained optimization problem is considerd, for which the optimal (S,s) inventory levels are found using a feasible direction algorithm in simulation, which also uses an IPA framework
4	Petrovic et al. (1998)	Used a fuzzy modeling and simulation of a supply chain in a stochastic environment to detrmine the stock levels and order quantities
5	Rao et al. (2000)	Developed an integrated model to analyze different models of Caterpiller's new line of compact construction equipment
6	Ettl Et al. (2000)	Developed supply network that took as input data (BOM, lead time, etc.) to come up with base-stock levels at each stocking location
7	Bollapragada et al. (2004)	Studied the supply performance in assembly systems with uncertain supply and demand in a multi-echelon system environment using an IPA framework
8	Daniel and Rajendran (2005)	The optimal base-stock levels for a single-product serial supply chain is found using GA

3. MODEL

In this chapter we will look at the multi-echelon systems considered, review the notations, discuss the sequence of activities, base-stock policy, and conduct a comprehensive analytical review of relationships among different stages and key performance measures.

3.1 Inventory system

The research describes multi-echelon inventory systems with an intermediate, external product demand in one or more upper echelons. Components are procured from external suppliers, are assembled into an intermediate product and a final product, and sold to respective customers. A lead time is associated with each stage, which corresponds to the ordering lead time and manufacturing lead time. Uncertainty is involved in both the demand and supply of components, intermediate product and final product. Let us initially consider a single-echelon model for brevity and continue with the three-echelon assembly system and followed by *m-echelon* assembly system.

3.1.1 Single-echelon Assembly System

In a single-echelon model an unlimited supply of components is considered, the components are modified and final products are sold to the customers. The single-echelon model is shown in figure 3.1. There is an l period ordering lead time, i.e. the components are obtained after l periods from the time the order is placed. The demand and the capacity are random in nature. A base-stock policy is used to replenish inventory of the final product.

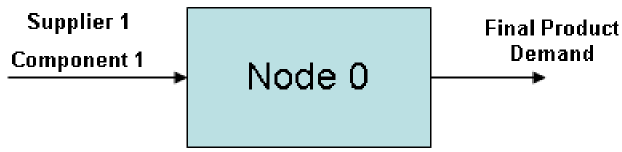

Figure 3.1: Single-echelon Model

3.1.2 Three-echelon Assembly System

A three-echelon assembly system is shown in figure 3.2. The three-echelon assembly system is represented by different nodes as shown in figure 3.2. Let $i = 0,1,\ldots, N-1$ where 0 denotes the last/end node (downstream), and $N-1$ denotes the first node (upstream). In a three-echelon assembly system the individual components (component 1 and 2) are purchased from external suppliers and processed at node 3 and node 2, assembled into an intermediate product at node 1 and further processed to obtain a final product at node 0. There is a l period ordering lead time, i.e. the components are obtained after k periods from the time the order is placed. There is also a k period manufacturing lead time associated between the echelons, i.e. the processed components at node 3 and node 2 will be available for further processing at node 1 only after l periods.

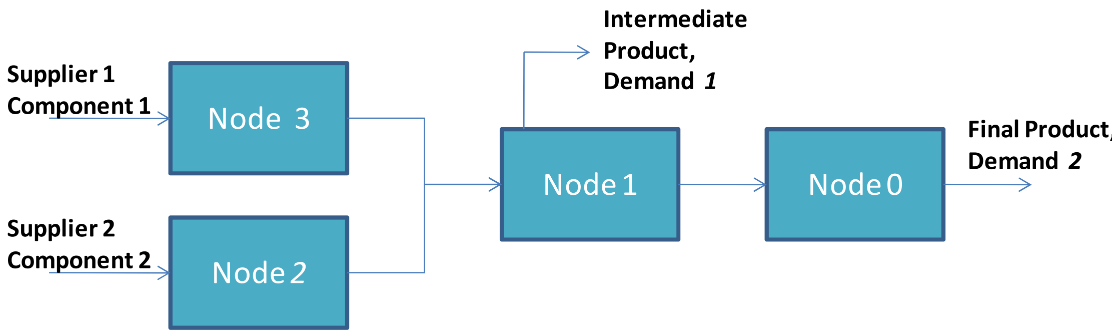

Figure 3.2: Three-echelon assembly system

3.1.3 m-echelon Assembly System

Similarly an *m-echelon* assembly system with several intermediate and final product demands is generalized and shown in Figure 3.3. An *m-echelon* assembly system follows the similar structure of three-echelon assembly system.

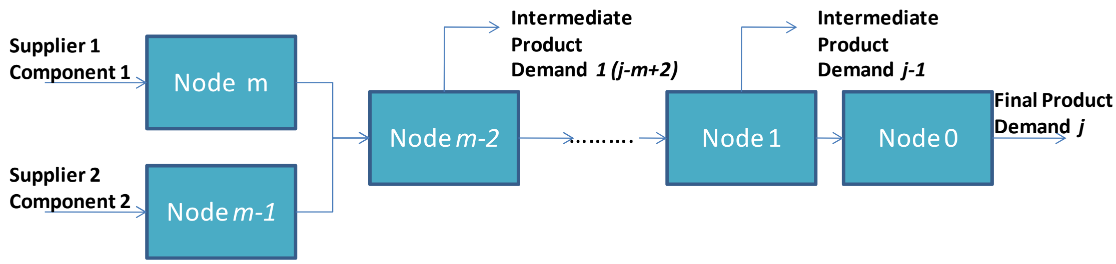

Figure 3.3: *m-echelon* assembly system

3.2 Notation

The lists of notations used are described below:

ξ_n^j : Product demand j in period n

l^i : Resupply/manufacturing lead time for component i

η_n^i : Realized capacity at stage i in period n

s^i : Base-stock level for item i

c^i : Cost per unit of item i

α : Required type-I service level

Y_n^i : Outstanding orders of item i in period n that have not been delivered due to limited capacity (shortfall of item i)

I_n^i : On-hand inventory level of item i in period n before the demand is realized

IP_n^i : It is the on-hand inventory plus the pipeline inventory for component i in period n

NI_n^i : The net inventory for component i in period n

DS_n^i : It is the downstream shortage at node i in period n

3.3 Sequence of Activities and Base-Stock Policy

All the events occur at the beginning of each period in the following sequence i) the outstanding orders are updated, i.e. items that have not been delivered in the previous period due to limited capacity ii) the on-hand inventory is updated, i.e. the physical inventory iii) demand is realized iv) capacity is realized. An order is placed with the supplier in response to demand ξ_n^j . The order can be received either in full or could be constrained by the node's capacity. Thus the amount received is $\min\left(Y_n^i + \xi_n^j, \eta_n^i\right)$, where η_n^i the realized capacity, and ξ_n^j is the demand facing that stage. The inventory allocation is determined when the outstanding orders are updated.

Since a base-stock policy is followed for each component, at the beginning of period n we have

$$I_n^i + Y_n^i - Y_n^0 = s^i, \text{for all } i \tag{3.1}$$

Where I_n^i is the on-hand inventory level for component i at the beginning of period n before the demand is realized. So the base stock level for component i is defined as the sum of on-hand inventory and the outstanding orders of component i in period n, Y_n^i minus the backlog at downstream node (in this case let us consider the downstream node as 0, it can change depending on the type of assembly model), Y_n^0. The on-hand inventory of component i in period n is $I_n^i = s^i - Y_n^i + Y_n^0$. Equation 3.1 does not take into account lead time of any type. If lead time is considered, the equation 3.1 is modified and is shown as equation 3.2.

$$IP^i_{n-l^i} + Y^i_{n-l^i} - Y^0_{n-l^i} = s^i, \text{for all } i \tag{3.2}$$

In equation 3.2 $IP^i_{n-l^i}$ represents component *i*'s on-hand inventory plus the pipeline inventory from the upstream node/supplier, and l^i represents lead time for component *i*. Because all in-transit inventory of component *i* in period $n-l^i$ will be delivered by period *n*, I^i_n equals all the $IP^i_{n-l^i}$ minus all the demand for the product that occurs during the lead time, i.e. from period $n-l^i$ to *n-1*. The total consumption can be written as similarly in Bollapragada et al., (2004): $\xi_{n-1} + \ldots + \xi_{n-l^i} + Y^0_{n-l^i}$. Thus the on-hand inventory level of component *i* in period *n* for a single source of demand, a single-stage:

$$\begin{aligned} I^i_n &= IP^i_{n-l^i} - \xi_{n-1} - \ldots - \xi_{n-l^i} - Y^0_{n-l^i} \\ &= s^i - Y^i_{n-l^i} - \xi_{n-1} - \ldots - \xi_{n-l^i} \end{aligned} \tag{3.2a}$$

3.4 Update Equations

Update equations for single-echelon, three-echelon, and multi-echelon inventory models are developed in this sub-section. Outstanding orders, on-hand inventory, net inventory (on-hand plus pipeline inventory) equations for single-echelon, three-echelon and *m*-echelon models are developed. The evolution of inventory over time is studied when a base-stock policy is used. As discussed in the earlier sections we characterize the relationship between the base stock level, outstanding orders, and on-hand inventory corresponding to a single-echelon, three-echelon and *m-echelon* assembly system. For brevity we start with a single-echelon (single-stage) system, and followed by the multi-echelon analytical review. These update equations describe the relation and interaction between various nodes, and these equations are updated every period using a simulation framework which we discuss in the later chapters.

3.4.1 Single-echelon Analytical Review

A single-echelon is a simplified version of the multi-echelon model, for brevity let us start with single-stage. The outstanding orders at the beginning of period *n+1* for component *i* can be written as:

$$Y_{n+1}^{i} = Y_{n}^{i} + \xi_{n} - \min\left\{Y_{n}^{i} + \xi_{n}, \eta_{n}^{i}\right\} \, where \, i \in \{0\} \tag{3.3}$$

The outstanding order is either zero or could be constrained by the node's capacity, η_{n}^{i} . ξ_{n}^{1}, is the demand 1 (there is only one demand in single-echelon problem) in period n. Equation 3.3 can be also written as shown below

$$Y_{n+1}^{i} = \max\left(0, Y_{n}^{i} + \xi_{n} - \eta_{n}^{i}\right) where \, i \in \{0\} \tag{3.4}$$

The net inventory for component i in period n, NI_{n}^{i} equals $IP_{n-l^{i}}^{i}$ minus the total consumption of component from period $n - l^{i}$ to period *n-1*. The total consumption is shown in equation (3.5) and the on-hand inventory is shown in equation (3.6). Since this is a single-echelon problem, $i \in 0$.

$$\text{Total Consumption} = \xi_{n-1} + ... + \xi_{n-l^{i}} + Y_{n-l^{i}}^{0} \text{ where } i \in 0 \tag{3.5}$$

$$NI_{n}^{i} = IP_{n-l^{i}}^{i} - \xi_{n-1} - ...... - \xi_{n-l^{i}} - Y_{n-l^{i}}^{0} \text{ where } i \in 0 \tag{3.6}$$

Substituting *i=0* in equation 3.2 we get: $IP_{n-l^{i}}^{i} = s^{i}$, substituting this result in equation 3.6 we get the net inventory as shown in equation (3.7)

$$NI_n^i = s^i - \xi_{n-1} - ......- \xi_{n-l^i} - Y^0_{n-l^i} \text{ Where } i \in 0 \tag{3.7}$$

The on-hand inventory or the physical inventory is a slightly modified form of the net-inventory equation (3.7) and is written as shown in equation (3.8). Since the physical inventory can never be less than zero, a max of zero and the net inventory is performed to obtain the on-hand inventory.

$$I_n^i = \max\left[0, s^i - Y^0_{n-2} - \xi_{n-1} - \xi_{n-2}\right], where\, i \in 0 \tag{3.8}$$

The single-stage problem can be formulated as shown in (3.9):

$$\begin{aligned} &\min_{s^i \geq 0} c^i s^i \;\; where\;\, i \in 0 \\ &s.t\;\; P\left[NI_n^0 \geq 0\right] > \alpha \end{aligned} \tag{3.9}$$

3.4.2 Three-echelon Analytical Review

Figure 3.2 shows the three-echelon assembly model, the two components (component 1 and component 2) have a constant delivery lead time from the suppliers. The supplier capacity is assumed to be unlimited, but the manufacturing capacity at the nodes limits the number of component that can be processed at a node. The availability of processed component inventory constrains the number of units that can be sent downstream for assembly. There is also a constant delivery lead time between the echelons. The manufacturing capacity constraints in node 1 and node 0 restrict the number of units that can be supplied downstream and satisfy intermediate and external demands. As shown in figure 3.2 some units are sold to an intermediate customer in addition to those sold as a final product at node 0. The multi-echelon system operates under a periodic review base-stock policy. Using this policy, at the start of each period a check on inventory position (*on-hand inventory + orders – backorders*) is performed and if it falls below the base-stock level an order is placed to bring the inventory position back to the base-stock level.

The mathematical equations developed in the three-echelon assume that intermediate product demand receives priority. The intermediate product demand in period n is denoted as, ξ_n^1 and similarly the final product demand in period n is denoted as ξ_n^2.

The net inventory in period n for node i (node 3,2 and 1) NI_n^i equals $IP_{n-l^i}^i$ minus the total consumption of from period $n-l^i$ to period n-1. The total consumption for nodes 3 and 2 is shown in equation (3.10).

$$\text{Total Consumption for Node 2 and 3} = \begin{cases} \xi_{n-1}^1 + \ldots + \xi_{n-l^i}^1 + \xi_{n-1}^2 + \ldots \\ + \xi_{n-l^i}^2 - DS_{n-1}^1 + Y_{n-l^i}^1 \end{cases} \text{where } i \in 2,3 \qquad (3.10)$$

The net inventory for node 2 and 3 is shown in equation (3.11)

$$NI_n^i = \begin{cases} IP_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 \\ -\xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^1 - Y_{n-l^i}^1 \end{cases}, where\, i \in 2,3 \qquad (3.11)$$

ξ_n^1, is the demand 1, and ξ_n^2 is demand 2 in period n. DS_{n-1}^1 is the downstream shortage, which occurs due to constrained manufacturing capacity. It is defined as shown in equation (3.12)

$$DS_{n-1}^1 = max\left\{\xi_{n-1}^1 + \xi_{n-1}^2 - \eta_{n-1}^1, 0\right\} \qquad (3.12)$$

We know that equation 3.2 represents the base-stock policy for a general multi-echelon system. Modifying (3.2) for node 2 and 3 in three-echelon model we get equation (3.13).

$$IP_{n-l^i}^i + Y_{n-l^i}^i - Y_{n-l^i}^1 = s^i, \text{where } i \in \{2,3\} \qquad (3.13)$$

$Y_{n-l^i}^1$ represents outstanding orders/backlog at downstream node 1, the node 1 is considered as the backlog instead of node 0 because node 1 is the last downstream node that accounts for the two demands.

Substituting equation (3.13) in (3.11), the net inventory for node 2 and 3 is obtained as shown below in equation (3.13)

$$NI_n^i = \begin{cases} s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 \\ -\xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^1 \end{cases}, where\, i \in 2,3 \tag{3.13}$$

The on-hand inventory $\left(I_n^{2,3}\right)$ for node 2 and 3, which is the modified form of equation (3.13) can be written as shown in (3.14)

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^1\right), where\, i \in 2,3 \tag{3.14}$$

Similar to node 2 and 3 the net inventory and the on-hand inventory equations for node 1 can be deduced. The total consumption for nodes 1, i.e. from period $n-l^i$ to period *n-1* is shown in equation (3.15).

$$\text{Total Consumption for Node 1} = \begin{cases} \xi_{n-1}^1 + \ldots + \xi_{n-l^i}^1 + \xi_{n-1}^2 + \ldots \\ +\xi_{n-l^i}^2 - DS_{n-1}^0 + Y_{n-l^i}^0 \end{cases} \text{where}\, i \in 1 \tag{3.15}$$

The net inventory for node 1 is shown in equation (3.16)

$$NI_n^i = \begin{cases} IP_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 \\ -\xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^0 - Y_{n-l^i}^0 \end{cases}, where\, i \in 1 \tag{3.16}$$

DS_{n-1}^0 is the downstream shortage, which occurs due to constrained manufacturing capacity in node 0. It is defined as shown in equation (3.17)

$$DS_{n-1}^0 = max\left\{\xi_{n-1}^2 - \eta_{n-1}^0, 0\right\} \tag{3.17}$$

We know that equation 3.2 represents the base-stock policy for a general multi-echelon system. Modifying (3.2) for node 1 in a three-echelon model we get equation (3.18).

$$IP_{n-l^i}^i + Y_{n-l^i}^i - Y_{n-l^i}^0 = s^i, \text{where}\, i \in \{1\} \tag{3.18}$$

$Y^0_{n-l^i}$, represents outstanding orders/backlog at downstream node 0.

Substituting equation (3.18) in (3.16), the net inventory for node 2 and 3 is obtained as shown below in equation (3.19)

$$NI^i_n = \begin{cases} s^i - Y^i_{n-l^i} - \xi^1_{n-1} - \ldots - \xi^1_{n-l^i} \\ -\xi^2_{n-1} - \ldots - \xi^2_{n-l^i} + DS^0_{n-1} \end{cases}, \text{where } i \in 1 \tag{3.19}$$

The on-hand inventory $\left(I^1_n\right)$ for node 1, which is the modified form of equation (3.19) can be written as shown in (3.20)

$$I^i_n = max\left(0\ ,\ s^i - Y^i_{n-l^i} - \xi^1_{n-1} - \ldots - \xi^1_{n-l^i} - \xi^2_{n-1} - \ldots - \xi^2_{n-l^i} + DS^0_{n-1}\right), \text{where } i \in 1 \tag{3.20}$$

Similarly the net inventory and the on-hand inventory equations for node 0 can be written as shown in (3.21) and (3.22) respectively.

$$NI^i_n = \left\{ s^i - Y^i_{n-l^i} - \xi^2_{n-1} - \ldots - \xi^2_{n-l^i}\right., \text{where } i \in 0 \tag{3.21}$$

$$I^i_n = max\left(0\ ,\ s^i - Y^i_{n-l^i} - \xi^2_{n-1} - \ldots - \xi^2_{n-l^i}\right), \text{where } i \in 0 \tag{3.22}$$

Note that there is only demand 2 (ξ^2) in the last node, i.e. the final demand, and since node 0 is the last node, so the downstream shortage is not applicable to the node 0.

The outstanding orders at the beginning of period $n+1$ for component i where $i \in \{2,3\}$ can be developed iteratively as follows:

$$Y^i_{n+1} = Y^i_n + \xi^1_n + \xi^2_n - DS^1_n - \min\left\{Y^i_n + \xi^1_n + \xi^2_n - DS^1_n, \eta^i_n\right\} \; where\, i \in \{2,3\} \tag{3.23}$$

The outstanding order is either zero or could be constrained by the nodes (i.e. node 2 and node 3) manufacturing capacity, η^i_n . ξ^1_n, is the demand 1, and ξ^2_n is demand 2. Equation (3.10) can be also written as shown below

$$Y^i_{n+1} = \max\left(0, Y^i_n + \xi^1_n + \xi^2_n - DS^1_n - \eta^i_n\right) \text{ where } i \in 2,3 \tag{3.24}$$

The iterative relationship for the outstanding orders of components 1 and 2 in equation (3.24) is similar to the inventory shortfall in a single-echelon system, similar to the shortfall defined in Glasserman and Tayur (1995). Similarly the outstanding orders at the beginning of period *n+1* for intermediate product (numbered item 1 at node 1) and final product (numbered item 0 at node 0) in Figure 3.2 can be defined iteratively, as in equations (3.25) and (3.26):

$$Y_{n+1}^{1} = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0, s^3 - Y_{n-2}^3 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 \\ -\xi_{n-l^i+1}^2 - .. - \xi_{n-1}^2, s^2 - Y_{n-2}^2 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 \\ -\xi_{n-l^i+1}^2 - .. - \xi_{n-1}^2, \eta_n^1 \end{pmatrix} \tag{3.25}$$

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^2 - \min\begin{pmatrix} Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-l^i}^1 - \ldots - \xi_{n-1}^1 \\ -\xi_{n-l^i+1}^2 - \ldots - \xi_{n-1}^2, \eta_n^0 \end{pmatrix} \tag{3.26}$$

The outstanding orders in (3.25) are determined on the basis of the manufacturing capacity of item 1 at node $1\left(\eta_n^1\right)$, available inventory of item 3 in previous period $\left(s^3 - Y_{n-2}^3 - \xi_{n-l^i-1}^1 - .. - \xi_{n-1}^1 - \xi_{n-l^i-1}^2 - .. - \xi_{n-1}^2\right)$, and on-hand inventory of item 2 in previous period$\left(s^2 - Y_{n-2}^2 - \xi_{n-l^i-1}^1 - .. - \xi_{n-1}^1 - \xi_{n-l^i-1}^2 - .. - \xi_{n-1}^2\right)$. Similarly (3.26) the outstanding orders are determined on the basis of the manufacturing capacity of item $0\left(\eta_n^0\right)$, and available inventory of item 1 in previous period $\left(s^1 - Y_{n-2}^1 - \xi_{n-l^i}^1 - \ldots - \xi_{n-1}^1 - \xi_{n-l^i-1}^2 - \ldots - \xi_{n-1}^2\right)$. Note that since priority is given to intermediate product demand, equation 3.26 has an additional intermediate demand term (i.e., demand 1starts from $\xi_{n-l^i}^1$).

The three-echelon problem can be formulated as shown in (3.27):

$$\min_{s^i \geq 0} \sum_{i=0}^{3} c^i s^i \quad s.t\ P\left[NI_n^i \geq 0\right] \geq \alpha^i, where\, i \in \{0,1,2,3\} \tag{3.27}$$

We assume the following conditions hold to ensure the existence of a rational problem:

$$E\left[\xi_n^1 + \xi_n^2\right] \leq E\left[\eta_n^{3,2,1}\right],\ E\left[\xi_n^3\right] \leq E\left[\eta_n^0\right] \tag{3.28}$$

The objective function in (3.27) indirectly penalizes holding inventory at each location, since higher s^i corresponds to more inventory of item *i* (Bollapragada, R. 2004). The constraints are formulated on the basis of a type-I service level. The constraints ensure that sufficient inventory is held to meet demands with a high level of certainty.

3.4.3 m-echelon Analytical Review

The outstanding orders at the beginning of period *n+1* for component *i* where $i \in \{m, m-1\}$ can be developed as follows.

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^{j-(m-2)} + ... + \xi_n^j - DS_n^{m-2} - \eta_n^i\right) where\, i \in m, m-1 \tag{3.29}$$

Throughout the equations it is assumed that there are $j = m-2$ number of demands for an *m*-echelon model.

The outstanding orders for node *m-2* can be developed iteratively as shown in equation (3.30)

$$Y_{n+1}^i = Y_n^i + \xi_n^{j-(m-2)} + \ldots + \xi_n^j - DS_{n-1}^{i-1} - \min\begin{pmatrix} Y_n^i + \xi_n^{j-(m-2)} + \ldots + \xi_n^j - DS_{n-1}^{i-1}, s^{m-1} - Y_{n-2}^{m-1} - \xi_{n-l^i+1}^{j-(m-2)} - \\ ..- \xi_{n-1}^{j-(m-2)} - .. - \xi_{n-1}^j - .... - \xi_{n-l^i+1}^j, s^m - Y_{n-2}^m - \xi_{n-l^i+1}^{j-(m-2)} \\ -.... - \xi_{n-1}^{j-(m-2)} - \ldots - \xi_{n-l^i+1}^j - .... - \xi_{n-1}^j,\ \eta_n^i \end{pmatrix}$$

$$where\, i \in m-2 \tag{3.30}$$

The outstanding orders for node *m-3, m-4….1,0* have similar structure, except the number of demands considered reduces or stay the same as we go further downstream, which depends on the number of intermediate demands considered. The outstanding order for *m-3, m-4…m-k* is shown in equations (3.31), (3.32), and (3.33) respectively.

$$Y_{n+1}^{i} = Y_n^i + \xi_n^{j-(m-3)} + ... + \xi_n^1 - DS_{n-1}^{i-1} - \min \begin{pmatrix} Y_n^i + \xi_n^{j-(m-3)} + ... + \xi_n^{j-1} - DS_{n-1}^{i-1},\ s^{m-2} \\ -Y_{n-2}^{m-2} - \xi_{n-l^i+1}^{j-(m-2)} ... - \xi_{n-1}^{j-(m-2)} - ..... \\ -\xi_{n-l^i+1}^{j-1} ... - \xi_{n-1}^{j-1},\ \eta_n^i \end{pmatrix}$$

$$where\, i \in m-3 \tag{3.31}$$

$$Y_{n+1}^{i} = Y_n^i + \xi_n^{j-(m-4)} + ... + \xi_n^j - DS_{n-1}^{i-1} - \min \begin{pmatrix} Y_n^i + \xi_n^{j-(m-4)} + ... + \xi_n^{j} - DS_{n-1}^{i-1},\ s^{m-3} \\ -Y_{n-2}^{m-3} - \xi_{n-l^i+1}^{j-(m-3)} - ... - \xi_{n-1}^{j-(m-3)} - ..... \\ -\xi_{n-l^i+1}^{j} ... - \xi_{n-1}^{j}, \eta_n^i \end{pmatrix} \tag{3.32}$$

$$where\, i \in m-4$$

$$Y_{n+1}^{i} = Y_n^i + \xi_n^{j-(m-k)} + ... + \xi_n^j - DS_{n-1}^{i-1} - \min \begin{pmatrix} Y_n^i + \xi_n^{j-(m-k)} + ... + \xi_n^{j} - DS_{n-1}^{i-1},\ s^{m-k-1} \\ -Y_{n-2}^{m-k-1} - \xi_{n-l^i+1}^{j-(m-k-1)} - ... - \xi_{n-1}^{j-(m-k-1)} - ... \\ -\xi_{n-l^i+1}^{1} .. - \xi_{n-1}^{1},\ \eta_n^i \end{pmatrix} \tag{3.33}$$

$$where\, i \in m-k$$

The relationships for outstanding orders can be written in a similar fashion for other stages downstream. The outstanding order for stage 0 is same as a three-echelon. The relationships for on-hand inventory can be written as shown in (3.34), (3.35), (3.36).

$$I_n^i = \max \begin{bmatrix} 0, s^i - Y_{n-2}^i - \xi_{n-1}^{j-(m-2)} ..... - \xi_{n-1}^{j} - .. \\ .. - \xi_{n-l^i}^{j-(m-2)} ..... - \xi_{n-l^i}^{j} + DS_{n-1}^{m-2} \end{bmatrix}, where\, i \in m, m-1 \tag{3.34}$$

$$I_n^i = \max \begin{bmatrix} 0, s^i - Y_{n-2}^i - \xi_{n-1}^{j-(m-2)} ..... - \xi_{n-1}^{j} \\ - \xi_{n-l^i}^{j-(m-2)} ..... - \xi_{n-l^i}^{j} + DS_{n-1}^{m-3} \end{bmatrix}, where\, i \in m-2 \tag{3.35}$$

$$I_n^i = \max \begin{bmatrix} 0, s^i - Y_{n-2}^i - \xi_{n-1}^{j-(m-k)} ..... - \xi_{n-1}^{j} \\ - \xi_{n-l^i}^{j-(m-k)} ..... - \xi_{n-l^i}^{j} + DS_{n-1}^{m-k-1} \end{bmatrix}, where\, i \in m-k \tag{3.36}$$

Where, DS_{n-1}^{m-2}, DS_{n-1}^{m-3} and DS_{n-1}^{m-1-k} are shown in equations (3.37), (3.38), and (3.39)

$$DS_{n-1}^{m-2} = max\left\{\xi_{n-1}^{j-(m-2)} + ... + \xi_{n-1}^{j} - \eta_{n-1}^{m-2}, 0\right\} \tag{3.37}$$

$$DS_{n-1}^{m-3} = max\left\{\xi_{n-1}^{j-(m-3)} + ... + \xi_{n-1}^{j} - \eta_{n-1}^{m-3}, 0\right\} \tag{3.38}$$

$$DS_{n-1}^{m-k-1} = max\left\{\xi_{n-1}^{j-(m-k-1)} + ... + \xi_{n-1}^{j} - \eta_{n-1}^{m-k-1}, 0\right\} \tag{3.39}$$

The m-echelon problem can be formulated as shown in (3.27):

$$\min_{s^i \geq 0} \sum_{i=0}^{m} c^i s^i \quad s.t\ P\left[NI_n^i \geq 0\right] \geq \alpha^i, where\, i \in \{0....m\} \tag{3.40}$$

We assume the following conditions hold to ensure the existence of a rational problem:

$$E\left[\sum_{i}^{0} \xi_n^{j-i}\right] \leq E\left[\eta_n^i\right], where\, i \in \{m-2, m-1, ...0\} \tag{3.41}$$

4. SIMULATION OPTIMIZATION USING *Opt Quest*

In an earlier chapter we studied the evolution of inventory over time when a base-stock policy is used. As discussed in the chapter 3 we characterize the relationship between the base stock level, outstanding orders, and on-hand inventory corresponding to a single-echelon, three-echelon and *m*-echelon assembly system. We consider only two components in either systems, i.e. three-echelon or *m*-echelon assembly system. A constant lead time of l periods between echelons and supply of components to initial nodes (i.e. to node 2 and 3 in case of a three-echelon system, node m and $m-1$ in case of a *m*-echelon system). *ARENA* (simulation software) is used to update the variables in the update equations (outstanding orders, on-hand inventory etc.) periodically.

In this chapter we examine how best found base-stock levels are obtained using simulation-based inventory optimization carried out in *OptQuest* (a tool in the *ARENA* simulation software) for several scenarios. The following sub-sections will discuss how simulation optimization is used in conjunction with *OptQuest* to find best found base-stock levels for three-echelon model, four-echelon model, five-echelon model using *OptQuest*, and finally discuss the simulation optimization results.

4.1 Optimization with *OptQuest*

A simulation based inventory optimization is carried out using the combination of *OptQuest* and *ARENA*. *OptQuest* is a stand-alone optimization software routine that is used along with a number of commercial simulation environments like *ARENA* and Crystal Ball (Fu, 2001). The *OptQuest* Callable Library (OCL) is the optimization engine of the *OptQuest* system, which is based on a heuristic known as *scatter search* (Laguna and Marti, 2002). Like many other stand alone optimization software routines *OptQuest* also follows a "black box" approach, figure 4.1 represents a complex system as a black box (April et al., 2001). A potential disadvantage of this approach would be that, the optimization procedure is generic and it does not know anything about the process that goes inside the box. A clear advantage of such approach would be the use of the optimization software for many complex systems.

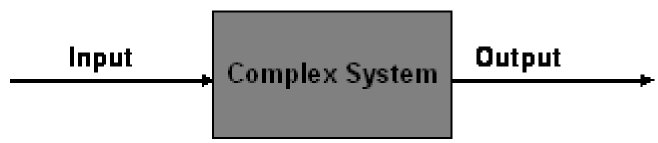

Figure 4.1: Complex System as a black box

OptQuest allows the user to input the problem structure through its graphical user interface. The objective function is specified along with the constraints. Upper and lower bounds of variables have to be specified, and some of them can be restricted to a discrete value with arbitrary step size. The OCL uses this information to determine the search area. OCL is designed to search for the following class of problems (Glover et al., 2000) shown in equation 4.1. The structure of the inventory problem considered in this research is also similar to the one shown in the equation below.

$$
\begin{array}{lll}
\text{Max or Min} & F(x) & \\
\text{Subject to} & Ax \le b & (\text{Constraints}) \\
 & g_l \le G(x) \le g_u & (\text{Requirements}) \\
 & l \le x \le u & (\text{Bounds})
\end{array}
\qquad (4.1)
$$

The coefficient matrix *A* and the right-hand side value *b* must be known. The requirement, i.e. the simple upper and/or lower bounds imposed on the function can be either linear or non-linear. The values of the bounds g_l and g_u must be known constants. When *OptQuest* is used as an optimization procedure in coordination with a system evaluator (it could be a model in simulation package, *ARENA* in our case), the output from the system evaluator is fed as input to the optimization procedure, and vice-versa. The update equations (outstanding orders and on-hand inventory equations for different nodes) are written in the simulation software *ARENA*, and *OptQuest* which is an optimization package sends in the base-stock values for each node to the simulation software *ARENA*. The value of base-stock is based on the upper and lower bounds provided in the *OptQuest* interface, this concept would be clearer after the subsequent sub-section which discusses the steps that OCL follows in performing its search.

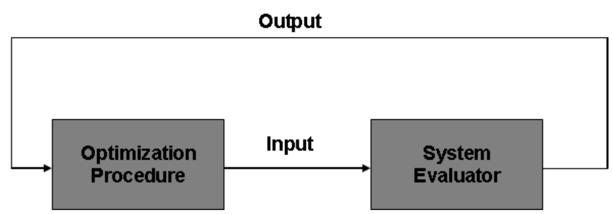

Figure 4.2: Coordination between Optimization and System Evaluator

The optimization procedure of OCL is designed to carry out a special "non-monotonic search", where the successively generated input values produce varying evaluations. All of the evaluations are not necessarily improving, but over time they provide good solution. It is done to maintain diversity in the search process. The user written application interacts with the OCL and defines complete optimization problem, and starts a search for best found values of the decision variables. Figure 4.3 provides the concept of how OCL can be used to search for best found solutions to complex optimization problems (Laguna and Marti, 2002).

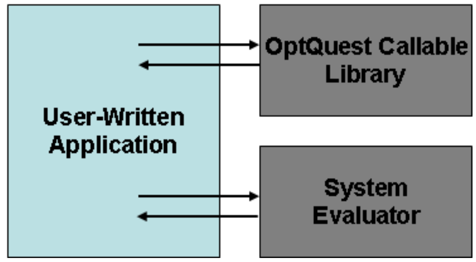

Figure 4.3: OCL linked to user-written application

4.1.1 Steps Used for Search by OCL (Laguna and Marti, 2002)

As stated earlier the optimization technology embedded in OCL is a metaheuristic known as scatter search. Scatter search has some common features of the popular genetic algorithm (GA). Both search methods are population based metaheuristic's. Unlike GA, scatter search operates on fewer sets of points, these sets of points are called as reference points. These reference points contain good starting solutions, are determined on the basis of user specification called "most likely", or based on previous solution efforts. The approach systematically generates combinations of reference points to create new reference which are mapped into feasible points. Scatter search in OCL employs the following steps in search:

Step 1: Applying a diversification generation method to build a starting set of solutions. Define the best points as a reference set (determined based on quality and diversity). This is accomplished by dividing the range of each variable into four sub-ranges of equal size. Then the solution is found in two steps:

- A sub-range is randomly picked among the four. The selection probability of a sub-range is inversely proportional to the frequency count (which keeps track of how many times the sub-range has been selected).

- After a sub-range has been selected the value is randomly chosen from the selected sub-range. The starting set of points might also include the following:
    - All variables are set to the lower bound
    - All variables are set to upper bound
    - All variables are set to the midpoint $x = l + (u - l)/2$
    - Other solutions that are user suggested

  The variables here refer to the base-stock value of different nodes, number of echelons considered determine the number of nodes.

Step 2: While the stopping criteria has not yet occurred combination of subsets of reference points are used to generate new points. The combination is done so as to produce points both inside and outside the convex region. Once the reference set is created, the combinations of new solutions are based following three types:

$$
\begin{aligned}
& x = x' - d \\
& x = x' + d \qquad \text{where, } d = r\frac{x'' - x'}{2} \\
& x = x'' - d \qquad r \text{ is a random number in range } (0,1)
\end{aligned}
\tag{4.2}
$$

When best of two reference solutions are combined up to five new solutions are generated, when the worst of the two solutions are combined only one new solution is generated. OCL uses Euclidean distance measure to determine how close a potential point is from the existing set of reference points, so as to be included or discarded from the reference set.

The points are further modified by mapping process to yield feasible points based on the constraints in the problem, both due to linearity and integrality constraints. OCL solves an LP or a mixed-integer problem with a goal of finding a feasible solution x^* that minimizes the deviation between x (infeasible point) and x^*.

Step 3: The reference set is continuously updated with points that improve the quality/diversity of the set until the termination condition occurs.

Step 4: if no new combinations are being explored then the collection of best solutions are used as a starting points for a new diversification generation method.

So far it was assumed that the feasibility check for new solutions is being performed outside the black box, but there are cases where the feasibility can be done only inside the black box. In such cases the feasibility test results are communicated as one of the output, figure 4.4 depicts the situation. One of the measures which is used as objective function value *F(X*)* will be able to provide differentiation between good and bad solutions. *G(X*)* is associated with the performance of the system and is defined by the requirements. Based on the bounds for the requirements the solution is termed as feasible or infeasible. OCL does not discard the requirement infeasible terms but instead uses a penalty function *P(X*)* that penalizes requirement violations. The penalty is not static but is determined on the basis of the degree of violation.

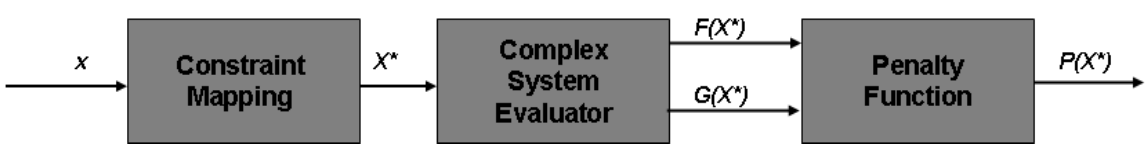

Figure 4.4: Solution Evaluation

These four steps describe above provide an overview of the underlying mechanisms involved in the working of the *OptQuest* optimization engine. In many business and engineering problems *OptQuest* can be used to find solutions close to best found very early in the search.

4.2 Simulation Optimization of *m*-Echelon Assembly Using *Opt Quest*

The simulation tool *ARENA* is used to update the equations of a *m*-echelon assembly system periodically, and an infinite horizon case is considered. Figure 4.5 show the block diagram of various activities that takes place every period, and each simulation run in *ARENA*.

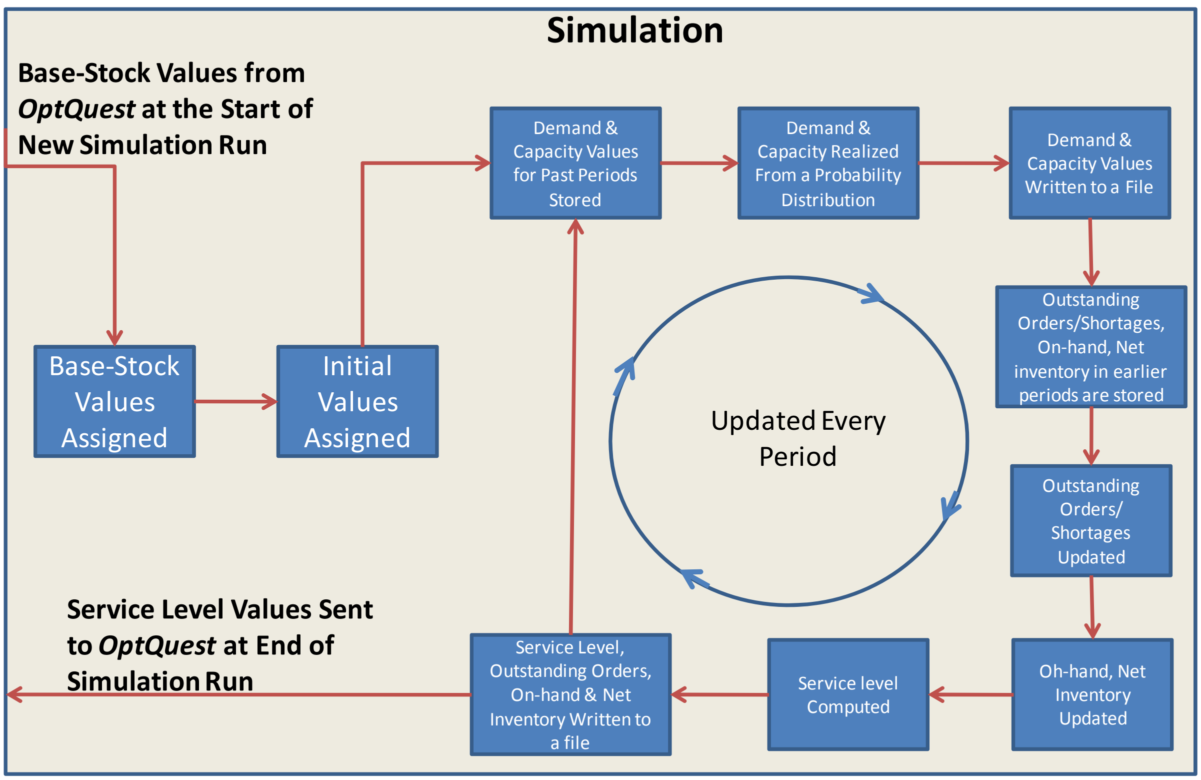

Figure 4.5: Simulation in *ARENA* within *OptQuest* Framework

The following activities take place every period:

- Demand and capacity values of earlier periods for all nodes are stored

- Demand and capacity values for all nodes are derived from probability distribution (the distribution might be specific to each node)
- The values of demand and capacity for all nodes are written to a file
- Outstanding orders (shortages of inventory in previous period), on-hand inventory, net inventory of previous periods for all nodes are stored
- Outstanding orders/shortage equations for all nodes are updated
- On-hand inventory, and net inventory equations for all nodes updated
- Based on the updated equations service level for all nodes is obtained
- The outstanding orders, net inventory, on-hand inventory for the current period for all the nodes are written to a file

The following activities take place during each simulation run:

- Base-stock values for all the nodes are assigned from *OptQuest*
- Initial values (On-hand inventory etc.) are assigned by *ARENA*
- The final service level for all nodes is sent to *OptQuest*

Each simulation run consists of a pre-specified number of periods, during which the equations (i.e. on-hand inventory, outstanding order, etc.) are updated, and service level is determined for each period. A new base-stock value is sent to *ARENA* from *OptQuest* at the beginning of a new simulation run. At the end of each simulation run the service level value is sent to *OptQuest*. Based on the service level constraints in *OptQuest* the feasibility of the solution (base-stock value) is determined, i.e. if the required service level is achieved for the base-stock value assigned at the start of the simulation run by *OptQuest*. All the activities described under each period and every simulation run takes place continuously till the termination condition is reached. Number of simulations required to obtain an best found base-stock level is determined on the termination condition. The termination condition can be either subjective (pre-specifying the number of simulation runs in *OptQuest*) or could be based on the specified tolerance/sensitivity of the solution (base-stock value) achieved. In figure 4.5 all the blocks except the first two blocks from the left side are updated every period, whereas the first two blocks on the left side (i.e. base-stock values assigned, and initial values assigned) are updated at the start of a new simulation run.

The description of each block in figure 4.5 is provided below:

Base-stock Values Assigned: At the start of every simulation run the base-stock values of each node are sent to *ARENA* from *OptQuest*. The base-stock values of each node that are sent to *ARENA* depends on the upper and lower bounds of the base-stock value specified in *OptQuest* as discussed in the earlier section 4.1.

Initial Values Assigned: All the variables that are used in update equations are assigned an initial each time at the start of the simulation run. These variables include:

- On-hand inventory variables for each node
- Net Inventory variables for each node
- Outstanding orders/ shortages for each node
- Demand values, which depends on the number of demands (intermediate, and final products demand) for an echelon-system
- Manufacturing Capacity for each node
- Cost of each item

Demand and Capacity Values of Past Periods: Once the base-stock values and the initial values are assigned the demand and capacity values in the past periods are stored.

The number of periods in the past that has to be stored is determined on the basis of supply and manufacturing lead time. As the lead time increases the amount of data (demand and capacity for each node in past) that needs to be stored also increases.

Demand and Capacity is Realized: The value of demands for the current period is determined based on the probability distribution function specified. Similarly each node's manufacturing capacity for current period is determined from a probability distribution function. The demand and capacity values of the current period are written to a file, i.e. spreadsheet or a notepad.

Past outstanding orders/shortages and Inventory Stored: Due to the complex relations among the variables previous period's values for outstanding orders, on-hand inventory, and net inventory are stored. For example, in order to compute the current net inventory equation needs information of outstanding order from three periods earlier.

Outstanding orders/shortages and Inventory Stored: Now that there is information regarding the demands, capacity of each node, base-stock values, and initial values. The outstanding orders, net inventory, and the on-hand inventory are updated respectively in the same sequence for each node.

Service Level is Computed: After the outstanding orders, and inventory updated the service level for each node is computed. Since the service level computed is a type-II service level, current period also accounts for earlier periods. At the end of simulation run the final service value which reflects all the periods is sent to *OptQuest*.

All the update variables, i.e. On-hand inventory, net inventory, service level, and outstanding orders/shortages for each node are written to a file, i.e. spreadsheet or notepad. The entire process is repeated over for number of simulation runs to find the optimum base-stock level.

The objective function and constraints are written in *OptQuest*. Figure 4.6 show the snapshot of *OptQuest* solution window for a three-echelon assembly problem. The lower portion of the window shows the graphical representation of the objective function value improving as the number of simulation runs increase. The upper portion of the window has three tables. The first table from top (minimize) shows the best objective found till the current simulation run, and also shows the objective value of the current simulation run. The second table from the top (controls) shows the best base-stock value for each node (*S0, S1, S2, and S3*; since it is a three-echelon there are only four nodes) found till the current simulation, and the current base-stock used for the simulation run. The third table (constraints) shows the status (feasible or infeasible) of the constraint in the current simulation run.

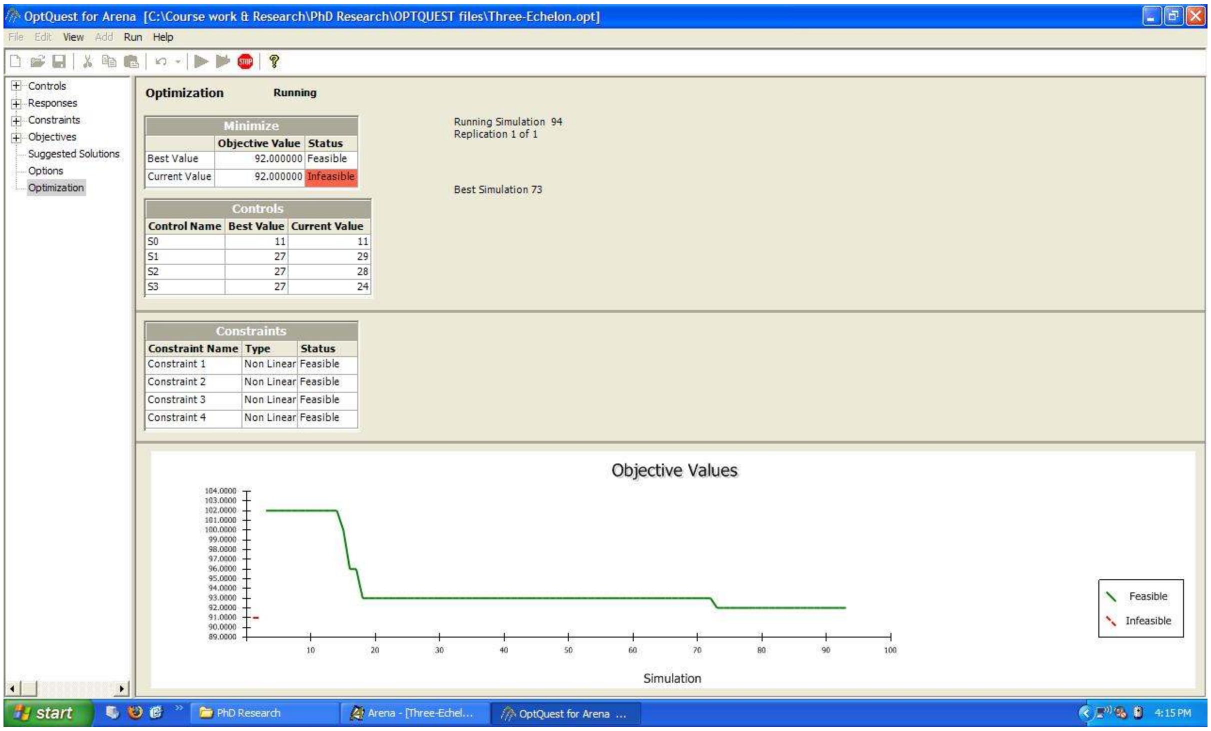

Figure 4.6: Snapshot of *OptQuest* Solution Window

<u>4.3 Three-Echelon Assembly Model</u>

Let us recall the assembly structure, and the update equations of the three-echelon model. Figure 4.7 shows the three-echelon assembly structure.

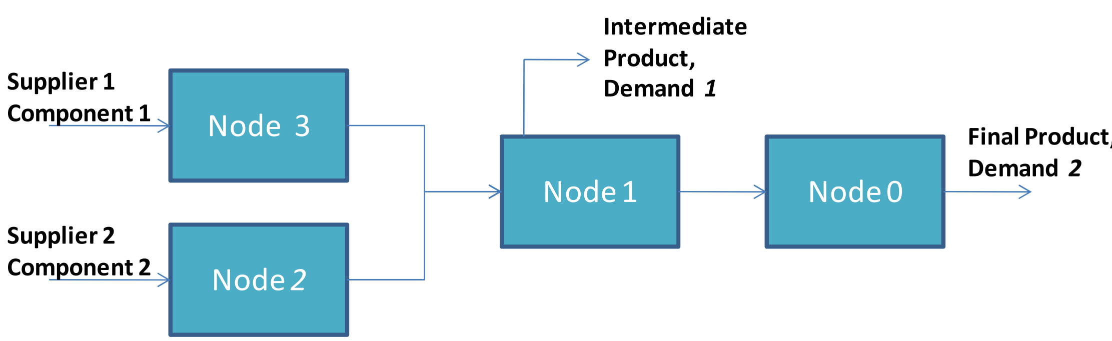

Figure 4.7: Three-echelon Assembly Structure

The on-hand inventory equations for three-echelon assembly system are listed below:

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^1\right), where\, i \in 2,3 \qquad (4.1)$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \qquad (4.2)$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2\right), \text{where } i \in 0 \qquad (4.3)$$

The outstanding order equations for three-echelon assembly system are listed below:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 - DS_n^1 - \eta_n^i\right) \text{ where } i \in 2,3 \qquad (4.4)$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0, s^3 - Y_{n-2}^3 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 \\ -\xi_{n-l^i+1}^2 - .. - \xi_{n-1}^2, s^2 - Y_{n-2}^2 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 \\ -\xi_{n-l^i+1}^2 - .. - \xi_{n-1}^2, \eta_n^1 \end{pmatrix} \qquad (4.5)$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-l^i}^1 - \ldots - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - \ldots - \xi_{n-1}^2, \eta_n^0\right) \qquad (4.6)$$

A simulation-based inventory optimization was carried out in *OptQuest* for eight different scenarios. Each scenario is simulated for a length 20 periods for an appropriate number of simulation runs (usually 300 to 500), where the number of simulation runs depends on how quickly the best found base-stock value is found for a given scenario. Fewer simulation runs are sufficient when the search bounds are tight. The cost per unit item is held constant, equal to 1. A 90% customer service level is used for all nodes, i.e. the value of $\alpha = 0.9$. A lead time of 2 periods is used in the model, i.e. the supply lead time for node 2 and 3, and manufacturing lead time between nodes. Of the eight scenarios considered, the first four consist of deterministic values for demand and capacity, whereas the values in last four scenarios are derived from a normal distribution. Table 4.1 provides the demand and capacity values used in the different simulation scenarios.

Table 4.1: Demand and Capacity Values of Three-echelon Used in Simulation*

	Intermediate Product, Demand 1	Final Product, Demand 2	Capacity at Node 3	Capacity at Node 2	Capacity at Node 1	Capacity at Node 0
Scenario 1	4	4	9	9	9	5
Scenario 2	4	4	11	11	11	6
Scenario 3	8	4	13	13	13	5
Scenario 4	8	4	17	17	17	6
Scenario 5	Norm (4,1)	Norm (4,1)	Norm (9,1)	Norm (9,1)	Norm (9,1)	Norm (5,1)
Scenario 6	Norm (4,1)	Norm (4,1)	Norm (11,1)	Norm (11,1)	Norm (11,1)	Norm (6,1)
Scenario 7	Norm (8,1)	Norm (4,1)	Norm (13,1)	Norm (13,1)	Norm (13,1)	Norm (5,1)
Scenario 8	Norm (8,1)	Norm (4,1)	Norm (17,1)	Norm (17,1)	Norm (17,1)	Norm (6,1)

**Norm (Value1, Value2) represents Normal Distribution (Mean, Standard Deviation)*

Table 4.2: Best Found Base-stock Levels and Capacity Utilization of Three-echelon**

		Optimal Base-Stock Levels**					Utilization of capacity			
		S0	S1	S2	S3	Objective Function	Node 0	Node 1	Node 2	Node 3
Deterministic Case	Scenario 1	10	19	23	23	75	0.8	0.89	0.89	0.89
	Scenario 2	11	21	23	23	78	0.67	0.72	0.72	0.72
	Scenario 3	12	35	35	35	117	0.89	0.92	0.92	0.92
	Scenario 4	12	35	35	35	117	0.73	0.71	0.71	0.71
Probabilistic case	Scenario 5	10	25	24	24	83	0.86	0.98	0.93	0.94
	Scenario 6	11	25	25	24	85	0.72	0.8	0.77	0.77
	Scenario 7	15	35	35	35	120	0.92	0.98	0.95	0.96
	Scenario 8	13	26	26	27	92	0.75	0.75	0.73	0.73

*** S0, S1, S2, and S3 represent* s^0, s^1, s^2, s^3 *of three-echelon respectively*

From Table 4.1 we can observe that scenarios 1 through 4 are deterministic, and scenarios 5 through 8 are probabilistic. In scenarios 1 and 3 the demands require 90% of the total capacity in each node. In scenarios 2 and 4 the demands require 70% of the total capacity in each node. The demand of the final product in scenarios 3 and 4 is exactly twice of scenarios 1 and 2. A similar approach is used for determining the average values of demand and capacity in scenarios 5 through 8, where the values of demand and capacity come from a normal distribution. Table 4.2 provides the optimal base-stock levels and capacity utilization for each node and scenario. From table 2 we can clearly observe a higher objective function value (OBV) for scenarios 3 and 4 when compared to scenarios 1 and 2. An increase in the value of the demand results in a higher base-stock levels and thus higher objective cost. For the same reason the OBV of scenarios 7 and 8 is higher than 5 and 6. Figure 4.8 and 4.9 show the graphical representation of the best found base-stock levels and the capacity utilization for scenarios 1,3,5 and 7 for better understanding.

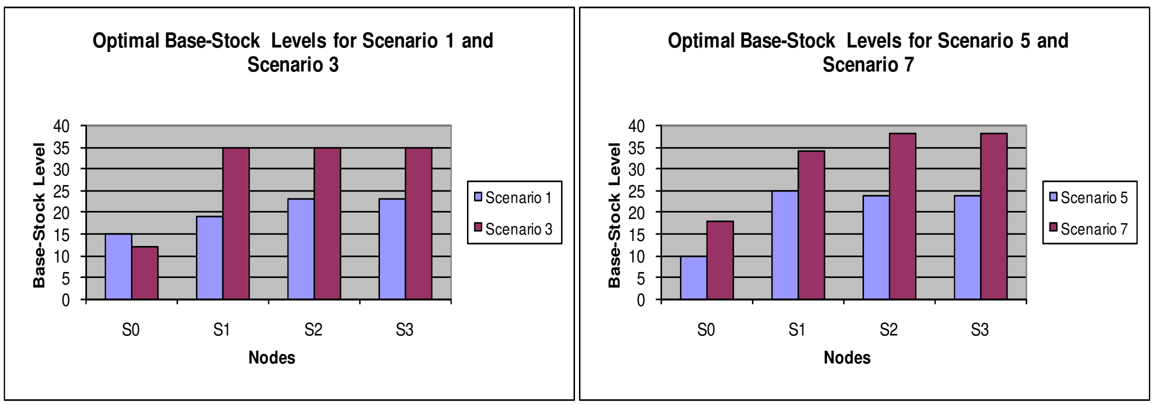

Figure 4.8: Best Found Base-stock Level for Scenarios 1, 3, 5, and 7

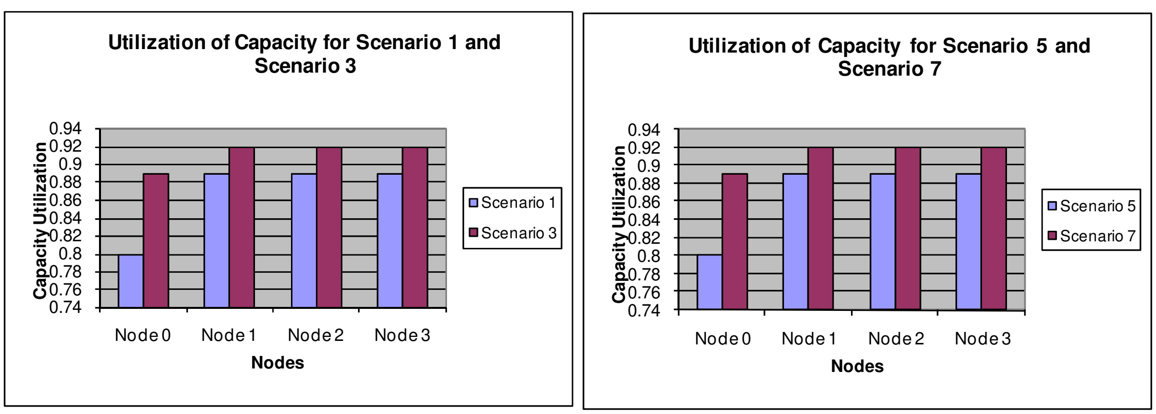

Figure 4.9: Capacity Utilization for Scenarios 1, 3, 5, and 7

4.4 Four-Echelon Assembly Model

Figure 4.10 shows the four-echelon assembly system.

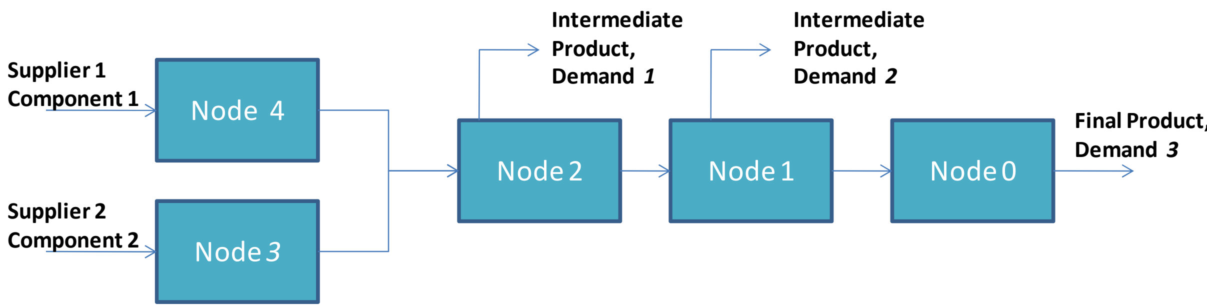

Figure 4.10: Four-echelon Assembly System

The on-hand inventory equations for the four-echelon assembly system are stated below:

$$I_n^i = max\left(0\ ,\begin{bmatrix} s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots \\ -\xi_{n-l^i}^2 - \xi_{n-1}^3 - \ldots - \xi_{n-l^i}^3 + DS_{n-1}^2 \end{bmatrix}\right), where\, i \in 4,3 \tag{4.7}$$

$$I_n^i = max\left(0\ ,\begin{bmatrix} s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots \\ -\xi_{n-l^i}^2 - \xi_{n-1}^3 - \ldots - \xi_{n-l^i}^3 + DS_{n-1}^1 \end{bmatrix}\right), \text{where } i \in 2 \tag{4.8}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 - \xi_{n-1}^3 - \ldots - \xi_{n-l^i}^3 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{4.9}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^3 - \ldots - \xi_{n-l^i}^3\right), \text{where } i \in 0 \tag{4.10}$$

The outstanding order equations for the four-echelon assembly system with two period lead time are stated below:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 + \xi_n^3 - DS_n^2 - \eta_n^i\right) \text{ where } i \in 4,3 \tag{4.11}$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^1 + \xi_n^2 + \xi_n^3 - DS_{n-1}^1 - \min\begin{pmatrix} Y_n^2 + \xi_n^1 + \xi_n^2 + \xi_n^3 - DS_{n-1}^1,\ s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \\ -\xi_{n-1}^2 - \xi_{n-1}^3,\ s^4 - Y_{n-2}^4 - \xi_{n-1}^1 - \xi_{n-1}^2 - \xi_{n-1}^3, \eta_n^2 \end{pmatrix} \tag{4.12}$$

$$Y_{n+1}^{1} = Y_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} - DS_{n-1}^{0} - \min\begin{pmatrix} Y_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} - DS_{n-1}^{0}, & s^{2} - Y_{n-2}^{2} - \xi_{n-1}^{1} \\ -\xi_{n-1}^{2} - \xi_{n-1}^{3} - \xi_{n-2}^{1}, & \eta_{n}^{1} \end{pmatrix} \tag{4.13}$$

$$Y_{n+1}^{0} = Y_{n}^{0} + \xi_{n}^{3} - \min\left(Y_{n}^{0} + \xi_{n}^{3}, s^{1} - Y_{n-2}^{1} - \xi_{n-1}^{2} - \xi_{n-1}^{3} - \xi_{n-2}^{2}, \eta_{n}^{0}\right) \tag{4.14}$$

A total of twelve scenarios are considered for the four-echelon assembly system. Scenarios 9 thru 14 consider deterministic values of demand and capacity at each node. The values of demand and capacity for Scenarios 15 thru 20 are based on a normal distribution. The parameters of demand and capacity for both deterministic and probabilistic scenarios are shown in Table 3.

Table 4.3: Demand and Capacity Values of Four-echelon used in Simulation

	Intermediate Product Demand 1	Intermediate Product Demand 2	Final Product Demand	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 9	4	4	4	13	13	13	9	5
Scenario 10	4	4	4	15	15	15	11	6
Scenario 11	8	8	4	22	22	22	13	5
Scenario 12	8	8	4	25	25	25	15	6
Scenario 13	8	4	4	18	18	18	9	5
Scenario 14	8	4	4	20	20	20	11	6
Scenario 15	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(13,1)	Norm(13,1)	Norm(13,1)	Norm(9,1)	Norm(5,1)
Scenario 16	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(15,1)	Norm(15,1)	Norm(15,1)	Norm(11,1)	Norm(6,1)
Scenario 17	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(22,1)	Norm(22,1)	Norm(22,1)	Norm(13,1)	Norm(5,1)
Scenario 18	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(25,1)	Norm(25,1)	Norm(25,1)	Norm(15,1)	Norm(6,1)
Scenario 19	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(9,1)	Norm(5,1)
Scenario 20	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(20,1)	Norm(20,1)	Norm(20,1)	Norm(11,1)	Norm(6,1)

Table 4.4: Best Found Base-stock Levels and Capacity Utilization of Four-echelon

		Optimal Base-Stock Levels						Utilization of capacity				
		S0	S1	S2	S3	S4	Objective Function	Node 0	Node 1	Node 2	Node 3	Node 4
Deterministic Case	**Scenario 9**	8	16	24	24	24	72	0.8	0.89	0.92	0.92	0.92
	Scenario 10	8	16	24	24	24	96	0.67	0.73	0.8	0.8	0.8
	Scenario 11	8	24	40	40	40	152	0.8	0.92	0.9	0.9	0.9
	Scenario 12	8	24	40	40	40	152	0.67	0.8	0.8	0.8	0.8
	Scenario 13	8	16	32	32	32	120	0.8	0.89	0.89	0.89	0.89
	Scenario 14	8	16	32	32	32	120	0.67	0.73	0.8	0.8	0.8
Probabilistic case	**Scenario 15**	10	21	38	33	33	135	0.8	0.89	0.92	0.92	0.92
	Scenario 16	10	21	37	34	34	136	0.67	0.73	0.8	0.8	0.8
	Scenario 17	10	29	54	58	58	209	0.8	0.92	0.9	0.9	0.9
	Scenario 18	10	29	54	58	58	209	0.67	0.8	0.8	0.8	0.8
	Scenario 19	10	21	42	47	47	167	0.8	0.89	0.89	0.89	0.89
	Scenario 20	10	21	43	47	47	168	0.67	0.73	0.8	0.8	0.8

From table 4.3 observe that first six scenarios (9-14) are deterministic, and last six scenarios (15-20) are probabilistic. In scenarios 9, 11 and 13 the demands on node 1 thru 4 require 90% of the total capacity in each node, where as the demand on node 0 is 80% of the total capacity at node 0. In scenarios 10, 12 and 14 the demands on node 1 thru 4 require 80% of the total capacity in each node, where as demand on node 0 requires 70% of the total capacity at node 0. A similar approach is used for determining the average values of demand and capacity in scenarios 15 through 20, where the values of demand and capacity in each period come from a normal distribution. Table 4 provides the best found base-stock levels and capacity utilization for each node and scenario. From Table 4 we can notice clearly that there is a higher objective function value (OBV) for scenarios 11 and 12 when compared to scenarios 9 and 10. Similarly we can also observe that the OBV for scenarios 11 and 12 is higher when compared to scenarios 13 and 14. An increase in the value of the demand results in a higher base-stock levels and thus higher objective cost. For the same reason the OBV of scenarios 17, 18, 19 and 20 is higher than 15 and 16.

There is no uncertainty or other variability in the demands or capacity for scenarios 1 thru 4 and 9 thru 14, which implies that safety-stock, is not required for these scenarios. In scenarios 5 thru 8 and 15 thru 20 there is uncertainty in the demands and capacities, and some nodes have a high utilization of capacity. This suggests that significant safety-stock must be included in the base-stock levels in order to meet the service level constraints. The behavior could be well understood from the results in Figure3. Figure 3 shows the safety stocks for various scenarios at each node in the four-echelon system when a 90% customer service level is required at each node. We can also observe that, as the demand at a given node increases the amount of safety stock at the node seems to increase proportionally.

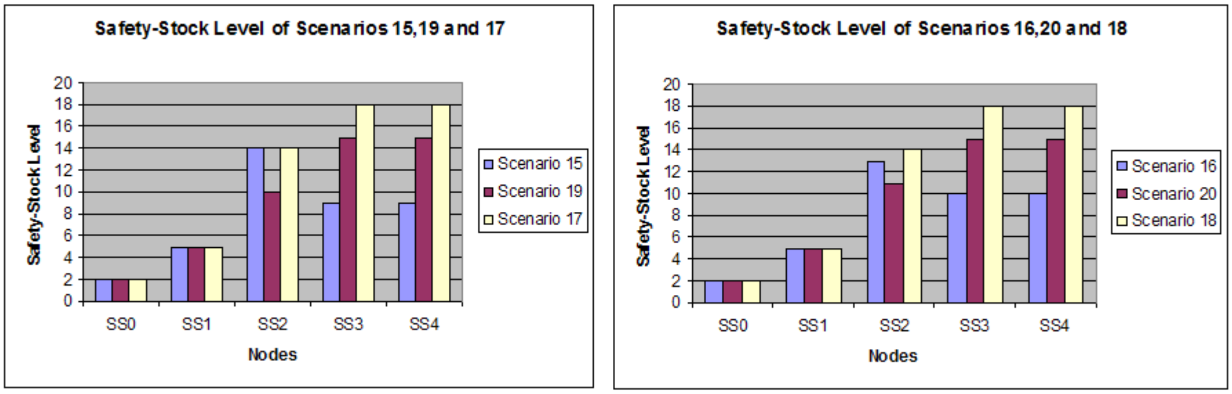

Figure 4.11: Safety-Stock for Four-echelon Assembly System (SS0-SS4 denotes safety stock at node 0-4)

4.5 Five-echelon Assembly Model

Figure 4.12 shows a five-echelon model with three intermediate product demands and a final product demand.

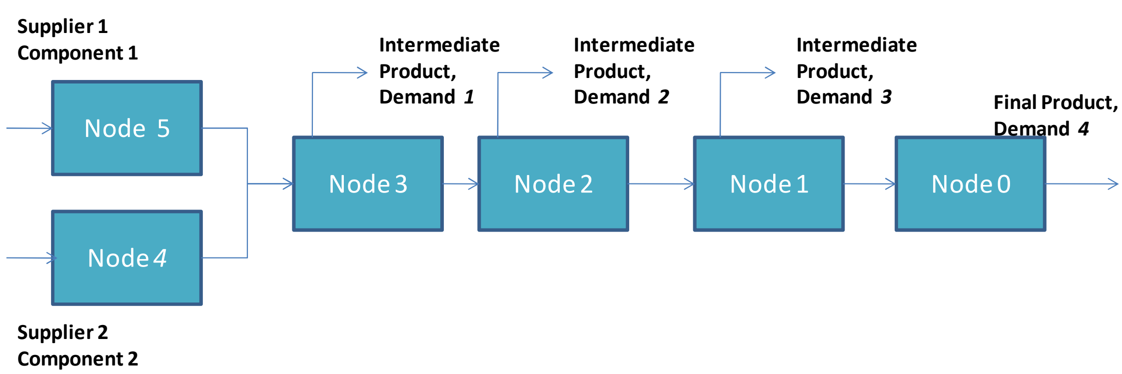

Figure 4.12: Five-echelon Assembly System

The on-hand inventory equations for the five-echelon assembly system are stated below:

$$I_n^i = max\left(0\ ,\begin{bmatrix} s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - .. - \xi_{n-l^i}^1 - \xi_{n-1}^2 - .. - \xi_{n-l^i}^2 \\ -\xi_{n-1}^3 - ... - \xi_{n-l^i}^3 - \xi_{n-1}^4 - ... - \xi_{n-l^i}^4 + DS_{n-1}^3 \end{bmatrix}\right), where\, i \in 4,5 \tag{4.15}$$

$$I_n^i = max\left(0\ ,\begin{bmatrix} s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - ... - \xi_{n-l^i}^1 - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 \\ -\xi_{n-1}^3 - ... - \xi_{n-l^i}^3 - \xi_{n-1}^4 - ... - \xi_{n-l^i}^4 + DS_{n-1}^2 \end{bmatrix}\right), \text{where } i \in 3 \tag{4.16}$$

$$I_n^i = max\left(0\ ,\begin{bmatrix} s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 - \xi_{n-1}^3 - \\ ... - \xi_{n-l^i}^3 - \xi_{n-1}^4 - ... - \xi_{n-l^i}^4 + DS_{n-1}^1 \end{bmatrix}\right), \text{where } i \in 2 \tag{4.17}$$

$$I_n^i = max\left(0\ ,\left[s^i - Y_{n-l^i}^i - \xi_{n-1}^3 - ... - \xi_{n-l^i}^3 - \xi_{n-1}^4 - ... - \xi_{n-l^i}^4 + DS_{n-1}^0 \right]\right), \text{where } i \in 1 \tag{4.18}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^4 - ... - \xi_{n-l^i}^4 \right), \text{where } i \in 0 \tag{4.19}$$

The outstanding order equations for the five-echelon assembly system are stated below:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_n^3 - \eta_n^i\right) \text{ where } i \in 5,4 \tag{4.20}$$

$$Y_{n+1}^3 = Y_n^3 + \xi_n^1 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^2 - \min\begin{pmatrix} Y_n^3 + \xi_n^1 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^2, \\ s^5 - Y_{n-2}^5 - \xi_{n-1}^1 - \xi_{n-1}^2 - \xi_{n-1}^3 - \xi_{n-1}^4, \\ s^4 - Y_{n-2}^4 - \xi_{n-1}^1 - \xi_{n-1}^2 - \xi_{n-1}^3 - \xi_{n-1}^4,\ \eta_n^3 \end{pmatrix} \tag{4.21}$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{pmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1,\ s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \\ -\xi_{n-1}^2 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^1,\ \eta_n^2 \end{pmatrix} \tag{4.22}$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0,\ s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \\ \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^2,\ \eta_n^1 \end{pmatrix} \tag{4.23}$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^4 - \min\left(Y_n^0 + \xi_n^4,\ s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^3,\ \eta_n^0\right) \tag{4.24}$$

A total of sixteen scenarios are considered for the five-echelon assembly system. Scenarios 21 thru 28 consider deterministic values of demand and capacity at each node. The values of demand and capacity for Scenarios 29 thru 36 are based on a normal distribution. The parameters of demand and capacity for both deterministic and probabilistic scenarios are shown in table 4.5. The best found base-stock values and the capacity utilization are shown in table 4.6

Table 4.5: Demand and Capacity Values of Five-echelon used in Simulation

	Demand 1	Demand 2	Demand 3	Demand 4	Capacity 5	Capacity 4	Capacity 3	Capacity 2	Capacity 1	Capacity 0
Scenario 21	4	4	4	4	18	18	18	13	9	5
Scenario 22	4	4	4	4	20	20	20	15	10	6
Scenario 23	8	8	8	4	31	31	31	22	13	5
Scenario 24	8	8	8	4	35	35	35	25	15	6
Scenario 25	8	8	4	4	27	27	27	18	9	5
Scenario 26	8	8	4	4	30	30	30	20	10	6
Scenario 27	8	4	4	4	22	22	22	13	9	5
Scenario 28	8	4	4	4	25	25	25	15	10	6
Scenario 29	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(13,1)	Norm(9,1)	Norm(5,1)
Scenario 30	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(20,1)	Norm(20,1)	Norm(20,1)	Norm(15,1)	Norm(10,1)	Norm(6,1)
Scenario 31	Norm(8,1)	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(31,1)	Norm(31,1)	Norm(31,1)	Norm(22,1)	Norm(13,1)	Norm(5,1)
Scenario 32	Norm(8,1)	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(35,1)	Norm(35,1)	Norm(35,1)	Norm(25,1)	Norm(15,1)	Norm(6,1)
Scenario 33	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(27,1)	Norm(27,1)	Norm(27,1)	Norm(18,1)	Norm(9,1)	Norm(5,1)
Scenario 34	Norm(8,1)	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(30,1)	Norm(30,1)	Norm(30,1)	Norm(20,1)	Norm(10,1)	Norm(6,1)
Scenario 35	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(22,1)	Norm(22,1)	Norm(22,1)	Norm(13,1)	Norm(9,1)	Norm(5,1)
Scenario 36	Norm(8,1)	Norm(4,1)	Norm(4,1)	Norm(4,1)	Norm(25,1)	Norm(25,1)	Norm(25,1)	Norm(15,1)	Norm(10,1)	Norm(6,1)

Table 4.6: Best Found Base-stock level and Capacity Utilization for Five-echelon System

		Optimal Base-Stock Levels							Utilization of capacity					
		S0	S1	S2	S3	S4	S5	Objective Function	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5
Deterministic Case	Scenario 21	8	16	24	32	32	32	144	0.80	0.89	0.92	0.89	0.89	0.89
	Scenario 22	8	16	24	32	32	32	144	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 23	8	24	40	56	56	56	240	0.80	0.92	0.91	0.90	0.90	0.90
	Scenario 24	8	24	40	56	56	56	240	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 25	8	16	32	48	48	48	200	0.80	0.89	0.89	0.89	0.89	0.89
	Scenario 26	8	16	32	48	48	48	200	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 27	8	16	24	40	40	40	168	0.80	0.89	0.92	0.91	0.91	0.91
	Scenario 28	8	16	24	40	40	40	168	0.67	0.80	0.80	0.80	0.80	0.80
Probabilistic case	Scenario 29	11	23	29	36	36	36	171	0.80	0.89	0.92	0.89	0.89	0.89
	Scenario 30	10	23	27	39	36	36	171	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 31	14	31	44	63	59	59	270	0.80	0.92	0.91	0.90	0.90	0.90
	Scenario 32	11	28	47	62	62	62	272	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 33	10	24	37	53	52	54	230	0.80	0.89	0.89	0.89	0.89	0.89
	Scenario 34	12	21	36	54	54	54	231	0.67	0.80	0.80	0.80	0.80	0.80
	Scenario 35	10	24	28	46	43	43	194	0.80	0.89	0.92	0.91	0.91	0.91
	Scenario 36	12	21	28	45	45	45	196	0.67	0.80	0.80	0.80	0.80	0.80

From the above discussions and results in table 4.1-4.6 the following statements can be made i) Costs increase when best found base-stock levels must be increased due to uncertainty and high capacity utilization ii) The total best found system inventory level increases when capacity is tight and there is uncertainty; iii) In the multi-echelon environment, individual base-stock levels may increase or decrease in response to uncertainty and high capacity utilization. iv) When uncertainty and capacity utilization increase a best found allocation of inventory moves safety stock closer to the nodes where demand occurs.

5. GRADIENT ESTIMATION FOR IPA

In this chapter gradient estimation technique of perturbation analysis is used to derive the sample path estimators for dY_n^i / ds^i, dI_n^i / ds^i, where $i \in \{node\}$. These estimators will be used to provide estimates for dL / ds^i, where L is a Lagrange function. For brevity the gradient estimation of a single-echelon model is shown initially, followed by three-echelon model and *m*-echelon model.

5.1 Gradient Estimation of a Single-echelon Model

The outstanding orders at the beginning of period $n+1$ for component i can be developed iteratively as follows:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n - \eta_n^i\right) where\, i \in \{0\} \tag{5.1}$$

The relationships for on-hand inventory can be written as shown in (2)

$$I_n^i = \max\left[0, s^i - Y_{n-2}^0 - \xi_{n-1} - .. - \xi_{n-l}\right], where\, i \in 0 \tag{5.2}$$

The inventory position is defined as shown in (3)

$$NI_n^i = s^i - Y_{n-2}^0 - \xi_{n-1} ... - \xi_{n-l}\, , where\, i \in 0 \tag{5.3}$$

The single-stage problem can be formulated as shown in (4):

$$\begin{aligned} &\min_{s^i \geq 0} c^i s^i \;\; where\;\; i \in 0 \\ &s.t\;\; P\left[NI_n^0 \geq 0\right] > \alpha^i \end{aligned} \tag{5.4}$$

A simulation optimization algorithm, which is discussed in the chapter 6, is used to obtain the best found base-stock level. A Lagrangian approach is used to handle the constraint.

Let the constraint in equation (4) be denoted using *v*. Equation (5) is obtained by substituting (3) in the constraint.

$$v = E\left[V_N\right] = E\left[N^{-1}\sum_{n=1}^{N} 1\left\{\xi_{n-1} + \ldots + \xi_{n-l} \le s^0 - Y_{n-2}^0\right\}\right] > \alpha^i \tag{5.5}$$

Where V_N is the service level attained at the end of *N* periods.

Let f_n be the density of demand $\left(\xi_{n-1} + \xi_{n-2}\right)on\left(0, \infty\right)$, corresponding cumulative distribution function, F_n is

$$F_n\left(x\right) = \int_0^x f_n\left(t\right) dt \tag{5.6}$$

Equation (5.5) can also be represented as shown in (5.7)

$$\tilde{V}_N = N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) > \alpha \tag{5.7}$$

The Lagrange function $L(s^0, u)$ can be written as shown in equation (5.8), where *u* is a Lagrange multiplier.

$$L(s^0, u) = c^0 s^0 - u * \left[N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) - \alpha\right] \tag{5.8}$$

We ignore the slack variable and consider it to be zero.

The gradient estimation technique of perturbation analysis is used to derive sample path estimators for dL/ds^0, and dL/du. The sample path estimators are shown in equation (5.9) and (5.10)

$$\frac{dL}{ds^0} = c^0 - \frac{u}{N}\sum_{n=1}^{N} 1\left\{s^0 + Y_{n-2}^0 > 0\right\} f_n\left(s^0 + Y_{n-2}^0\right)\left(s^0 + Y_{n-2}^0\right)' \tag{5.9}$$

$$\frac{dL}{du} = -N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) - \alpha^i = 0 \tag{5.10}$$

The second term in Equation (5.9) is obtained as a result of $\left(\tilde{V}_N\right)'$. The first order derivation equation of Y_{n+1}^0, I_n^0 is given as shown in equation (5.11) and (5.12) respectively.

$$\frac{dY_{n+1}^0}{ds^0} = \begin{cases} 0 & if\ Y_{n+1}^0 = 0 \\ \dfrac{dY_n^0}{ds^0} & if\ Y_{n+1}^0 = Y_{n+1}^0 + \xi_n - \eta_n^0 \end{cases} \tag{5.11}$$

$$dI_n^0 \Big/ ds^0 = \begin{cases} 0 & \text{if } NI_n^0 \le 0 \\ 1 + Y_{n-2}^0 \Big/ ds^0 & \text{otherwise} \end{cases} \tag{5.12}$$

5.2 Gradient Estimation of a Three-echelon Model

For the gradient estimation of a three-echelon assembly model let us recall the objective function and constraints of the model. Based on the objective function and constraints, a Lagrange function is framed. The objective function of a three-echelon model can be stated as shown in equation (5.13), the service level constraints are shown in equations (5.14) – (5.17), one for each node.

Objective function:-

$$\min_{s^i \ge 0} c^0 s^0 + c^1 s^1 + c^2 s^2 + c^3 s^3 \ \ where\ i \in \{0,1,2,3\} \tag{5.13}$$

Constraints:-

$$N^{-1}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^2 + \ldots + \xi_{n-l^0}^2 \le s^0 - Y_{n-2}^0\right\} \ge \alpha^0 \tag{5.14}$$

$$N^{-1}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{2}+\ldots+\xi_{n-l^{1}}^{2}+\xi_{n-1}^{1}+\ldots+\xi_{n-l^{1}}^{1} \leq s^{1}-Y_{n-2}^{1}+DS_{n-1}^{0}\right\} \geq \alpha^{1} \tag{5.15}$$

$$N^{-1}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{2}+\ldots+\xi_{n-l^{2}}^{2}+\xi_{n-1}^{1}+\ldots+\xi_{n-l^{2}}^{1} \leq s^{2}-Y_{n-2}^{2}+DS_{n-1}^{1}\right\} \geq \alpha^{2} \tag{5.17}$$

$$N^{-1}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{2}+\ldots+\xi_{n-l^{3}}^{2}+\xi_{n-1}^{1}+\ldots+\xi_{n-l^{3}}^{1} \leq s^{3}-Y_{n-2}^{3}+DS_{n-1}^{1}\right\} \geq \alpha^{3} \tag{5.18}$$

Let f_n be the density of demand $\left(\xi_{n-1}^{1}+\ldots+\xi_{n-l^{0}}^{1}\right),\left(\xi_{n-1}^{1}+\ldots+\xi_{n-l^{1,2,3}}^{1}+\xi_{n-1}^{2}+\ldots+\xi_{n-l^{1,2,3}}^{2}\right)$ on $\left(0,\infty\right)$, corresponding cumulative distribution function, F_n. Based on this, equations (5.14) - (5.18) can be rewritten as (5.19) – (5.22).

$$N^{-1}\sum_{n=1}^{N} F_{n}\left(s^{0}-Y_{n-2}^{0}\right) \geq \alpha^{0} \tag{5.19}$$

$$N^{-1}\sum_{n=1}^{N} F_{n}\left(s^{1}-Y_{n-2}^{1}-DS_{n-1}^{0}\right) \geq \alpha^{1} \tag{5.20}$$

$$N^{-1}\sum_{n=1}^{N} F_{n}\left(s^{2}-Y_{n-2}^{2}-DS_{n-1}^{1}\right) \geq \alpha^{2} \tag{5.21}$$

$$N^{-1}\sum_{n=1}^{N} F_{n}\left(s^{3}-Y_{n-2}^{3}-DS_{n-1}^{1}\right) \geq \alpha^{3} \tag{5.22}$$

The Lagrange function $L(\vec{s}^{i},\vec{u}^{i})$ can be written as shown in equation (5.23), where $\vec{s}^{i}=\left\{s^{0},s^{1},s^{2},s^{3}\right\}$, $\vec{u}^{i}=\left\{u^{0},u^{1},u^{2},u^{3}\right\}$.

$$L(\vec{s}^{\,i},\vec{u}^{\,i}) = \sum_{i=0}^{3} c^i s^i - \left\{ \begin{array}{l} u^0 * \left[N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) - \alpha^0 \right] \\ +u^1 * \left[N^{-1}\sum_{n=1}^{N} F_n\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right) - \alpha^1 \right] \\ +u^2 * \left[N^{-1}\sum_{n=1}^{N} F_n\left(s^2 - Y_{n-2}^2 + DS_{n-1}^1\right) - \alpha^2 \right] \\ +u^3 * \left[N^{-1}\sum_{n=1}^{N} F_n\left(s^3 - Y_{n-2}^3 + DS_{n-1}^1\right) - \alpha^3 \right] \end{array} \right\} \tag{5.23}$$

We ignore the slack variable and consider it to be zero. The first-order equations of the Lagrange function with respect to the base-stock levels, and Lagrange multiplier are stated in equations (5.24) – (5.31).

$$\frac{dL}{ds^0} = c^0 - \frac{u^0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left(s^0 - Y_{n-2}^0\right)' \tag{5.24}$$

$$\frac{dL}{ds^1} = c^1 - \left[\begin{array}{l} \frac{u^1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 + DS_{n-1}^0 > 0\right\} f_n\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right)\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right)' \\ + \frac{u^0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left\{-{dY_{n-2}^0}/{ds^1}\right\} \end{array} \right] \tag{5.25}$$

$$\frac{dL}{ds^2} = c^2 - \left[\begin{array}{l} \frac{u^2}{N}\sum_{n=1}^{N} 1\left\{s^2 - Y_{n-2}^2 + DS_{n-1}^1 > 0\right\} f_n\left(s^2 - Y_{n-2}^2 + DS_{n-1}^1\right)\left(s^2 - Y_{n-2}^2 + DS_{n-1}^1\right)' \\ + \frac{u^1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 + DS_{n-1}^0 > 0\right\} f_n\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right)\left\{-{dY_{n-2}^1}/{ds^2}\right\} \\ + \frac{u^0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left\{-{dY_{n-2}^0}/{ds^2}\right\} \end{array} \right] \tag{5.26}$$

$$\frac{dL}{ds^3} = c^3 - \begin{bmatrix} \frac{u^3}{N}\sum_{n=1}^{N} 1\left\{s^3 - Y_{n-2}^3 + DS_{n-1}^1 > 0\right\} f_n\left(s^3 - Y_{n-2}^3 + DS_{n-1}^1\right)\left(s^3 - Y_{n-2}^3 + DS_{n-1}^1\right)' \\ + \frac{u^1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 + DS_{n-1}^0 > 0\right\} f_n\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right)\left\{-\frac{dY_{n-2}^1}{ds^2}\right\} \\ + \frac{u^0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left\{-\frac{dY_{n-2}^0}{ds^2}\right\} \end{bmatrix} \tag{5.27}$$

$$\frac{dL}{du^0} = -N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) - \alpha^0 = 0 \tag{5.28}$$

$$\frac{dL}{du^1} = -N^{-1}\sum_{n=1}^{N} F_n\left(s^1 - Y_{n-2}^1 + DS_{n-1}^0\right) - \alpha^1 = 0 \tag{5.29}$$

$$\frac{dL}{du^2} = -N^{-1}\sum_{n=1}^{N} F_n\left(s^2 - Y_{n-2}^2 + DS_{n-1}^1\right) - \alpha^2 = 0 \tag{5.30}$$

$$\frac{dL}{du^3} = -N^{-1}\sum_{n=1}^{N} F_n\left(s^3 - Y_{n-2}^3 + DS_{n-1}^1\right) - \alpha^3 = 0 \tag{5.31}$$

5.2.1 Mean and Standard Deviation for Density of Demand

Let us suppose that a normal distribution for demand is considered for the three-echelon assembly system. Node 0 considers only one demand i.e. final product demand (demand 2), whereas all the other nodes (1,2,3) consider two demands, final and intermediate product (demand 1 and demand 2).

Node 0

Let us consider the density of demand for node 0, i.e. $f_n\left(s^0 - Y_{n-2}^0\right)$ as stated in equation (5.24). The probability density function $f_n\left(s^0 - Y_{n-2}^0\right)$ can be expanded and written for normal distribution as shown below in equation (5.32):

$$f_n\left(s^0 - Y_{n-2}^0\right) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)} e^{-\frac{(s^0 - Y_{n-2}^0 - \mu)^2}{2\sigma^2}} \tag{5.32}$$

The value of the mean (μ) and standard deviation (σ) in the normal density function are dependent on the lead time. For instance let us consider that the supply/manufacturing lead time between nodes is two periods. The mean of the normal distribution in equation (5.32) for a two period lead time is shown in equation (5.33):

$$\mu = \left(\xi_{n-1}^2 + \xi_{n-2}^2\right) = 2 * \xi^2 \tag{5.33}$$

Whereas, the standard deviation of a normal distribution for a two period lead time can be determined as follows:

$$Var\left(\xi_{n-1}^2 + \xi_{n-2}^2\right) = Var\left(\xi_{n-1}^2\right) + Var\left(\xi_{n-2}^2\right) - cov\left(\xi_{n-1}^2, \xi_{n-2}^2\right) \tag{5.34}$$

Since covariance in equation (5.33) is zero the equation can be written as show in equation (5.35)

$$Var\left(\xi_{n-1}^2 + \xi_{n-2}^2\right) = 2 * Var\left(\xi^2\right) \tag{5.35}$$

So the standard deviation of the normal distribution in equation (5.32) for a two period lead time can be written as shown in (5.36)

$$\sigma = SD = \sqrt{Var\left(\xi_{n-1}^2 + \xi_{n-2}^2\right)} = \sqrt{2 * Var\left(\xi^2\right)} = \sqrt{2} * SD(\xi^2) \tag{5.36}$$

From equation (5.33) and (5.36) for a l^0 period lead time the mean and the standard deviation of a normally distributed demand can be written as shown in equation (5.37) and (5.38) respectively:

$$\mu = \left(\xi_{n-1}^2 + \ldots + \xi_{n-l^0}^2\right) = l^0 * \xi^2 \tag{5.37}$$

$$\sigma = SD = \sqrt{Var\left(\xi_{n-1}^2 + \ldots + \xi_{n-l^0}^2\right)} = \sqrt{l^0 * Var\left(\xi^2\right)} = \sqrt{l^0} * SD(\xi^2) \tag{5.38}$$

Node 1,2, and 3

let us consider the density of demand for node 1,2,3 all of which have a similar structure, and all the three nodes consider two demands (intermediate and final product demand) i.e. $f_n\left(s^i - Y^i_{n-2} + DS^{0,1}_{n-1}\right), where\, i \in \{1,2,3\}$ as stated in equation (5.25) – (5.27). The probability density function $f_n\left(s^i - Y^i_{n-2} + DS^{0,1}_{n-1}\right), where\, i \in \{1,2,3\}$ can be expanded and written for normal distribution as shown below in equation (5.39):

$$f_n\left(s^i - Y^i_{n-2} + DS^{0,1}_{n-1}\right) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)} e^{-\frac{(s^i - Y^i_{n-2} + DS^{0,1}_{n-1} - \mu)^2}{2\sigma^2}} \tag{5.39}$$

let us consider that the supply/manufacturing lead time and the between nodes is two periods. The mean of the normal distribution in equation (5.39) for a two period lead time is shown in equation (5.40):

$$\mu = \left(\xi^1_{n-1} + \xi^1_{n-2} + \xi^2_{n-1} + \xi^2_{n-2}\right) = 2*\xi^1 + 2*\xi^2 \tag{5.40}$$

Whereas, the standard deviation of a normal distribution for a two period lead time can be determined as follows:

$$Var\left(\xi^1_{n-1} + \xi^1_{n-2} + \xi^2_{n-1} + \xi^2_{n-2}\right) = Var\left(\xi^1_{n-1}\right) + Var\left(\xi^1_{n-2}\right) + Var\left(\xi^2_{n-1}\right) + Var\left(\xi^2_{n-2}\right) \tag{5.41}$$

All the covariance terms in equation (5.41) has been ignored since all the demand variables are independent. Equation (5.41) can be written as show in equation (5.42)

$$Var\left(\xi^1_{n-1} + \xi^1_{n-2} + \xi^2_{n-1} + \xi^2_{n-2}\right) = 2*Var\left(\xi^1\right) + 2*Var\left(\xi^2\right) \tag{5.42}$$

So the standard deviation of the normal distribution in equation (5.39) for a two period lead time can be written as shown in (5.43)

$$\sigma = SD = \sqrt{Var\left(\xi_{n-1}^1 + \xi_{n-2}^1 + \xi_{n-1}^2 + \xi_{n-2}^2\right)} = \sqrt{2*Var\left(\xi^1\right) + 2*Var\left(\xi^2\right)} = \sqrt{2}*SD(\xi^1) + \sqrt{2}*SD(\xi^2) \tag{5.43}$$

From equation (5.40) and (5.43) for a l^0 period lead time the mean and the standard deviation of a normally distributed demand can be written as shown in equation (5.44) and (5.45) respectively:

$$\mu = \left(\xi_{n-1}^1 + ... + \xi_{n-l^i}^1 + \xi_{n-1}^2 + ... + \xi_{n-l^i}^2\right) = l^i * \xi^1 + l^i * \xi^2 \quad where\, i \in \{1,2,3\} \tag{5.44}$$

$$\sigma = SD = \sqrt{Var\left(\xi_{n-1}^1 + ... + \xi_{n-l^i}^1 + \xi_{n-1}^2 + ... + \xi_{n-l^i}^2\right)} = \sqrt{l^i * Var\left(\xi^1\right) + l^i * Var\left(\xi^2\right)} \quad where\, i \in \{1,2,3\} \quad = \sqrt{l^i} * SD(\xi^1) + \sqrt{l^i} * SD(\xi^2) \tag{5.45}$$

5.2.2 First Order Update Equations

First order on-hand inventory and outstanding orders for Node 3

The on-hand inventory equation for node 3 is stated ion equation (5.46) below

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - ... - \xi_{n-l^i}^1 - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 + DS_{n-1}^1\right), where\, i \in 3 \tag{5.46}$$

Differentiating equation (5.46) with respect to the base-stock level for node 3, is shown in equation (5.47), and (5.48).

$$\frac{dI_n^3}{ds^3} = \begin{cases} 0 & \text{if } NI_n^3 \le 0 \\ 1 - \dfrac{dY_{n-2}^3}{ds^3} & \text{otherwise} \end{cases} \tag{5.47}$$

$$\frac{dI_{n+1}^3}{ds^3} = \begin{cases} 0 & \text{if } NI_n^3 \le 0 \\ 1 - \dfrac{dY_{n-1}^3}{ds^3} & \text{otherwise} \end{cases} \tag{5.48}$$

Similarly equation (5.49) and (5.50) show the first order equation of on-hand inventory for node 3 with respect to base-stock level of node 0,1,2

$$\frac{dI_n^3}{ds^i}=\begin{cases}0 & \text{if } NI_n^3\le 0\\ -\dfrac{dY_{n-2}^3}{ds^i} & \text{otherwise}\end{cases}\quad where\, i\in\{2,1,0\} \tag{5.49}$$

$$\frac{dI_{n+1}^3}{ds^i}=\begin{cases}0 & \text{if } NI_n^3\le 0\\ -\dfrac{dY_{n-1}^3}{ds^i} & \text{otherwise}\end{cases}\quad where\, i\in\{2,1,0\} \tag{5.50}$$

Equation (5.51) is a generalized form of first order on-hand inventory equation for node 3, i.e. differentiation of on-hand inventory for node 3 with respect to base-stock levels. This is obtained by considering (5.47) – (5.50)

$$\frac{dI_{n+1}^3}{ds^i}=\begin{cases}0 & \text{if } NI_{n+1}^3\le 0\\ \dfrac{dI_n^3}{ds^i}+\dfrac{dY_{n-2}^3}{ds^i}-\dfrac{dY_{n-1}^3}{ds^i} & \text{otherwise}\end{cases}\quad where\, i\in\{0,1,2,3\} \tag{5.51}$$

The outstanding order equation for node 3 is stated below in equation (5.52)

$$Y_{n+1}^i=\max\left(0,Y_n^i+\xi_n^1+\xi_n^2-DS_n^1-\eta_n^i\right)\text{ where } i\in 3 \tag{5.52}$$

Differentiating equation (5.52) with respect to base-stock level for node 3, 2, 1, and 0 i.e. s^i, where $i\in\{0,1,2,3\}$ results in equation (5.53) shown below.

$$\frac{dY_{n+1}^3}{ds^i}=\begin{cases}0 & \text{if } Y_{n+1}^3\le 0\\ \dfrac{dY_n^3}{ds^i} & \text{otherwise}\end{cases} \tag{5.53}$$

First order on-hand inventory and outstanding orders for Node 2

The on-hand inventory equation for node 2 is stated in equation (5.54) below

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - ... - \xi_{n-l^i}^1 - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 + DS_{n-1}^1\right), where\, i \in 2 \tag{5.54}$$

Differentiating equation (5.54) with respect to the base-stock level for node 2 is shown in equation (5.55), and (5.56).

$$\frac{dI_n^2}{ds^2} = \begin{cases} 0 & \text{if } NI_n^2 \le 0 \\ 1 - \dfrac{dY_{n-2}^2}{ds^2} & \text{otherwise} \end{cases} \tag{5.55}$$

$$\frac{dI_{n+1}^2}{ds^2} = \begin{cases} 0 & \text{if } NI_n^2 \le 0 \\ 1 - \dfrac{dY_{n-1}^2}{ds^2} & \text{otherwise} \end{cases} \tag{5.56}$$

Similarly equation (5.55) and (5.56) show the first order equation of on-hand inventory for node 3 with respect to base-stock level of node 3, 1, 0

$$\frac{dI_n^2}{ds^i} = \begin{cases} 0 & \text{if } NI_n^2 \le 0 \\ -\dfrac{dY_{n-2}^2}{ds^i} & \text{otherwise} \end{cases} \quad where\, i \in \{3,1,0\} \tag{5.57}$$

$$\frac{dI_{n+1}^2}{ds^i} = \begin{cases} 0 & \text{if } NI_n^2 \le 0 \\ -\dfrac{dY_{n-1}^2}{ds^i} & \text{otherwise} \end{cases} \quad where\, i \in \{3,1,0\} \tag{5.58}$$

Equation (5.59) is a generalized form of first order on-hand inventory equation for node 2, i.e. differentiation of on-hand inventory for node 2 with respect to base-stock levels. This is obtained by considering (5.55) – (5.58)

$$\frac{dI_{n+1}^{3}}{ds^{i}}=\begin{cases}0 & \text{if } NI_{n+1}^{3}\leq 0\\ \frac{dI_{n}^{3}}{ds^{i}}+\frac{dY_{n-2}^{3}}{ds^{i}}-\frac{dY_{n-1}^{3}}{ds^{i}} & \text{otherwise}\end{cases}\quad where\, i\in\{0,1,2,3\} \tag{5.59}$$

The outstanding order equation for node 2 is stated below in equation (5.60)

$$Y_{n+1}^{i}=\max\left(0,Y_{n}^{i}+\xi_{n}^{1}+\xi_{n}^{2}-DS_{n}^{1}-\eta_{n}^{i}\right)\text{ where } i\in 2 \tag{5.60}$$

Differentiating equation (5.60) with respect to base-stock level for node 3, 2, 1, and 0 i.e. s^{i}, where $i\in\{0,1,2,3\}$ results in equation (5.61) shown below.

$$\frac{dY_{n+1}^{2}}{ds^{i}}=\begin{cases}0 & \text{if } Y_{n+1}^{3}\leq 0\\ \frac{dY_{n}^{2}}{ds^{i}} & \text{otherwise}\end{cases} \tag{5.61}$$

First order on-hand inventory and outstanding orders for Node 1

The on-hand inventory equation for node 1 is stated in equation (5.62) below

$$I_{n}^{i}=max\left(0\ ,\ s^{i}-Y_{n-l^{i}}^{i}-\xi_{n-1}^{1}-\ldots-\xi_{n-l^{i}}^{1}-\xi_{n-1}^{2}-\ldots-\xi_{n-l^{i}}^{2}+DS_{n-1}^{0}\right),\text{where } i\in 1 \tag{5.62}$$

Differentiating equation (5.62) with respect to the base-stock level for node 1 is shown in equation (5.63), and (5.64).

$$\frac{dI_{n}^{1}}{ds^{1}}=\begin{cases}0 & \text{if } NI_{n}^{1}\leq 0\\ 1-\frac{dY_{n-2}^{1}}{ds^{1}} & \text{otherwise}\end{cases} \tag{5.63}$$

$$\frac{dI_{n+1}^{1}}{ds^{1}}=\begin{cases}0 & \text{if } NI_{n}^{1}\leq 0\\ 1-\frac{dY_{n-1}^{1}}{ds^{1}} & \text{otherwise}\end{cases} \tag{5.64}$$

Similarly equation (5.65) and (5.66) show the first order equation of on-hand inventory for node 3 with respect to base-stock level of node 3, 2, 0

$$\frac{dI_n^1}{ds^i} = \begin{cases} 0 & \text{if } NI_n^1 \le 0 \\ -\frac{dY_{n-2}^1}{ds^i} & \text{otherwise} \end{cases} \quad where\, i \in \{3,2,0\} \tag{5.65}$$

$$\frac{dI_{n+1}^1}{ds^i} = \begin{cases} 0 & \text{if } NI_n^1 \le 0 \\ -\frac{dY_{n-1}^1}{ds^i} & \text{otherwise} \end{cases} \quad where\, i \in \{3,2,0\} \tag{5.66}$$

Equation (5.67) is a generalized form of first order on-hand inventory equation for node 1, i.e. differentiation of on-hand inventory for node 1 with respect to base-stock levels. This is obtained by considering (5.63) – (5.66)

Equation is written by considering

$$\frac{dI_{n+1}^1}{ds^i} = \begin{cases} 0 & \text{if } NI_{n+1}^1 \le 0 \\ \frac{dI_n^1}{ds^i} + \frac{dY_{n-2}^1}{ds^i} - \frac{dY_{n-1}^1}{ds^i} & \text{otherwise} \end{cases} \quad where\, i \in \{0,1,2,3\} \tag{5.67}$$

The outstanding order equation for node 1 is stated below in equation (5.68)

$$Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_n^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_n^0, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \xi_{n-1}^2, \\ s^2 - Y_{n-2}^2 - \xi_{n-1}^1 - \xi_{n-1}^2, \eta_n^1 \end{pmatrix} \tag{5.68}$$

Differentiating equation (5.68) with respect to base-stock level for node 3 i.e. s^3, which results in equation (5.69) shown below.

$$\frac{dY_{n+1}^1}{ds^3} = \begin{cases} 0 & \text{if } Y_{n+1}^1 = 0 \\ \frac{dY_n^1}{ds^3} + \frac{dY_{n-2}^3}{ds^3} - 1 & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^3 \\ +Y_{n-2}^3 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \frac{dY_n^1}{ds^3} + \frac{dY_{n-2}^2}{ds^3} & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^2 \\ +Y_{n-2}^2 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \frac{dY_n^1}{ds^3} & \text{if } Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - \eta_n^1 \end{cases} \tag{5.69}$$

Differentiating equation (5.68) with respect to base-stock level for node 2 i.e. s^2, which results in equation (5.70) shown below.

$$\frac{dY_{n+1}^1}{ds^2} = \begin{cases} 0 & \text{if } Y_{n+1}^1 = 0 \\ \dfrac{dY_n^1}{ds^2} + \dfrac{dY_{n-2}^3}{ds^2} & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^3 \\ +Y_{n-2}^3 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \dfrac{dY_n^1}{ds^2} + \dfrac{dY_{n-2}^2}{ds^2} - 1 & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^2 \\ +Y_{n-2}^2 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \dfrac{dY_n^1}{ds^2} & \text{if } Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - \eta_n^1 \end{cases} \tag{5.70}$$

Differentiating equation (5.68) with respect to base-stock level for node 1, and 0 i.e. s^i, $i \in \{0,1\}$ which results in equation (5.71) shown below.

$$\frac{dY_{n+1}^1}{ds^i} = \begin{cases} 0 & \text{if } Y_{n+1}^1 = 0 \\ \dfrac{dY_n^1}{ds^i} + \dfrac{dY_{n-2}^3}{ds^i} & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^3 \\ +Y_{n-2}^3 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \dfrac{dY_n^1}{ds^i} + \dfrac{dY_{n-2}^2}{ds^i} & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - s^2 \\ +Y_{n-2}^2 + \xi_{n-1}^2 + \xi_{n-1}^1 \end{bmatrix} \\ \dfrac{dY_n^1}{ds^i} & \text{if } Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - \eta_n^1 \end{cases} \tag{5.71}$$

First order on-hand inventory and outstanding orders for Node 0

The on-hand inventory equation for node 0 is stated ion equation (5.72) below

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2\right), \text{where } i \in 0 \tag{5.72}$$

Differentiating equation (5.72) with respect to the base-stock level for node 0 is shown in equation (5.73), and (5.74).

$$\frac{dI_n^0}{ds^0}=\begin{cases}0 & \text{if } NI_n^0 \le 0\\ 1-\frac{dY_{n-2}^0}{ds^0} & \text{otherwise}\end{cases} \tag{5.73}$$

$$\frac{dI_{n+1}^0}{ds^0}=\begin{cases}0 & \text{if } NI_n^0 \le 0\\ 1-\frac{dY_{n-1}^0}{ds^0} & \text{otherwise}\end{cases} \tag{5.74}$$

Similarly equation (5.75) and (5.76) show the first order equation of on-hand inventory for node 3 with respect to base-stock level of node 3, 2, 1

$$\frac{dI_n^0}{ds^i}=\begin{cases}0 & \text{if } NI_n^0 \le 0\\ -\frac{dY_{n-2}^0}{ds^i} & \text{otherwise}\end{cases} \quad where\, i \in \{3,2,1\} \tag{5.75}$$

$$\frac{dI_{n+1}^0}{ds^i}=\begin{cases}0 & \text{if } NI_n^0 \le 0\\ -\frac{dY_{n-1}^0}{ds^i} & \text{otherwise}\end{cases} \quad where\, i \in \{3,2,1\} \tag{5.76}$$

Equation (5.77) is a generalized form of first order on-hand inventory equation for node 0, i.e. differentiation of on-hand inventory for node 0 with respect to base-stock levels. This is obtained by considering (5.73) – (5.76)

$$\frac{dI_{n+1}^0}{ds^i}=\begin{cases}0 & \text{if } NI_{n+1}^0 \le 0\\ \frac{dI_n^0}{ds^i}+\frac{dY_{n-2}^0}{ds^i}-\frac{dY_{n-1}^0}{ds^i} & \text{otherwise}\end{cases} \quad where\, i \in \{0,1,2,3\} \tag{5.77}$$

The outstanding order equation for node 0 is stated below in equation (5.78)

$$Y_{n+1}^0 = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-1}^2, \eta_n^0\right) \tag{5.78}$$

Differentiating equation (5.78) with respect to base-stock level for node 1 i.e. s^1, which results in equation (5.79) shown below.

$$\frac{dY_{n+1}^{0}}{ds^{1}}=\begin{cases}0 & \text{if } Y_{n+1}^{0}\le 0\\ \dfrac{dY_{n}^{0}}{ds^{1}}+\dfrac{dY_{n-2}^{1}}{ds^{1}}-1 & \text{if } Y_{n+1}^{0}=\begin{bmatrix}s^{1}-Y_{n-2}^{1}-\xi_{n-1}^{1}\\ -\xi_{n-2}^{1}-\xi_{n-1}^{2}\end{bmatrix}\\ \dfrac{dY_{n}^{0}}{ds^{1}} & \text{if } Y_{n+1}^{0}=Y_{n}^{0}+\xi_{n}^{2}-\eta_{n}^{0}\end{cases} \tag{5.79}$$

Differentiating equation (5.78) with respect to base-stock level for node 0, 2, and 3 i.e. s^{i}, $i\in\{0,2,3\}$ which results in equation (5.80) shown below.

$$\frac{dY_{n+1}^{0}}{ds^{i}}=\begin{cases}0 & \text{if } Y_{n+1}^{0}\le 0\\ \dfrac{dY_{n}^{0}}{ds^{i}}+\dfrac{dY_{n-2}^{1}}{ds^{i}} & \text{if } Y_{n+1}^{0}=\begin{bmatrix}s^{1}-Y_{n-2}^{1}-\xi_{n-1}^{1}\\ -\xi_{n-2}^{1}-\xi_{n-1}^{2}\end{bmatrix}\\ \dfrac{dY_{n}^{0}}{ds^{i}} & \text{if } Y_{n+1}^{0}=Y_{n}^{0}+\xi_{n}^{2}-\eta_{n}^{0}\end{cases} \tag{5.80}$$

5.3 *m*-echelon First Order Update Equations

The objective function of an *m*-echelon model can be stated as shown in equation (5.81). There are *m* constraints, one for each node. Since all the nodes have a similar structure, only four constraints are shown in equations (5.82) – (5.85), i.e. nodes *0. m-k, m-1, m*.

Objective function:-

$$\min_{s^{i}\ge 0} c^{i}s^{i} \;\; \textit{where } i\in\{0,1,2,\ldots m-1,m\} \tag{5.81}$$

Constraints:-

$$N^{-1}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{j}+\ldots+\xi_{n-l^{0}}^{j}\le s^{0}-Y_{n-2}^{0}\right\}\ge\alpha^{0} \tag{5.82}$$

$$N^{-1}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{j-(m-k)}+..+\xi_{n-1}^{j}+..+\xi_{n-l^{m-k}}^{j-(m-k)}+..+\xi_{n-l^{m-k}}^{j}\le s^{m-k}-Y_{n-2}^{m-k}+DS_{n-1}^{m-k-1}\right\}\ge\alpha^{m-k} \tag{5.83}$$

$$N^{-1}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{j-(m-2)}+...+\xi_{n-1}^{j}+..+\xi_{n-l^{m-1}}^{j-(m-2)}+...+\xi_{n-l^{m-1}}^{j}\le s^{m-1}-Y_{n-2}^{m-1}+DS_{n-1}^{m-2}\right\}\ge\alpha^{m-1} \tag{5.84}$$

$$N^{-1}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{j-(m-2)}+...+\xi_{n-1}^{j}+...+\xi_{n-l^{m}}^{j-(m-2)}+...+\xi_{n-l^{m}}^{j}\le s^{m}-Y_{n-2}^{m}+DS_{n-1}^{m-2}\right\}\ge\alpha^{m} \tag{5.85}$$

Let f_n be the density of demand, i.e. $\left(\xi_{n-1}^{j}+...+\xi_{n-l^0}^{j}\right)$, and $\left(\xi_{n-1}^{j-(m-k)}.....+\xi_{n-1}^{j}+...+\xi_{n-l^i}^{j-(m-k)}.....+\xi_{n-l^i}^{j}\right)$ on $(0,\infty)$, where $i\in\{1...m-1,m\}$ corresponding cumulative distribution function, F_n. Based on this, equations (5.82) - (5.85) can be rewritten as (5.86) – (5.89).

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^0-Y_{n-2}^{0}\right)\ge\alpha^0 \tag{5.86}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{m-k}-Y_{n-2}^{m-k}+DS_{n-1}^{m-k-1}\right)\ge\alpha^{m-k} \tag{5.87}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{m-1}-Y_{n-2}^{m-1}+DS_{n-1}^{m-2}\right)\ge\alpha^{m-1} \tag{5.88}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{m}-Y_{n-2}^{m}+DS_{n-1}^{m-2}\right)\ge\alpha^{m} \tag{5.89}$$

The Lagrange function $L(\vec{s}^{\,i},\vec{u}^{\,i})$ can be written as shown in equation (5.90), where $\vec{s}^{\,i}=\left\{s^0,....,s^m\right\}$, $\vec{u}^{\,i}=\left\{u^0,...,u^m\right\}$.

$$L(\vec{s}^{\,i},\vec{u}^{\,i}) = \sum_{n=0}^{3} c^i s^i - \left\{ \begin{array}{l} u^0 * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0\right) - \alpha^0 \right] + \dots\dots\dots + \\ u^{m-k} * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^{m-k} - Y_{n-2}^{m-k} + DS_{n-1}^{m-k-1}\right) - \alpha^{m-k} \right] \\ + \dots . + u^{m-1} * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^{m-1} - Y_{n-2}^{m-1} + DS_{n-1}^{m-2}\right) - \alpha^{m-1} \right] \\ + u^3 * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^3 - Y_{n-2}^3 + DS_{n-1}^1\right) - \alpha^3 \right] \end{array} \right\} \tag{5.90}$$

We ignore the slack variable and consider it to be zero. The first-order equations of the Lagrange function with respect to the base-stock levels, and Lagrange multiplier are stated in equations (5.91) – (5.98).

$$\frac{dL}{ds^0} = c^0 - \frac{u^0}{N} \sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left(s^0 - Y_{n-2}^0\right)' \tag{5.91}$$

:

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$$\frac{dL}{ds^{m-k}} = c^{m-k} - \left[\begin{array}{l} \frac{u^{m-k}}{N} \sum_{n=1}^{N} 1\left\{s^{m-k} - Y_{n-2}^{m-k} + DS_{n-1}^{m-k-1} > 0\right\} \\ f_n\left(s^{m-k} - Y_{n-2}^{m-k} + DS_{n-1}^{m-k-1}\right)\left(s^{m-k} - Y_{n-2}^{m-k} + DS_{n-1}^{m-k-1}\right)' + \dots \\ \dots\dots + \frac{u^0}{N} \sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left\{ - {dY_{n-2}^0}\big/{ds^{m-k}} \right\} \end{array} \right] \tag{5.92}$$

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$$\frac{dL}{ds^{m-1}} = c^{m-1} - \left[\begin{array}{l} \frac{u^{m-1}}{N} \sum_{n=1}^{N} 1\left\{ s^{m-1} - Y_{n-2}^{m-1} + DS_{n-1}^{m-2} > 0 \right\} \\ f_n\left(s^{m-1} - Y_{n-2}^{m-1} + DS_{n-1}^{m-2} \right)\left(s^{m-1} - Y_{n-2}^{m-1} + DS_{n-1}^{m-2} \right)' + ... \\ ... + \frac{u^0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n\left(s^0 - Y_{n-2}^0 \right)\left\{ -{dY_{n-2}^0}\big/{ds^{m-1}} \right\} \end{array} \right] \tag{5.93}$$

$$\frac{dL}{ds^{m}} = c^{m} - \left[\begin{array}{l} \frac{u^{m}}{N} \sum_{n=1}^{N} 1\left\{ s^{m} - Y_{n-2}^{m} + DS_{n-1}^{m-2} > 0 \right\} \\ f_n\left(s^{m} - Y_{n-2}^{m} + DS_{n-1}^{m-2} \right)\left(s^{m} - Y_{n-2}^{m} + DS_{n-1}^{m-2} \right)' + ... \\ ... + \frac{u^0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n\left(s^0 - Y_{n-2}^0 \right)\left\{ -{dY_{n-2}^0}\big/{ds^{m}} \right\} \end{array} \right] \tag{5.94}$$

$$\frac{dL}{du^0} = -N^{-1} \sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^0 \right) - \alpha^0 = 0 \tag{5.95}$$

$$\frac{dL}{du^{m-k}} = -N^{-1} \sum_{n=1}^{N} F_n\left(s^{m-k} - Y_{n-2}^{m-k} + DS_{n-1}^{m-k-1} \right) - \alpha^{m-k} = 0 \tag{5.96}$$

$$\frac{dL}{du^{m-1}} = -N^{-1} \sum_{n=1}^{N} F_n\left(s^{m-1} - Y_{n-2}^{m-1} + DS_{n-1}^{m-2} \right) - \alpha^{m-1} = 0 \tag{5.97}$$

$$\frac{dL}{du^{m}} = -N^{-1} \sum_{n=1}^{N} F_n\left(s^{m} - Y_{n-2}^{m} + DS_{n-1}^{m-2} \right) - \alpha^{m} = 0 \tag{5.98}$$

The first order update equations for *m*-echelon assembly system are similar to the three-echelon system.

5.4 Variations in Lagrange Function

In order to get a better convergence on the best found base-stock level some variations were made to the original Lagrange function, for brevity let us consider Lagrange function for single-stage, equation (5.8). Three variations are made to the Lagrange function:

- Squared Lagrange Function
- Squared & Multiplied Function
- Discontinuous Lagrange Function

5.4.1 Squared Lagrange function

The constraint in the Lagrange function is squared, by doing so the objective function is penalized more when the base-stock $\left(s^0\right)$ is far from the best found base-stock $\left(\hat{s}^0\right)$, as the base-stock gets closer to the best found answer the penalty reduces. The squared Lagrange function is shown in equation (5.99)

$$L(s^0,u) = c^0 s^0 - u * \left[N^{-1} \sum_{n=1}^{N} F_n \left(s^0 + Y_{n-2}^0 \right) - \alpha \right]^2 \tag{5.99}$$

5.4.2 Squared & Multiplied Lagrange function

The constraint in the Lagrange function is squared and also multiplied with a large constant (Q), along with the Lagrange multiplier, which is shown in equation (5.100). By doing so the objective function is penalized on a greater magnitude than compared to the previous case. When the base-stock $\left(s^0\right)$ is far from the best found base-stock $\left(\hat{s}^0\right)$ there is a larger penalty, whereas, as the base-stock gets closer to the best found answer the penalty decreases. The squared & multiplied Lagrange function is shown in equation (5.100)

$$L(s^0,u) = c^0 s^0 - u * Q * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^0 + Y_{n-2}^0\right) - \alpha \right]^2 \tag{5.100}$$

5.4.3 Discontinuous Lagrange Function

The objective function is penalized heavily when the base-stock level provides a service level lower than the desired service level(α) and penalized lightly when the base-stock level provides a service level higher than the desired service level. The Lagrange function is discontinuous at the desired service level. The function is shown in equation (5.101) below.

$$L(s^0,u) = \begin{cases} c^0 s^0 - u * Q * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^0 + Y_{n-2}^0\right) - \alpha \right] & if\ N^{-1} \sum_{n=1}^{N} F_n\left(s^0 + Y_{n-2}^0\right) < \alpha \\ c^0 s^0 - u * \left[N^{-1} \sum_{n=1}^{N} F_n\left(s^0 + Y_{n-2}^0\right) - \alpha \right] & otherwise \end{cases} \tag{5.101}$$

Q is a large constant.

5.5 Justification of Derivatives

In this section, all the equations discussed in section 5.2.2 are shown as valid and that the sample-path derivatives they generate are unbiased estimators of derivative of expectations. To accomplish this we show that the conditions in proposition 1 and proposition 2 that ensure the following properties are satisfied:

- The outstanding orders, on-hand inventory, and the net inventory are differentiable with probability one, with respect to all base-stock levels
- The expectation and derivatives are interchangeable for outstanding orders, on-hand inventory, and net-inventory
- Show that the Lagrange function is Lipchitz continuous

In order to show that the Lagrange function is Lipchitz continuous we initially prove that all the components of the function are continuous, which include the demand, capacity, outstanding orders, on-hand inventory, and net inventory (Glasserman et al., 1995). Since the demand and capacity are derived from the probability distributions, it is known that they are continuous. The outstanding orders, on-hand inventory and the net-inventory are also relations which are based on the demand and capacity for period 0. The important step in verifying that the derivative estimates based on the Lagrange function are unbiased is showing that the on-hand inventory, outstanding orders, and net inventory, with probability one are Lipchitz functions and have integrable moduli. This requires according to generalized mean value theorem, and dominated convergence theorem to show that expectation and derivatives are interchangeable.

We assume that for period-*n* and demand j, ξ_n^j has a density on $(0,\infty)$, i.e. $x \to P\left(\xi_n^j \le x\right)$ is absolutely continuous on all $x > 0$. Similarly we assume that for period-*n* and capacity for node i, η_n^i has a density on $(0,\infty)$, i.e. $y \to P\left(\eta_n^i \le y\right)$ for all $y > 0$. We also consider the possibility of demand and capacity being equal to zero, i.e. $P\left(\xi_n^j = 0\right) > 0$, and $P\left(\eta_n^i = 0\right) > 0$ respectively.

Proposition 1

If $\left\{\xi_n^j, n = 1,2,...N; j = 1,2,...,m-2\right\}$, $\left\{\eta_n^i, n = 1,2,...N; i = 1,2,...,m\right\}$ are independent, and each ξ_n^j, η_n^i has a density on $(0,\infty)$, and then the following hold:

iii) For $i = 1,....,m$ and $n = 1,2,...N$ each Y_n^i, I_n^i, NI_n^i are differentiable, with probability one, at s^i with respect to s^i, $i = 1,2,...,m$

iv) If $\left[\sum_{j=1}^{m-2} \xi_n^j\right] < \infty$, and $\eta_n^i < \infty$ for all n, then $E\left[Y_n^i\right]', E\left[NI_n^i\right]', \text{and}\, E\left[I_n^i\right]'$ exist and equal $E\left[\left(Y_n^i\right)'\right], E\left[\left(NI_n^i\right)'\right], \text{and}\, E\left[\left(I_n^i\right)'\right]$

Proof of part (i):- The state variables are differentiable for any sequence of demands up to period *N* if the minimax in (5.102) and (5.103) is uniquely attained in all *N* periods.

The outstanding orders at the beginning of period *n+1* for component *i* where $i \in \{m, m-1\}$ can be developed as follows.

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^{j-(m-2)} + ... + \xi_n^j - DS_n^{m-2} - \eta_n^i\right) where\, i \in m, m-1 \tag{5.102}$$

For the *m*-echelon analytical review it is assumed that there are $j = m-2$ demands. The outstanding orders for node *m-2* can be developed iteratively as shown in (3.30)

$$Y_{n+1}^{i} = Y_{n}^{i} + \xi_{n}^{j-(m-2)} + \ldots + \xi_{n}^{j} - DS_{n-1}^{i-1} - \min \begin{pmatrix} Y_{n}^{i} + \xi_{n}^{j-(m-2)} + \ldots + \xi_{n}^{j} - DS_{n-1}^{i-1}, s^{m-1} - Y_{n-2}^{m-1} - \xi_{n-l^{i}+1}^{j-(m-2)} - \\ .. - \xi_{n-1}^{j-(m-2)} - .. - \xi_{n-1}^{j} - \ldots. - \xi_{n-l^{i}+1}^{j}, s^{m} - Y_{n-2}^{m} - \xi_{n-l^{i}+1}^{j-(m-2)} \\ - \ldots. - \xi_{n-1}^{j-(m-2)} - \ldots - \xi_{n-l^{i}+1}^{j} - \ldots. - \xi_{n-1}^{j}, \ \eta_{n}^{i} \end{pmatrix}$$
$$where\, i \in m-2$$

(5.103)

We assume that the values of the demand and capacity are between $[0,\infty)$. Since the values of the demand are obtained from the probability distribution, they are continuous, ties have almost zero probability except when the available inventory from the two upstream nodes (node *m* and *m-1*) in (5.103) have a tie. This would only be possible, in particular if both have the same base-stock levels and outstanding orders. The possibility of both having the same outstanding orders depends on the capacity and demand, which is based of probability distributions, and we know that they are continuous. So, ties for (5.103) have almost zero probability.

Similarly the values of NI_{n}^{i}, and I_{n}^{i} are dependent on the demand, outstanding orders, and downstream shortages. We know that the outstanding orders and the demand are continuous. The values of the downstream shortages depend on the capacity and demand, which we know are continuous. Thus, with probability one, differentiability is preserved in each period. The result for proposition 1 - part (ii), (i.e., the expectation and derivatives are interchangeable) is based of the result from proposition 2 and is shown after the proof for proposition 2. ■

Proposition 2

Let $\{L(\vec{s},\vec{u}), \vec{s} \in S, \vec{u} \in U\}$ be a random function with $S,\ U \subseteq \mathbb{R}$. If $E[L(\vec{s},\vec{u})] < \infty$ for all $\vec{s} \in S, \vec{u} \in U$. Assume that L is differentiable at $s^{i} \in S$ and $u^{i} \in U$ with probability one, and that L defined on set S, and U is almost surely Lipschitz continuous with modulus M_L satisfying $E[M_L] < \infty$. Then $E\left[L(\vec{s},\vec{u})\right]'$ exists and equals $E\left[L'(\vec{s},\vec{u})\right]$.

Proof:- let us start the with the definition of Lipschitz continuous

Definition: A function L defined on a set $S,U \subseteq \square$ is said to be Lipschitz continuous on S, U if there exists an M so that

$$\frac{\left|L(\vec{s}+\delta,\vec{u}+\delta)-L(\vec{s},\vec{u})\right|}{\delta} \le M \tag{5.104}$$

For all $\vec{s}$ in S and $\vec{u}$ in U such that $\delta \ge 0$ (Reed, 1998)

The proof is provided by contradiction. Suppose that L is not Lipschitz continuous. Then, there exists a M_0 so that we cannot choose any $\delta \ge 0$ with the property as stated in the definition.

Set $(\vec{s}+\delta,\vec{u}+\delta) = x$ and $(\vec{s},\vec{u}) = c$ for brevity of the proof

Let us consider $\delta_n = 2^{-n}$. If we consider $S,U \in [0,\infty)$ (base-stock values, and Lagrange multipliers are considered positive) then for each n there are points x_n and c_n in $[0,\infty)$ so that

$$\frac{\left|L(x_n)-L(c_n)\right|}{\left|x_n-c_n\right|} \ge M_0 \tag{5.105}$$

Since $[0,\infty)$ is a finite interval, the Bolzano-Weiestrass theorem guarantees that there exists a subsequence $\{x_{n_k}\}$ that converges to a point d in $[0,\infty)$ (Reed, 1998). Furthermore,

$$\begin{aligned}\left|c_{n_k}-d\right| &\le \left|c_{n_k}-x_{n_k}\right| + \left|x_{n_k}-d\right| \\ &\le 2^{-n_k} + \left|x_{n_k}-d\right|\end{aligned} \tag{5.106}$$

So $c_{n_k} \to d$ also as $k \to \infty$. Since *f* is continuous, $L(x_{n_k}) \to L(d)$ and $L(c_{n_k}) \to L(d)$ as $k \to \infty$. This is impossible since $\left|L(x_{n_k}) - L(c_{n_k})\right| \geq M_0$ for each *k*. Thus *L* is uniformly continuous on $[0,\infty]$.

The composition of Lipschitz functions with moduli M_0 and M_1 is Lipschitz with modulus $M_0 M_1$. A random function is Lipschitz with probability one if there exists a random variable M that serves the path-wise modulus (Glasserman, 1995).

The right side of (5.104) is integrable, by proof provided earlier. From the dominated convergence theorem and generalized mean value theorem (Glasserman, 1991), we have

$$\begin{aligned} E\left[L'(\vec{s},\vec{u})\right] &= E\left[\lim_{\delta\to 0}\frac{\left|L(\vec{s}+\delta,\vec{u}+\delta)-L(\vec{s},\vec{u})\right|}{(\delta,\delta)}\right] \\ &= \lim_{\delta\to 0} E\left[\frac{\left|L(\vec{s}+\delta,\vec{u}+\delta)-L(\vec{s},\vec{u})\right|}{(\delta,\delta)}\right] \\ &= E\left[L(\vec{s},\vec{u})\right]' \end{aligned} \tag{5.107}$$

■

Proof for part (ii) of proposition 1:- if the $\left[\sum_{j=1}^{m-2}\xi_n^j\right] < \infty$ for all *n*, then the outstanding orders (Y_n^i), on-hand inventory (I_n^i), and net inventory (NI_n^i) have finite expectation as well. At time $(n=0)$, each of the state variables (update equations) are linear, hence are considered as Lipschitz continuous with modulus *M*. The operations min, max, and addition are Lipschitz so each of $Y_n^i, I_n^i,$ and NI_n^i is a composition of Lipschitz functions and therefore Lipschitz continuous.

From the dominated convergence theorem, and generalized mean value theorem, we know that for uniform integrability the interchange of derivative and expectation is possible (Glasserman, 1991).

$$E\left[\left(Y_n^i\right)'\right]=E\left[\left(Y_n^i\right)\right]'\ E\left[\left(I_n^i\right)'\right]=E\left[\left(I_n^i\right)\right]\ E\left[\left(NI_n^i\right)'\right]=E\left[\left(NI_n^i\right)\right] \quad (5.108)$$

A sufficiently small change of δ in base-stock level $\left(s^i\right)$ will not change outstanding orders, on-hand inventory, and net inventory by more than δ, for $i=\{1,2,...,m\}$. The result in (5.108) follows proposition 2.

■

6. SIMULATION OPTIMIZATION FRAMEWORK USING IPA

In the previous chapter we derived the first order equations that would be used in gradient estimation. All the equations derived in chapter 5 are used to obtain the best found base-stock level in the current chapter using the simulation optimization. In this chapter we will discuss the simulation optimization framework using Infinitesimal Perturbation Analysis (IPA) used to obtain the best found base-stock levels. The simulation optimization is carried out in conjunction of *ARENA* (simulation software), Visual Basic, and Xpress (optimization software). In the following sub-sections will review the simulation optimization framework using IPA, role of *ARENA* in IPA, role of Visual Basic in IPA, role of Xpress in IPA, and initial results.

6.1 Simulation Optimization Framework for IPA

As mentioned earlier the simulation optimization framework uses a combination of *ARENA*, Visual Basic (VB), and Xpress. *ARENA* is used to update the equations (on-hand inventory, outstanding orders etc.), first-order equations, and service level equations. VB is used for the feasible direction algorithm and line search. A modified Zoutendijk's feasible direction algorithm is used, and a golden section algorithm is used for line search. The feasible direction algorithm and the line search will be discussed in detail in subsequent sub-sections. Figure 6.1 show the flowchart of the simulation based Inventory optimization framework (SIO) for IPA.

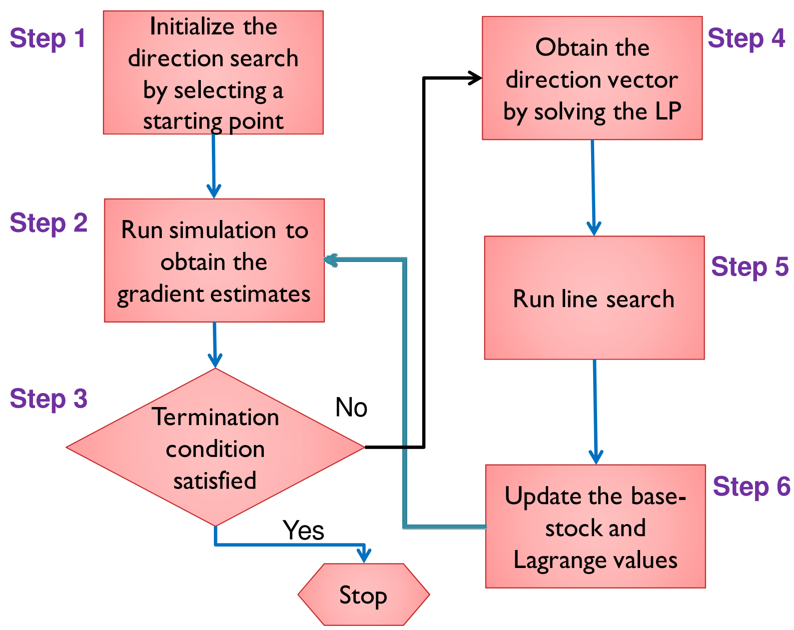

Figure 6.1 Flowchart of SOF for IPA

From the flowchart shown in figure 6.1 we can observe six primary steps that the simulation optimization framework takes to obtain the best found base-stock levels. Each step is discussed briefly below:

- *Step 1*: The feasible direction search is initialized by providing a starting point. The starting point mentioned in both *ARENA* and also in VB. The starting points are the base-stock levels and Lagrange multipliers, i.e. in case of a three-echelon model they would be $s^0, s^1, s^2,$ and s^3.
- *Step 2:* Once the starting points are assigned the update equations, first-order equations, and the service level equations in *ARENA* are updated, i.e. the simulation is run for a pre-specified number of periods.
- *Step 3:* Based on the simulation run earlier the service levels for the base-stock levels are obtained. Theses service levels are used in termination

condition to determine if the best found answer is achieved. Usually the process will not stop without a single cycle is finished (i.e. step 1 - step 6).

- *Step 4:* If the termination condition is not satisfied the process continues to step 4. A linear program (LP) is solved to obtain the direction vectors. LP is written in Xpress. A class is written in VB which runs the Xpress and returns the values back to VB. This process is part of the Zoutendijk's feasible direction algorithm
- *Step 5:* Once the direction is known, the line search (golden section algorithm) determines distance that needs to be travelled in the direction. Several iterations of line search are allowed to occur. The line search takes several iterations to determine the distance.
- *Step 6:* Based on the direction, and line search new values of base-stock and Lagrange multipliers are determined. These new values are used in simulation.

The process continues until the termination condition is satisfied.

6.2 Role of *ARENA* in IPA Framework

As stated earlier, *ARENA* is used for updating the on-hand inventory equations, outstanding order equations, net inventory equations, first order equations and service level equations every period. Figure 6.2 show the block diagram of simulation in *ARENA* within IPA framework, the block diagram show the various activities that are performed every period.

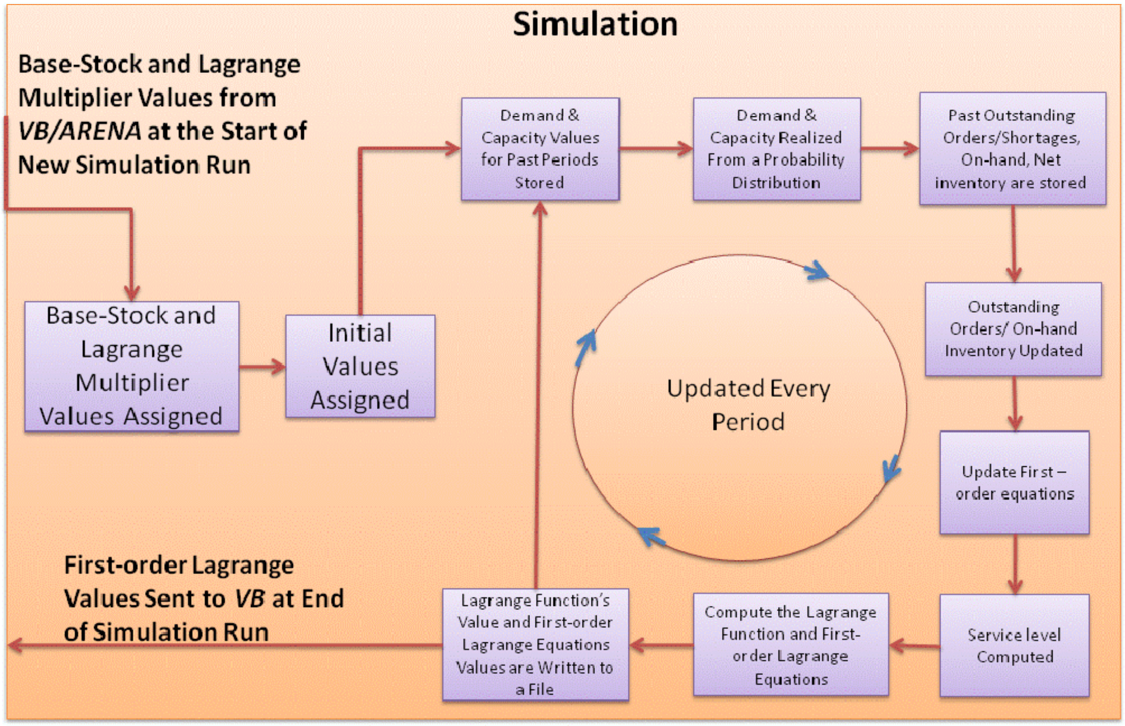

Figure 6.2 Block Diagram of Simulation in *ARENA* within IPA framework

The block diagram shown in figure 6.2 is almost similar to the simulation in *ARENA* with *OptQuest* framework. The base-stock and the Lagrange multiplier values are initially obtained from VB and *ARENA*. Initial values which are used in the update equations, first order equations, service level equations are assigned. Once the initial values are assigned, the simulation runs for a pre-defined number of periods.

The following activities take place each period:

- Demand and capacity values of the past are stored in *ARENA* as past variables
- Demand and capacity values are realized from a probability distribution
- Past outstanding orders/shortages, on-hand inventory, net inventory equations are stored in *ARENA* as past variables

- Update equations are updated, i.e. outstanding orders and on-hand inventory equations
- Based on the updated equations (i.e. outstanding orders, on-hand inventory etc.) the first-order equations are determined
- The service levels for each node are computed
- Lagrange function and first order Lagrange function (i.e. differentiation of Lagrange function with-respect-to base-stock levels and Lagrange multipliers) are computed
- These values are stored in a text file or an Excel sheet

The process continuous till the pre-defined number of periods is completed. At the end of the simulation run the estimates of the derivatives, i.e. Lagrange function and the first-order Lagrange equations are sent to VB. A snapshot of *ARENA* is shown in figure 6.3. In Figure 6.3 we can see three rows of blocks which are used to assign base-stock values, assign initial values, store demand and capacity values from past periods, and assign new demand and capacity values to variables in current period based of a probability distribution. The white colored block with a little arrow mark symbol represents a sub-system which further consists of several blocks.

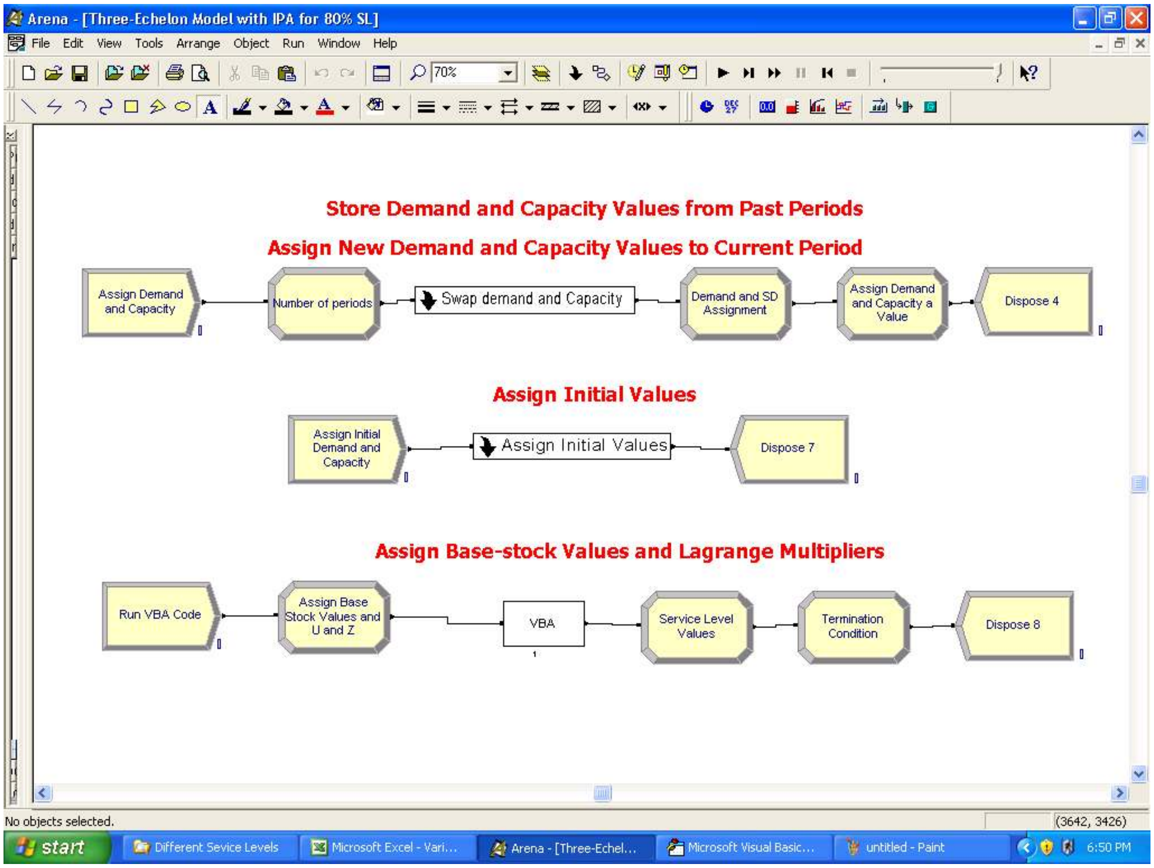

Figure 6.3 Snapshot of Simulation in *ARENA*

6.2 Role of VB in IPA Framework

As stated in the earlier sub-section, VB plays an important role in the simulation optimization framework using IPA. Figure 6.4 explicitly shows how the VB block is connected with the other blocks used in the IPA framework. The information regarding the update equations, first order equations, service-level of all nodes, and Lagrange function is sent to the VB block from *ARENA*. The code for the feasible direction algorithm is written in VB, i.e. a modified Zoutendijk's feasible direction algorithm along with the line search which uses a Golden Section Algorithm (GSA). A linear program (LP) is written in Xpress (which is discussed in the next sub-section) for obtaining the direction vectors. The base-stock values are updated in VB and sent to *ARENA*, which is used to update the equations (on-hand inventory, outstanding orders etc.), and first order equations etc.

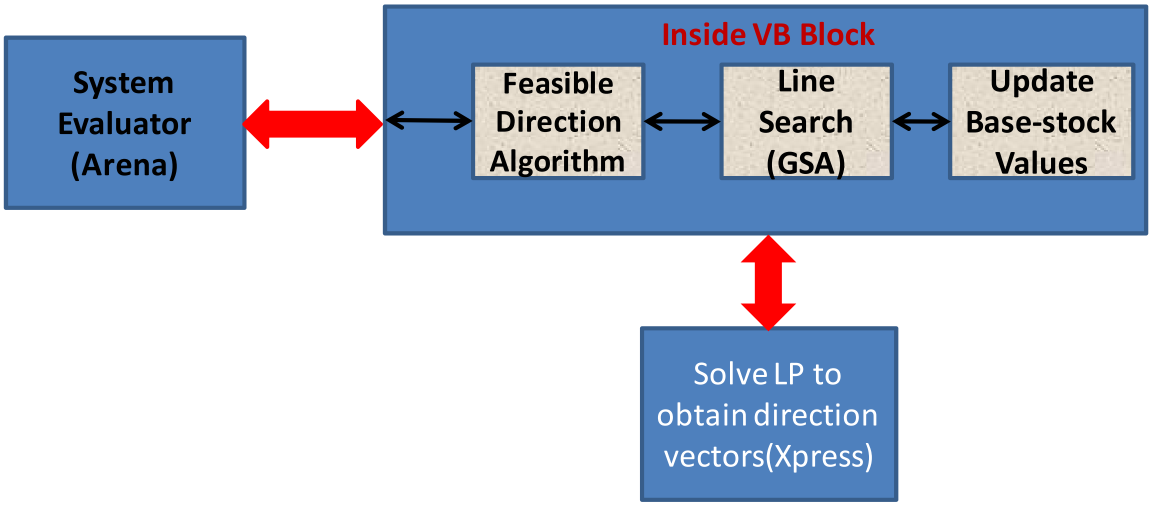

Figure 6.4 VB and Adjoining Blocks

6.2.1 Modified Zoutendijk's Feasible Direction Algorithm

In this sub-section we will look initially at the modified Zoutendijk's feasible-direction algorithm and will also visit the original Zoutendijk's feasible direction algorithm. The feasible direction method is modified since the Lagrange function, equation (5.23) is minimized instead of directly using the objective function, equation (3.27). Since the original problem converts to an unconstrained one, the limits for the line search are subjective. A three-echelon problem is considered to describe the steps of the algorithm.

Steps in the modified Zoutendijk's feasible direction algorithm:

Initialization Step: Choose a starting point, i.e. a base-stock level, and Lagrange multipliers for each node. In a three-echelon model we have four nodes, so we initialize four base-stock levels, and Lagrange multipliers, $(\vec{s}_t, \vec{u}_t) = (s^0, s^1, s^2, s^3; u^0, u^1, u^2, u^3)$. Let $t = 1$ and start with step 1

Step 1: Solve the following problem

$$\begin{aligned} &\text{Minimize} && Z \\ &s.t && \vec{\nabla}L(\vec{s}_t, \vec{u}_t)d \; - Z \leq 0; \\ & && -1 \leq d_j \leq 1; \quad d_j \geq 0, where\ j \in \{1, 2, ...8\} \end{aligned} \tag{6.1}$$

$\vec{\nabla}L(\vec{s}_t, \vec{u}_t)$ is the gradient of the Lagrange function (the new objective function) at $(\vec{s}_t, \vec{u}_t)$, i.e. base-stock level and Lagrange multiplier. The values of $\vec{\nabla}L(\vec{s}_t, \vec{u}_t)$ are obtained from simulation and IPA. Whereas d_t is an improving feasible direction which is found by solving the linear program (discussed in next section 6.3), $j \in \{1, 2, ...8\}$ since we have four base-stock levels and four Lagrange multipliers, one for each node. let (Z_t, d_t) be the best found solution. If $Z_t = 0$, stop; $(\vec{s}_t, \vec{u}_t)$ is a Fritz John point (Bazaraa et al., 2004). If $Z_t < 0$, go to step 2.

Step 2: λ_t be the best found solution to the following line search problem:

$$\begin{aligned} &\text{Minimize} && f\left[(\vec{s}_t, \vec{u}_t) + \lambda d_t\right] \\ &s.t && \lambda min \leq \lambda \leq \lambda max \end{aligned} \tag{6.2}$$

Where $\lambda min, \lambda max =$ are arbitrary values, usually between (-0.5, 0.5) and (-2, 2). The value is picked depending on how far the best found answer might be from the starting point. Let $(\vec{s}_{t+1}, \vec{u}_{t+1}) = \left[(\vec{s}_t, \vec{u}_t) + \lambda_t d_t\right]$ replace t with $t + 1$, and return to step 1.

Steps in the Zoutendijk's feasible direction algorithm:

Consider a problem to minimize *f(x)* shown in equation (6.3) subject to $g_i\left(x\right)\le 0$ for $i=1,...,m$. Let *x* be a feasible solution.

Initialization Step: Choose a starting point, x_1 such that $g_i\left(x\right)\le 0$. Let *k* = 1 and start with step 1. *t* stands for transpose here, but it does not indicate the iteration.

Step 1: Let $I=\left\{i: g_i\left(x_k\right)=0\right\}$ and solve the following problem

$$
\begin{aligned}
&\text{Minimize} && Z \\
&s.t && \nabla f(x_k)^t d - Z \le 0; \\
& && \nabla g_i(x_k)^t d - Z \le 0 \quad for\, i\in I \\
& && -1\le d_j \le 1; \quad d_j \ge 0, where\ j\in\{1,2,...,n\}
\end{aligned}
\tag{6.3}
$$

Let $(Z_{k,}\, d_k)$ be an best found solution. If $Z_k=0$, stop; x_k is a Fritz John point (Bazaraa et al., 2004). If $Z_k<0$, go to step 2.

Step 2: λ_t be the best found solution to the following line search (discussed next) problem:

$$
\begin{aligned}
&\text{Minimize} && f\left[x_k+\lambda d_k\right] \\
&s.t && 0\le\lambda\le\lambda max
\end{aligned}
\tag{6.4}
$$

Where $\lambda max = \sup\left\{\lambda : g_i\left(x_k+\lambda d_k\right)\le 0 \text{ for } i=1,...,m\right\}$ Let $x_{k+1}=\left[x_k+\lambda_k d_k\right]$ replace *t* with *t +1*, and return to step 1.

6.2.2 Golden Section Algorithm (Line Search)

This sub-section describes the summary of golden section algorithm, a method of minimizing a quasi convex function over an interval. Let us assume the interval is $\left[\lambda min_1, \lambda max_1\right]$. From equation (6.2) we have:

Minimize $f\left[(\vec{s}_t, \vec{u}_t) + \lambda d_t\right]$

s.t $\lambda min \le \lambda \le \lambda max$

Initialization Step: A length is chosen between $\left[\lambda min_1, \lambda max_1\right]$, let the length of uncertainty be equal to $l > 0$, and let $\tau_1 = \lambda min_1 + (1-\alpha)(\lambda max_1 - \lambda min_1)$ and $\omega_1 = \lambda min_1 + (\alpha)(\lambda max_1 - \lambda min_1)$, where $\alpha = 0.618$. Evaluate $f\left[\tau_1\right]$ and $f\left[\omega_1\right]$, let $k = 1$, and go to step 1

Step 1: If $\lambda max_k - \lambda min_k < l$, stop; the best found solution lies in the interval $\left[\lambda min_k, \lambda max_k\right]$. Otherwise, if $f\left[\tau_k\right] > f\left[\omega_k\right]$, go to step 2, and if $f\left[\tau_k\right] \le f\left[\omega_k\right]$, go to step 3.

Step 2: Let $\lambda min_{k+1} = \tau_k$ and $\lambda max_{k+1} = \lambda max_k$. Furthermore, let $\tau_{k+1} = \omega_k$, and let $\omega_{k+1} = \lambda min_{k+1} + \alpha(\lambda max_{k+1} - \lambda min_{k+1})$. Evaluate $f\left[\omega_{k+1}\right]$ and go to step 4 .

Step 3: Let $\lambda min_{k+1} = \lambda min_k$ and $\lambda max_{k+1} = \omega_k$. Furthermore, let $\omega_{k+1} = \tau_k$, and let $\tau_{k+1} = \lambda min_{k+1} + (1-\alpha)(\lambda max_{k+1} - \lambda min_{k+1})$. Evaluate $f\left[\tau_{k+1}\right]$ and go to step 4.

Step 4: Replace k by $k+1$ and go to step 1.

The value of $f\left[\tau_k\right]$ and $f\left[\omega_k\right]$ are estimated using simulation in *ARENA*.

6.2.3 Termination Condition

The termination condition is straight forward, and can be easily understood from the section which discusses the Zoutendijk's feasible direction algorithm. As per the Zoutendijk's feasible direction algorithm, the termination occurs when the value of $Z \le 0$, but it is quite difficult to achieve, or to put it in other words it would require several thousand additional simulations runs to obtain the value of $Z \le 0$ in the feasible direction algorithm. So the termination condition that is used in this research is slightly relaxed, and the simulation stops when $Z \leq 5$ occurs. It is also ensured at the same time that the desired service level is achieved for each node. If the desired service level is not achieved, then we allow the algorithm to keep running for another complete cycle (a complete cycle would be to determine the direction vectors, run the feasible direction algorithm, run the line search, and obtain the new base-stock values) and so on till the desired service level is achieved. The termination condition would be sensitive to the structure of Lagrange function used (discussed in chapter 5). This kind of termination condition is used when a squared & multiplied Lagrange function is used, where the multiplier (Q) is equal to a constant value of 10.

6.2.4 Pseudo Code in VB

The pseudo code would help in understanding the basic flow of the algorithm (Zoutendijk's feasible direction, and line search).

/ Declare Global Variables*

{

Declare Lagrange multipliers, and base-stock variables;

Declare Iteration, count variables, termination condition variables;

Declare variables for line search;

Declare direction vectors (obtains the values from LP);

}

/ Create a class for updating variables before the simulation starts (also known as RunBeginSimulation ( ) class in VB)*

Start the class

Declare variables that can make connection between *ARENA* and VB;

Declare initial values of base-stock, and Lagrange multipliers;

Declare initial variables for line search;

Declare Step Number = 0 (the entire cycle is divided into number of steps for programming convenience);

Send the initial values (Lagrange multipliers, base-stock values etc. from VB to *ARENA*;

End class

/ Create a class for updating variables when an entity passes through VB block in ARENA (also known as fire ( ) class in VB)*

Start the class

Declare variables that can make connection between *ARENA* and VB;

Declare variables for that are used for first-order Lagrange equations, (i.e. derivation of Lagrange function with-respect to all the decision variables);

Declare variables that perform read-write operation to a notepad file;

/ End of declarations*

************************Start Main Program**********************

Update the step number = step number + 1;

If (step number = 2 and $Z > 5$) Then

{

Declare variables that can make connection between *ARENA* and VB;

Obtain the value of first order estimates from simulation in *ARENA*;

Initialize the upper, lower bound, interval of uncertainty values used in line search;

reduce the interval of uncertainty by 10% every time a new cycle (Iteration) starts;

Continue to reduce for 20 Iterations, after which the interval of uncertainty is no longer reduced;

Write all the values of first-order Lagrange variables to a text file, each value is written to a separate text file;

*************Run LP from Xpress by Integrating Mosel*************

Call the class that Mosel module;

Call the supporting modules that are required for Mosel to run through VB;

************************End of LP*************************

Read the direction vectors from text file (obtained after the LP is solved);

Read the all direction vectors into variables in VB;

}

End If;

If (Step Number = = 2) Then

{

Declare variables that can make connection between *ARENA* and VB;

Compute the value of τ_k ;

Obtain the base-stocks values, and Lagrange multipliers from *ARENA*;

/ * *Update the base-stock variables based on the direction vectors and* τ_k

New base-stock value = old base-stock value + (direction vector for the base-stock * τ_k) ;

/ * *Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

End If

If (Step Number = = 3) Then

{

Declare variables that can make connection between *ARENA* and VB;

Obtain the value of $f\left(\tau_k\right)$ from *ARENA*;

Compute the value of ω_k ;

Obtain the base-stocks values, and Lagrange multipliers from *ARENA*;

/ * *Update the base-stock variables based on the direction vectors and* ω_k

New base-stock value = old base-stock value + (direction vector for the base-stock * ω_k) ;

/ * *Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

If (Step Number = = 4) Then

{

Declare variables that can make connection between *ARENA* and VB;

Obtain the value of $f\left(\omega_k\right)$ from *ARENA*;

Obtain the base-stocks values, and Lagrange multipliers from *ARENA*;

If (Step 2 described in section 6.2.2 (line search) is selected) Then

{

/ * *Update the base-stock variables based on the direction vectors and* τ_k

New base-stock value = old base-stock value + (direction vector for the base-stock * τ_k) ;

/ * *Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

End if;

If (Step 3 described in section 6.2.2 (line search) is selected) Then

{

/ * *Update the base-stock variables based on the direction vectors and* ω_k

New base-stock value = old base-stock value + (direction vector for the base-stock * ω_k) ;

/ * *Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

End if;

}

End if

If (Step Number > 4 and $\lambda max_k - \lambda min_k >$ interval of uncertainty) Then

{

Declare variables that can make connection between *ARENA* and VB;

Obtain/ update the value of $f(\tau_k)$, and $f(\omega_k)$ from *ARENA*;

Obtain the base-stocks values, and Lagrange multipliers from *ARENA*;

If (Step 2 described in section 6.2.2 (line search) is selected) Then

{

/ * *Update the base-stock variables based on the direction vectors and* τ_k

New base-stock value = old base-stock value + (direction vector for the base-stock * τ_k) ;

/ * *Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

End if;

If (Step 3 described in section 6.2.2 (line search) is selected) Then

{

/ * *Update the base-stock variables based on the direction vectors and* ω_k

New base-stock value = old base-stock value + (direction vector for the base-stock * ω_k) ;

*/ * Repeat the process for base-stock values, and Lagrange multipliers of all the nodes*

}

}

End if

*/ * Get the service levels from ARENA*

Declare variables that can make connection between *ARENA* and VB;

Obtain the service level values for all the nodes;

*/ * Increase the number of simulation runs (periods) to obtain an estimate from simulation as the service level of one of the nodes reaches the desired service level*

*/ * Check the termination condition*

If ($Z <= 5$) Then

{

Check the service level variables for desired service-level;

*/ * If satisfied terminate the process*

Else

*/ * Assign an arbitrary large Z value, and override the terminating condition that uses Z value during the next cycle (iteration), and terminate only on the basis of service level constraint from subsequent iterations*

Z = large value;

}

End if

If ($\lambda max_k - \lambda min_k \leq$ interval of uncertainty) Then

{

Set all the line variables to initial values, step value;

Declare variables that can make connection between *ARENA* and VB;

Obtain the values of λmax_k, and λmin_k from *ARENA*

Optimal length $= \left(\lambda max_k + \lambda min_k\right) / 2$;

Update all the base-stock variables and Lagrange multipliers based on the best found solution obtained from line search;

}

End of Class

6.3 Role of Xpress in IPA Framework

As mentioned in earlier sub-sections the role of Xpress is to solve a linear program (LP). The linear program is used to obtain the direction vectors by solving equation (6.3) (a part of feasible direction method). The solution from the LP will provide the search direction for the base-stock levels and Lagrange multipliers. The pseudo code for solving the LP in Xpress is shown in sub-section below.

6.3.1 Pseudo Code in Xpress

Start-Model

/* Define Parameters

Get first order Lagrange Variables (Lagrange with-respect-to base-stock level and multipliers) from the files;

Define the path for storing the direction variables (obtained after solving LP);

/* End-Parameters

/* Decelerations

Define direction variables;

Define objective function variables (*Z*, first order Lagrange variables);

/* End-Decelerations

/* Read all the eight files file's (since there are eight first-order variables, one for each variable)

Fopen (first order variable)

Read (into variable declared in Xpress)

Fclose (command)

/* continue the process for all eight files

/* Define the objective

Objective : = (First order variable 1 * direction vector 1+…….+ First order variable 8 * Direction vector) < = *Z*

/* Define constraints

Direction vector 1 > = -1

Direction vector 1 < = 1

/* Continue for all the eight direction vectors

Minimize (*Z*);

/* Write the direction vectors to a text file

Fopen (file name, path)

write (to text file)

Fclose (command)

```
/* Continue for all eight directions

End Model;
```

7. COMPUTATIONAL RESULTS FOR THREE-ECHELON ASSEMBLY SYSTEM

In this chapter we will discuss the results from two simulation-optimization frameworks discussed in the earlier chapters, although more emphasis will be given to results obtained using the IPA framework. A comparison between the results obtained from the two frameworks is discussed. The computational results and initial implications from seven different cases using the IPA framework are discussed in this chapter. All the results in this chapter pertain to the three-echelon assembly system shown in chapter 3.

7.1 Results from *OptQuest* Framework

A total of 14 scenarios are run using *OptQuest* as a part of initial study. The scenarios are shown in table 7.1. The first six scenarios have deterministic capacity and random demand, whereas the last eight scenarios have random demand and capacity. A two-period supply lead time and a two period lead time between the nodes is assumed. The normal distribution is used for random demand and capacity.

Table 7.1 Demand and Capacity Values for *OptQuest* Framework

	Intermediate product, Demand 1	Final Product, Demand 2	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 1	Norm(8,1)	Norm(4,1)	18	18	18	9
Scenario 2	Norm(8,2)	Norm(4,2)	18	18	18	9
Scenario 3	Norm(8,3)	Norm(4,3)	18	18	18	9
Scenario 4	Norm(8,4)	Norm(4,4)	18	18	18	9
Scenario 5	Norm(8,1)	Norm(4,1)	14	14	14	6
Scenario 6	Norm(8,2)	Norm(4,2)	14	14	14	6
Scenario 7	Norm(8,1)	Norm(4,1)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(9,1)
Scenario 8	Norm(8,1)	Norm(4,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(9,3)
Scenario 9	Norm(8,2)	Norm(4,2)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(9,1)
Scenario 10	Norm(8,2)	Norm(4,2)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(9,3)
Scenario 11	Norm(8,1)	Norm(4,1)	Norm(14,1)	Norm(14,1)	Norm(14,1)	Norm(6,1)
Scenario 12	Norm(8,2)	Norm(4,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(6,2)
Scenario 13	Norm(12,1)	Norm(8,1)	Norm(24,1)	Norm(24,1)	Norm(24,1)	Norm(12,1)
Scenario 14	Norm(12,2)	Norm(8,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(12,2)

**Norm (Value1, Value2) represents Normal Distribution (Mean, Standard Deviation)*

Each scenario listed in table 7.1 is run for a length of 20 periods, for a customer service level of 90% for each node. The best found base-stock levels, objective function value and the utilization of capacity are listed in table 7.2.

Table 7.2 Best Found Base-stock Levels for *OptQuest* Framework

Optimal Base-Stock Levels**						Utilization of capacity			
	S0	S1	S2	S3	Total Cost	Node 0	Node 1	Node 2	Node 3
Scenario 1	11.3	26.97	26.4	26.23	**90.9**	0.45	0.67	0.67	0.67
Scenario 2	13.53	29.54	29.39	29.02	**101.48**	0.45	0.67	0.67	0.67
Scenario 3	14.87	34.82	32.45	31.29	**113.43**	0.45	0.67	0.67	0.67
Scenario 4	23.69	36.92	35.71	35.53	**131.85**	0.45	0.67	0.67	0.67
Scenario 5	10.41	28.63	25.88	25.47	**90.39**	0.67	0.86	0.86	0.86
Scenario 6	10.79	26.65	28.84	26.87	**93.15**	0.67	0.86	0.86	0.86
Scenario 7	14.82	28.48	28.5	28.79	**100.59**	0.45	0.67	0.67	0.67
Scenario 8	14.27	29.5	30	30	**103.77**	0.45	0.67	0.67	0.67
Scenario 9	10.88	27.04	26.85	26.56	**91.33**	0.45	0.67	0.67	0.67
Scenario 10	14.96	31	31	31.38	**108.34**	0.45	0.67	0.67	0.67
Scenario 11	12.27	26.33	26.84	26.96	**92.4**	0.67	0.86	0.86	0.86
Scenario 12	13.74	35.27	35.78	35.54	**120.33**	0.67	0.86	0.86	0.86
Scenario 13	18.74	42	42.58	44.51	**147.83**	0.67	0.85	0.85	0.85
Scenario 14	22.13	46.2	45.16	49.83	**163.32**	0.67	0.85	0.85	0.85

*** S0, S1, S2, and S3 represent* $s^0, s^1, s^2,$ and s^3 *of three-echelon respectively*

From the best found base-stock levels in table 7.2 we can see that as the standard deviation of demand increases the best found base-stock level also increases at each node, i.e. for scenarios 1- 4 we can see that as the standard deviation of demand increases the base-stock level at each node also increases, this is shown in figure 7.1. We can also observe from table 7.2 that as the capacity becomes tighter the base-stock values at each node also increase, and this is shown in figure 7.2

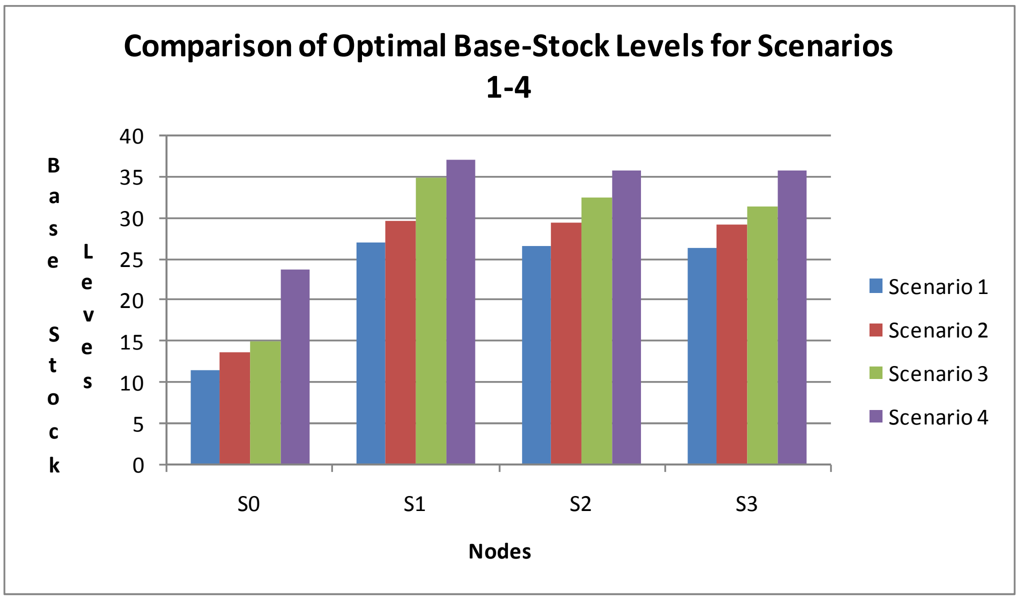

Figure 7.1: Best Found Base-stock Levels for Scenarios 1-4

From figure 7.1 we can observe that there is a clear increase in the base-stock levels as the standard deviation of demand increases for scenarios 1 to 4 from Normal (4, 1) to Normal (4, 4) for final product demand, and Normal (8, 1) to Normal (8,4) for intermediate product demand.

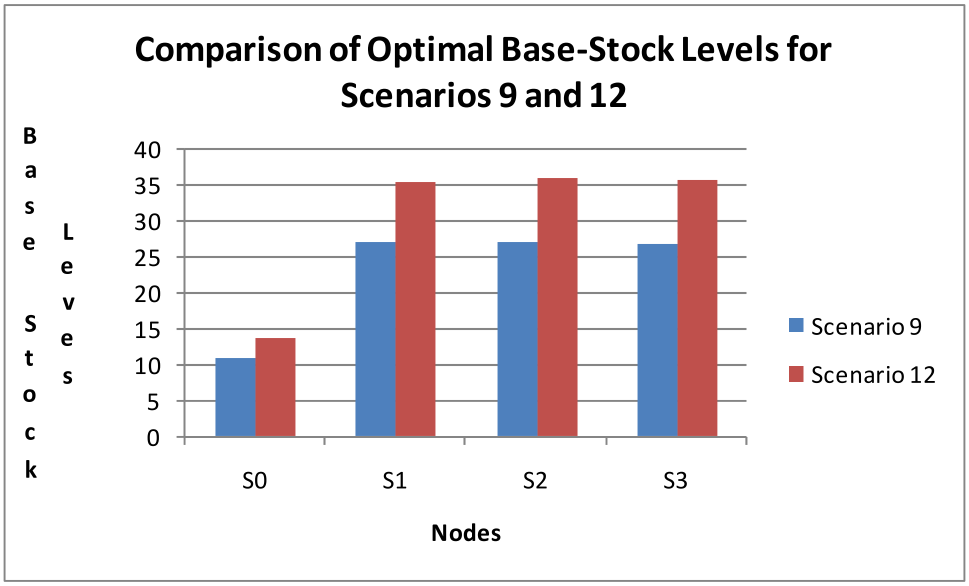

Figure 7.2 Best Found Base-stock Levels for Scenarios 9 and 12

In figure 7.2 a comparison is made between scenarios 9 and 12. We can observe an increase in base-stock values across the nodes as the capacity gets tighter, i.e. a reduction of 4 units from Normal (18, 1) to Normal (14, 2) for node 3 to 1, and Normal (9, 1) to Normal (6, 2).

7.2 Results from IPA Framework

A total of 14 scenarios, similar to *OptQuest* framework are run in IPA framework as a part of initial study. The scenarios are shown in table 7.3, they are same as table 7.1 but the scenario numbers are changed. The first six scenarios have deterministic capacity and random demand, whereas the last eight scenarios have random demand and capacity. A two period supply lead time and a two period lead time between the nodes is assumed. Normal distribution is used for random demand and capacity.

Table 7.3 Demand and Capacity Values for IPA Framework

	Intermediate product, Demand 1	Final Product, Demand 2	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 15	Norm(8,1)	Norm(4,1)	18	18	18	9
Scenario 16	Norm(8,2)	Norm(4,2)	18	18	18	9
Scenario 17	Norm(8,3)	Norm(4,3)	18	18	18	9
Scenario 18	Norm(8,4)	Norm(4,4)	18	18	18	9
Scenario 19	Norm(8,1)	Norm(4,1)	14	14	14	6
Scenario 20	Norm(8,2)	Norm(4,2)	14	14	14	6
Scenario 21	Norm(8,1)	Norm(4,1)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(9,1)
Scenario 22	Norm(8,1)	Norm(4,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(9,3)
Scenario 23	Norm(8,2)	Norm(4,2)	Norm(18,1)	Norm(18,1)	Norm(18,1)	Norm(9,1)
Scenario 24	Norm(8,2)	Norm(4,2)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(9,3)
Scenario 25	Norm(8,1)	Norm(4,1)	Norm(14,1)	Norm(14,1)	Norm(14,1)	Norm(6,1)
Scenario 26	Norm(8,2)	Norm(4,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(6,2)
Scenario 27	Norm(12,1)	Norm(8,1)	Norm(24,1)	Norm(24,1)	Norm(24,1)	Norm(12,1)
Scenario 28	Norm(12,2)	Norm(8,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(12,2)

The best found base-stock levels for scenarios 15-28 obtained using the IPA framework, and the utilization for each node is shown in table 7.4. A service level of 90% is used for each node. The simulation is run for 500 periods, i.e. to obtain one estimate, and the simulation is run for 3000 periods when the service level estimate for one of the nodes is close to the required service level of 90%. This process carries on till the termination condition is satisfied.

Table 7.4 Best Found Base-stock Levels for IPA Framework

Optimal Base-Stock Levels						Utilization of capacity			
	S0	S1	S2	S3	Objective Function	Node 0	Node 1	Node 2	Node 3
Scenario 15	10.79	26.79	26.79	26.79	**91.16**	0.45	0.67	0.67	0.67
Scenario 16	13.37	29.3	29.3	29.3	**101.27**	0.45	0.67	0.67	0.67
Scenario 17	15	31	31	31	**108**	0.45	0.67	0.67	0.67
Scenario 18	18.23	34.24	34.24	34.24	**120.95**	0.45	0.67	0.67	0.67
Scenario 19	10.23	26.97	26.97	26.97	**91.14**	0.67	0.86	0.86	0.86
Scenario 20	11.3	27.31	27.31	27.31	**93.23**	0.67	0.86	0.86	0.86
Scenario 21	13.07	29.02	29.02	29.02	**100.13**	0.45	0.67	0.67	0.67
Scenario 22	13.57	29.57	29.57	29.57	**102.28**	0.45	0.67	0.67	0.67
Scenario 23	10.94	26.95	26.95	26.95	**91.79**	0.45	0.67	0.67	0.67
Scenario 24	13	29	29	29	**100**	0.45	0.67	0.67	0.67
Scenario 25	11.06	26.86	26.86	26.86	**91.64**	0.67	0.86	0.86	0.86
Scenario 26	14.95	31	31	31	**107.95**	0.67	0.86	0.86	0.86
Scenario 27	18.89	42.89	42.89	42.89	**147.56**	0.67	0.85	0.85	0.85
Scenario 28	21.75	44.75	44.75	44.75	**156**	0.67	0.85	0.85	0.85

The results shown in table 7.4 have a similar trend to that of table 7.2. We can see that as the standard deviation of demand increases the best found base-stock level also increases at each node, i.e. for scenarios 15 and 18 we can see that as the standard deviation of demand increases the base-stock level at each node also increases, this is shown in figure 7.3. We can also observe from table 7.2 that as the capacity becomes tighter the base-stock values at each node also increase, and this is shown in figure 7.4

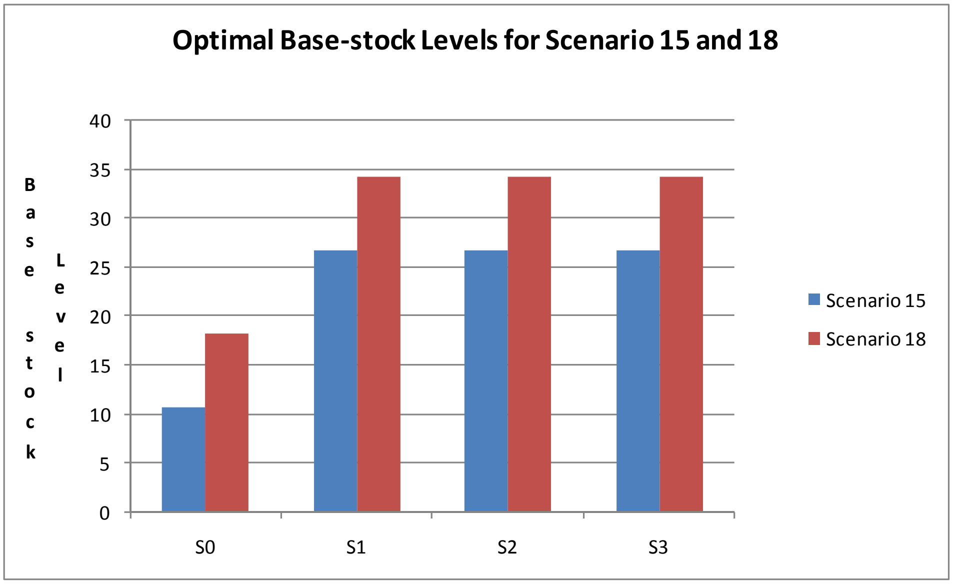

Figure 7.3: Best Found Base-stock Levels for Scenarios 15 and 18

From figure 7.3 we can observe that there is a clear increase in the base-stock levels for scenario 18 compared to 15. As the standard deviation of demand increases form from Normal (4, 1) to Normal (4, 4) for final product demand, and Normal (8, 1) to Normal (8,4) for intermediate product demand we can observe an increase in the base-stock level.

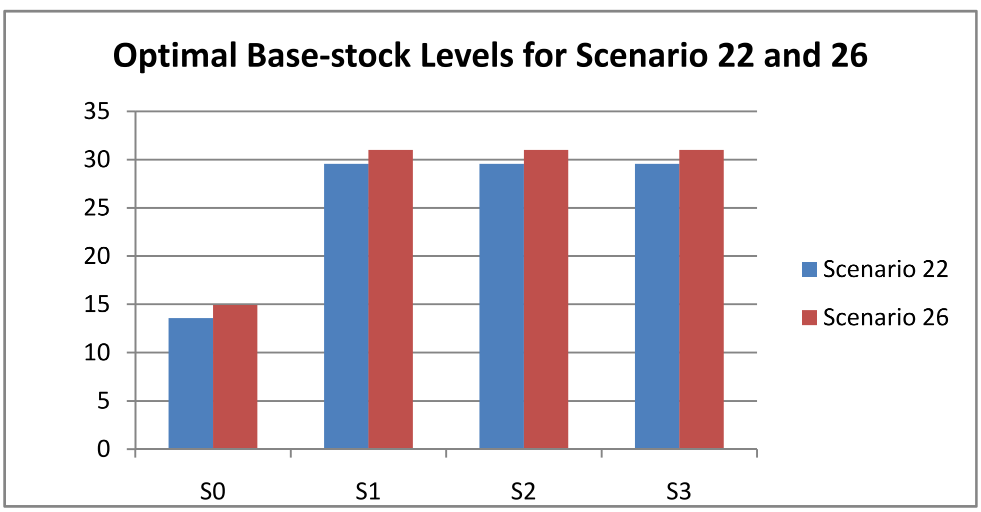

Figure 7.4: Best Found Base-stock Levels for Scenarios 22 and 26

In figure 7.4 a comparison is made between scenarios 22 and 26. We can observe an increase in base-stock values across the nodes as the capacity gets tighter, i.e. a reduction of 4 units from Normal (18, 1) to Normal (14, 2) for node 3 to 1, and Normal (9, 1) to Normal (6, 2).

7.3 Comparison between Results from IPA and *OptQuest*

A comparison of performance between the two frameworks used, i.e. *OptQuest* and IPA is studied. The study involves the comparison of the quality of solution and time taken to obtain the best found answer. Table 7.5 and 7.6 show the results from 14 different cases.

Table 7.5: Comparison of Computational Time and Base-Stock Values Using Two Frameworks (Case's 1- 8)

	Case 1		Case 2		Case 3		Case 4	
Final Demand	Norm(4,1)		Norm(4,2)		Norm(4,3)		Norm(4,4)	
Intermediate Demand	Norm(8,1)		Norm(8,2)		Norm(8,3)		Norm(8,4)	
Stage 0 Capacity	9		9		9		9	
Stage 1 Capacity	18		18		18		18	
Stage 2 Capacity	18		18		18		18	
Stage 3 Capacity	18		18		18		18	
Time (OptQuest)	6:16		7:40:00		7:41		8:01:00	
Time (IPA)	0:15		0:33		1:34		2:40	
	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA
S0	11.3	10.79	13.53	13.37	14.87	15	23.69	18.23
S1	26.97	26.79	29.54	29.3	34.82	31	36.92	34.24
S2	26.4	26.79	29.39	29.3	32.45	31	35.71	34.24
S3	26.23	26.79	29.02	29.3	31.29	31	35.53	34.24
Total Cost	90.9	91.16	101.48	101.27	113.43	108	131.85	120.95

	Case 5		Case 6		Case 7		Case 8	
Final Demand	Norm(4,1)		Norm(4,1)		Norm(4,2)		Norm(4,2)	
Intermediate Demand	Norm(8,1)		Norm(8,1)		Norm(8,2)		Norm(8,2)	
Stage 0 Capacity	Norm(9,1)		Norm(9,3)		Norm(9,1)		Norm(9,3)	
Stage 1 Capacity	Norm(18,1)		Norm(18,3)		Norm(18,1)		Norm(18,3)	
Stage 2 Capacity	Norm(18,1)		Norm(18,3)		Norm(18,1)		Norm(18,3)	
Stage 3 Capacity	Norm(18,1)		Norm(18,3)		Norm(18,1)		Norm(18,3)	
Time (OptQuest)	7:30		15:57		10:11		10:07	
Time (IPA)	0:14		0:26		0:38		0:35	
	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA
S0	10.41	10.23	10.79	11.3	14.82	13.07	14.27	13.57
S1	28.63	26.97	26.65	27.31	28.48	29.02	29.5	29.57
S2	25.88	26.97	28.84	27.31	28.5	29.02	30	29.57
S3	25.47	26.97	26.87	27.31	28.79	29.02	30	29.57
Total Cost	90.39	91.14	93.15	93.23	100.59	100.13	103.77	102.28

Table 7.6: Comparison of Computational Time and Base-Stock Values Using Two Frameworks (Case's 9-14)

	Case 9		Case 10		Case 11		Case 12	
Final Demand	Norm(4,1)		Norm(4,2)		Norm(4,1)		Norm(4,2)	
Intermediate Demand	Norm(8,1)		Norm(8,2)		Norm(8,1)		Norm(8,2)	
Stage 0 Capacity	6		6		Norm(6,1)		Norm(6,2)	
Stage 1 Capacity	14		14		Norm(14,1)		Norm(14,2)	
Stage 2 Capacity	14		14		Norm(14,1)		Norm(14,2)	
Stage 3 Capacity	14		14		Norm(14,1)		Norm(14,2)	
Time (OptQuest)	5:24:00		3:38		7:05:00		7:36	
Time (IPA)	0:15		0:41		0:15		1:30	
	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA	OptQuest	IPA
S0	10.88	10.94	14.96	13	12.27	11.06	13.74	14.95
S1	27.04	26.95	31	29	26.33	26.86	35.27	31
S2	26.85	26.95	31	29	26.84	26.86	35.78	31
S3	26.56	26.95	31.38	29	26.96	26.86	35.54	31
Total Cost	91.33	91.79	108.34	100	92.4	91.64	120.33	107.95

	Case 13		Case 14	
Final Demand	Norm(8,1)		Norm(8,2)	
Intermediate Demand	Norm(12,1)		Norm(12,2)	
Stage 0 Capacity	Norm(12,1)		Norm(12,2)	
Stage 1 Capacity	Norm(24,1)		Norm(24,2)	
Stage 2 Capacity	Norm(24,1)		Norm(24,2)	
Stage 3 Capacity	Norm(24,1)		Norm(24,2)	
Time (OptQuest)	6:50		8:08	
Time (IPA)	0:11		0:52	
	OptQuest	IPA	OptQuest	IPA
S0	18.74	18.89	22.13	21.75
S1	42	42.89	46.2	44.75
S2	42.58	42.89	45.16	44.75
S3	44.51	42.89	49.83	44.75
Total Cost	147.83	147.56	163.32	156

From tables 7.5 and 7.6 we can see that in almost all the cases the quality of the solution obtained using IPA framework is much better than the *OptQuest* framework. In case 1, 5, and 6 the *OptQuest* solution is fractionally better than the IPA, but if we ignore the fraction and round the number to the nearest integer the solution of IPA and *OptQuest* are same. All the other cases except 1, 5, and 6 IPA performs a lot better when compared to the *OptQuest*. If we compare the amount of time taken by the two frameworks to get to the solution, IPA outperforms *OptQuest* by several minutes. IPA in not only provides a better solution quality, but also provides the answer quicker.

The starting points for the IPA framework are twice the mean demand for each node. The base-stock level for each node start from twice mean demand since the lead time considered for all the case's is two periods. The capacity for each node is made large enough so that the node is not constrained from capacity. In case of *OptQuest* the lower bound for the base-stock level is twice the mean demand for the node, and the upper bound considered is 95th percentile of the demand.

7.4 Detailed Computational Results

In this sub-section we will discuss the computational results of three-echelon assembly system in much detail. The study involves four cases, the inferences from each case is discussed. The hypothesis statements made in this sub-section are not statistically proven; they are only based of numerical observations.

7.4.1 Case 1

Description of Case 1: Study the impact on total cost & safety-stock cost under varying demand CV (coefficient of variation) and service levels.

Hypothesis 1a: As the demand CV increases the base-stock level and safety stock for each node, and hence the total cost increases.

Hypothesis 1b: As the service level for each node increases the base-stock level, and safety-stock for the respective nodes increases, and hence the total cost increases.

The mean demand, and capacity values used for computation is shown in table 7.7. Normal distribution is used for demand and capacity. Mean capacity is kept constant, and the standard deviation of the capacity depends on the CV.

Table 7.7: Mean Demand and Capacity Values for Case 1

	Capacity	Mean Demand
Node 3	15	12
Node 2	15	12
Node 1	15	12
Node 0	10	4

The impact of demand CV and service level on the value of total cost & safety-stock costs for a one, two, three period lead time is shown in table 7.8, 7.9 and 7.10.

Table 7.8: Impact of Demand CV on Total Cost and Safety Stock Costs (1 Period Supply/Manufacturing Lead Time)

Impact of Demand CV on Total Costs				Impact of Demand CV on Safety Stock Costs			
	Demand CV				Demand CV		
Service Level	0.2	0.6	1	Service Level	0.2	0.6	1
95%	51.79 (100%)	85.28 (100%)	127.52 (100%)	95%	11.79 (100%)	45.28 (100%)	87.52 (100%)
90%	51.32 (99%)	72.16 (84.61)	123.72 (97%)	90%	11.32 (96%)	32.16 (71.02)	83.72 (95%)
85%	49.08 (94%)	69.48 (81%)	109.2 (85%)	85%	9.08 (77%)	29.48 (65%)	69.2 (79%)
80%	47.28 (91%)	68.16 (79%)	105.2 (82%)	80%	7.28 (61%)	28.16 (62%)	65.2 (74%)

Table 7.9: Impact of Demand CV on Total Cost and Safety Stock Costs (2 Period Supply/Manufacturing Lead Time)

Impact of Demand CV on Total Costs				Impact of Demand CV on Safety-Stock Costs			
	Demand CV				Demand CV		
Service Level	0.2	0.6	1	Service Level	0.2	0.6	1
95%	97.04 (100%)	140.8 (100%)	208.84 (100%)	95%	17.04 (100%)	60.8 (100%)	128.84 (100%)
90%	94.04 (96%)	131.64 (93%)	200.6 (96%)	90%	14.04 (82%)	51.64 (84%)	120.6 (93%)
85%	92.52 (95%)	126.12 (89%)	152.44 (73%)	85%	12.52 (73%)	46.12 (75%)	72.44 (56%)
80%	90.84 (93%)	122.36 (86%)	145.6 (69%)	80%	10.84 (63%)	42.36 (69%)	65.6 (50%)

Table 7.10: Impact of Demand CV on Total Cost and Safety Stock Costs (3 Period Supply/Manufacturing Lead Time)

Impact of Demand CV on Total Costs				Impact of Demand CV on Safety Stock Costs			
	Demand CV				Demand CV		
Service Level	0.2	0.6	1	Service Level	0.2	0.6	1
95%	140.24 (100%)	192.72 (100%)	265.04 (100%)	95%	20.24 (100%)	72.72 (100%)	145.04 (100%)
90%	138.02 (98%)	190.56 (98%)	258.12 (97%)	90%	18.02 (89%)	70.56 (97%)	138.12 (95%)
85%	135.11 (96%)	180 (93%)	250.64 (94%)	85%	15.11 (74%)	60 (82%)	130.64 (90%)
80%	134.08 (95%)	176.68 (91%)	240.8 (90%)	80%	14.08 (69%)	56.68 (77%)	120.8 (83%)

By observing table 7.8, 7.9, and 7.10 the following inferences can be made:

- As the demand CV increases from 0.2 to 1 the total cost increases greatly
- As the service level increases the total cost within a specific demand CV, and across demand CV's increases
- The rate of change (percentage) increase in total cost is greater when the demand CV = 1 than compared to demand CV = 0.2
- Similar statements also hold true for the safety-stock costs

From the above statements and the computational results shown in tables provided above we can state that the Hypothesis 1a and 1b hold true (based of only observations).

7.4.1 Case 2

Description of Case 2: Study the impact on total cost & safety-stock cost under varying capacity CV (coefficient of variation) and service levels.

Hypothesis 2a: As the capacity CV increases the base-stock level and safety stock for each node, and hence the total cost increases.

Hypothesis 2b: As the service level for each node increases the base-stock level, and safety-stock for the respective nodes increases, and hence the total cost increases.

The demand, and mean capacity values used for computation is shown in table 7.11. Normal distribution is used for demand. The mean capacity is kept constant, and normal distribution is used for capacity. The standard deviation of the capacity depends on the coefficient of variation (CV).

Table 7.11: Demand and Mean Capacity Values for Case 2

	Mean Capacity	Demand
Node 3	15	Norm (12,1)
Node 2	15	Norm (12,1)
Node 1	15	Norm (12,1)
Node 0	10	Norm (4,1)

The impact of capacity CV and service level on the value of total cost & safety-stock costs for a one, two, three period lead time is shown in table 7.12, 7.13 and 7.14.

Table 7.12: Impact of Capacity CV on Total Cost and Safety Stock Costs (1 period lead time)

Impact of Capacity CV on Total Costs				Impact of Capacity CV on Safety Stock Costs			
	Capacity CV				Capacity CV		
Service Level	0.1	0.3	0.6	Service Level	0.1	0.3	0.6
95%	49.68 (100%)	76.16 (100%)	119.2 (100%)	95%	9.68 (100%)	39.72 (100%)	79.2 (100%)
90%	48.72 (98%)	74.21 (98%)	113.2 (94%)	90%	8.72 (90%)	38.88 (97%)	73.2 (92%)
85%	48.24 (97%)	71.20 (91%)	108.8 (91%)	85%	8.24 (85%)	33.32 (83%)	68.8 (86%)
80%	46.84 (94%)	68.4 (86%)	96.96 (81%)	80%	6.84 (70%)	28.4 (71%)	56.96 (71%)

Table 7.13: Impact of Capacity CV on Total Cost and Safety Stock Costs (2 period lead time)

Impact of Capacity CV on Total Costs				Impact of Capacity CV on Safety Stock Costs			
	Capacity CV			Capacity CV			
Service Level	0.1	0.3	0.6	Service Level	0.1	0.3	0.6
95%	93.64 (100%)	117.12 (100%)	168.16 (100%)	95%	13.64 (100%)	37.12 (100%)	88.16 (100%)
90%	91.96 (98%)	114.28 (97%)	162.28 (96%)	90%	11.96 (87%)	34.28 (92%)	82.28 (93%)
85%	91.48 (97%)	111.2 (94%)	153.2 (91%)	85%	11.48 (84%)	31.2 (84%)	73.2 (83%)
80%	88.4 (94%)	109.36 (93%)	147.2 (87%)	80%	8.4 (61%)	29.36 (79%)	67.2 (76%)

Table 7.14: Impact of Capacity CV on Total Cost and Safety Stock Costs (3 period lead time)

Impact of Capacity CV on Total Costs				Impact of Capacity CV on Safety Stock Costs			
	Capacity CV				Capacity CV		
Service Level	0.1	0.3	0.6	Service Level	0.1	0.3	0.6
95%	148.64 (100%)	160.16 (100%)	235.56 (100%)	95%	28.64 (100%)	40.16 (100%)	115.56 (100%)
90%	136.2 (91%)	155.28 (96%)	226.16 (96%)	90%	16.2 (56%)	35.28 (87%)	106.16 (91%)
85%	133.04 (89%)	153.12 (95%)	218.16 (92%)	85%	13.04 (45%)	33.12 (82%)	98.16 (84%)
80%	131.2 (88%)	151.2 (94%)	209.6 (88%)	80%	11.2 (39%)	31.2 (77%)	89.6 (77%)

The impact of capacity CV and service level on the value of total cost & safety-stock costs is shown in tables 7.12, 7.13, and 7.14. By observing tables the following can be inferred:

- As the capacity CV increases from 0.1 to 0.3 we can see an increase in the total cost
- As the service level increases the total cost within a specific capacity CV, and across capacity CV's increases

- The rate of change (percentage) increase in total cost is greater when the capacity CV = 0.3 than compared to demand CV = 0.1
- Similar statements also hold true for the safety-stock costs

From the above statements and the computational results shown in tables above we can state that the Hypothesis 2a and 2b hold true.

7.4.3 Case 3

Description of Case 3: Study the impact on safety-stock cost under varying lead time (supply/manufacturing lead times), demand CV, and service levels.

Hypothesis 3a: As the lead time increases the safety stock cost for each node also increases, and hence the total safety stock cost also increases.

Hypothesis 3b: As the service levels and demand CV increases the safety stock costs increase significantly.

Table 7.15, and 7.16 shows the impact of service level, and lead time on the safety-stock costs for two different demand CV's. The demand follows a normal distribution, whereas capacity is deterministic. The demand and capacity values used for computation are same as table 7.7.

Table 7.15 Impact of Lead Time on Safety Stock Costs for Demand CV = 0.2

Effect of Lead Time on Safety Stock Costs for Different Service Levels, Demand CV = 0.2				
Lead Time (# of Periods)	Service Level			
	80%	85%	90%	95%
1	7.28	9.08	11.32	11.79
2	10.84	12.52	14.04	17.04
3	14.08	15.11	18.02	20.24

Table 7.16 Impact of Lead Time on Safety Stock Costs for Demand CV = 1

Effect of Lead Time on Safety Stock Costs for Different Service Levels, Demand CV = 1				
Lead Times (# of Periods)	Service Level			
	80%	85%	90%	95%
1	65.2	69.2	83.72	87.52
2	65.6	72.44	120.6	128.84
3	120.8	130.64	138.12	145.04

The following inferences can be made from tables 7.15 and 7.16:

- The safety-stock costs increases as the lead time increases irrespective of the service level considered
- As the service level increases the rate of increase in safety stock costs within a specific service level for two different lead time's increases greatly. For instance in table 7.16 the difference between safety-stock costs for one period and two period lead time within 85 % service level is 3.24, whereas difference between safety-stock costs for one period and two period lead time within 95 % service level is 41.32.
- The rate of change for safety stock cost increases as the demand CV increases. It can be seen by observing the last column in table 7.15 and 7.16.

From the above statements and the computational results in tables provided above we can state that the Hypothesis 3a and 3b hold true.

7.4.4 Case 4

Description of Case 4: Study the impact on safety-stock cost under varying lead time (supply/manufacturing lead times), capacity CV, and service levels.

Hypothesis 4: As the service levels and capacity CV increases the safety stock costs increases significantly.

Table 7.17, and 7.18 show the impact of service level, and lead time on the safety-stock costs for two different capacity CV's. The capacity and demand follow a normal distribution. The demand and mean capacity values used for computation are same as table 7.11.

Table 7.17 Impact of Lead Time on Safety Stock Costs for Capacity CV = 0.3

Effect of Lead Time on Safety Stock Costs for Different Service Levels, Capacity CV = 0.3				
Lead Time (# of Periods)	Service Level			
	80%	85%	90%	95%
1	28.4	31.2	34.21	36.6
2	29.36	31.2	34.28	37.12
3	31.2	33.12	35.28	40.16

Table 7.18 Impact of Lead Time on Safety Stock Costs for Capacity CV = 0.6

Effect of Lead Time on Safety Stock Costs for Different Service Levels, Capacity CV = 0.6				
Lead Time (# of Periods)	Service Level			
	80%	85%	90%	95%
1	56.96	68.8	73.2	79.2
2	67.2	73.2	82.28	88.16
3	89.6	98.16	106.16	115.56

Similar inferences as that of case 3 can also be made from the tables 7.17 and 7.18 respectively.

The following inferences can be made from table 7.17 and 7.18:

- The rate of change for safety stock cost increases as the capacity CV increases. It can be seen by observing the last column in table 7.15 and 7.16.
- The safety-stock costs increases as the lead time increases irrespective of the service level considered
- As the service level increases the rate of increase in safety stock costs within a specific service level for two different lead time’s increases greatly.

From the above statements and the computational results in tables provided above we can state that the Hypothesis 4 holds true.

Additional computational results can be found in appendix A5.

8. INVENTORY ALLOCATION POLICIES

In this chapter we discuss some alternative inventory allocation policies to use in the multi-echelon inventory systems. Specifically the allocation policies studied here demonstrate how inventory optimization in a supply chain requires attention to base-stock levels as well as the allocation policy. The update equations for four different inventory allocations polices used in the three and five-echelon assembly models is discussed, followed by numerical results for the three and five-echelon assembly models. Statistical inferences and hypothesis testing is done to derive some implications based on the five-echelon assembly model.

The inventory allocation policy decides how the inventory is allocated between different sources of demand. For instance in a three-echelon assembly system it would be between downstream demand and the intermediate demand. We discussed several inventory allocation policies in chapter 2 which have been used in the recent literature. For the multi-echelon problem we use four inventory allocation policies. Three of the four inventory allocation polices are used in practice (Cachon and Lariviere, 1999).

The update equations for the multi-echelon problems discussed in earlier chapters are based on the inventory allocation policy where the priority is given to the intermediate product demand, in other words lexicographic allocation with priority to intermediate product demand. Let us discuss the allocation policy currently used, and a few other inventory allocation policies that are used in this chapter in detail. The following are the inventory allocation policies:

- Lexicographic Allocation (Priority to Intermediate Product Demand)

- Lexicographic Allocation (Priority to Downstream Demand)
- Predetermined Proportional Allocation
- Proportional Allocation

The four inventory allocation policies are used in the three-echelon assembly system, five-echelon assembly system, and the larger networks. This chapter discusses results for the three and the five-echelon assembly system, whereas the larger multi-echelon networks are discussed in chapter 9.

8.1 Description of Inventory Allocation Policies

8.1.1 Lexicographic Allocation (Priority to Intermediate Product Demand) (LAPI)

The Lexicographic allocation policy ranks the sources of demand in some manner independent of their order size, and based on the ranking the each source receives the amount of supplier capacity. In the three-echelon problem we consider that there are two sources of demand, 1) intermediate product demand and 2) downstream demand, and we assign the priority to the intermediate product demand. The available supplier (upstream) capacity is first used to satisfy the entire intermediate product's demand, and the remaining capacity (if any) is used to serve downstream product demand. The update equations developed for the three-echelon assembly system reflect this allocation policy. This allocation policy is used for the problems pertaining to three and five-echelon assembly models. By assigning priority to intermediate products the following are observed:

- Decrease in the total cost when the upstream nodes have a greater proportion of demand when compared to the downstream nodes.

- Decrease in total safety-stock cost when the upstream nodes have a greater proportion of demand when compared to the downstream nodes.

Detailed insights are provided later in the chapter.

8.1.2 Lexicographic Allocation (Priority to Downstream Product Demand) (LAPD)

In this lexicographic allocation policy the priority is assigned to the downstream product demand. The available supplier (upstream) capacity is used to satisfy the entire downstream product's demand, and the remaining capacity is used to serve intermediate product demand. The update equations need to be modified to reflect this change in the allocation policy. This allocation policy is used for the problems pertaining to three and five-echelon assembly models. By assigning priority to downstream products the following are observed:

- Decrease in the total cost when the downstream nodes have a greater proportion of demand when compared to the upstream nodes.
- Decrease in total safety-stock cost when the downstream nodes have a greater proportion of demand when compared to the upstream nodes.
- Decrease in the total cost and safety-stock cost when there is a high demand CV for all the demands (intermediate and final products) and high capacity CV.

Detailed insights are provided later in the chapter.

8.1.3 Predetermined Proportional Allocation (PPA)

In a predetermined proportional allocation if the supplier's capacity is less than the sum of all the demand, then the supplier could use a predetermined proportional allocation mechanism. Each source of demand receives not more than the pre-determined ratio of the available inventory on-hand. In the three-echelon problem we consider that there are two sources of demands, 1) intermediate product demand and 2) downstream demand, and we assign a predetermined ratio, say for instance 0.5, which means each source of demand obtains 50% of the available on-hand inventory. We use this allocation policy in the three and five-echelon problems and gain additional insights. By using a PPA inventory policy we observe the following:

- In most cases using this inventory allocation policy leads to higher total cost compared to when other (LAPI, LAPD, and PA) inventory allocation policies are used.

Detailed insights are provided later in the chapter.

8.1.4 Proportional Allocation (PA)

In a proportional allocation, when upstream capacity is insufficient, each sources of demand receives an equal proportion of his current order. Unlike in the PPA where the ratio is fixed for each period here the ratio dynamically changes according to the demand in each period. We use this allocation policy for the three and five-echelon problems and gain additional insights. By using proportional inventory allocation we observe:

- Decrease in the total cost and safety-stock cost when there is a high demand CV for all the demands (intermediate and final products).

Detailed insights are provided later in the chapter.

8.2 Inventory Allocation Policies for Three-echelon Assembly System

The update equations discussed in the earlier chapters for the three-echelon assembly system have been based of the default inventory allocation policy, i.e. the LAPI. Let us recall all the update equations for the outstanding orders and on-hand inventory using LAPI.

8.2.1 Lexicographic Allocation (priority to intermediate demand) (LAPI)

The update equations for outstanding orders at the beginning of period *n+1* for node 3 & 2, 1, and 0 are as stated in equations (8.1) – (8.3), whereas the update equations for on-hand inventory at the beginning of period *n* for node 2 & 3, 1, and 0 are as stated in equation (8.4) – (8.6).

$$Y_{n+1}^{i} = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 - DS_n^1 - \eta_n^i\right) \text{ where } i \in 2,3 \tag{8.1}$$

$$Y_{n+1}^{1} = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0, \;\; s^3 - Y_{n-2}^3 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - . \\ . - \xi_{n-1}^2, \; s^2 - Y_{n-2}^2 - \xi_{n-l^i+1}^1 - .. - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - .. - \xi_{n-1}^2, \; \eta_n^1 \end{pmatrix} \tag{8.2}$$

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-l^i}^1 - \ldots - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - \ldots - \xi_{n-1}^2, \eta_n^0\right) \tag{8.3}$$

$$I_n^i = \max\left(0 \,,\, s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^1\right), \text{where } i \in 2,3 \tag{8.4}$$

$$I_n^i = \max\left(0 \,,\, s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{8.5}$$

$$I_n^i = \max\left(0 \,,\, s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2\right), \text{where } i \in 0 \tag{8.6}$$

The outstanding orders in (8.2) are determined on the basis of the realized capacity of item 1, available inventory of item 2, and on-hand inventory of item 3. Similarly in equation 4 the outstanding orders are determined on the basis of realized capacity of item 0, and on-hand inventory of item 1. This allocation policy would be used when demand from component parts is given more importance compared to the final product. The possible reasons for using this allocation might be that component parts sales are more lucrative than final product sales, or if the component consumer market significantly bigger than the final product.

8.2.2 Lexicographic Allocation (priority to downstream demand) (LAPD)

The update equations must be modified to reflect the LAPD inventory allocation policy. The equation for outstanding orders and on-hand inventory for node 0 and 1 shown in equations (8.7) and (8.9) are modified, respectively. The new update equation that represents the outstanding orders due to intermediate demand (Y_{n+1}^{1In}) is stated as equation (8.8).

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - \ldots - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - \ldots - \xi_{n-1}^2, \eta_n^0\right) \tag{8.7}$$

$$Y_{n+1}^{1In} = Y_n^0 + \xi_n^1 - \min\left(Y_n^0 + \xi_n^1, s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - \ldots - \xi_{n-1}^1 - \xi_{n-l^i}^2 - \ldots - \xi_{n-1}^2\right) \tag{8.8}$$

$$I_n^i = \max\left(0, s^i - Y_{n-l^i}^i - Y_{n-l^i}^{iIn} - \xi_{n-1}^1 - \ldots - \xi_{n-l^i}^1 - \xi_{n-1}^2 - \ldots - \xi_{n-l^i}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{8.9}$$

Equation (8.8) i.e. Y_{n+1}^{1In} is used to reflect the priority to downstream demand, i.e. the demand for downstream product is realized initially followed by the intermediate product. The allocation policy prioritizes the final product compared to the component parts. A competitive market with stringent delivery commitments for the final product is usually the reason for implementation of this allocation policy.

8.2.3 Predetermined Proportional Allocation (PPA)

In a predetermined proportional allocation policy each source of demand cannot receive more than the predetermined proportion of the available inventory. All the equations for the outstanding orders and on-hand inventory remain the same as in section 8.2.1, except outstanding order for node 0 and on-hand inventory for node 1. In equation (8.10) and (8.11) γ denotes the predetermined proportion, the value of γ is always between 0 and 1, Y_{n+1}^{1In} represents the outstanding orders due to intermediate product.

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2,\ \gamma * \left(s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - ... - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - ... - \xi_{n-1}^2\right),\ \eta_n^0\right) \tag{8.10}$$

$$Y_{n+1}^{1In} = Y_n^0 + \xi_n^1 - \min\left(Y_n^0 + \xi_n^1,\ (1-\gamma) * \left(s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - ... - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - ... - \xi_{n-1}^2\right)\right) \tag{8.11}$$

$$I_n^i = \max\left(0\ ,\ s^i - Y_{n-l^i}^i - Y_{n-l^i}^{iIn} - \xi_{n-1}^1 - ... - \xi_{n-l^i}^1 - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{8.12}$$

8.2.4 Proportional Allocation (PA)

In proportional allocation each source of demand receives an equal proportion of the order. In this policy, the proportion $\left(\xi_n^i \big/ \left(\xi_n^1 + \xi_n^2\right) \right)$ represents the proportion corresponding to demand source *i.* All the update equations stay the same except the outstanding order equation, and the on-hand inventory for node 0 and 1 respectively.

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2,\ \left(\xi_n^2 \big/ \left(\xi_n^1 + \xi_n^2\right) \right) * \left(s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - ... - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - ... - \xi_{n-1}^2\right),\ \eta_n^0 \right) \tag{8.13}$$

$$Y_{n+1}^{1In} = Y_n^0 + \xi_n^1 - \min\left(Y_n^0 + \xi_n^1,\ \left[1 - \left(\xi_n^2 \big/ \left(\xi_n^1 + \xi_n^2\right) \right) \right] * \left(s^1 - Y_{n-2}^1 - \xi_{n-l^i+1}^1 - ... - \xi_{n-1}^1 - \xi_{n-l^i+1}^2 - ... - \xi_{n-1}^2\right) \right) \tag{8.14}$$

$$I_n^i = \max\left(0\ ,\ s^i - Y_{n-l^i}^i - Y_{n-l^i}^{iIn} - \xi_{n-1}^1 - ... - \xi_{n-l^i}^1 - \xi_{n-1}^2 - ... - \xi_{n-l^i}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{8.15}$$

8.3 Computational Results for Three-echelon Inventory Allocation Policy

The results for the four allocation policies are based on the following input parameters shown in table 1. A two period supply and manufacturing lead time is used between the nodes. γ of 0.5 is used for PPA. Based on the input values in table 8.1 best found base-stock levels were found for each allocation policy. There are two cases used for PPA and PA policy which are listed below:

- *Case A:* The PA and PPA policy is used in every period, i.e. even when there is sufficient on-hand inventory to satisfy all the sources of demand.
- *Case B:* The PA and PPA policy is employed only for periods when there is insufficient on-hand inventory, whereas in the event of sufficient on-hand inventory all demands for the period are met.

Table 8.2 provides the total system cost, which is a direct measure of the best found base-stock level. These costs are the result of using best found base-stock levels, as determined by the IPA based search.

Table 8.1: Demand and Capacity Values Used in Simulation*

	Intermediate product, Demand 1	Final Product, Demand 2	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 1	Norm(8,1)	Norm(4,1)	18	18	18	8
Scenario 2	Norm(4,1)	Norm(8,1)	18	18	18	15
Scenario 3	Norm(8,4)	Norm(4,4)	18	18	18	8
Scenario 4	Norm(4,4)	Norm(8,4)	18	18	18	15
Scenario 5	Norm(8,2)	Norm(4,2)	14	14	14	6
Scenario 6	Norm(4,2)	Norm(8,2)	14	14	14	12
Scenario 7	Norm(8,1)	Norm(4,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(8,3)
Scenario 8	Norm(4,1)	Norm(8,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(15,3)
Scenario 9	Norm(8,2)	Norm(4,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(6,2)
Scenario 10	Norm(4,2)	Norm(8,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(12,2)
Scenario 11	Norm(12,2)	Norm(8,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(12,2)
Scenario 12	Norm(8,2)	Norm(12,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(18,2)

*Norm (8, 1) refers to Normal (mean, standard deviation)

Table 8.2: Total System Cost for Four Allocation Policies Based on Case A

Total System Cost				
	LAPI	LAPD	PPA	PA
Scenario 1	91.04	92.4	100.12	92.04
Scenario 2	97.83	97.53	105.56	106.14
Scenario 3	127.36	127.36	132.44	130.03
Scenario 4	144.88	141.16	144.04	142.6
Scenario 5	106.46	106.55	115.03	104.95
Scenario 6	129.11	126.44	128.44	127
Scenario 7	92.52	92.56	101.6	92.88
Scenario 8	121.43	120	120	119.28
Scenario 9	112	110.56	118.88	114.12
Scenario 10	129	128.8	130	128.88
Scenario 11	159.27	160.8	168.68	162.23
Scenario 12	183.32	181.68	184.16	182.84

The results for PPA and PA policy in table 8.2, and 8.3 are based on case A defined earlier at the start of 8.3, and figures 8.1, 8.2, and 8.3 are based on table 8.3. The following observations can be made from table 8.2, 8.3:

- We can observe that scenarios with increased mean demand for intermediate product (i.e., scenario's 1, 3, 5... 11), a lower total system cost and total safety-stock cost is obtained when LAPI policy is employed in most scenarios when compared to other inventory allocation policies. Figure 8.1 depicts this result.
- Similarly we can observe that scenarios with increased mean demand for final product (i.e., scenario's 2, 4, 6... 12), a lower total system cost and total safety-stock cost is obtained when LAPD policy is employed when compared to other inventory allocation policies. Figure 8.2 depicts this result.
- From table 3 we can observe that for all scenarios where the mean demand for the final product is higher than the mean demand for the intermediate product (even

numbered scenarios), a significant increase in the total safety stock when compared to other scenarios, this holds true for all allocation policies studied in this chapter. When the downstream demand forms the significant portion of the total demand, we see an significant increase in the total safety-stock cost compared to vice-versa. This significant increase is due to the lead time, since it takes more periods to satisfy the final product demand when compared to intermediate product's demand. Figure 8.3 depicts this result.

- The total system cost does not behave rationally when a PA is used, i.e. the total system costs of PA are more often greater than LAPI and LAPD. This is because the proportional allocation is run every period despite in some periods the available on-hand inventory might be sufficient to satisfy both the intermediate and final product demand. This result shows that it is difficult to interpret the behavior of optimized inventory for poor allocation policies.
- When PPA is employed for all the scenarios we can observe an increased total system compared to all the other allocation polices. That would be because only a 50% of the on-hand inventory is available to satisfy intermediate and final product each period, despite the mean demand for intermediate product is double the final product in six scenarios and vice-versa.

Table 8.3: Total System Safety Stock Cost Based on Case A**

Total System Safety Stock Cost*				
	LAPI	**LAPD**	**PPA**	**PA**
Scenario 1	8.28	9.3	15.09	9.03
Scenario 2	7.13	6.83	15.67	16.03
Scenario 3	35.52	35.52	39.33	37.53
Scenario 4	48.66	45.87	48.03	46.95
Scenario 5	20.46	20.55	26.28	18.72
Scenario 6	36.84	34.83	36.33	35.25
Scenario 7	9.39	9.42	16.2	9.66
Scenario 8	31.08	30	30	29.46
Scenario 9	24	22.92	29.16	25.59
Scenario 10	36.75	36.6	37.5	36.66
Scenario 11	17.46	18.6	24.51	19.68
Scenario 12	35.49	34.26	36.12	35.13

**does not include safety stock cost of node 0

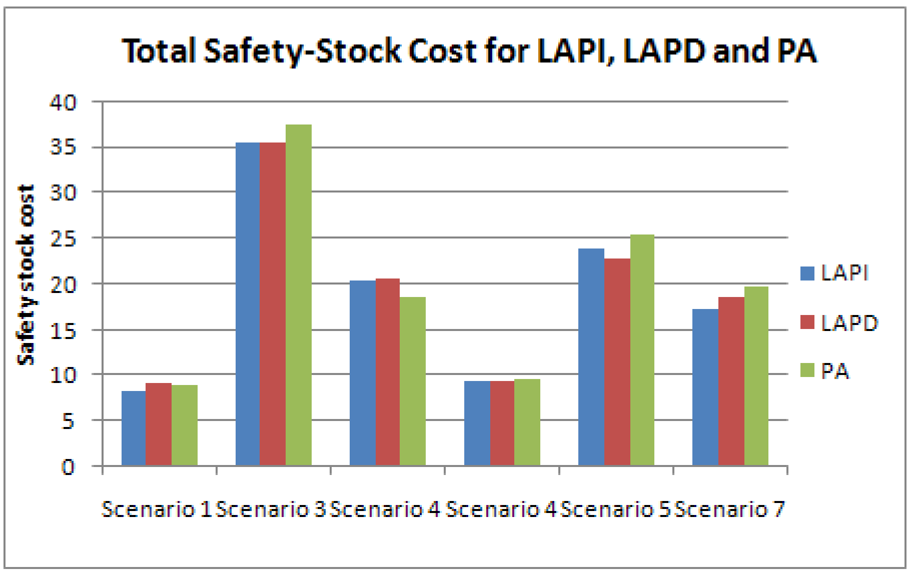

Figure 8.1: Scenarios with Higher Demand for Intermediate Product (Case A)

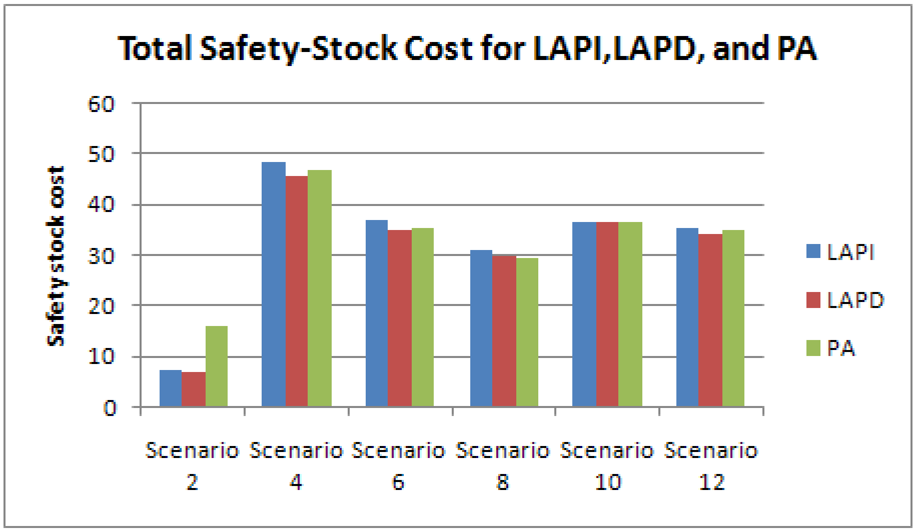

Figure 8.2: Scenarios with Higher Demand for Final Product (Case A)

Figure 8.3 shows a graph which compares the average safety-stock cost (averaged over all inventory allocation policies) for all the scenarios with increased intermediate product demand (odd numbered scenarios in table 8.2) with scenarios which have increased final product demand (even numbered scenarios in table 8.2).

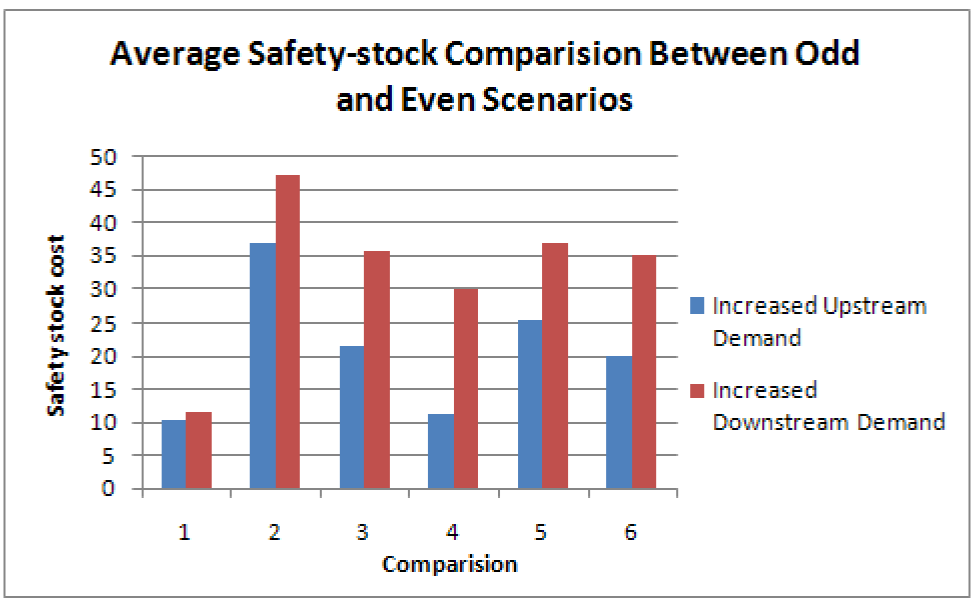

Figure 8.3: Comparison Between Even and Odd Scenarios (Case A)

Table 8.4: Total System Cost for Four Allocation Policies Based on Case B

Total System Cost				
	LAPI	LAPD	PPA	PA
Scenario 1	91.04	92.4	91.75	92.12
Scenario 2	97.83	97.53	99.4	98.84
Scenario 3	127.36	127.36	127.32	127.32
Scenario 4	144.88	141.16	135.1	135.88
Scenario 5	106.46	106.55	106.6	104.95
Scenario 6	129.11	126.44	115.04	115.2
Scenario 7	92.52	92.56	91.4	93.48
Scenario 8	121.43	120	102.56	101.48
Scenario 9	112	110.56	107.91	109.64
Scenario 10	129	128.8	121.64	120.4
Scenario 11	159.27	160.8	160.6	160.68
Scenario 12	183.32	181.68	168.56	168.72

Table 8.4 and 8.5 provides the total system cost, and safety-stock cost for case B, the values for LAPI and LAPD policy are essentially same. The figures 8.4 and 8.5 are based of table 8.5. The following observations can be made from table 8.4 and 8.5:

- We can observe that scenarios with increased mean demand for intermediate product (i.e., scenario's 1, 3, 5... 11), a lower total system cost and total safety-stock cost is obtained when LAPI policy is employed in three out of six scenarios. In the other three scenarios where there is an increased demand variance PA policy results in lower cost. Figure 8.4 depicts this result. Since PA policy has a chance of working better when there is insufficient on-hand inventory at the node by allocating the higher proportion of inventory to the source with high demand.
- Similarly we can observe that scenarios with increased mean demand for final product (i.e., scenario's 2, 4, 6... 12), a lower total system cost and total safety-stock cost is obtained when a PA policy is used as compared to other inventory allocation policies. Figure 8.5 depicts this result.

- The total system cost behaves rationally when a PA policy is used, i.e. the total system costs of PA are marginally equal or lower than LAPI and LAPD in most scenarios. This is because the proportional allocation is run only in periods when the available on-hand inventory is less than the demand.
- PA is most effective because it reduces the effective variance of supply for downstream nodes. Sometimes with LAPI, LAPD, the downstream nodes can receive very little when supply is short.

Table 8.5: Total System Safety Stock Cost Based on Case B**

Total System Safety Stock Cost*				
	LAPI	LAPD	PPA	PA
Scenario 1	8.28	9.3	8.82	9.09
Scenario 2	7.13	6.83	8.55	8.13
Scenario 3	35.52	35.52	35.49	35.49
Scenario 4	48.66	45.87	35.42	35.91
Scenario 5	20.46	20.55	19.95	18.72
Scenario 6	36.84	34.83	20.28	20.4
Scenario 7	9.39	9.42	8.55	10.11
Scenario 8	31.08	30	10.92	10.11
Scenario 9	24	22.92	20.94	22.23
Scenario 10	36.75	36.6	25.23	24.3
Scenario 11	17.46	18.6	18.45	18.51
Scenario 12	35.49	34.26	18.42	18.54

**does not include safety stock cost of node 0

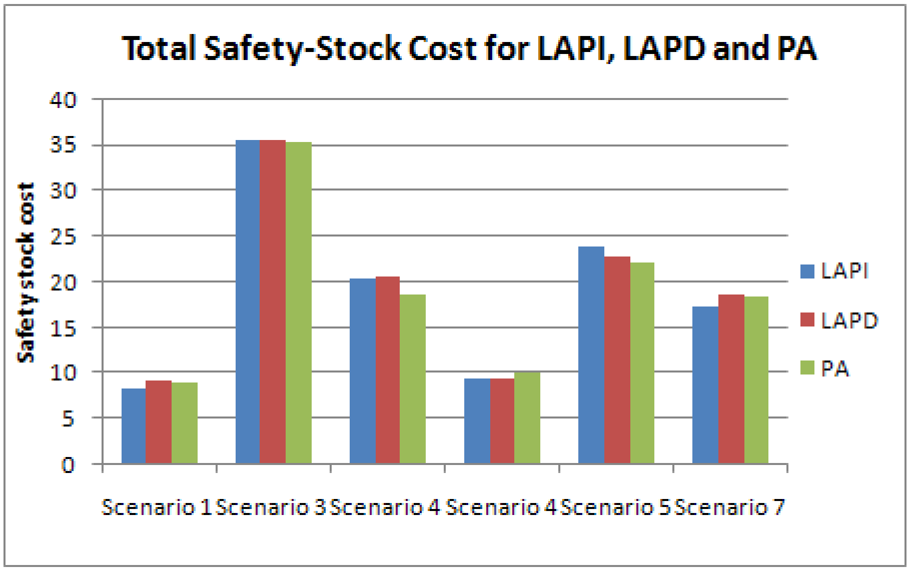

Figure 8.4: Scenarios with Higher Demand for Intermediate Product (Case B)

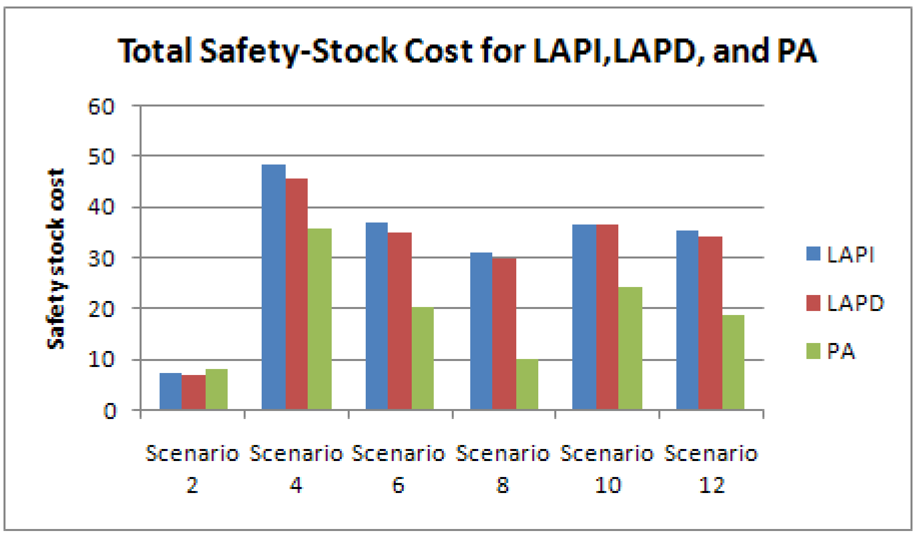

Figure 8.5: Scenarios with Higher Demand for Final Product (Case B)

Additional results for the inventory allocation of the three-echelon assembly system are provided in the appendix A6.

8.4 Inventory Allocation Policies for Five-echelon Assembly System

Let us consider a five-echelon assembly system as shown in figure 8.6, similar to the three-echelon assembly system the five-echelon assembly consists of an assembly process in the upstream portion, followed by a serial system, intermediate sources of demand, and a final product demand. Instead of two sources of demand, we now have four sources of demand.

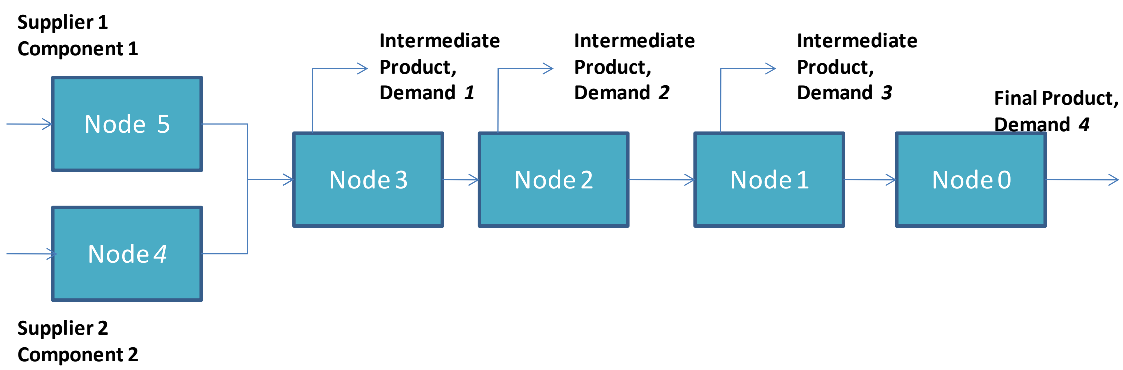

Figure 8.6: Five-echelon Assembly System

Three nodes (node 3, 2, and 1) in the five-echelon assembly model have multiple sources of demand. A decision on inventory allocation has to be made at nodes where more than one demand occurs. The decision is based on the four allocation policies that have been already discussed in the previous section. Ten different models, each model having a different set of inventory allocation policies are used for the numerical analysis. Table 8.6 shows the allocation policies used in nodes 3, 2, and 1 of the five-echelon assembly model. Before the numerical results are discussed, the update (on-hand inventory and outstanding order) equations for the first four models are described in the following sub-sections, and the update equations for the other models are provided in the appendix. A two period lead time is assumed for the equations.

Table 8.6: Inventory Allocation Polices Used in Models

Model #	Nodes		
	3	2	1
1	LAPI	LAPI	LAPI
2	LAPD	LAPD	LAPD
3	PPA	PPA	PPA
4	PA	PA	PA
5	PPA	LAPD	PA
6	LAPD	LAPI	PA
7	LAPD	LAPI	LAPI
8	LAPI	LAPD	LAPD
9	LAPD	PA	PA
10	PA	LAPI	LAPI

8.4.1Update Equations for Model #1 (LAPI)

Model 1 uses lexicographic allocation with priority to the intermediate product (local demand) to all the nodes with multiple sources of demand. The on-hand inventory equations for this model are similar to the *m*-echelon assembly model, are listed below:

$$I_n^i = \max\left[0, s^i - Y_{n-2}^i - \xi_{n-1}^1 - \xi_{n-2}^1 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^3\right] \text{where } i \in \{4,5\} \tag{8.16}$$

$$I_n^3 = \max\left[0, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \xi_{n-2}^1 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^2\right] \tag{8.17}$$

$$I_n^2 = \max\left[0, s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \xi_{n-2}^2 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1\right] \tag{8.18}$$

$$I_n^1 = \max\left[0, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-2}^3 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^0\right] \tag{8.19}$$

$$I_n^0 = \max\left[0, s^0 - Y_{n-2}^0 - \xi_{n-1}^4 - \xi_{n-2}^4\right] \tag{8.20}$$

The downstream shortage equations for model #1 are listed below:

$$DS_{n-1}^3 = \max\left[\xi_{n-1}^1 + \xi_{n-1}^2 + \xi_{n-1}^3 + \xi_{n-1}^4 - \eta_{n-1}^3, 0\right] \tag{8.21}$$

$$DS_{n-1}^2 = \max\left[\xi_{n-1}^2 + \xi_{n-1}^3 + \xi_{n-1}^4 - \eta_{n-1}^2, 0\right] \tag{8.22}$$

$$DS_{n-1}^1 = \max\left[\xi_{n-1}^3 + \xi_{n-1}^4 - \eta_{n-1}^1, 0\right] \tag{8.23}$$

$$DS_{n-1}^{0} = \max\left[\xi_{n-1}^{4} - \eta_{n-1}^{0}, 0\right] \tag{8.24}$$

The outstanding order equations for model #1are listed below:

$$Y_{n+1}^{i} = \max\left[0, Y_{n}^{i} + \xi_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - \eta_{n}^{5} - DS_{n-1}^{3}\right] \text{where } i \in \{4,5\} \tag{8.25}$$

$$Y_{n+1}^{3} = Y_{n}^{3} + \xi_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{2} - \min\begin{bmatrix} Y_{n}^{3} + \xi_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{2}, \\ s^{5} - Y_{n-2}^{5} - \xi_{n-1}^{1} - \ldots - \xi_{n-1}^{4}, \\ s^{4} - Y_{n-2}^{4} - \xi_{n-1}^{1} - \ldots - \xi_{n-1}^{4}, \ \eta_{n}^{3} \end{bmatrix} \tag{8.26}$$

$$Y_{n+1}^{2} = Y_{n}^{2} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1} - \min\begin{bmatrix} Y_{n}^{2} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1}, s^{3} - Y_{n-2}^{3} \\ -\xi_{n-1}^{1} \ldots - \xi_{n-1}^{4} - \xi_{n-2}^{1}, \ \eta_{n}^{2} \end{bmatrix} \tag{8.27}$$

$$Y_{n+1}^{1} = Y_{n}^{1} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{0} - \min\begin{bmatrix} Y_{n}^{1} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{0}, s^{2} - Y_{n-2}^{2} \\ -\xi_{n-1}^{2} \ldots - \xi_{n-1}^{4} - \xi_{n-2}^{2}, \ \eta_{n}^{1} \end{bmatrix} \tag{8.28}$$

$$Y_{n+1}^{0} = Y_{n}^{0} + \xi_{n}^{4} - \min\begin{bmatrix} Y_{n}^{0} + \xi_{n}^{4}, s^{1} - Y_{n-2}^{1} \\ -\xi_{n-1}^{3} - \xi_{n-1}^{4} - \xi_{n-2}^{3}, \ \eta_{n}^{0} \end{bmatrix} \tag{8.29}$$

8.4.2Update Equations for Model #2 (LAPD)

Model 2 uses lexicographic allocation with priority to the final product (downstream demand) to all the nodes with multiple sources of demand. Some of the update equations for LAPD are the same as specified in the earlier sub-section; only the equations which have a change are specified here. There are three new terms introduced $\left(Y_{n+1}^{3i}, Y_{n+1}^{2i}, \text{and}, Y_{n+1}^{1i}\right)$ which denote the outstanding orders of the intermediate product (local demand) at nodes 3, 2 and 1 respectively. The modified outstanding orders and the new outstanding order equations are listed below:

$$Y_{n+1}^{3i} = Y_{n}^{3i} + \xi_{n}^{1} - \min\left[Y_{n}^{3i} + \xi_{n}^{1}, s^{3} - Y_{n-2}^{3} - \xi_{n-1}^{1} - \ldots - \xi_{n-1}^{4} - \xi_{n-2}^{2} - \xi_{n-2}^{3} - \xi_{n-2}^{4}\right] \tag{8.30}$$

$$Y_{n+1}^{2} = Y_{n}^{2} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1} - \min\begin{bmatrix} Y_{n}^{2} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1}, s^{3} - Y_{n-2}^{3} \\ -\xi_{n-1}^{1} \ldots - \xi_{n-1}^{4}, \ \eta_{n}^{2} \end{bmatrix} \tag{8.31}$$

$$Y_{n+1}^{2i} = Y_n^{2i} + \xi_n^2 - \min\left[Y_n^{2i} + \xi_n^2, s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^3 - \xi_{n-2}^4\right] \quad (8.32)$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, s^2 - Y_{n-2}^2 \\ -\xi_{n-1}^2 \ldots - \xi_{n-1}^4, \eta_n^1 \end{bmatrix} \quad (8.33)$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^4\right] \quad (8.34)$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^4 - \min\left[Y_n^0 + \xi_n^4, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4, \eta_n^0\right] \quad (8.35)$$

The modified on-hand inventory equations are listed below:

$$I_n^3 = \max\left[0, s^3 - Y_{n-2}^3 - Y_{n-2}^{3i} - \xi_{n-1}^1 - \xi_{n-2}^1 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^2\right] \quad (8.36)$$

$$I_n^2 = \max\left[0, s^2 - Y_{n-2}^2 - Y_{n-2}^{2i} - \xi_{n-1}^2 - \xi_{n-2}^2 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1\right] \quad (8.37)$$

$$I_n^1 = \max\left[0, s^1 - Y_{n-2}^1 - Y_{n-2}^{1i} - \xi_{n-1}^3 - \xi_{n-2}^3 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^0\right] \quad (8.38)$$

8.4.3Update Equations for Model #3 (PPA)

Model 3 uses the predetermined proportional allocation at all the nodes with multiple sources of demand. The changes in the update equations according to the PPA policy are listed below:

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, ratio\,3 * \left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \ldots - \xi_{n-1}^4\right)\right] \quad (8.39)$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{bmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, (1 - ratio\,3) * \\ \left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \ldots - \xi_{n-1}^4\right), \eta_n^2 \end{bmatrix} \quad (8.40)$$

$$Y_{n+1}^{2i} = Y_n^{2i} + \xi_n^2 - \min\left[Y_n^{2i} + \xi_n^2, ratio\,2 * \left(s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \ldots - \xi_{n-1}^4\right)\right] \quad (8.41)$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, (1 - ratio\,2) * \\ \left(s^2 - Y_{n-2}^2 - \xi_{n-1}^2 \ldots - \xi_{n-1}^4\right), \eta_n^1 \end{bmatrix} \quad (8.42)$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, ratio\,1 * \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right)\right] \quad (8.43)$$

$$Y_{n+1}^{0}=Y_{n}^{0}+\xi_{n}^{4}-\min\begin{bmatrix} Y_{n}^{0}+\xi_{n}^{4},(1-ratio1)* \\ \left(s^{1}-Y_{n-2}^{1}-\xi_{n-1}^{3}-\xi_{n-1}^{4}\right),\ \eta_{n}^{0} \end{bmatrix} \tag{8.44}$$

$$0\le ratio\,i\le 1 \text{ where } i\in\{1,2,3\} \tag{8.45}$$

The on-hand inventory equations are as described in model 2.

8.4.4Update Equations for Model #4 (PA)

Model 4 uses the proportional allocation at all the nodes with multiple sources of demand. The outstanding order and on-hand inventory equations are same as described in model 3, but the ratio varies according to the demand. The equations for the ratios are listed below:

$$ratio3=\xi_{n}^{1}\Big/\left(\xi_{n}^{1}+\xi_{n}^{2}+\xi_{n}^{3}+\xi_{n}^{4}\right);ratio2=\xi_{n}^{2}\Big/\left(\xi_{n}^{2}+\xi_{n}^{3}+\xi_{n}^{4}\right);ratio1=\xi_{n}^{3}\Big/\left(\xi_{n}^{3}+\xi_{n}^{4}\right) \tag{8.46}$$

The update equations for models 5-10 are described in appendix A7.

8.5 Computational Results for Five-echelon Inventory Allocation Policies

This section presents the computational results based on the inventory allocation policies discussed in earlier sub-section. In order to gain a deeper understanding of the allocation policies we consider a five-echelon assembly system as opposed to the three-echelon assembly system that we have been discussing in previous chapters. The five-echelon assembly system is shown in figure 8.6. As the five-echelon assembly system has three nodes with more than one demand, i.e. in the form of intermediate product and downstream demand, the five-echelon assembly system requires three inventory allocation policies, one at each node where multiple sources of demands are present.

In order to determine which inventory allocation policy would help in minimizing the total cost and safety-stock cost across the entire multi-echelon network, four instances; each consisting of several scenarios are solved to optimality. The values of the demand and capacity for all the scenarios are provided in the appendix A8. The instances are described in the table 8.7, which show the name of the instance and coefficient of variance for the demand and capacity. The capacity is denoted as "average" capacity when the mean capacity utilization is between 65% and 75%, and the capacity is defined as "tight" capacity when the mean capacity utilization is between 85% and 95%. The demand is categorized as "high" and "low" demand. If the values of the mean demand is between 8 and 10 it is referred as high demand, whereas low demand is defined between 3.5 and 5. The variance for the demand and capacity is defined as high and low if the coefficient of variance is 0.3 and 0.1 respectively.

Table 8.7: Instances for Five-echelon Assembly System

Instance #	Name of Instance	CV for Capacity	CV for Demand
1	Average Capacity (AC)	0.1	0.1
2	Tight Capacity (TC)	0.1	0.1
3	High Demand Variance with Average Capacity (HDVAC)	0.3	0.3
4	High Demand Variance with Tight Capacity (HDVTC)	0.3	0.3

As mentioned earlier each instance consists of several scenarios (shown in appendix) that are solved to optimality. The scenarios are developed on the basis of three criteria which are mentioned in table 8.8. Each criterion is specifically used to study the effect of the inventory allocation policy on the total cost and safety-stock cost. Criteria 1 is designed to study the effect of the inventory allocation policies when a great percentage of mean demand occurs upstream. Criteria 2 is designed to study the effect of inventory allocation policies when a larger fraction of total mean demand occurs for the final product. Criteria 3 is designed to study the effect of inventory allocation policies when a majority fraction of mean demand occurs at two sources, i.e. 70% of the total mean demand is shared between demand 3 and 4. The values for each scenario are randomly picked.

Table 8.8: Criteria for Developing Scenarios

Criteria #	Description of Criteria
1	The Mean Demand of Final Product consists of only 10% to 17% of the Total Demand
2	The Mean Demand of Final Product consists of 35% to 49% of the Total Demand
3	The sum of mean demand for two sources contributes 62% to 75% of total demand

Ten different models are initially analyzed to gain insight. Each model has a combination of different inventory allocation policies. Table 8.9 show the inventory allocation policies associated with each model. Note that only nodes 1, 2, and 3 require inventory allocation. Each model is run under all four instances, and scenarios satisfying all three criteria. The first four models have the same allocation policy for all the three nodes, whereas other models have a combination of different inventory allocation policies across the five-echelon assembly system. As stated earlier all ten models are solved to optimality for four instances, with each instance consisting of several scenarios. Initial insights are described on the basis of numerical results from these ten models. Additional insights and hypothesis are stated and investigated in the next section.

Table 8.9: Inventory Allocation Polices Used in Models

Model #	Nodes		
	3	2	1
1	LAPI	LAPI	LAPI
2	LAPD	LAPD	LAPD
3	PPA	PPA	PPA
4	PA	PA	PA
5	PPA	LAPD	PA
6	LAPD	LAPI	PA
7	LAPD	LAPI	LAPI
8	LAPI	LAPD	LAPD
9	LAPD	PA	PA
10	PA	LAPI	LAPI

8.5.1 Average Best Found Safety-stock Cost

Table 8.10 show the best found safety-stock cost averaged over all scenarios in each instance for the ten models shown in table 8.9. Each instance consists of at least one scenario that satisfies the criteria stated in table 8.8. Table 8.10 also shows the percentage increase of an average best found safety-stock cost from the lowest average best found safety-stock cost in a given instance.

Instance 1 (AC). From the numerical results for instance 1 listed in table 8.10 we can observe that, model 8 results in the lowest average best found safety-stock cost, closely followed by model 1 and model 10. In all three models a LAPI policy is prominently used, except model 8 where LAPI policy is used in only node 3. We can also observe that the model 2 and 4 which use only LAPD and PA policy has an average best found safety-stock cost greater by 4.46% and 14.67%. If only one allocation policy is to be used across the entire supply chain network, LAPI policy works as the best inventory allocation in case of average capacity (instance 1) closely followed by LAPD policy. Specifically except PPA policy all the other three allocation policies provide an average best found safety-stock cost closer to the lowest average best found safety-stock cost.

Table 8.10: Average Best Found Safety-stock Cost for All Scenarios

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10
Instance 1	17.46	18.17	21.17	19.95	22	18.52	17.94	17.4	20.31	17.59
(AV)	(0.36%)	(4.46%)	(21.70%)	(14.67%)	(26.44%)	(6.46%)	(3.10%)	(0%)	(16.74%)	(1.11%)
Instance 2	33.75	28.69	41.68	34.67	36.94	30.27	32.07	31.56	30.8	33.45
(TC)	(17.64%)	(0%)	(45.27%)	(20.84%)	(28.74%)	(5.51%)	(11.78%)	(9.99%)	(7.35%)	(16.57%)
Instance 3	102.44	93.95	101.54	91.54	103.71	95.19	99.78	95.22	91.21	93.99
(HDVAC)	(12.31%)	(2.98%)	(11.33%)	(0.37%)	(13.70%)	(4.37%)	(9.39%)	(4.4%)	(0%)	(3.05%)
Instance 4	205.15	192.08	210.85	208.79	217.12	210.91	210.45	204.11	207.74	216.35
(HDVTC)	(6.8%)	(0%)	(9.77%)	(8.7%)	(13.03%)	(9.8%)	(9.57%)	(6.26%)	(8.15%)	(12.62%)

Instance 2 (TC). From the numerical results for instance 2 listed in table 8.10 we can observe that, model 2 provides the lowest average best found safety-stock cost, from table 8.9 we know that model 2 has LAPD policy applied to its nodes. Model 6 provides the second lowest average best found safety-stock cost, which is about 5.5% greater than model 2. If only one allocation policy is to be used across the entire supply chain, LAPD allocation works as the best inventory allocation policy in reducing the safety-stock cost in case of tight capacity instance.

Instance 3 (HDVAC). From the numerical results for instance 3 listed in table 8.10 we can observe that, model 9 provides the lowest average best found safety-stock cost, very closely followed by model 4. From table 8.9 we know that model 9 and model 4 uses PA policy significantly. Model 4 has an average best found safety-stock about 0.37% greater than model 9. If only one allocation policy is to be used across the entire supply chain, PA allocation works as the best inventory allocation policy in reducing the safety-stock cost in case of a high demand variance average capacity instance.

Instance 4 (HDVTC). From the numerical results for instance 4 listed in table 8.10 we can observe that, model 2 provides the lowest average best found safety-stock cost, from table 8.9 we know that model 2 has LAPD policy applied to its nodes. Model 8 provides the second lowest average best found safety-stock cost, which is about 6.26% greater than model 2. If only one allocation policy is to be used across the entire supply chain, LAPD allocation works as the best inventory allocation policy in reducing the safety-stock cost in case of high demand variance tight capacity instance.

8.5.2 Safety-stock Cost for Increased Downstream Demand

Table 8.11 show the best found safety-stock cost averaged over scenarios which satisfy criteria 2, i.e. having increased downstream demand. The observation from table 8.11 specifically addresses the situation with increased downstream demand.

In case of instance 1 most of the models provide an average best found safety-stock cost close to the lowest average best found safety-stock cost. Model 10 has the lowest safety-stock cost, closely followed by model 9, 1, and 4, which suggests that LAPI policy in combination with PA policy will help in reduction of the safety-stock cost. The models that involve nodes having PPA policy as the allocation results in largest average best found safety-stock cost, i.e. 35%-39% higher than the lowest safety-stock cost.

LAPD policy provides the lowest average best found safety-stock in case of instance 2 and instance 4. Model 2 provides the safety-stock cost in either case. This suggests that LAPD policy if used across the entire supply chain will significantly reduce the safety-stock cost compared to the other inventory allocation policies. In case of instance 3, model 9 provides the lowest average best found safety-stock cost. Model 9 has a combination of PA policy and LAPD, with two of the three nodes having a PA policy. Assigning PA policy to the nodes downstream, with LAPD policy for the upstream nodes will help in reducing the safety-stock cost in case of instance 3, HDVAC. If the supply chain would use only one inventory allocation policy, using the PA policy for all the nodes will reduce the safety-stock cost.

Table 8.11: Average Safety-stock Cost for Scenarios Satisfying Criteria 2

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10
Instance 1	17.79	18.34	23.76	17.85	24.34	18.5	18.53	17.99	17.65	17.54
(AV)	(1.41%)	(4.54%)	(35.44%)	(1.79%)	(38.75%)	(5.45%)	(5.65%)	(2.58%)	(0.65%)	(0%)
Instance 2	38.19	32.12	49.78	41.92	44.41	35.75	38.39	36.05	35.69	38.16
(TC)	(18.90%)	(0%)	(54.98%)	(30.51%)	(38.28%)	(11.30%)	(19.52%)	(12.24%)	(11.11%)	(18.79%)
Instance 3	102.91	91.12	107.53	90.91	110.78	93.75	98.68	91.25	86.28	94.03
(HDVAC)	(19.28%)	(5.62%)	(24.63%)	(5.37%)	(28.40%)	(8.66%)	(14.38%)	(5.76%)	(0%)	(8.98%)
Instance 4	228.21	214.56	231.55	229.23	239.3	228.95	227.9	220.4	229.29	236.95
(HDVTC)	(6.36%)	(0%)	(7.92%)	(6.84%)	(11.53%)	(6.71%)	(6.22%)	(2.72%)	(6.87%)	(10.44%)

8.5.3 Safety-stock Cost for Increased Local/Intermediate Product Demand

Table 8.12 show the best found safety-stock cost averaged over scenarios which satisfy criteria 1, i.e. having increased local (intermediate product) demand. The observations from table 8.12 specifically address the situation with increased local demand.

Under instance 1, model 8 provides the lowest average best found safety stock cost closely followed by model 7 and model 1. Model 8 has a LAPI policy for node 3 and LAPD policy for other nodes. Model 7 and 1 is 1.46% and 2.37% greater than the lowest average best found safety stock cost, both models 7 and 1 have LAPI policy for majority of the nodes. Model 2 which has LAPD policy has an average best found safety-stock cost almost 10% greater than lowest average best found safety-stock cost, i.e. model 8. Based on these observations we can determine that either LAPI policy will help in reducing the safety-stock cost under instance 1, average capacity. LAPD policy can also be used in combination with LAPI over the supply chain to minimize the safety-stock cost. For instance 2, model 6 provides the lowest average safety-stock cost. Model 6 uses three different inventory allocation policies. In case of instance 3 and 4 from table 8.12 we can observe that model 4 and model 2 reduces the average best found safety-stock cost, i.e. PA policy and LAPD policy help in reducing the safety-stock cost.

Table 8.12: Average Safety-stock Cost for Scenarios Satisfying Criteria 1

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10
Instance 1 (AV)	16.83 (2.37%)	18.07 (9.95%)	17.59 (7.03%)	23.87 (45.24%)	18.98 (15.49%)	18.54 (12.81%)	16.68 (1.46%)	16.44 (0%)	24.36 (48.22%)	17.07 (3.86%)
Instance 2 (TC)	25.1 (10.11%)	24.76 (8.62%)	28.5 (25.03%)	26.17 (14.82%)	26.19 (14.89%)	22.8 (0%)	25.17 (10.4%)	25.33 (11.12%)	25.35 (11.21%)	25.92 (13.71%)
Instance 3 (HDVAC)	101.08 (10.55%)	96.58 (5.63%)	93.26 (2%)	91.43 (0%)	94.65 (3.52%)	96.68 (5.74%)	102.68 (12.30%)	98.3 (7.52%)	96.99 (6.08%)	92.37 (1.03%)
Instance 4 (HDVTC)	173.19 (2.86%)	168.37 (0%)	184.88 (9.81%)	179.32 (6.50%)	190.62 (13.21%)	187.29 (11.23%)	183.33 (8.88%)	178.99 (6.31%)	177.42 (5.37%)	188.69 (12.07%)

8.5.4 Safety-stock Cost analysis for Criteria 3

Table 8.13 shows the average best found safety-stock cost for two special cases, i.e. IUDD (increase in upstream and downstream demand) and IDIN (increase in demand for intermediate product), and the two special cases follow criteria 3. The scenarios are averaged for all instances across all the ten models shown in table 8.9. In case of IUDD, the sum of demand 1 and 4 combine to form 62% to 75% of the total demand, with one of them having at least 27% of total demand. Similarly in case of IDIN, the sum of demand 2 and 3 combine to form 62% to 75% of the total demand, with one of them having at least 27% of total demand. The two special cases were created to gain insight associated to inventory allocation on special situations where there might be more local demand in only some nodes as opposed to all nodes. From the numerical results in table 8.13 we can see that model 2 provides the lowest average best found safety-stock cost in either special case, i.e. LAPD policy will help maintaining the required service level across all the nodes with minimum safety-stock cost.

Table 8.13: Average Safety-stock Cost for Scenarios Satisfying Criteria 3

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10
Safety-stock Cost for IUDD	89.79 (14.66%)	78.31 (0%)	91.29 (16.58%)	87.09 (11.21%)	90.31 (15.32%)	86.98 (11.07%)	88.81 (13.41%)	88.63 (13.18%)	86.34 (10.25%)	90 (14.93%)
Safety-stock Cost for IDIN	83.81 (15.68%)	72.45 (0%)	83.3 (14.98%)	77.06 (6.36%)	85.13 (17.50%)	75.31 (3.95%)	78.74 (8.68%)	79.31 (9.46%)	74.5 (2.83%)	80.06 (10.50%)

Additional information on the numerical results for the five-echelon model is provided in the appendix.

8.6 Hypothesis Testing

In this section we state some important implications based of the numerical results derived from the five-echelon assembly system. The statements are supported with a hypothesis, and are proven statistically using a *student t-test: paired two sample for mean*. Microsoft Excel 2007 is used to perform the student t-test. The following formula is used to determine the test statistic value *t*.

$$t_0\,(t\,\text{stat}) = \frac{\overline{x} - \overline{y} - \Delta_0}{S_{x\,y}\sqrt{\dfrac{2}{m}}} \tag{8.47}$$

Where $\overline{x}$ the sample is mean for the first population, $\overline{y}$ is the sample mean for the second population. Δ_0 is the hypothesized mean difference, a value of zero indicates sample means hypothesized to be equal. $S_{x\,y}$ is the pooled standard deviation. m represent number of observations. The formula used to calculate the pooled standard deviation is shown in (8.48). The degrees of freedom used for the calculation is *m-1*.

$$S_{x\,y} = \sqrt{\frac{m(S_x^2 + S_y^2)}{2m-2}} \tag{8.48}$$

Statement 1. If only one type of inventory allocation policy is used across the entire supply chain (five-echelon assembly system), using LAPI inventory allocation policy results in the lowest safety-stock cost compared to models which use other inventory allocation polices under average capacity instance (instance 1)

Null Hypothesis (H0): For all scenarios that occur under instance 1 in a five-echelon assembly system, the best found safety-stock cost for models using only LAPI inventory allocation policy is not significantly different from the best found safety-stock cost for models which use only one type of inventory allocation policy

Alternate Hypothesis (H1): For all scenarios that occur under instance 1 in a five-echelon assembly system, the best found safety-stock cost for models using only LAPI inventory allocation policy is significantly different from the best found safety-stock cost for models which use only one type of inventory allocation policy

$$H0:\mu_1 = \mu_2;\ H1:\mu_1 \neq \mu_2 \tag{8.49}$$

The hypothesis can also be stated as in equation (8.49). For the statistical test we compare means of the best found safety-stock for model 1 (μ_1) and model's 2, 3, and 4 averaged (μ_2) for all scenarios that come under instance 1. From table 8.9 we know that model 1 uses LAPI inventory allocation policy, whereas model's 2-4 use other inventory allocation policies. Models 1-4 use only one type of inventory allocation policy across their supply chain.

Table 8.14: Statistical Results for Statement 1

t Stat	t Critical two-tail	alpha	P-value two-tail
-2.662	2.015	0.100	0.045

Table 8.14 shows the statistical results based on the *t-test*. From the table 8.14 we can observe that $t_0\text{(t Stat)} = \text{-2.6619} < \text{-}t_c\text{(t Critical two-tail)} = \text{-2.015}$. Based on alpha = 0.1 we reject the null hypothesis. So we state that, when only one type of inventory allocation policy is used across the five-echelon assembly system LAPI inventory allocation policy results in the lowest safety-stock cost for instance 1 when compared to models 2-4 which

use only LAPD or PPA or PA inventory allocation policies across the five-echelon assembly system.

Statement 2. If a combination of inventory allocation policies are used across the entire supply chain (five-echelon assembly system), using LAPI & LAPD inventory allocation policy in combination will results in the lowest safety-stock cost, compared to models which use other combinations of inventory allocation polices under average capacity instance (instance 1)

Null Hypothesis (H0): For all scenarios that occur under instance 1 in a five-echelon assembly system, the best found safety-stock cost for models using a combination of LAPI & LAPD inventory allocation policy is not significantly different from the best found safety-stock cost for models which use a different combination of inventory allocation policy (i.e. other than LAPI & LAPD)

Alternate Hypothesis (H1): For all scenarios that occur under instance 1 in a five-echelon assembly system, the best found safety-stock cost for models using a combination of LAPI & LAPD inventory allocation policy is significantly different from the best found safety-stock cost for models which use a different combination of inventory allocation policy

A similar hypothesis statement as described in 6.1 can be stated, where the statistical test is conducted by comparing means of the best found safety-stock for model's 7, and 8 averaged (μ_1) and model's 5, 6, 9, and 10 averaged (μ_2) for all scenarios that come under instance 1. From table 8.6 we know that model's 7 and 8 uses a combination of LAPI & LAPD inventory allocation policies, whereas model's 5, 6, 9, and 10 use combination of other inventory allocation policies.

Table 8.15: Statistical Results for Statement 2

t Stat	t Critical two-tail	alpha	P-value two-tail
-2.419	2.015	0.100	0.060

Table 8.15 shows the statistical results based on the *t-test*. From the table 8.15 we can observe that t_0(t Stat) = -2.419 < $-t_c$ (t Critical two-tail) = -2.015. Based on alpha = 0.1 we reject the null hypothesis. So we can state that, when a combination of inventory allocation policies is used across the five-echelon assembly system LAPI in combination with LAPD inventory allocation policy results in the lowest safety-stock cost for instance 1 when compared to models 5, 6, 9, and 10 which use different combinations (other than LAPD and LAPI) of inventory allocation policies across the five-echelon assembly system.

Statement 3. LAPD inventory allocation policy results in the lowest safety-stock cost across the supply chain (five-echelon assembly system) under tight capacity instance (instance 2).

Null Hypothesis (H0): For all scenarios that occur under instance 2 in a five-echelon assembly system, the best found safety-stock cost for models using LAPD inventory allocation policy is not significantly different from the best found safety-stock cost from all the other models

Alternate Hypothesis (H1): For all scenarios that occur under instance 2 in a five-echelon assembly system, the best found safety-stock cost for models using LAPD inventory allocation policy is significantly different from the best found safety-stock cost form all the other models

Two statistical tests are performed to provide evidence for the hypothesis test. The first test is conducted by comparing means of the best found safety-stock for model 2

(μ_1) and model's 1, 3, and 4 averaged (μ_2) for all scenarios that come under instance 2. The second test is conducted by comparing means of the best found safety-stock for model 2 (μ_1) and model's 5 to 10 averaged (μ_2) for all scenarios that come under instance 2. The first test is used to determine the best policy statistically when only one type of inventory allocation policy is used across the supply chain. The second test is used to determine statistically if only one type or a combination of inventory allocation policies should be used across the supply chain.

Table 8.16: Statistical Results for Statement 3

Test #	t Stat	t Critical two-tail	alpha	P-value two-tail
1	-3.390	2.015	0.100	0.019
2	-2.89	2.015	0.100	0.034

Table 8.16 shows the statistical results based on the *t-test*. From the table 8.16 we can observe that for test 1, $t_0(\text{t Stat}) = \text{-}3.39 < \text{-}t_c\,(\text{t Critical two-tail}) = \text{-}2.015$. Similarly for test 2, we can observe that $t_0(\text{t Stat}) = \text{-}2.89 < \text{-}t_c\,(\text{t Critical two-tail}) = \text{-}2.015$. For alpha = 0.1, based of the two test conducted we reject the null hypothesis. On the basis of this statistical test we can infer that using LAPD policy in a tight capacity instance results in lowest safety-stock cost. Also, using only LAPD policy across the entire supply chain is suggestible rather than using a combination of different inventory allocation policies.

Statement 4. If only one type of inventory allocation policy is used under high demand average capacity instance (instance 3) across the entire supply chain (five-echelon assembly system), using PA inventory allocation policy results in the lowest safety-stock cost compared to models which use only one type of inventory allocation policy

Null Hypothesis (H0): For all scenarios that occur under instance 3 in a five-echelon assembly system, the best found safety-stock cost for models using only PA inventory allocation policy is not significantly different from the best found safety-stock cost for models which use only one type of inventory allocation policy

Alternate Hypothesis (H1): For all scenarios that occur under instance 3 in a five-echelon assembly system, the best found safety-stock cost for models using only PA inventory allocation policy is significantly different from the best found safety-stock cost for models which use only one type of inventory allocation policy

For the statistical test we compare means of the best found safety-stock for model 4 (μ_1) and model's 1, 2, and 3 averaged (μ_2) for all scenarios that come under instance 3. From table 8.6 we know that model 4 uses PA inventory allocation policy, whereas model's 2-4 use other inventory allocation policies.

Table 8.17: Statistical Results for Statement 4

t Stat	t Critical two-tail	alpha	P-value two-tail
-6.138	2.015	0.100	0.002

Table 8.17 shows the statistical results based on the *t-test*. From table 8.17 we can observe that $t_0\text{(t Stat)} = \text{-}6.13 < \text{-}t_c\text{(t Critical two-tail)} = \text{-}2.015$. Based on a alpha = 0.1 we reject the null hypothesis. So we state that, when only one type of inventory allocation policy is used across the five-echelon assembly system PA inventory allocation policy results in the lowest safety-stock cost for instance 3 when compared to models 1-3 which use only LAPI or LAPD or PPA inventory allocation policies across the five-echelon assembly system.

Statement 5. If a combination of inventory allocation policies are used across the entire supply chain (five-echelon assembly system), using LAPD & PA inventory allocation policy in combination will results in the lowest safety-stock cost, compared to models which use other combinations of inventory allocation polices under high demand variance average capacity instance (instance 3)

Null Hypothesis (H0): For all scenarios that occur under instance 3 in a five-echelon assembly system, the best found safety-stock cost for models using a combination of LAPD & PA inventory allocation policy is not significantly different from the best found safety-stock cost for models which use a different combination of inventory allocation policy (i.e. other than LAPD & PA)

Alternate Hypothesis (H1): For all scenarios that occur under instance 3 in a five-echelon assembly system, the best found safety-stock cost for models using a combination of LAPD & PA inventory allocation policy is significantly different from the best found safety-stock cost for models which use a different combination of inventory allocation policy (i.e. other than LAPD & PA)

The statistical test is conducted by comparing means of the best found safety-stock for model 9 $\left(\mu_1\right)$ and model's 5 - 8 and 10 averaged $\left(\mu_2\right)$ for all scenarios that come under instance 3. From table 8.6 we know that model 9 uses a combination of LAPD & PA inventory allocation policies, whereas model's 5-8 and 10 use combination of other inventory allocation policies.

Table 8.18: Statistical Results for Statement 5

t Stat	t Critical two-tail	alpha	P-value two-tail
-2.644	2.015	0.100	0.046

Table 8.18 shows the statistical results based on the *t-test*. From the table 8.18 we can observe that $t_0(\text{t Stat}) = \text{-}2.650 < \text{-}t_c(\text{t Critical two-tail}) = \text{-}2.015$. Based on alpha = 0.1 we reject the null hypothesis. So it can be stated that, when a combination of inventory allocation policies is used across the five-echelon assembly system LAPD in combination with PA inventory allocation policy results in the lowest safety-stock cost for instance 3 when compared to models 5-8, and 10.

Statement 6. LAPD inventory allocation policy results in the lowest safety-stock cost across the supply chain (five-echelon assembly system) under high demand variance tight capacity instance (instance 4).

Null Hypothesis (H0): For all scenarios that occur under instance 4 in a five-echelon assembly system, the best found safety-stock cost for models using LAPD inventory allocation policy is not significantly different from the best found safety-stock cost from all the other models

Alternate Hypothesis (H1): For all scenarios that occur under instance 4 in a five-echelon assembly system, the best found safety-stock cost for models using LAPD inventory allocation policy is significantly different from the best found safety-stock cost from all the other models

Table 8.19: Statistical Results for Statement 6

Test #	t Stat	t Critical two-tail	alpha	P-value two-tail
1	-4.046	2.015	0.100	0.010
2	-4.76129567	2.015	0.100	0.005

Similar to statement 3 two statistical tests are performed to provide evidence for the hypothesis test. The models for comparison are same as the ones used for statement 3. Table 8.16 shows the statistical results based on the *t-test*. From the table 8.19 we can observe that for test 1, $t_0(\text{t Stat}) = \text{-}4.046 < \text{-}t_c\,(\text{t Critical two-tail}) = \text{-}2.015$. Similarly for test 2, we can observe that $t_0(\text{t Stat}) = \text{-}4.76 < \text{-}t_c\,(\text{t Critical two-tail}) = \text{-}2.015$. For alpha = 0.1, based of the two test conducted we reject the null hypothesis. On the basis of this statistical test we can infer that using LAPD policy in a high demand variance tight capacity instance results in lowest safety-stock cost. Also, using only LAPD policy across the entire supply chain is suggestible rather than using a combination of different inventory allocation policies.

9. MULTI-ECHELON NETWORKS

Two contemporary network models were evaluated in order to show that the multi-echelon inventory model developed in this research is applicable on a wide range of multi-echelon network models. This investigation shows the robustness, and practicality of the multi-echelon inventory analysis discussed in earlier chapters. A numerical analysis is performed using the two networks. Based on these numerical results a few implications are deduced. The two networks also demonstrate the ability to apply the products of this research to larger models with complex interactions. The two networks studied were:

- *Network 1:* Multiple suppliers – representing a manufacturing industry with demand for spare parts
- *Network 2:* Representing a system with multiple manufacturers/suppliers – warehouse - distribution center – retailer interactions

In this chapter we discuss each network, its update equations and its first order differential equation equations, followed by some numerical results.

9.1 Network 1

Figure 9.1 shows a network structure with three suppliers, and a manufacturer. In a real world scenario a supplier can be procuring raw material from an external supplier (not shown in the figure), process the raw material and send components to more than one manufacturer. The manufacturer might assemble different products based on the components procured from different suppliers and sell the result as a final product.

Let us consider an illustrative example of multiple suppliers - manufacturing industry for the network structure shown in figure 9.1. Suppose the manufacturing firm is an automotive firm, and consider the three suppliers who supply different parts to the automotive assembly line. The assembly of the parts takes place at the manufacturer's location, and the automobile is further processed and sold to the end customer. Each supplier might not just satisfy the manufacturing firm's demand but can also be supplying spare parts to a distributor or an automotive dealer.

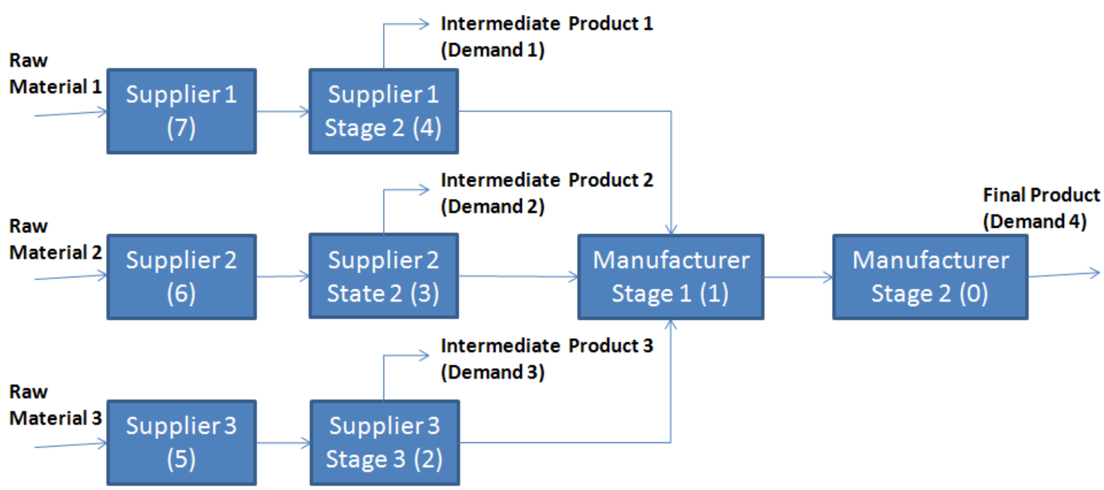

Figure 9.1: Multiple Suppliers-Manufacturing Industry Setup

The network model shown in figure 9.1 attempts to capture a real world scenario, and show that the multi-echelon inventory model discussed in this research can be extended to more general models. One question that often arises is why is it necessary to study the entire network instead of just focusing on supplier or manufacturer. From the Toyota Production System and lean manufacturing principles it is well known that in order to have accurate stock levels and reduced manufacturing costs the entire system needs to be accounted, i.e. Tier 1- Tier 3 suppliers, and the manufacturer itself. In the network shown in figure 10.1 only tier 1 suppliers and the manufacturing process are considered, but can certainly be expanded to a much larger network. In figure 9.1 The numbers in the bracket represent the node number, and are used for the update equations in the following sub-sections.

9.1.1 Update Equations for Network 1

A two-period lead time is considered for all the update equations. The default inventory allocation policy is considered for nodes 2, 3, and 4 (LAPI). The on-hand inventory equations are listed below:

$$I_n^7 = \max\left[0, s^7 - Y_{n-2}^7 - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^4\right] \tag{9.1}$$

$$I_n^6 = \max\left[0, s^6 - Y_{n-2}^6 - \xi_{n-1}^2 - \xi_{n-2}^2 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^3\right] \tag{9.2}$$

$$I_n^5 = \max\left[0, s^5 - Y_{n-2}^5 - \xi_{n-1}^3 - \xi_{n-2}^3 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^2\right] \tag{9.3}$$

$$I_n^4 = \max\left[0, s^4 - Y_{n-2}^4 - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1\right] \tag{9.4}$$

$$I_n^3 = \max\left[0, s^3 - Y_{n-2}^3 - \xi_{n-1}^2 - \xi_{n-2}^2 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1\right] \tag{9.5}$$

$$I_n^2 = \max\left[0, s^2 - Y_{n-2}^2 - \xi_{n-1}^3 - \xi_{n-2}^3 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1\right] \tag{9.6}$$

$$I_n^1 = \max\left[0, s^1 - Y_{n-2}^1 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^0\right] \tag{9.7}$$

$$I_n^0 = \max\left[0, s^0 - Y_{n-2}^0 - \xi_{n-1}^4 - \xi_{n-2}^4\right] \tag{9.8}$$

The equations for the downstream shortages are listed below:

$$DS_{n-1}^4 = \max\left[0,\ \xi_{n-1}^1 + \xi_{n-1}^4 - \eta_{n-1}^4\right] \tag{9.9}$$

$$DS_{n-1}^3 = \max\left[0,\ \xi_{n-1}^2 + \xi_{n-1}^4 - \eta_{n-1}^3\right] \tag{9.10}$$

$$DS_{n-1}^2 = \max\left[0,\ \xi_{n-1}^3 + \xi_{n-1}^4 - \eta_{n-1}^2\right] \tag{9.11}$$

$$DS_{n-1}^1 = \max\left[0,\ \xi_{n-1}^4 - \eta_{n-1}^1\right] \tag{9.12}$$

$$DS_{n-1}^{0} = \max\left[0,\ \xi_{n-1}^{4} - \eta_{n-1}^{0}\right] \tag{9.13}$$

The outstanding orders are listed below:

$$Y_{n+1}^{7} = \max\left\{0, Y_n^7 + \xi_n^1 + \xi_n^4 - \eta_n^7 - DS_{n-1}^4\right\} \tag{9.14}$$

$$Y_{n+1}^{6} = \max\left\{0, Y_n^6 + \xi_n^2 + \xi_n^4 - \eta_n^6 - DS_{n-1}^3\right\} \tag{9.15}$$

$$Y_{n+1}^{5} = \max\left\{0, Y_n^5 + \xi_n^3 + \xi_n^4 - \eta_n^5 - DS_{n-1}^2\right\} \tag{9.16}$$

$$Y_{n+1}^{5} = \max\left\{0, Y_n^5 + \xi_n^3 + \xi_n^4 - \eta_n^5 - DS_{n-1}^2\right\} \tag{9.17}$$

$$Y_{n+1}^{4} = Y_n^4 + \xi_n^1 + \xi_n^4 - DS_{n-1}^1 - \min\left\{Y_n^4 + \xi_n^1 + \xi_n^4 - DS_{n-1}^1, s^7 - Y_{n-2}^7 - \xi_{n-1}^1 - \xi_{n-1}^4, \eta_n^4\right\} \tag{9.18}$$

$$Y_{n+1}^{3} = Y_n^3 + \xi_n^2 + \xi_n^4 - DS_{n-1}^1 - \min\left\{Y_n^3 + \xi_n^2 + \xi_n^4 - DS_{n-1}^1, s^6 - Y_{n-2}^6 - \xi_{n-1}^2 - \xi_{n-1}^4, \eta_n^3\right\} \tag{9.19}$$

$$Y_{n+1}^{2} = Y_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\left\{Y_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, s^5 - Y_{n-2}^5 - \xi_{n-1}^3 - \xi_{n-1}^4, \eta_n^2\right\} \tag{9.20}$$

The outstanding order equation (9.21) for node 1 is slightly different from the other outstanding order equations in network 1. It is an assembly of three different products, product from node 4, 3 and 2. The shortages in the node can occur due to unavailability of inventory at nodes 2, 3 and 4, or shortages can occur due to the manufacturing capacity of the node itself. The available inventory from nodes 2, 3 and 4 represented in equation (9.21) inside the min term also takes the default inventory allocation policy into account (LAPI).

$$Y_{n+1}^{1} = Y_n^1 + \xi_n^4 - DS_{n-1}^0 - \min\left\{\begin{array}{l} Y_n^1 + \xi_n^4 - DS_{n-1}^0, s^4 - Y_{n-2}^4 - \xi_{n-1}^1 - \xi_{n-1}^4 - \xi_{n-2}^1, s^3 - Y_{n-2}^3 \\ -\xi_{n-1}^2 - \xi_{n-1}^4 - \xi_{n-2}^2, s^2 - Y_{n-2}^2 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^3, \eta_n^1 \end{array}\right\} \tag{9.21}$$

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^4 - \min\left\{Y_n^0 + \xi_n^4, s^1 - Y_{n-2}^1 - \xi_{n-1}^4, \eta_n^0\right\} \tag{9.22}$$

The Lagrange equation, first-order Lagrange equations with respect to all the decision variables (Lagrange multipliers, and base-stock levels), and the first-order outstanding orders are provided in appendix, sections A9.1 - A9.3.

9.2 Network 2

The second multi-echelon network considered is shown in figure 10.2, this network focuses on manufacturing - retail industry as opposed to the pure manufacturing industry accounted for in the earlier section. The network comprises of manufacturers/supplier, warehouse, distribution center, and retailers. The manufacturers store the finished goods in the warehouse, from the warehouse the product is sent to the distribution center, and to retail outlets.

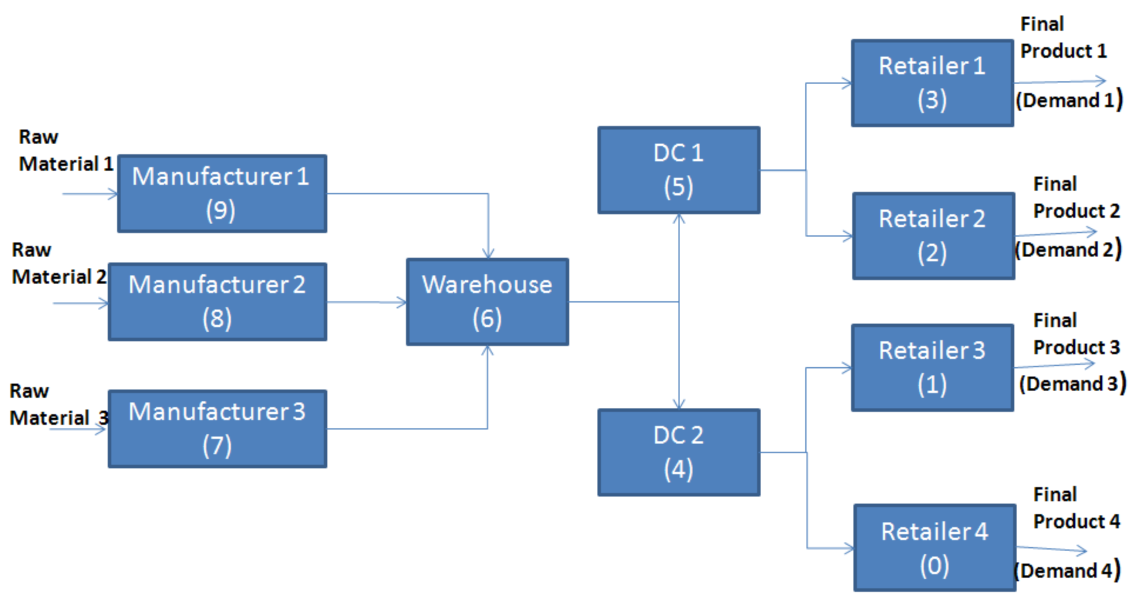

Figure 9.2: Manufacturers – Warehouse - Distribution Center – Retailer

Let us consider an illustrative example of the network structure shown in figure 10.2. The network structure considered here resembles a typical Wal-Mart or any other retail store. Demand occurs at the retail store is satisfied by the manufacturers (P & G etc.), the product is sent through a warehouse (usually a part of manufacturer) and distribution center (usually a part of retailer).

9.2.1 Update Equations for Network 2

A two-period lead time is considered for all the update equations. Lexicographic inventory allocation policy with a priority to node 5 (DC 1) is used at node 6 (warehouse). The proportional allocation (PA) policy is used at node 5 and 4 (DC 1 and DC 2). The on-hand inventory equations are listed below:

$$I_n^i = \max\left[0, s^i - Y_{n-2}^i - \xi_{n-1}^1 - \xi_{n-2}^1 - .... - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^6\right] \text{ where } \mathrm{i} \in \{9,8,7\} \tag{9.23}$$

$$I_n^6 = \max\left[0, s^6 - Y_{n-2}^6 - \xi_{n-1}^1 - \xi_{n-2}^1 - .... - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^5 + DS_{n-1}^4\right] \tag{9.24}$$

$$I_n^5 = \max\left[0, s^5 - Y_{n-2}^5 - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-1}^2 - \xi_{n-2}^2 + DS_{n-1}^3 + DS_{n-1}^2\right] \tag{9.25}$$

$$I_n^4 = \max\left[0, s^4 - Y_{n-2}^4 - \xi_{n-1}^3 - \xi_{n-2}^3 - \xi_{n-1}^4 - \xi_{n-2}^4 + DS_{n-1}^1 + DS_{n-1}^0\right] \tag{9.26}$$

$$I_n^3 = \max\left[0, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \xi_{n-2}^1\right] \tag{9.27}$$

$$I_n^2 = \max\left[0, s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \xi_{n-2}^2\right] \tag{9.28}$$

$$I_n^1 = \max\left[0, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-2}^3\right] \tag{9.29}$$

$$I_n^0 = \max\left[0, s^0 - Y_{n-2}^0 - \xi_{n-1}^4 - \xi_{n-2}^4\right] \tag{9.30}$$

The downstream shortage equations are listed below:

$$DS_{n-1}^6 = \max\left[\xi_{n-1}^1 + \xi_{n-1}^2 + \xi_{n-1}^3 + \xi_{n-1}^4 - \eta_n^6, 0\right] \tag{9.31}$$

$$DS_{n-1}^5 = \max\left[\xi_{n-1}^1 + \xi_{n-1}^2 - \eta_n^5, 0\right] \tag{9.32}$$

$$DS_{n-1}^4 = \max\left[\xi_{n-1}^3 + \xi_{n-1}^4 - \eta_n^4, 0\right] \tag{9.33}$$

$$DS_{n-1}^3 = \max\left[\xi_{n-1}^1 - \eta_n^3, 0\right]; DS_{n-1}^2 = \max\left[\xi_{n-1}^2 - \eta_n^2, 0\right] \tag{9.34}$$

$$DS_{n-1}^{1} = \max\left[\xi_{n-1}^{3} - \eta_{n}^{1}, 0\right]; DS_{n-1}^{0} = \max\left[\xi_{n-1}^{4} - \eta_{n}^{0}, 0\right] \tag{9.35}$$

The outstanding order equations for network 2 are listed below:

$$Y_{n+1}^{i} = \max\left\{0, Y_{n}^{i} + \xi_{n}^{1} + \xi_{n}^{2} + \xi_{n}^{3} + \xi_{n}^{4} - \eta_{n}^{i} - DS_{n-1}^{6}\right\} \, where\, i \in \left\{9,8,7\right\} \tag{9.36}$$

Similar to node 1 in network 1, node 6 in network 2 consists of an assembly of three products from nodes 9, 8, and 7; also consists of two sources of demand from nodes 5 and 4.

$$Y_{n+1}^{6} = Y_{n}^{6} + \sum_{j=1}^{4} \xi_{n}^{j} - DS_{n-1}^{5} - DS_{n-1}^{4} - \min\begin{Bmatrix} Y_{n}^{6} + \sum_{j=1}^{4} \xi_{n}^{j} - DS_{n-1}^{5} - DS_{n-1}^{4}, s^{9} - Y_{n-2}^{9} - \sum_{j=1}^{4} \xi_{n-1}^{j}, \\ s^{8} - Y_{n-2}^{8} - \sum_{j=1}^{4} \xi_{n-1}^{j}, s^{7} - Y_{n-2}^{7} - \sum_{j=1}^{4} \xi_{n-1}^{j}, \eta_{n}^{6} \end{Bmatrix} \tag{9.37}$$

$$Y_{n+1}^{5} = Y_{n}^{5} + \xi_{n}^{1} + \xi_{n}^{2} - DS_{n-1}^{3} - DS_{n-1}^{2} - \min\begin{Bmatrix} Y_{n}^{5} + \xi_{n}^{1} + \xi_{n}^{2} - DS_{n-1}^{3} - DS_{n-1}^{2}, \\ s^{6} - Y_{n-2}^{6} - \sum_{j=1}^{4} \xi_{n-1}^{j}, \eta_{n}^{5} \end{Bmatrix} \tag{9.38}$$

$$Y_{n+1}^{4} = Y_{n}^{4} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1} - DS_{n-1}^{0} - \min\begin{Bmatrix} Y_{n}^{4} + \xi_{n}^{3} + \xi_{n}^{4} - DS_{n-1}^{1} - DS_{n-1}^{0}, \\ s^{6} - Y_{n-2}^{6} - \sum_{j=1}^{4} \xi_{n-1}^{j} - \xi_{n-2}^{1} - \xi_{n-2}^{2}, \eta_{n}^{4} \end{Bmatrix} \tag{9.39}$$

Node 5 consists of two sources of demand (two retailers); a proportional allocation is used to split the available inventory to nodes 3 and 2. Similarly proportional allocation is used at node 4 to split the available inventory to nodes 1 and 0.

$$Y_{n+1}^{3} = Y_{n}^{3} + \xi_{n}^{1} - \min\left\{Y_{n}^{3} + \xi_{n}^{1}, ratio1 * \left(s^{5} - Y_{n-2}^{5} - \xi_{n-1}^{1} - \xi_{n-1}^{2}\right), \eta_{n}^{3}\right\} \tag{9.40}$$

$$Y_{n+1}^{2} = Y_{n}^{2} + \xi_{n}^{2} - \min\left\{Y_{n}^{2} + \xi_{n}^{2}, \left(1 - ratio1\right) * \left(s^{5} - Y_{n-2}^{5} - \xi_{n-1}^{1} - \xi_{n-1}^{2}\right), \eta_{n}^{2}\right\} \tag{9.41}$$

$$Y_{n+1}^{1} = Y_{n}^{1} + \xi_{n}^{3} - \min\left\{Y_{n}^{1} + \xi_{n}^{3}, ratio2 * \left(s^{4} - Y_{n-2}^{4} - \xi_{n-1}^{3} - \xi_{n-1}^{4}\right), \eta_{n}^{1}\right\} \tag{9.42}$$

$$Y_{n+1}^{0} = Y_n^0 + \xi_n^4 - \min\left\{Y_n^0 + \xi_n^4, \left(1 - ratio\,2\right) * \left(s^4 - Y_{n-2}^4 - \xi_{n-1}^3 - \xi_{n-1}^4\right), \eta_n^0\right\} \tag{9.43}$$

$$ratio\,1 = \frac{\xi_n^1}{\xi_n^1 + \xi_n^2}; \quad ratio\,2 = \frac{\xi_n^3}{\xi_n^3 + \xi_n^4} \tag{9.44}$$

The objective function and constraints, Lagrange function, first-order Lagrange equations, and the first order outstanding orders for network 2 are provided in the appendix, sections A9.4-A9.6

9.3 Computational Results

The computational results for both the networks are discussed here. Similar to the five-echelon computational results, we look at four instances and each instance consists of two to three scenarios of random demand and capacity values. The demand and capacity are classified into two types as defined in the table 9.1 and 9.2 respectively. In each scenario the demands have a combination of high and low demands. For instance in scenario #1 from table 9.4, demands 1-3 have high demand and demand 4 has a low demand value, but both the values are randomly picked by Microsoft excel from the classification table 9.1.

Table 9.1: Classification of Demand

Demand Type	Random Values Selected
High Demand	Between 8-10
Low Demand	Between 3.5- 5

Table 9.2: Classification of Capacity

Capacity Type	Random Capacity Utilization Values
Normal Capacity	Between 65 %-75%
Tight Capacity	Between 85%-95%

The instances created for the both the networks are listed below in table 9.3.

Table 9.3: Instances for Network 1 and Network 2

[redacted]

9.3.1 Computational Results from Network 1

Initially let us look at the results from network 1. The demand and capacity values for the four scenarios are provided below in table 9.4 and 9.5 respectively. ***The scenario numbers and the demand pattern numbers are limited only to this sub-section, i.e. to network 1***. A two period lead time is used to compute the results, and a LAPI policy is used at nodes 4,3, and 2. All the demand and capacity values are from a normal distribution, and the CV is provided in table 9.3. Scenarios # $\{1,\ 3,\ 5,\ 7\}$ (demand pattern 1) and $\{2,\ 4,\ 6,\ 8\}$ (demand pattern 2) have same demand patterns, i.e. high and low demand values for specific demand.

Table 9.4: Demand and Capacity Values for Instance 1 and 2 Network 1

Instance 1 : Normal Capacity

	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 1	8.2	8.8	8.6	3.5	16.6	17.3	18.1	17.4	16.9	16.5	4.8	4.8
Scenario 2	3.9	4.4	4	9.7	20.7	20.2	19	20.9	20.1	20.1	13.8	13.2

Instance 2: Tight Capacity

	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 3	8.1	9.9	9.8	4.9	13.8	16.2	15.6	14.6	17.1	17	5.4	5.2
Scenario 4	3.8	4.6	4.5	10	15.8	15.6	15.5	15.7	15.9	17	11	11.1

Table 9.5: Demand and Capacity Values for Instance 3 and 4 Network 1

Instance 3: Higher Demand Variance under Normal Capacity												
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 5	8.3	8.5	8.3	4.8	18	18	18.9	19.7	18.2	19	7.1	6.5
Scenario 6	4.5	4.9	3.9	9.8	19.9	20.2	18.3	19.8	20.4	18.4	14.7	14.5
Instance 4: Higher Demand Variance with Tighter Capacity (Worst Case Scenario)												
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 7	9.5	9.8	9.8	4.4	15.1	16.4	15.7	15.4	15.8	15.3	4.7	4.7
Scenario 8	3.7	5	4.1	10	15.1	16.4	16.1	15.8	17.2	16	10.7	10.8

Table 9.6: Total Safety-stock for Network 1 Scenarios Under Four Instances

Scenario #	Total Safety-Stock
1	24.64
2	19.52
3	42.08
4	42
5	96.96
6	108.86
7	225.6
8	192.24

Figure 9.3 and 9.4 are based on the results show in table 9.6. Figure 9.3 compares all the instances for demand pattern #1 (i.e. scenarios 1, 3, 5, and 7), and likewise figure 9.4 compares all the instances for demand pattern #2 (i.e. scenarios 2, 4, 6, and 8). In both the figures 9.3 and 9.4 we can observe that the safety-stock is lowest for instance 1 and highest for instance 4. Since instance 1 has a lower CV and capacity utilization value compared to other three instances, we observe a lower safety-stock value for scenario 1 and 2 in figures 9.3 and 9.4 respectively. Similarly instance 4 has the highest CV and utilization which results in the highest safety-stock assigned, i.e. scenarios 7 and 8 in figure 9.3 and 9.4 respectively.

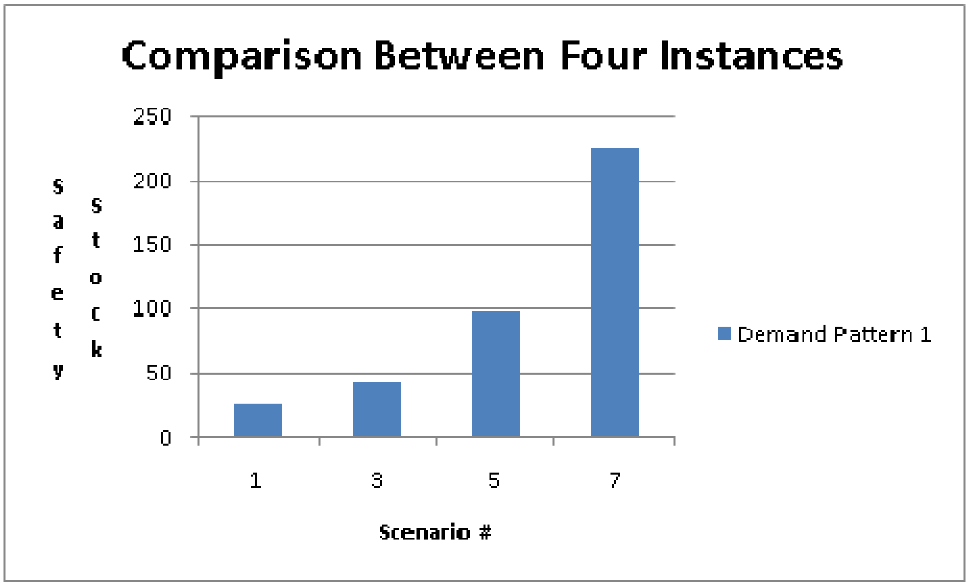

Figure 9.3: Comparison Between Instances for Network 1 Demand Pattern 1

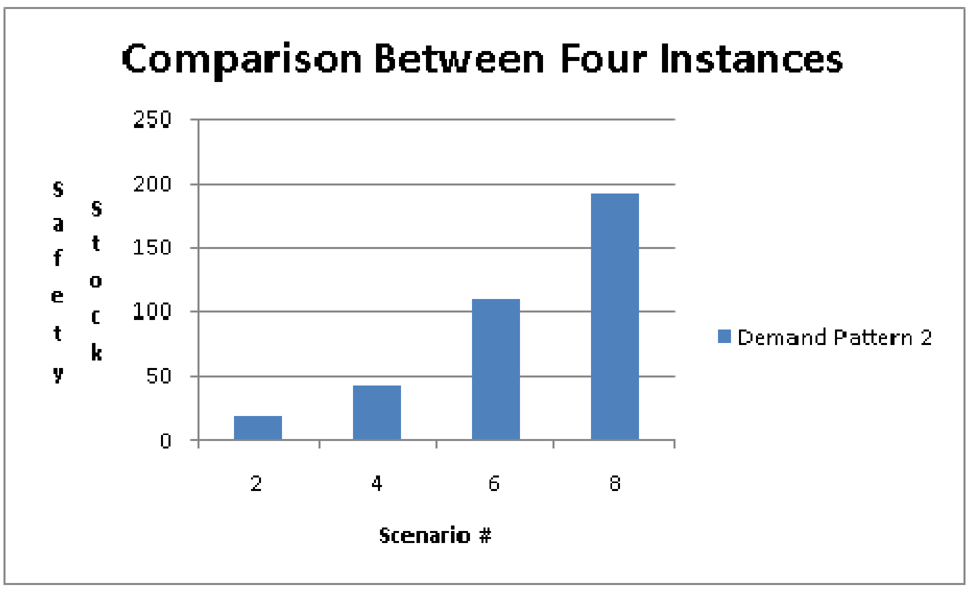

Figure 9.4: Comparison Between Instances for Network 1 Demand Pattern 2

9.3.2 Computational Results from Network 2

Let us look at the results from network 2. The demand and capacity values for the four scenarios are provided below in table 9.7 and 9.8 respectively. ***The scenario numbers and the demand pattern numbers are with respect only to this sub-section, i.e. to network 2***. A two period lead time is used to compute the results. Three inventory allocation policies are used at node 6: 1) lexicographic allocation policy with priority to node 5 (PT5), 2) lexicographic allocation policy with priority to node 4 (PT4), 3) proportional allocation (PA). Nodes 5 and 4 always use proportional allocation (PA). All the demand and capacity values are use a normal distribution, and the CV is provided in table 9.3. Scenarios # $\{1, 4, 7, 10\}$(demand pattern 1), $\{2, 5, 8, 11\}$(demand pattern 2), $\{3, 6, 9, 12\}$(demand pattern 3)and have same demand patterns, i.e. high and low demand values for specific demand.

Table 9.7: Demand and Capacity Values for Instance 1 and 2 Network 2

Instance 1 : Normal Capacity

	Demand 1	Demand 2	Demand 3	Demand 4	Capacity At Node 9	Capacity At Node 8	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 1	8.2	8.8	8.6	3.5	40.2	42.8	44.4	42.6	25.6	16.8	11.6	13.5	12.5	4.9
Scenario 2	4.4	4.4	9.1	8.8	40.9	35.9	37.1	40	12.5	26.7	6.5	6.6	13.3	11.8
Scenario 3	8.5	9.3	3.6	4	34.6	35.1	34.9	34.4	24.2	11.3	12.7	12.9	5.5	5.8

Instance 2: Tight Capacity

	Demand 1	Demand 2	Demand 3	Demand 4	Capacity At Node 9	Capacity At Node 8	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 4	8.1	9.9	9.8	4.9	37.8	34.5	35	34.9	19.8	15.6	9.4	10.7	10.6	5.7
Scenario 5	4.3	3.6	9.6	8.6	27.6	27.7	29.4	28.6	9.1	21.1	4.6	4	11.2	9.9
Scenario 6	9.3	8	4	3.7	28.1	27	27.4	27.8	19.2	8.2	9.9	9	4.3	4.3

Table 9.8: Demand and Capacity Values for Instance 3 and 4 Network 2

Instance 3: Higher Demand Variance under Normal Capacity

	Demand 1	Demand 2	Demand 3	Demand 4	Capacity At Node 9	Capacity At Node 8	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 7	8.3	8.5	8.3	4.8	41.7	45.1	44.3	43.2	24.9	17.9	12.5	11.9	11.6	6.5
Scenario 8	4.2	4.3	8.2	8.5	36.1	36.9	36.2	38.1	12.9	23.4	5.9	6.3	11.5	11.7
Scenario 9	9.3	9.4	4.6	3.6	41.1	38.3	40.3	37.7	25.7	11.1	12.9	13.1	6.9	5.1

Instance 4: Higher Demand Variance with Tighter Capacity (Worst Case Scenario)

	Demand 1	Demand 2	Demand 3	Demand 4	Capacity At Node 9	Capacity At Node 8	Capacity At Node 7	Capacity At Node 6	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 10	9.5	9.8	9.8	4.4	37.5	36.8	36.1	36.1	22.4	15.8	10.7	11.5	10.9	4.8
Scenario 11	4.1	4.7	9	9.5	32	32.1	29.3	30.5	9.4	19.6	4.5	5	10.4	10.1
Scenario 12	8.7	8.8	4.2	3.9	27	28.1	27.7	27.5	18.8	9.3	10.1	9.6	4.9	4.5

Table 9.9 show the total safety-stock values averaged across all scenarios under each instance. From the table 9.9 we can observe that PT5 allocation results in lower average total safety-stock cost under instance 1, PT4 under instance 2, PA under instance 3, and PT4 under instance 4. In order to understand the system better let us look at scenarios having same demand patterns.

Table 9.9: Average Total Safety-Stock Values for Network 2

	Average Safety-Stock		
Instance #	PT4	PT5	PA
1	30.03	28.67	40.19
2	76.44	82.43	80.13
3	171.32	176.38	165.87
4	369	370.6	372.83

Table 9.10 consists of safety-stock values from scenarios that come under demand pattern 1. We can observe that for scenarios 1 and 4 PT5 allocation policy provides the lowest safety-stock, and PA allocation policy provides the lowest safety-stock cost for scenarios 7 and 10. Under demand pattern 1 there is a higher demand at node 5 (demand 1 and 2) as opposed to node 4 (demand 3 and 4), similar to the other multi-echelon systems we see that lower total safety-stock cost results when an allocation priority is assigned to node with greater demand. In case of instances where increased demand and capacity variance (instance 3 and 4) exists we see that proportional allocation works marginally better than PT5 allocation.

Table 9.10: Safety-Stock Values for Network 2 Demand Pattern 1

	Safety-Stock		
Scenario #	PT4	PT5	PA
1	30	27.8	59.98
4	97	94	97
7	181.96	179.4	176.1
10	387	387	386.7

Table 9.11 provides the safety-stock values for scenarios under demand pattern 2. From the table 9.11 we can observe that the PT4 provides the significantly lowest safety-stock cost compared to other two allocation policy. Under demand pattern 2 there is a higher demand at node 4 (demand 3 and 4) as opposed to node 5 (demand 1 and 2), similar to the demand pattern 1 we observe that lower total safety-stock cost results when an allocation priority is assigned to node with greater demand.

Table 9.11: Safety-Stock Values for Network 2 Demand Pattern 2

	Safety-Stock		
Scenario #	**PT4**	**PT5**	**PA**
2	30.1	30.5	30.6
5	65.92	86.9	75.88
8	151	169.48	165.5
11	361	361	362.4

Table 9.12 provides the safety-stock values for scenarios under demand pattern 3. Similarly we can observe from table 9.12 that the PT5 provides the lowest safety-stock cost compared to other two allocation policy. Under demand pattern 3 there is a higher demand at node 5 (demand 1 and 2) as opposed to node 4 (demand 3 and 4), similar to the other multi-echelon systems we see that lower total safety-stock cost results when an allocation priority is assigned to node with greater demand. Except for scenario 9, which belongs to instance 3 in all the other scenarios shown in table 9.12 PT5 provides a lower safety stock.

In case of an increased demand variance (scenario 9 and 7, instance 3) we can observe from table 9.10 and 9.12 that PA works significantly better that PT5.

Table 9.12: Safety-Stock Values for Network 2 Demand Pattern 3

	Safety-Stock		
Scenario #	**PT4**	**PT5**	**PA**
3	30	27.7	30
6	66.4	66.4	67.5
9	181	180.26	156
12	359	359	369.4

Based on the results from the larger networks, we can infer that it is suggestible to assign priority to the node or source of demand which has higher proportion of the total demand, with an exception for instance 3 where the demand variance is large, PA policy should be used. Additional results for the two networks can be found in the appendix A10.

10. HEURISTIC STARTING POINTS

In this chapter we develop two heuristic approaches that determine target base-stock levels for each component, intermediate products, and final products in the multi-echelon inventory system for a given service level. The approach will be based on the following:

- *Rule based approach* : Determining good initial points/starting points for the search based on a set of rules
- *Decomposition approach* : Determine the initial base-stock levels for each node based on a decomposition approach

The basic idea is to develop heuristic starting points, and determine target base-stock levels for each node much quicker and closer to the traditional starting points (mean demand during lead time) used for the frameworks earlier. Each approach is discussed in detail following the analysis of computational results for the multi-echelon inventory systems.

10.1 Rule Based Approach

Determining good initial/starting points indicates providing the optimization framework with good starting base-stock values for each node which are close to the best found answer. The question arises how this can be achieved. Based on the numerical cases that have been presented in earlier sections using the IPA framework a unique set of rules for the various scenarios were developed and are stated below. All the scenarios determine an initial base-stock level for each node. The best found base-stock level depends on $\xi, \eta, \text{and service level for a node}$, so the determination of an initial point (base-stock level) should account for all of these factors.

Scenario 1: Demand is stochastic and capacity is deterministic in nature

*Initial point for a node: = Mean demand during the lead time for the node + K1 * coefficient of variance (C.V) of demand+ K2 – K3*

Scenario 2: Demand is deterministic and capacity is stochastic in nature

*Initial point for a node: = Mean demand during the lead time for the node + K1 * coefficient of variance (C.V) of capacity+ K2 – K3*

Scenario 3: Demand is stochastic and capacity is stochastic in nature

*Initial point for a node: = Mean demand during the lead time for the node + K1 * coefficient of variance (C.V) of demand + K1 * coefficient of variance (C.V) of capacity + K2 – K3*

Since the demand and capacity being deterministic or stochastic would affect the way the initial point for each node is determined, it is evident that we need three sets of rules, one for each scenario. Here *K1, K2, K3* and *K4* are constants. The definitions for *K1, K2*, K3 and *K4* are described below:

$$K1 = \left(Z - value\right) * \left(Discount\ factor\right) * \left(Inflation\ factor\right)$$

$$K2 = Echelon\ factor * echelon's\ away * mean\ demand\ During\ lead\ time$$

$$K3 = Non\ linear\ lead\ time\ factor * Mean\ demand\ during\ lead\ time$$

$Z - value$ = The value is obtained from the standard normal tables for the given service level. For instance, the value of *Z* for a 0.9 (or in other words 90% service level) is 1.3.

Discount factor = The discount factor is used to reduce the value of the starting point (initial base-stock level) on the basis of capacity utilization. Based on the past observations it is found that at lower capacity utilizations the large demand variance has only a small effect on the base-stock level.

$$Discount\ factor\ = \begin{cases} 0.4 & -if\ capacity\ utilization \le 75\% \\ 0.6 & -if\ capacity\ utilization\ is\ 75\% - 85\% \\ 0.8 & -if\ capacity\ utilization\ is\ 85\% - 90\% \\ 1(no\ discount) & -if\ capacity\ utilization\ is > 90\% \end{cases}$$

Inflation factor = The inflation factor is used to increase the value of the starting point (initial base-stock level) on the basis of demand or capacity CV. There are two sets of inflation factors used, one for multi-echelon inventory system with fewer than 6 nodes and another for systems with more than 6 nodes and fewer than 12.

Inflation factor for system with nodes less than 6:-

$$\text{If } CV \le 0.1\ -1(\text{no inflation})$$

$$\text{If } 0.1 < CV \le 0.3 = \begin{cases} 1.5 & -if\ capacity\ utilization\ 65\% - 75\% \\ 1.7 & -if\ capacity\ utilization\ 75\% - 85\% \\ 2 & -if\ capacity\ utilization > 85\% \end{cases}$$

Similarly *Inflation factor* for system with nodes greater than 6 and less than 12:

$$\text{If } CV \le 0.1\ -1(\text{no inflation})$$

$$\text{If } 0.1 < CV \le 0.3 = \begin{cases} 1.8 & -if\ capacity\ utilization\ 65\% - 75\% \\ 2 & -if\ capacity\ utilization\ 75\% - 85\% \\ 2.2 & -if\ capacity\ utilization > 85\% \end{cases}$$

Echelon factor = Depending on the structure of the node, i.e. multiple sources of demand or single source of demand a constant value is added to the starting point.

$$Echelon\ factor = \begin{cases} Multiple\ sources\ of\ demand : -0.03 \\ Single\ sources\ of\ demand\ \ : -0.01 \end{cases}$$

Echelons away = Depending on a node's position in the supply chain, i.e. number of echelons away from the supplier, base-stock level at a node is slightly higher or lower.

The echelon factor is multiplied with the number of echelons away from the source (supplier's node).

$$Echelons\, away :- \{1, 2, 3 \ldots m\}$$

Non-linear lead time factor: As lead time between echelons increases the safety-stock for a node increases in a non-linear fashion. In order to address this non-linear increase in safety-stock level the non-linear lead time factor is introduced. The starting point increases linearly, and in a significant way if this factor is not subtracted from the initial point as the lead time between echelons increases.

$$Non-linear\, lead\, time\, factor = \begin{cases} 1\, period\, lead\, time : \begin{bmatrix} Nodes\, with\, multiple\, sources - : 0.01 \\ Nodes\, with\, single\, source \quad - : 0.02 \end{bmatrix} \\ 2\, period\, lead\, time : \begin{bmatrix} Nodes\, with\, multiple\, sources - : 0.02 \\ Nodes\, with\, single\, source \quad - : 0.04 \end{bmatrix} \\ 3\, period\, lead\, time : \begin{bmatrix} Nodes\, with\, multiple\, sources - : 0.04 \\ Nodes\, with\, single\, source \quad - : 0.06 \end{bmatrix} \end{cases}$$

10.2 Decomposition Approach

In decomposition approach each node in a multi-echelon inventory system is considered as an individual node (single-echelon). From the computational results, it has been observed that the best found base-stock level for a node when considered as an individual node is, less than or equal to the best found base-stock level of a node when it is a part of a multi-echelon. This has prompted to look for a decomposition approach which provides heuristic starting points for the IPA based search. The best found base-stock level for each node in an *m*-echelon problem is obtained by solving *m* or more single-echelon problems. This base-stock value for the single-echelon is used as a starting point for the node in the multi-echelon inventory system. By solving the single-echelon problem we can obtain good starting points for the component, intermediate, final product nodes.

10.3 Computational Results

The computational results for the two approaches, 1) rule based approach 2) decomposition approach are studied for the three-echelon, five-echelon, and the two multi-echelon networks discussed in chapter 9. *All the case numbers and instance numbers are with respect to chapter 10 only*. Definitions of the terms used in the following sub-sections are described below:

$$\textit{Percentage Relative Error}\left(PRE\right): \left[\frac{\text{Total Heuristic Cost-Total BestFound Cost}}{\text{Total Optimal Cost}}\right]*100\%$$

Total Heuristic Cost: It is the sum of all the target base-stock levels obtained from either using the rule based heuristic starting points or using the decomposition approach.

Total Best found Cost: It is the sum of all the best found base-stock levels that are obtained using the traditional starting points (mean demand during lead time) for each node.

Instances: Three instances used for the computational results are: 1) ***instance 1***-: random demand and deterministic capacity, 2) ***instance 2***-: deterministic demand and random capacity, 3) ***instance 3*** -: random demand and random capacity

Heuristic: - It refers to the total cost from either 1) rule based approach or 2) the decomposition approach. The base-stock levels obtained for each node by using the heuristic starting points are also referred to as *target base-stock levels* throughout the chapter and in the appendix.

A 90% service level is used for all the nodes, and all the units for the time are in minutes. A normal distribution is used for both the demand and capacity depending on instances. The starting points for the search using both the approaches are provided in the appendix A11.

10.3.1 Heuristic Performance for Three-echelon Assembly System

The demand, capacity, lead time, and CV values used for the three-echelon assembly system are provided in the table 10.1. The values in table 10.1 are used to develop the heuristic starting points for the rule based approach. The heuristic starting points (initial base-stock levels for each node) are based on the set of rules discussed in section 10.1.

Table 10.1: Three-Echelon Assembly System Input Values Used For Performance Evaluation of Heuristic

	Demand 1	Demand 2	Node 3	Node 2	Node 1	Node 0	Lead time (periods)	CV
Case 1	17.19	9.32	36.1	40.38	37.79	14.34	1	0.1
Case 2	17.93	8.57	38.5	39.27	38.94	13.18	1	0.1
Case 3	12.55	6.9	26.03	27.73	27	10.62	1	0.3
Case 4	13.67	12.17	36	39.74	36.48	18.72	2	0.1
Case 5	10.25	5.16	20.96	23.32	23.01	7.94	2	0.3
Case 6	18.85	7.09	39.19	37.43	36.78	10.91	2	0.3
Case 7	14.88	11.96	40.89	36.68	39.33	18.4	3	0.1
Case 8	18.85	7.15	35.23	38.81	35.58	11	3	0.1
Case 9	14.95	7.94	30.9	31.1	34.6	12.22	3	0.3

The performance results for instance 1 using the rule based approach are shown in table 10.2. All the demand and capacity values are based on table 10.1, in which the demand values are random and the capacity values are deterministic. From the results we can observe that the using a rule-based approach reduces the amount of time taken for search to terminate. Specifically for case 6 and 9 the time taken for the traditional approach is close to 15 min, whereas the time taken using the rule-base approach is less than 5 sec. We can also observe that the PRE (percentage relative error) for 7 out of 9 cases is less than 2%.

Table 10.2: Performance of Rule Based Approach For Three-echelon Instance 1

	Traditional Approach		Rule Based Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 1	**92.72**	0.84	**94.35**	0.10	1.76	1	0.1
Case 2	**91.94**	0.58	**93.52**	0.06	1.72	1	0.1
Case 3	**103.76**	11.85	**108.37**	8.4	4.44	1	0.3
Case 4	**183.25**	0.54	**187.43**	0.06	2.28	2	0.1
Case 5	**125.42**	4.96	**126.11**	0.06	0.55	2	0.3
Case 6	**207.83**	14.26	**208.38**	0.05	0.26	2	0.3
Case 7	**281.31**	0.64	**285.3**	0.11	1.42	3	0.1
Case 8	**259.48**	0.56	**262.81**	0.1	1.28	3	0.1
Case 9	**272.34**	14.52	**277.41**	0.06	1.86	3	0.3

The performance results for instance 1 using the decomposition approach are shown in table 10.3. For the decomposition approach only two period lead times are considered. Though there is a noticeable reduction in time taken for the search to terminate, the PRE value is larger than the rule based approach.

Table 10.3: Performance of Decomposition Approach For Three-echelon Instance 1

	Traditional Approach		Decomposition Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 4	**183.25**	0.54	**196.04**	0.10	6.98	2	0.1
Case 5	**125.42**	4.96	**131.53**	0.07	4.87	2	0.3
Case 6	**207.83**	14.26	**219.04**	0.5	5.39	2	0.3

The performance results for instance 2 using the rule based approach are shown in table 10.4. All the demand and capacity values are based on table 10.1, in which the demand values are deterministic and the capacity values are random. Except for case 3, we do observe a significant time difference taken for the search to terminate between the traditional approach and the rule based approach. The PRE for 6 out of 9 cases is less than 2%.

Table 10.4: Performance of Rule Based Approach For Three-echelon Instance 2

	Traditional Approach		Rule Based Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 1	92.72	0.84	94.35	0.10	1.76	1	0.1
Case 2	91.94	0.58	93.52	0.06	1.72	1	0.1
Case 3	103.76	11.85	106.21	8.83	2.36	1	0.3
Case 4	181.1	0.54	187.43	0.06	3.5	2	0.1
Case 5	125.42	4.96	128.35	0.4	2.34	2	0.3
Case 6	207.83	14.26	208.3	0.4	0.23	2	0.3
Case 7	281.31	0.64	284.35	0.11	1.08	3	0.1
Case 8	259.48	0.56	262.81	0.11	1.28	3	0.1
Case 9	272.34	14.52	277.41	0.06	1.86	3	0.3

The performance results for instance 2 using the decomposition approach are shown in table 10.5. Comparing the PRE in table 10.4 with 10.5, 1 out of 3 cases the decomposition approach provides lower PRE, but the rule based approach provides much lower search time compared to the decomposition approach.

Table 10.5: Performance of Decomposition Approach For Three-echelon Instance 2

	Traditional Approach		Decomposition Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 4	181.1	0.54	181.37	0.10	0.15	2	0.1
Case 5	125.42	4.96	129.02	5.4	2.87	2	0.3
Case 6	207.83	14.26	213.38	6.67	2.67	2	0.3

The performance results for instance 3 using the rule based approach are shown in table 10.6. All the demand and capacity values are based of table 10.1, in which the demand and the capacity values are random. Except for case 3, we do observe a significant time difference taken for the search to terminate between the traditional approach and the rule based approach. The PRE for 6 out of 9 cases (except lead time of two periods) is less than 2%.

Table 10.6: Performance of Rule Based Approach For Three-echelon Instance 3

	Traditional Approach		Rule Based Approach		Percentage Relative Error	Lead time (periods)	CV
	Total Cost	Time(min)	Total Cost	Time (Min)			
Case 1	**101.57**	1.59	**102.56**	0.42	0.97	1	0.1
Case 2	**100.79**	1.45	**101.7**	0.4	0.9	1	0.1
Case 3	**117.12**	24.77	**117.76**	4.6	0.55	1	0.3
Case 4	**194.73**	4.25	**200.34**	0.46	2.88	2	0.1
Case 5	**141.47**	6.59	**150.17**	0.06	6.15	2	0.3
Case 6	**238.85**	26.85	**247.06**	0.15	3.44	2	0.3
Case 7	**281.31**	0.6	**282.13**	0.06	0.29	3	0.1
Case 8	**259.41**	0.56	**259.8**	0.06	0.15	3	0.1
Case 9	**233.7**	0.56	**233.73**	0.06	0.01	3	0.3

The performance results for instance 3 using the decomposition approach are shown in table 10.7. Comparing the PRE in table 10.7 with 10.6, all the three cases the decomposition approach provides lower PRE, but the rule based approach provides much lower search time in 2 out of 3 cases compared to the decomposition approach.

Table 10.7: Performance of Decomposition Approach For Three-echelon Instance 3

	Traditional Approach		Decomposition Approach		Percentage Relative Error	Lead time (periods)	CV
	Total Cost	Time(min)	Total Cost	Time (Min)			
Case 4	**194.73**	4.25	**195.26**	0.05	0.27	2	0.1
Case 5	**141.47**	6.59	**142.1**	3.83	0.45	2	0.3
Case 6	**238.85**	26.85	**239.03**	9.23	0.08	2	0.3

10.3.2 Heuristic Performance for Five-echelon Assembly System

The demand, lead time, and CV values used for the five-echelon assembly system are provided in the table 10.8. The capacity values used for the five-echelon assembly system are provided in the table 10.9. The values in table 10.8 and 10.9 are used to develop the heuristic starting points for the rule based approach using the rules discussed in section 10.1. Only cases with two-period lead time and under instance 3 are considered.

Table 10.8 Demand, Lead Time and CV Values for Five-echelon Assembly System Used For Performance Evaluation of Heuristic

	Demand 1	Demand 2	Demand 3	Demand 4	Lead time	CV
Case 10	17.97	13.84	12.88	9.22	2	0.1
Case 11	13.55	14.02	8.16	8.06	2	0.1
Case 12	15.81	10.58	9.14	6.2	2	0.3

Table 10.9 Capacity Values for Five-echelon Assembly System Used For Performance Evaluation of Heuristic

	Node 5	Node 4	Node 3	Node 2	Node 1	Node 0
Case 10	73.99	73.09	72.98	52.03	32.19	12.33
Case 11	58.77	60.15	59.34	43.59	22.09	10.75
Case 12	57.34	57.31	63.17	36.31	20.48	9.44

The performance results for instance 3 using the rule based approach are shown in table 10.10. All the demand and capacity values are based on table 10.8 and 10.9, in which the demand and the capacity values are random. We do observe a significant time difference taken for the search to terminate between the traditional approach and the rule based approach. The PRE for all cases is less than 2.5%.

Table 10.10: Performance of Rule Based Approach For Five-echelon Instance 3

	Traditional Approach		Rule Based Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 10	**489.06**	6.02	**499.38**	0.06	2.11	2	0.1
Case 11	**396.83**	12.12	**406.74**	0.14	2.5	2	0.1
Case 12	**465.84**	65	**465.88**	4.49	0.01	2	0.3

The performance results for instance 3 using the decomposition approach are shown in table 10.11. The amount of time taken to terminate and the PRE are relatively close in both the heuristic approaches for the five-echelon assembly system under instance 3.

Table 10.11: Performance of Decomposition Approach For Five-echelon Instance 3

	Traditional Approach		Decomposition Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 10	**489.06**	6.02	**501.46**	0.10	2.54	2	0.1
Case 11	**396.83**	12.12	**406.75**	0.06	2.5	2	0.1
Case 12	**465.84**	65	**473.27**	1.67	1.59	2	0.3

10.3.3 Heuristic Performance for Multi-echelon Networks

The demand, lead time, and CV values used for network 1 and 2 are provided in the table 10.12 and 10.16 respectively. The capacity values used for network 1 and 2 are provided in the table 10.13 and 10.17 respectively. The values in table 10.12, 10.13, 10.16 and 10.17 are used to develop the heuristic starting points for the rule based approach using the rules discussed in section 10.1. Only cases with two-period lead time and under instance 3 are considered.

Table 10.12 Demand, Lead Time and CV Values for Network 1 Used For Performance Evaluation of Heuristic

	Demand 1	Demand 2	Demand 3	Demand 4	Lead time	CV
Case 13	12.8	11.51	10.83	6.67	2	0.1
Case 14	8.83	12.33	10.23	6.63	2	0.1
Case 15	11.1	12.31	12.97	9.47	2	0.3

Table 10.13 Capacity Values for Network 1 Used For Performance Evaluation of Heuristic

	Node 7	Node 6	Node 5	Node 4	Node 3	Node 2	Node 1	Node 0
Case 13	28.86	26.68	23.89	29.01	24.44	25.41	9.59	9.6
Case 14	20.94	26.01	25.53	23.77	26.88	24.1	10.11	9.99
Case 15	30.4	31.29	33.53	29.75	31.08	30.15	13.18	13.25

The performance results for instance 3 using the rule based approach are shown in table 10.14. All the demand and capacity values are based of table 10.12 and 10.13, in which the demand and the capacity values are random. We do observe a significant time difference taken for the search to terminate between the traditional approach and the rule based approach. This is slightly higher than what we have seen in the earlier multi-echelon systems.

Table 10.14: Performance of Rule Based Approach For Network 1 Instance 3

	Traditional Approach		Rule Based Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 13	272.3	2.25	279.24	0.80	2.55	2	0.1
Case 14	255.4	2.8	268.38	0.5	5.08	2	0.1
Case 15	448.6	39	466.92	1.06	4.08	2	0.3

The performance results for instance 3 using the decomposition approach are shown in table 10.15. Comparing the PRE in table 10.15 with 10.14 we can observe that in all the cases decomposition approach performs better than the rule based approach. For cases with CV = 0.1 the amount of time taken to terminate is lower for the decomposition approach when compared to the rule based approach. At higher CV, i.e. CV = 0.3 the time taken by rule based approach (1.06 min) is significantly lower than the decomposition approach (21.1 min).

Table 10.15: Performance of Decomposition Approach For Network 1 Instance 3

	Traditional Approach		Decomposition Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 13	272.3	2.25	278.14	0.12	2.14	2	0.1
Case 14	255.4	2.8	257.12	0.47	0.67	2	0.1
Case 15	448.6	39	449.09	21.1	0.11	2	0.3

Table 10.16 Demand, Lead Time and CV Values for Network 2 Used For Performance Evaluation of Heuristic

	Demand 1	Demand 2	Demand 3	Demand 4	Lead time	CV
Case 16	8.45	8.97	10.66	6.99	2	0.1
Case 17	9.13	9.52	11.06	5.61	2	0.1
Case 18	11.37	8.14	10.12	9.72	2	0.3

Table 10.17 Capacity Values for Network 2 Used For Performance Evaluation of Heuristic

	Node 9	Node 8	Node 7	Node 6	Node 5	Node 4	Node 3	Node 2	Node 1	Node 0
Case 16	48.81	47.88	49.08	46.79	25.44	26.9	12.02	13.05	14.78	10.16
Case 17	50.64	47.73	49.51	53.77	25.87	22.25	13.27	14.11	15.92	8.25
Case 18	54.03	59.36	53.49	53.42	29.48	29.35	17.04	11.36	15.32	14.51

The performance results for instance 3 using the rule based approach are shown in table 10.18. All the demand and capacity values are based of table 10.16 and 10.17, in which the demand and the capacity values are random. The performance results for instance 3 using the decomposition approach are shown in table 10.19. Similar inference made to network 1 can be made for network 2.

Table 10.18: Performance of Rule Based Approach For Network 2 Instance 3

	Traditional Approach		Rule Based Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 16	461.2	7.8	462.24	1.03	0.23	2	0.1
Case 17	477.83	7.4	478.06	3	0.05	2	0.1
Case 18	652.2	71.8	678.65	0.06	4.06	2	0.3

Table 10.19: Performance of Decomposition Approach For Network 2 Instance 3

	Traditional Approach		Decomposition Approach		Percentage	Lead time	
	Total Cost	Time(min)	Total Cost	Time (Min)	Relative Error	(periods)	CV
Case 16	461.2	7.8	462.56	0.06	0.29	2	0.1
Case 17	477.83	7.4	480.31	0.12	0.52	2	0.1
Case 18	652.2	71.8	655.58	34.32	0.52	2	0.3

Additional results can be found in the appendix A11.

10.4 Inferences Based of Heuristic Starting Points

Based on the numerical results for heuristic starting points we can infer the following:

- For a three-echelon assembly system

- Under instance 1- random demand and deterministic capacity and instance 2 - deterministic demand and random capacity, rule based approach works better than the decomposition approach in terms of time taken to obtain total cost and PRE.
- Under instance 3 – random demand and capacity, rule based approach works better than the decomposition approach in terms of time taken to obtain the total cost. But, decomposition approach provides much lower PRE compared to the rule based approach.
- In most cases rule based approach works better than decomposition approach (time, PRE) for a three-echelon assembly system.

- For a five-echelon assembly system and larger networks
  - The time taken to obtain the total cost using the decomposition approach is quicker than the rule based approach for lower demand and capacity variance.
  - The PRE for decomposition approach is lower than the rule based approach when for lower demand and capacity variance.
  - Under high demand and capacity variance rule based approach works better than the decomposition approach in terms of both time taken to obtain the total cost and PRE.

11. CONCLUSION AND FUTURE RESEARCH

In this research we studied several multi-echelon inventory systems with stochastic capacity and intermediate product demand. Specifically we analyzed at how the system would behave when several intermediate product demands occur in a supply chain. The analysis was three fold i) developed update (relational) equations for the multi-echelon assembly systems, ii) develop two simulation optimization approaches to obtain the best found base-stock level for each node in the assembly system that satisfy the required amount of service level, iii) extensive analysis of the numerical results for the multi-echelon inventory systems.

The update equations are developed for three-echelon, five-echelon, m-echelon, and large multi-echelon network models under four different inventory allocation policies. Two frameworks were used as a part of the simulation optimization, 1) *OptQuest* and 2) IPA (Infinitesimal Perturbation Analysis). *OptQuest* which is a tool in *ARENA* was used as a part of the initial study (only for three-echelon) to obtain the base-stock levels for each node which satisfy the desired service level. Gradient estimation in the multi-echelon inventory system supports simulation optimization using an IPA framework. The simulation optimization using IPA uses a combination of *ARENA*, Visual Basic, and Xpress. A detailed numerical analysis of the three-echelon, five-echelon, and large networks inventory system is conducted. A hypothesis testing experiment was conducted on the results from the five-echelon assembly system to gain additional insight. From the numerical results we identify what would lead to a lower total system cost and safety-stock cost under specific cases and instances. In particular we noted that for some instances a combination of inventory allocation policies across the supply chain might result in lower total system cost, whereas for other instances using one type inventory allocation policy across the supply chain will result in the lowest cost. We showed that adding multiple sources of demand to the traditional multi-echelon problem leads to an interesting allocation issue, complex mathematical equations, a different problem, and new inferences based of the numerical results.

Heuristic starting points using two approaches 1) rule based 2) decomposition are implemented to obtain the near best found base-stock levels much quicker than using the traditional starting points for the search. The two heuristic approaches are tested on larger networks to demonstrate the robustness. From the performance evaluation of the heuristic approaches we also noted that there can be a significant reduction in the search time if one of the two approaches is used.

The research done in this dissertation differs from earlier works, since it considers a complex (combination of serial and assembly systems) multi-period multi-echelon inventory system with several sources of demand (specifically intermediate product demands). We obtain the best found base-stock levels for each node in the system that satisfies the required customer service level. A SIO approach is used to obtain the best found base-stock level for the system under several inventory allocation policies. We consider a system which is closer to the actual world and can be used to solve contemporary issues like, 1) manufacturing firm that produces finished products as well as spare parts, 2) manufacturer – warehouse – distribution center – retail outlets etc. I am not aware of any work that studies the impact of inventory allocation polices for multi-period in a multi-echelon inventory system, and obtains best found base stock level for each node using an IPA framework. Moreover the best found base-stock level for each node is obtained under realistic conditions like stochastic demand, stochastic capacity, and lead time.

There are several interesting directions for the future research, which include developing a novel heuristic approach (for instance using nested partitions) to obtain the near best found base-stock levels that does not use IPA based search. Another research direction would be to introduce the aspect of multiple final products, where each final product needs to maintain a desired service and obtain the best found base-stock level for all the final products. A further extension would be to develop a multi-echelon inventory system with random lead time, by relaxing some of the assumptions made for the current multi-echelon inventory systems. One more tough yet interesting aspect for the future research would be to prove that the multi-echelon inventory systems are quasi-convex in nature.

12. REFERENCES

Agrawal, N., & Cohen, M. (2001). Optimal material control and performance evaluation in an assembly environment and component commonality. *Naval Research Logistics, 48*, 409-429.

Andersson, J., & Melchiors, P. (2001). A two-echelon inventory model with lost sales. *International Journal of Production Economics, 69*, 307-315.

Anupindi, R., & Akella, R. (1993). Diversification under supply uncertainty.*39*, 944-963.

April, J., Glover, F., Kelly, J., & Laguna, M. (2001). Simulation/Optimization using real-world applications. *Winter Simulation Conference,*

Arora, J. S. (2004). *Introduction to optimum design* (2nd ed.)Academic Press.

Arrow, K. J., Harris, T., & Marschak, J. (1951). Optimal inventory policy. *Econometrica, 19*, 250-272.

Axsäater, S. (1990). Simple solution procedures for a class of two-echelon inventory problems. *Operations research, 38*(1), 64-69.

Axsater, S., & Rosling, K. (1993). Notes: Installation vs. echelon stock policies for multilevel inventory control. *Management Science, 39*(10), 1274-1280.

Axsater, S., & Rosling, K. (1994). Multi-level production-inventory control: Material requirements planning or reorder point policies. *European Journal of Operations Research, 75*, 405-412.

Axsäter, S. (1993). Exact and approximate evaluation of batch-ordering policies for two-level inventory systems. *Operations research, 41*(4), 777-785.

Badinelli, R. D. (1992). A model for continuous-review pull policies in serial inventory systems. *Operations research, 40*(1), 142-156.

Banks, J. (1998). *Handbook of simulation: Principles, methodology, advances, applications, and practice* (1st ed.)John Wiley & Sons, Inc.

Bashyam, S., & Fu, M. C. (1998). Optimization of (s,S) inventory systems with random lead times and a sevice level constraint. *Management Science, 44*(12), 243-255.

Bassok, Y., & Akella, R. (1991). Combined component ordering and production decisions in manufacturing systems with supply quality and demand uncertainty. *Management Science, 37*(12), 1556-1574.

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2004). *Nonlinear programming - theory and algorithms* (2nd ed.)John Wiley & Sons, Inc.

Beamon, B.M., & Chen, V.C.P. (2001). Performance analysis of conjoined supply chains. *International Journal of Production Research, 39*, 3195–3218.

Bellman, R. (Ed.). (1957). *Dynamic programming*. Princeton, NJ: Princeton University Press.

Bish, E. K., Muriel, A., & Biller, S. (2006). Managing flexible capacity in a make-to-order environment. *Management Science, 51*(2), 167-180.

Bollapragada, R., Rao, U. S., & Zhang, J. (2004). Managing inventory and supply performance in assembly systems with random supply capacity and demand. *Management Science, 50*(12), 1729-1743.

Bollapragada, R., Rao, U. S., & Zhang, J. (2004a). Managing two-stage serial inventory systems under demand and supply uncertainty and customer service level requirements. *IIE Transactions, 36*, 73-85.

Box, G. E. P., & Draper, N. R. (1987). *Empirical model-building and response surfaces* (1st ed.)John Wiley & Sons, Inc.

Cachon, G., & Terwiesch, C. (2005). *Matching supply with demand: An introduction to operations management* (1st ed.)McGraw-Hill/Irwin.

Carr, S. M., & Karmarkar, U. S. (2005). Competition in multi-echelon assembly supply chains. *Management Science, 51*(1), 45-59.

Carson, Y., & Maria, A. (1997). Simulation optimization: Methods and applications Paper presented at the *Proceedings of the 1997 Winter Simulation Conference,*

Chang, H. K., Yushin, H., & Soo-Young, K. (1997). An extended optimal lot sizing model with an unreliable machine. *Production Planning & Control, 8*(6), 577-585.

Chen, F., & Zheng, Y. S. (1994). Evaluating echelon stock (R, nQ) policies in serial Production/Inventory systems with stochastic demand. *Management Science, 40*, 1262-1275.

Cheung, K. L., & Hausman, W. (1995). Multiple failures in a multi-item spares inventory model - IIE transactions. *IIE Transactions, 27*, 171-180. Retrieved 8/10/2007 from

Ciarallo, F. W., Akella, R., & Morton, T. E. (1994). A periodic review, production planning model with uncertain capacity and uncertain demand- optimality of extended myopic policies. *Management Science, 40*(3), 320-332.

Clark, A. J., & Scraf, H. (1960). Optimal policies for a multi-echelon inventory problem. *Management Science, 50*(12), 1782-1790.

Daniel, J.S.R., & Rajendran, C. (2005). A simulation-based genetic algorithm for inventory optimization in a serial supply chain. *International Transactions In Operational Research, 12*, 101-127.

De Bodt, M. A., & Graves, S. C. (1985). Continuous-review policies for a multi -echelon inventory problem with stochastic demand. *Management Science, 31*(10), 1286-1299.

Diaz, A. (2003). Modeling approaches to optimize spares in multi-echelon systems. *International Journal of Logistics: Research and Applications, 6*(1-2), 51-62.

Diks, E. B., de Kok, A. G., & Lagodimos, A. G. (1996). Multi-echelon systems: A service measure perspective. *European Journal of Operations Research, 95*, 241-263.

Disney, S.M, Naim, M.M., & Towill, D.R. (1997). Dynamic simulation modeling for lean logistics. *International Journal of Physical Distribution and Logistics Management, 27*, 174–196.

Erkip, N., Hausman, W., & Nahmias, S. (1990). Optimal centralized ordering policies in multi-echelon inventory systems with correlated demands. *Management Science, 36*, 381-392.

Ettl, M., Feigin, G. E., Lin, G. Y., & Yao, D. (2000). A supply network model with base-stock control and service requirements. *Operations Research,* (48), 216-232.

Ettl, M., Feigin, G. E., Lin, G. Y., & Yao, D. D. (2000). A supply network model with base-stock control and service requirements. *Operations research, 48*(2), 216.

Eylem, T., & Ihsan, S. (2004). Simulation optimization: A comprehensive review on the theory and applications. *IIE Transactions, 11*, 1067-1081.

Federgruen, A., & Zipkin, P. (1986). An inventory model with limited production capacity and uncertain demand II- The discounted-cost criterion. *Mathematics of Operations Research, 11*(2), 208-215.

Federgruen, A., & Zipkin, P. (1984). Computational issues in an infinite-horizon, multi-echelon inventory model. *Operations research, 32*(4), 818-836.

Fu, M. C. (1994). Optimization via simulation: A review. *Annals of operation research, 53*, 199-247.

Fu, M. C. (2002). Optimization for simulation: Theory vs. practice. *INFORMS journal on computing, 14*(3), 192-215.

Fu, M. C. (2001). Simulation optimization. *Winter Simulation Conference,*

Gallego, G., & Ozer, O. (2001). Integrating replenishment decisions with advance demand information. *Management Science, 47*(10), 1344-1360.

Gallego, G., & Ozer, O. (2003). Optimal replenishment policies for multi-echelon inventory problems under advance demand information. *Manufacturing & Service Operations Management, 5*(2), 157-175.

Gallien, J., & Wein, L. M. (2001). A simple and effective component procurement policy for stochastic assembly systems. *Queuing Systems, 38*(2), 221-248.

Gavirneni, S., Kapuscinski, R., & Tayur, S. (1999). Value of information in capacitated supply chains. *Management Science, 45*(1), 16-24.

Gerchak, Y. (1992). Order Point/Order quantity models with random yield. *International Journal of Production Economics, 26*, 297-298.

Gerchak, Y., Vickson, R. G., & Parlar, M. (1988). Periodic review production models with variables yield and uncertain demand. *IIE Transactions, 2*, 144-150.

Glasserman, P. (1991). *Gradient estimation via perturbation analysis* (1st ed.) Kluwer Academic Publishers.

Glasserman, P., & Tayur, S. (1995). Sensitivity analysis for base-stock levels in multi-echelon production-inventory systems. *Management Science, 41*(2), 263-281.

Glover, F., & Laguna, M. (2000). Fundamentals of scatter search and path relinking. *Control and Cybernetics, 29*(4), 653-684.

Graves, S. C. (1996). Multi-echelon inventory models with fixed replenishment intervals. *Management Science, 42*(1), 1-18.

Graves, S. C., & Willems, S. P. (2000). Optimizing strategic safety stock placement in supply chains. *Manufacturing & Service Operations Management, 2*, 68-83.

Graves, S. C. (1985). A multi-echelon inventory model for a repairable item with one-for-one replenishment. *Management Science, 31*(10), 1247-1256.

Graves, S. C., & Schwarz, L. B. (1977). Single Cycle Continuous Review Policies for Arborescent Production/ Inventory Systems. *Management Science*, 23, 529-540

Groenevelt, H., Pintelon, L., & Seidemann, A. (1992). Production batching with machine breakdowns and safety stocks. *Operations Research, 40*(5), 959-971.

Groenevelt, H., Pintelon, L., & Seidemann, A. (1992a). Production lot sizing with machine Breakdowns. *Management Science, 38*, 104-123.

Gurnani, H., Akella, R., & Lehoczky, J. (1996). Optimal order policies in assembly systems with random demand and random supplier delivery. *IIE Transactions, 28*, 865-878.

Gurnani, H., Akella, R., & Lehoczky, J. (2000). Supply management in assembly systems with random yield and random demand. *IIE Transactions, 32*, 701-714.

Hariharian, R., & Zipkin, P. (1995). Customer order information, lead times, and inventories. *Management Science, 41*, 1599-1607.

Hausman, W., Lee, H. L., & Zhang, A. X. (1998). Order response time reliability in multi-item inventory systems. *European Journal of Operations Research, 109*(3), 646-659.

Hopp, W., & Spearman, M. (2000). *Factory physics* (2nd ed.)McGraw-Hill/Irwin.

Inderfurth, K. (1993). Valuation of lead-time reduction in multi-stage production systems. *Operations Research in Production Planning ad Inventory Control,* , 413-427.

Inderfurth, K., & Minner, S. (1998). Safety-stocks in multi-stage inventory systems under different service measures. *European Journal of Operations Research, 106*, 57-73.

Jain, K., & Silver, E. A. (1995). The single period procurement problem where dedicated supplier capacity can be reserved. *Naval Research Logistics, 42*, 320-332.

Kaplan, R. S. (1970). A dynamic inventory model with stochastic lead times. *Management Science, 16*(7), 491-507.

Karlin, S. (1960a). Dynamic inventory policy with varying stochastic demand. *Management Science, 6*, 231-258.

Karlin, S. (1960b). Optimal policy for dynamic inventory process with stochastic demands subject to seasonal variations. *SIAM, 8*, 611-629.

Kelton, W. D., Sandowski, R. P., & Sandowski, D. A. (2002). *Simulation with ARENA* (2nd ed.)McGraw-Hill.

Kim, E., Lee, E., & Kang, S. (2006). Optimal purchasing policy in a two-component assembly system with different purchasing contracts for each component. *Math. Meth. Oper. Res., 63*, 301-327.

Kleijnen, J. P. C. (Marcel Dekker, Inc.). *Statistical tools for simulation practitioners* (1st ed.)

Kochel, P., & Nilander, U. (2005). Simulation-based optimization of multi-echelon inventory systems. *International Journal of Production Economics, 93-94*, 505-513.

Laguna, M., & Martí, R. (2002). *OptQuest* callable libraries. *Optimization Software,* , 193-218.

Langenhoff, L. J. G., & Zijm, W. H. M. (1990). An analytical theory of multi-echelon Production/Distribution systems. *Statistica Neerlandica, 44*(3), 149-174.

Lee, H.L., & Billington, C. (1992). Managing supply chain inventory: pitfalls and opportunities. *Sloan Management Review, 33*, 65–72.

Lee, H. L., & Billington, C. (1993). Material management in decentralized supply chains. *Operations Research, 7*, 835-847.

Lee, H. L., & Yano, C. A. (1988). Production control in multi-stage systems with variable yield losses. *Operations Research, 36*(2), 269-278.

Min, H., & Zhou, G. (2002). Supply chain modeling: past, present and future. *Computers and Industrial Engineering, 43*,231–249.

Moinzadeh, K., & Lee, H. L. (1986). Batch size and stocking levels in multi-echelon repairable systems. *Management Science, 32*(12), 1567-1581.

Montgomery, D. C. (2000). *Design and analysis of experiments* John Wiley & Sons, Inc.

Montgomery, D. C., & Runnger, G. C. (2003). *Applied statistics and probability for engineers* (3rd ed.)John Wiley & Sons, Inc.

Moreira, J. A. A., & Bispo, C. F. G. (1999). An echelon inventory-based single stage cost function for a two-station tandem system. *Production Planning & Control, 10*(7), 643-649.

Noori, H., & Keller, G. (1986). The lot-size reorder point model with upstream-downstream uncertainty. *Decision Science, 17*, 285-291.

OptQuest for ARENA User's guide, Rockwell Automation.

Rao, U., Scheller-Wolf, A., & Tayur, S. (2000). Development of a rapid-response supply chain at caterpillar. *Operations Research, 48*, 189–204.

Rao, U. S., Swaminathan, J. M., & Zhang, J. (2004). Multi-product inventory planning with downward substitution, stochastic demand and setup costs. *IIE Transactions, 36*, 59-71.

Reed, M. C. (1998). *Fundamental ideas of analysis* (1st ed.) John Wiley & Sons, Inc.

Rosling, K. (1989). Optimal inventory policies for assembly systems under random demands. *Operations Research, 37*(4), 565-579.

Roundy, R. (1986). A 98%--effective lot--sizing rule for a multi--product, multi--stage production/ inventory system. *Mathematics of Operations Research, 11*(4), 699-727.

Shang, K. H., & Song, J. (2006). A closed-form approximation for serial inventory systems and application to system design. *Manufacturing & Service Operations Management, 8*(4), 394-406.

Sherbrooke, C. C. (1968). Metric: A multi-echelon technique for recoverable item control. *Operations research, 16*(1), 122-141.

Silver, E. A. (1976). Establishing the reorder quantity when amount received is uncertain. *INFOR, 14*(1), 32-39.

Simchi-Levi, D., & Zhao, Y. (2005). Safety stock positioning in supply chains with stochastic lead times. *Manufacturing & Service Operations Management, 7*(4), 295-318.

Simpson, K. F. (1958). In-process inventories. *Operations Research,* , 863-873.

Sobel, M. J. (2004). Fill rates of single-stage and multistage supply systems. *Manufacturing & Service Operations Management, 6*(1), 41-52.

Song, J. (1998). On the order fill rate in a multi-item, base-stock inventory system. *Operations research, 46*(6), 831-845.

Song, J. (2002). Order-based backorders and their implications in multi-item inventory systems. *Management Science, 48*(4), 499-516.

Song, J., Xu, S., & Liu, B. (1999). Order-fulfillment performance measures in an assemble-to-order system with stochastic lead-times. *Management Science, 47*, 131-149.

Song, J., & Yao, D. (2002). Performance analysis and optimization of assemble-to-order systems with random lead times. *Operations Research, 50*, 889-903.

Song, J., & Zipkin, P. (1996). Inventory control with information about supply conditions. *Management Science, 42*, 1409-1419.

Souza, R.D., Zice, S., & Chaoyang, L. (2000). Supply chain dynamics and optimization. *Integrated Manufacturing Systems, 11*, 348–364.

Strader, T.J., Lin, F., & Shaw, M.J. (1998). Simulation of order fulfillment in divergent assembly supply chains. *Journal of Artificial Societies and Social Simulation, 1*.

Svoronos, A., & Zipkin, P. (1988). Estimating the performance of multi-level inventory systems. *Operations research, 36*(1), 57-72.

Tetsuo, I. (2002). A non-stationary periodic review production- inventory model with uncertain production capacity and uncertain demand. *European Journal of Operations Research, 149*, 670-683.

Towill, D.R., Naim, M.M., & Winker, J. (1992). Industrial dynamics simulation models in the design of supply chains. *International Journal of Physical Distribution and Logistics Management, 22*, 3–13.

van Houtum, G. J., & Zijm, W. H. M. (1991). Computational procedures for stochastic multi-echelon production systems. *International Journal of Production Economics, 23*, 223-237.

Wu, K. (2001). A mixed inventory model with variable lead time and random supplier capacity. *Production Planning & Control, 12*(4), 353-361.

Zhang, A. X. (1998). Demand fulfillment rates in an assemble-to-order system with multiple products and dependent demands. *Production and Operations Management Society,* , 177-192.

Zipkin, P. (1989). Critical numbers policies for inventory models with periodic data. *Management Science, 35*(1), 71-80.

Zipkin, P., & Federgruen, A. (1986). An inventory model with limited production capacity and uncertain demands I. the average-cost criterion. *Mathematics of Operations Research, 11*, 193-207.

12. APPENDIX

A.1 Best Found Base-stock Levels for Single-Echelon

The best found base-stock levels for a single-echelon inventory model are shown in figure A1. The results from eight different cases are shown below. The demand is stochastic, and the capacity is deterministic in nature.

Table A1: Best Found Base-stock Levels for Single-Echelon

Case #	Demand	Capacity	Optimal Base-Stock for a Single-Echelon
1	Norm(4,2)	10	11.46
2	Norm(4,2)	5	12.26
3	Norm(4,4)	12	14.89
4	Norm(4,4)	8	15.71
5	Norm(6,2)	15	15.46
6	Norm(6,2)	7	16.26
7	Norm(6,4)	15	18.55
8	Norm(6,4)	9	19.15

Single-echelon Base-stock Level vs. Service Level

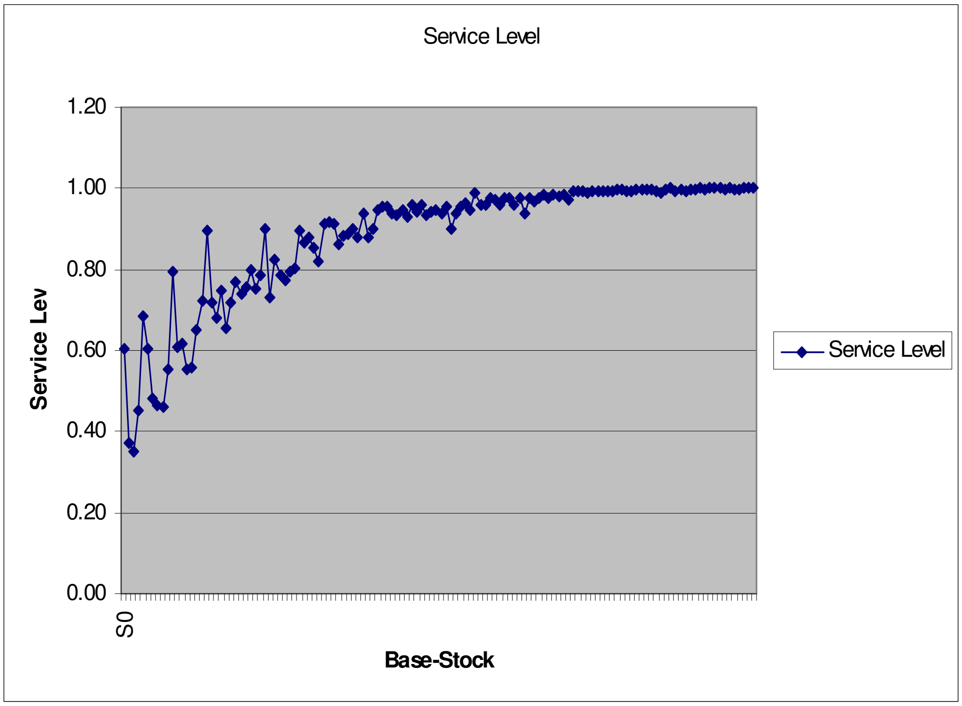

Figure A1: Single-echelon Base-stock Level vs. Service Level

Figure A1 shown above represents the service levels for different base-stock values. From the figure A1 it is clear that as the base-stock level increases the service-level also increases till the service level of 1 (100%) is attained thereafter the service level remains constant at 1. From the figure we can also observe that there is a lot of variance in the service level values when the base-stock values are small and relatively less variance when the base-stock values are large. This is due to the stochastic distribution of demand.

A.2 Variance and Standard Deviation of Service Level and Lagrange Multipliers

Table A2 shows the variance and standard deviation (SD) of the service level for all the four nodes in a three-echelon model. Each variance value is computed based on ten replications, where the number of periods for one replication is shown in the column of the table A2. We can see that as the number of periods for obtaining an estimate increases the variance also reduces. This would provide an idea on how many runs (periods in this case) would be sufficient for one estimate for the desired confidence interval.

Table A2: Variance and Standard Deviation of Service Level

	Variance of Sevice Level				SD of Sevice Level			
	S0	S1	S2	S3	S0	S1	S2	S3
500 Periods	0.0031	0.001306	0.000515	0.000515	0.055	0.036145	0.02269	0.02269
1000 Periods	0.003175	0.000852	0.000327	0.000327	0.05635	0.029184	0.018086	0.018086
2000 Periods	0.001588	0.000681	0.000293	0.000293	0.039855	0.026095	0.017117	0.017117
3000 Periods	0.000912	0.000392	0.000176	0.000176	0.030203	0.019807	0.013281	0.013281
4000 Periods	0.000697	0.000215	9.82E-05	9.82E-05	0.026407	0.014679	0.009911	0.009911
5000 Periods	0.000347	0.000109	5.35E-05	5.35E-05	0.018622	0.010429	0.007316	0.007316

A small analytical proof is shown, based of which we can find out the number of periods/runs that would be sufficient for an estimate based depending on the required confidence interval. Let Y_i be the service level variable estimated through the simulation, and let X_i be an indicator variable (i.e. the value of the variable is either 0 or 1), n be the number of periods or runs. The expression shown in equation A1 represents the equation used in simulation which is used compute the service level.

$$Y_i = \sum_{i=1}^{n} \frac{X_i}{n} \tag{A1}$$

Equation A1 can be rewritten as shown in A2.

$$nY_i = \sum_{i=1}^{n} X_i \tag{A2}$$

Computing the variance of the equation A2 we have

$$Var\left(nY_i\right) = Var\left(\sum_{i=1}^{n} X_i\right) \tag{A3}$$

The indicator variable on the right-hand side of equation A3 can be represented using a binomial distribution. The variance of a binomial distribution is given as $n * p * (1-p)$, where p is the probability, which would depend on the confidence interval required. Substituting the value of binomial distribution in equation A3, and further simplifying we have A4.

$$n^2 Var\left(Y_i\right) = np(1-p) \tag{A4}$$

Further simplifying to determine the number of simulation runs we get equation A5.

$$n = \frac{p(1-p)}{Var\left(Y_i\right)} \Rightarrow \sqrt{n} = \frac{\sqrt{p(1-p)}}{SD\left(Y_i\right)} \tag{A5}$$

Table A2 (a): Variance and Standard Deviation of Lagrange Multipliers*

	LS0		LU0		LS1		LU1		LS2	
	Variance	SD	Variance	SD	Variance	SD	Variance	SD	Variance	SD
50 Periods	9065.45	95.21	17779.24	133.34	3718.27	60.98	51479.59	226.89	2791.53	52.83
100 Periods	4822.95	69.45	5874.59	76.65	4429.43	66.55	41764.73	204.36	1731.45	41.61
150 Periods	2491.37	49.91	2784.79	52.77	2020.16	44.95	28931.86	170.09	1523.79	39.04
200 Periods	1504.87	38.79	1616.60	40.21	1404.35	37.47	20987.43	144.87	1028.11	32.06
350 Periods	536.64	23.17	549.33	23.44	452.67	21.28	14346.74	119.78	656.29	25.62
500 Periods	272.10	16.50	273.44	16.54	603.26	24.56	9333.27	96.61	318.53	17.85
1000 Periods	70.74	8.41	69.62	8.34	317.92	17.83	7574.84	87.03	293.73	17.14

	LU2		LS3		LU3		Added Value	
	Variance	SD	Variance	SD	Variance	SD	Variance	SD
50 Periods	4304.04	65.61	2791.53	52.83	4304.04	65.61	96233.70	310.22
100 Periods	2119.62	46.04	1731.45	41.61	2119.62	46.04	64593.82	254.15
150 Periods	1673.51	40.91	1523.79	39.04	1673.51	40.91	42622.77	206.45
200 Periods	1151.05	33.93	1028.11	32.06	1151.05	33.93	29871.57	172.83
350 Periods	803.31	28.34	656.29	25.62	803.31	28.34	18804.57	137.13
500 Periods	346.02	18.60	318.53	17.85	346.02	18.60	11811.17	108.68
1000 Periods	308.24	17.56	293.73	17.14	308.24	17.56	9237.07	96.11

*LS0 denotes dL/ds^0, and LU0 denotes dL/du^0

Table A2 (a) shows the variance and SD of the first order Lagrange function with respect to (w.r.t) base-stock levels, and Lagrange multipliers for all four nodes considered in a three-echelon inventory model. Each variance value is computed based on ten replications, where the number of periods for one replication is shown in the column of the table A2 (a). We can see that as the number of periods for obtaining an estimate increases the variance and SD also reduces, this is shown in figure A2 and A3 respectively.

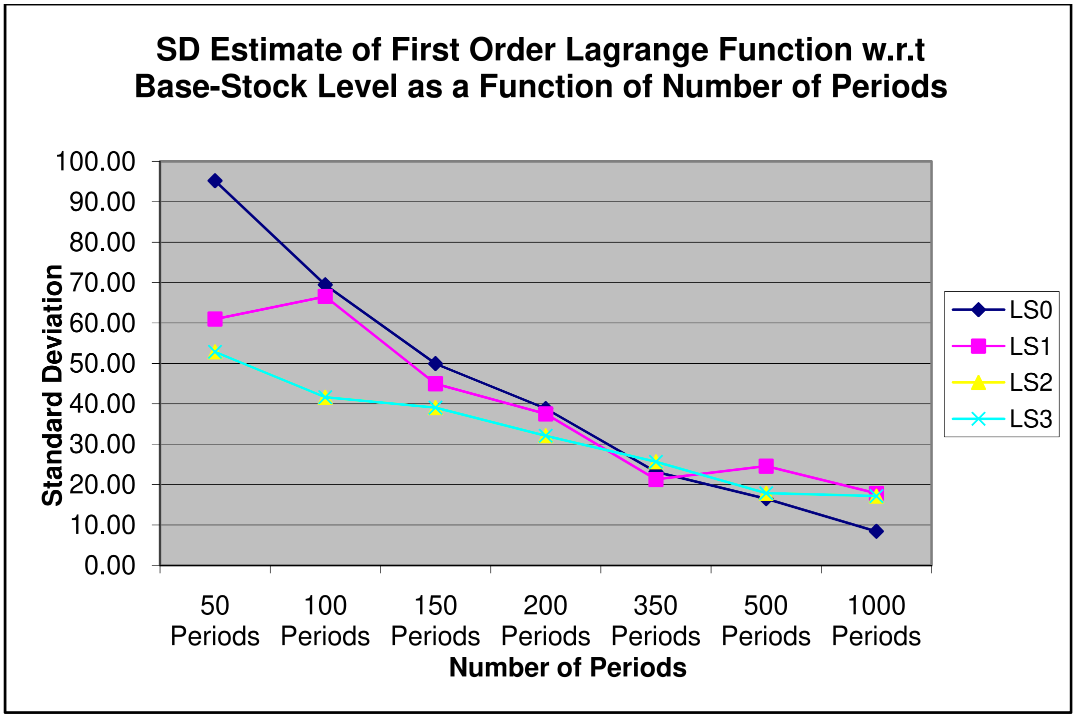

Figure A2: SD Estimate of First Order Lagrange Function w.r.t Base-stock Level

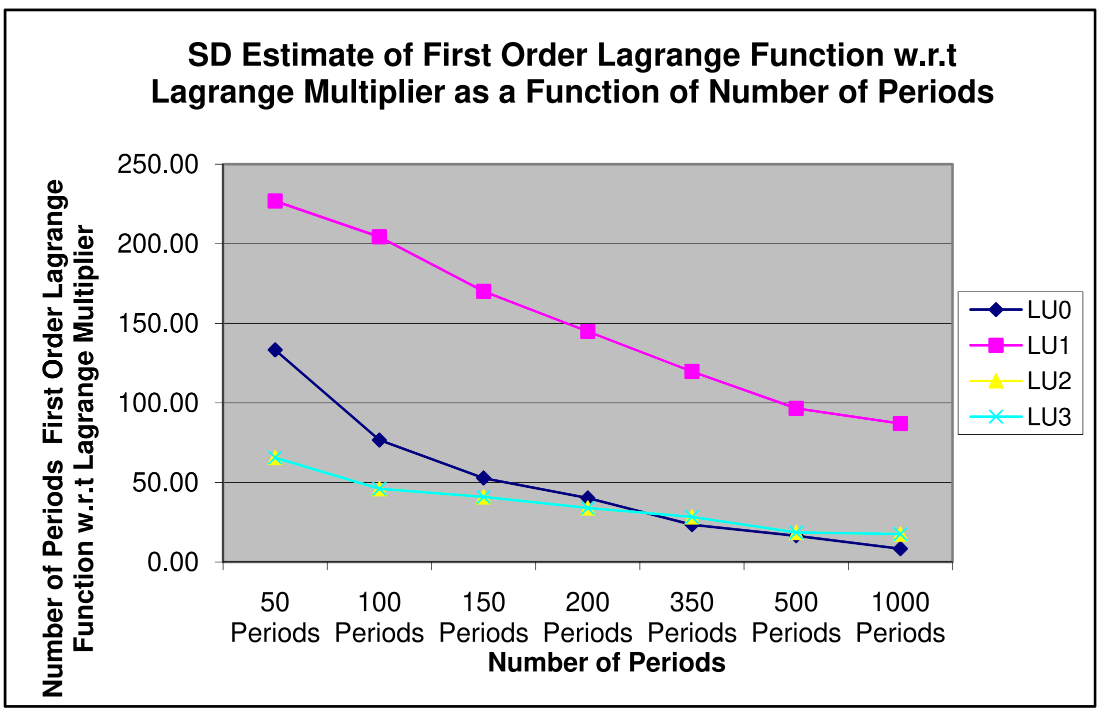

Figure A3: SD Estimate of First Order Lagrange Function w.r.t Lagrange Multiplier

A.3 Three-Echelon Inventory System with One-Period Lead-Time

The on-hand inventory equations for a three-echelon inventory system with one-period lead time are stated below:

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \xi_{n-1}^2 + DS_{n-1}^1\right), where\, i \in 2,3 \tag{A.6}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \xi_{n-1}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \tag{A.7}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^2\right), \text{where } i \in 0 \tag{A.8}$$

The outstanding order equations for a three-echelon inventory system with one-period lead time are stated below:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 - DS_n^1 - \eta_n^i\right) \text{ where } i \in 2,3 \tag{A.9}$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_{n-1}^0, s^3 - Y_{n-2}^3, \\ s^2 - Y_{n-2}^2, \eta_n^1 \end{pmatrix} \tag{A.10}$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-1}^1, \eta_n^0\right) \tag{A.11}$$

A.4 Three-Echelon Inventory System with Three-Period Lead-Time

The on-hand inventory equations for a three-echelon inventory system with three-period lead time are stated below:

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-3}^1 - \xi_{n-1}^2 - \xi_{n-2}^2 - \xi_{n-3}^2 + DS_{n-1}^1\right), where\, i \in 2,3 \qquad \text{(A.12)}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-3}^1 - \xi_{n-1}^2 - \xi_{n-2}^2 - \xi_{n-3}^2 + DS_{n-1}^0\right), \text{where } i \in 1 \qquad \text{(A.13)}$$

$$I_n^i = max\left(0\ ,\ s^i - Y_{n-l^i}^i - \xi_{n-1}^2 - \xi_{n-2}^2 - \xi_{n-3}^2\right), \text{where } i \in 0 \qquad \text{(A.14)}$$

The steady-state outstanding order inventory equations for three-echelon assembly system are listed below:

$$Y_{n+1}^i = \max\left(0, Y_n^i + \xi_n^1 + \xi_n^2 - DS_n^1 - \eta_n^i\right) \text{ where } i \in 2,3 \qquad \text{(A.15)}$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^1 + \xi_n^2 - DS_n^0 - \min\begin{pmatrix} Y_n^1 + \xi_n^1 + \xi_n^2 - DS_n^0, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \xi_{n-1}^2 - \xi_{n-2}^1 \\ -\xi_{n-2}^2,\ s^2 - Y_{n-2}^2 - \xi_{n-1}^1 - \xi_{n-1}^2 - \xi_{n-2}^1 - \xi_{n-2}^2, \eta_n^1 \end{pmatrix} \qquad \text{(A.16)}$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^2 - \min\left(Y_n^0 + \xi_n^2, s^1 - Y_{n-2}^1 - \xi_{n-1}^1 - \xi_{n-2}^1 - \xi_{n-3}^1 - \xi_{n-1}^2 - \xi_{n-2}^2, \eta_n^0\right) \qquad \text{(A.17)}$$

A5. Additional Computational Results for Three-echelon Assembly System

A5.1 One Period Supply/Manufacturing Lead Time (LT)

Table A3: Best Found Base-Stock Levels for Different Demand CV's (1 LT)*

Demand	Base-stock Levels for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	5.82	13.82	13.82	13.82	47.28
0.6	11.04	19.04	19.04	19.04	68.16
1	20.3	28.3	28.3	28.3	105.20
	Base-stock Levels for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	6.27	14.27	14.27	14.27	49.08
0.6	11.37	19.37	19.37	19.37	69.48
1	21.3	29.3	29.3	29.3	109.20
	Base-stock Levels for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	6.83	14.83	14.83	14.83	51.32
0.6	12.04	20.04	20.04	20.04	72.16
1	24.93	32.93	32.93	32.93	123.72
	Base-stock Levels for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	6.94	14.95	14.95	14.95	51.79
0.6	15.32	23.32	23.32	23.32	85.28
1	25.88	33.88	33.88	33.88	127.52

Table A4: Best Found Base-Stock Levels for Different Capacity CV's (1 LT)

Optimal Base-stock Level for Sevice Level - 80%					
Capacity CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	5.71	13.71	13.71	13.71	46.84
0.3	11.1	19.1	19.1	19.1	68.40
0.6	18.24	26.24	26.24	26.24	96.96
Optimal Base-stock Level for Sevice Level - 85%					
Capacity CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	6.06	14.06	14.06	14.06	48.24
0.3	11.8	19.8	19.8	19.8	71.20
0.6	21.2	29.2	29.2	29.2	108.80
Optimal Base-stock Level for Sevice Level - 90%					
Capacity CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	6.18	14.18	14.18	14.18	48.72
0.3	12.5	20.57	20.57	20.57	74.21
0.6	22.3	30.3	30.3	30.3	113.20
Optimal Base-stock Level for Sevice Level - 95%					
Capacity CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	6.42	14.42	14.42	14.42	49.68
0.3	13.04	21.04	21.04	21.04	76.16
0.6	23.8	31.8	31.8	31.8	119.20

* 1 LT refers to one period lead time

Table A5: Safety Stock Levels for Different Demand CV's (1 LT)

Demand	Safety Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	1.82	1.82	1.82	1.82	7.28
0.6	7.04	7.04	7.04	7.04	28.16
1	16.30	16.30	16.30	16.30	65.20
	Safety Stock for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	2.27	2.27	2.27	2.27	9.08
0.6	7.37	7.37	7.37	7.37	29.48
1	17.30	17.30	17.30	17.30	69.20
	Safety Stock for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	2.83	2.83	2.83	2.83	11.32
0.6	8.04	8.04	8.04	8.04	32.16
1	20.93	20.93	20.93	20.93	83.72
	Safety Stock for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	2.94	2.95	2.95	2.95	11.79
0.6	11.32	11.32	11.32	11.32	45.28
1	21.88	21.88	21.88	21.88	87.52

Table A6: Safety Stock Levels for Different Capacity CV's (1 LT)

Capacity	Safety Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	1.71	1.71	1.71	1.71	6.84
0.3	7.10	7.10	7.10	7.10	28.40
0.6	14.24	14.24	14.24	14.24	56.96
	Safety Stock for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	2.06	2.06	2.06	2.06	8.24
0.3	7.80	7.80	7.80	7.80	31.20
0.6	17.20	17.20	17.20	17.20	68.80
	Safety Stock for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	2.18	2.18	2.18	2.18	8.72
0.3	8.50	8.57	8.57	8.57	34.21
0.6	18.30	18.30	18.30	18.30	73.20
	Safety Stock for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	2.42	2.42	2.42	2.42	9.68
0.3	9.04	9.04	9.04	9.04	36.16
0.6	19.80	19.80	19.80	19.80	79.20

The demand and capacity values for tables A3 and A4, A5 and A6 are provided in table 7.7 and 7.11 respectively.

A5.2 Two Period Supply/Manufacturing Lead Time

Table A7: Best Found Base-Stock Levels for Different Demand CV's (2 LT)

Demand	Optimal Base-Stock Level for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	10.71	26.71	26.71	26.71	90.84
0.6	19.34	34.34	34.34	34.34	122.36
1	24.4	40.4	40.4	40.4	145.60
	Optimal Base-Stock Level for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	11.13	27.13	27.13	27.13	92.52
0.6	19.53	35.53	35.53	35.53	126.12
1	26.11	42.11	42.11	42.11	152.44
	Optimal Base-Stock Level for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	11.51	27.51	27.51	27.51	94.04
0.6	20.91	36.91	36.91	36.91	131.64
1	38.15	54.15	54.15	54.15	200.60
	Optimal Base-Stock Level for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	12.26	28.26	28.26	28.26	97.04
0.6	23.2	39.2	39.2	39.2	140.80
1	40.21	56.21	56.21	56.21	208.84

Table A8: Best Found Base-Stock Levels for Different Capacity CV's (2 LT)

	Optimal Base-Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	10.10	26.1	26.1	26.1	88.40
0.3	15.34	31.34	31.34	31.34	109.36
0.6	24.8	40.8	40.8	40.8	147.20
	Optimal Base-Stock for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	10.87	26.87	26.87	26.87	91.48
0.3	15.8	31.8	31.8	31.8	111.20
0.6	26.3	42.3	42.3	42.3	153.20
	Optimal Base-Stock for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	10.99	26.99	26.99	26.99	91.96
0.3	16.57	32.57	32.57	32.57	114.28
0.6	28.57	44.57	44.57	44.57	162.28
	Optimal Base-Stock for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	11.41	27.41	27.41	27.41	93.64
0.3	17.28	33.28	33.28	33.28	117.12
0.6	30.04	46.04	46.04	46.04	168.16

Table A9: Safety Stock Levels for Different Demand CV's (2 LT)

Demand	Safety Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	2.71	2.71	2.71	2.71	10.84
0.6	11.34	10.34	10.34	10.34	42.36
1	16.40	16.40	16.40	16.40	65.60
	Safety Stock for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	3.13	3.13	3.13	3.13	12.52
0.6	11.53	11.53	11.53	11.53	46.12
1	18.11	18.11	18.11	18.11	72.44
	Safety Stock for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	3.51	3.51	3.51	3.51	14.04
0.6	12.91	12.91	12.91	12.91	51.64
1	30.15	30.15	30.15	30.15	120.60
	Safety Stock for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	4.26	4.26	4.26	4.26	17.04
0.6	15.20	15.20	15.20	15.20	60.80
1	32.21	32.21	32.21	32.21	128.84

Table A10: Safety Stock Levels for Different Capacity CV's (2 LT)

Capacity	Safety Stock for Sevice Level - 80%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	2.10	2.10	2.10	2.10	8.40
0.3	7.34	7.34	7.34	7.34	29.36
0.6	16.80	16.80	16.80	16.80	67.20
	Safety Stock for Sevice Level - 85%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	2.87	2.87	2.87	2.87	11.48
0.3	7.80	7.80	7.80	7.80	31.20
0.6	18.30	18.30	18.30	18.30	73.20
	Safety Stock for Sevice Level - 90%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	2.99	2.99	2.99	2.99	11.96
0.3	8.57	8.57	8.57	8.57	34.28
0.6	20.57	20.57	20.57	20.57	82.28
	Safety Stock for Sevice Level - 95%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	3.41	3.41	3.41	3.41	13.64
0.3	9.28	9.28	9.28	9.28	37.12
0.6	22.04	22.04	22.04	22.04	88.16

A5.3 Three Period Supply/Manufacturing Lead Time

Table A11: Best Found Base-Stock Levels for Different Demand CV's (3 LT)

Demand	Optimal Base-Stock for Sevice Level - 80%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.2	15.52	39.52	39.52	39.52	134.08
0.6	26.17	50.17	50.17	50.17	176.68
1	42.2	66.2	66.2	66.2	240.80
	Optimal Base-Stock for Sevice Level - 85%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.2	15.78	39.79	39.79	39.75	135.11
0.6	27	51	51	51	180.00
1	44.66	68.66	68.66	68.66	250.64
	Optimal Base-Stock for Sevice Level - 90%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.2	16.43	40.53	40.53	40.53	138.02
0.6	29.64	53.64	53.64	53.64	190.56
1	46.53	70.53	70.53	70.53	258.12
	Optimal Base-Stock for Sevice Level - 95%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.2	17.06	41.06	41.06	41.06	140.24
0.6	30.18	54.18	54.18	54.18	192.72
1	48.26	72.26	72.26	72.26	265.04

Table A12: Best Found Base-Stock Levels for Different Capacity CV's (3 LT)

Capacity	Base-Stock Levels for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	14.80	38.8	38.8	38.8	131.20
0.3	19.8	43.8	43.8	43.8	151.20
0.6	34.4	58.4	58.4	58.4	209.60
	Base-Stock Levels for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	15.26	39.26	39.26	39.26	133.04
0.3	20.28	44.28	44.28	44.28	153.12
0.6	36.54	60.54	60.54	60.54	218.16
	Base-Stock Levels for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	16.05	40.05	40.05	40.05	136.20
0.3	20.82	44.82	44.82	44.82	155.28
0.6	38.54	62.54	62.54	62.54	226.16
	Base-Stock Levels for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	28.16	40.16	40.16	40.16	148.64
0.3	22.04	46.04	46.04	46.04	160.16
0.6	40.89	64.89	64.89	64.89	235.56

Table A13: Safety Stock Levels for Different Demand CV's (3 LT)

Demand	Safety Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	3.52	3.52	3.52	3.52	14.08
0.6	14.17	14.17	14.17	14.17	56.68
1	30.20	30.20	30.20	30.20	120.80
	Safety Stock for Sevice Level - 85%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	3.78	3.79	3.79	3.75	15.11
0.6	15.00	15.00	15.00	15.00	60.00
1	32.66	32.66	32.66	32.66	130.64
	Safety Stock for Sevice Level - 90%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	4.43	4.53	4.53	4.53	18.02
0.6	17.64	17.64	17.64	17.64	70.56
1	34.53	34.53	34.53	34.53	138.12
	Safety Stock for Sevice Level - 95%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.2	5.06	5.06	5.06	5.06	20.24
0.6	18.18	18.18	18.18	18.18	72.72
1	36.26	36.26	36.26	36.26	145.04

Table A14: Safety Stock Levels for Different Demand CV's (3 LT)

Capacity	Safety Stock for Sevice Level - 80%				
CV	Node 0	Node 1	Node 2	Node 3	Total
0.1	2.80	2.80	2.80	2.80	11.20
0.3	7.80	7.80	7.80	7.80	31.20
0.6	22.40	22.40	22.40	22.40	89.60
	Safety Stock for Sevice Level - 85%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	3.26	3.26	3.26	3.26	13.04
0.3	8.28	8.28	8.28	8.28	33.12
0.6	24.54	24.54	24.54	24.54	98.16
	Safety Stock for Sevice Level - 90%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	4.05	4.05	4.05	4.05	16.20
0.3	8.82	8.82	8.82	8.82	35.28
0.6	26.54	26.54	26.54	26.54	106.16
	Safety Stock for Sevice Level - 95%				
CV	**Node 0**	**Node 1**	**Node 2**	**Node 3**	**Total**
0.1	16.16	4.16	4.16	4.16	28.64
0.3	10.04	10.04	10.04	10.04	40.16
0.6	28.89	28.89	28.89	28.89	115.56

A6. Additional Computational Results for Three-echelon Allocation Policies

Table A15: Best Found Base-stock Levels for LAPD Policy

Optimal Base-Stock Levels for LAPD					
	S0	S1	S2	S3	Objective Function
Scenario 1	11.1	27.1	27.1	27.1	**92.4**
Scenario 2	17.8	33.8	33.8	33.8	**119.2**
Scenario 3	19.84	35.84	35.84	35.84	**127.36**
Scenario 4	23.29	39.29	39.29	39.29	**141.16**
Scenario 5	14	30.85	30.85	30.85	**106.55**
Scenario 6	19.61	35.61	35.61	35.61	**126.44**
Scenario 7	11.14	27.14	27.14	27.14	**92.56**
Scenario 8	18	34	34	34	**120**
Scenario 9	15.64	31.64	31.64	31.64	**110.56**
Scenario 10	20.2	36.2	36.2	36.2	**128.8**
Scenario 11	22.2	46.2	46.2	46.2	**160.8**
Scenario 12	27.42	51.42	51.42	51.42	**181.68**

Table A16: Best Found Base-stock Levels for LAPI Policy

Optimal Base-Stock Levels for LAPI					
	S0	S1	S2	S3	Objective Function
Scenario 1	10.76	26.76	26.76	26.76	**91.04**
Scenario 2	17.85	33.85	33.85	33.85	**119.4**
Scenario 3	19.84	35.84	35.84	35.84	**127.36**
Scenario 4	24.22	40.22	40.22	40.22	**144.88**
Scenario 5	14	30.82	30.82	30.82	**106.46**
Scenario 6	20.27	36.28	36.28	36.28	**129.11**
Scenario 7	11.13	27.13	27.13	27.13	**92.52**
Scenario 8	18.35	34.36	34.36	34.36	**121.43**
Scenario 9	16	32	32	32	**112**
Scenario 10	20.25	36.25	36.25	36.25	**129**
Scenario 11	21.81	45.82	45.82	45.82	**159.27**
Scenario 12	27.83	51.83	51.83	51.83	**183.32**

Table A17: Best Found Base-stock Levels for PPA Policy for Case A

Optimal Base-Stock Levels for PPA for Case A					
	S0	**S1**	**S2**	**S3**	**Objective Function**
Scenario 1	13.03	29.03	29.03	29.03	**100.12**
Scenario 2	17.89	33.89	33.89	33.89	**119.56**
Scenario 3	21.11	37.11	37.11	37.11	**132.44**
Scenario 4	24.01	40.01	40.01	40.01	**144.04**
Scenario 5	16.75	32.76	32.76	32.76	**115.03**
Scenario 6	20.11	36.11	36.11	36.11	**128.44**
Scenario 7	13.4	29.4	29.4	29.4	**101.6**
Scenario 8	18	34	34	34	**120**
Scenario 9	17.72	33.72	33.72	33.72	**118.88**
Scenario 10	20.5	36.5	36.5	36.5	**130**
Scenario 11	24.17	48.17	48.17	48.17	**168.68**
Scenario 12	28.04	52.04	52.04	52.04	**184.16**

Table A18: Best Found Base-stock Levels for PPA Policy for Case B

Optimal Base-Stock Levels for PPA for Case B					
	S0	**S1**	**S2**	**S3**	**Objective Function**
Scenario 1	10.93	26.94	26.94	26.94	**91.75**
Scenario 2	18.85	26.85	26.85	26.85	**99.4**
Scenario 3	19.83	35.83	35.83	35.83	**127.32**
Scenario 4	27.68	36.06	35.68	35.68	**135.1**
Scenario 5	14.65	30.65	30.65	30.65	**106.6**
Scenario 6	22.76	30.76	30.76	30.76	**115.04**
Scenario 7	10.85	26.85	26.85	26.85	**91.4**
Scenario 8	19.64	27.64	27.64	27.64	**102.56**
Scenario 9	14.97	30.98	30.98	30.98	**107.91**
Scenario 10	24.41	32.41	32.41	32.41	**121.64**
Scenario 11	22.15	46.15	46.15	46.15	**160.6**
Scenario 12	30.14	46.14	46.14	46.14	**168.56**

Table A19: Best Found Base-stock Levels for PA Policy for Case A

Optimal Base-Stock Levels for PA for Case A					
	S0	S1	S2	S3	Objective Function
Scenario 1	11.01	27.01	27.01	27.01	**92.04**
Scenario 2	18.01	34.01	34.01	34.01	**120.04**
Scenario 3	20.5	36.51	36.51	36.51	**130.03**
Scenario 4	23.65	39.65	39.65	39.65	**142.6**
Scenario 5	14.23	30.24	30.24	30.24	**104.95**
Scenario 6	19.75	35.75	35.75	35.75	**127**
Scenario 7	11.22	27.22	27.22	27.22	**92.88**
Scenario 8	17.82	33.82	33.82	33.82	**119.28**
Scenario 9	16.53	32.53	32.53	32.53	**114.12**
Scenario 10	20.22	36.22	36.22	36.22	**128.88**
Scenario 11	22.55	46.56	46.56	46.56	**162.23**
Scenario 12	27.71	51.71	51.71	51.71	**182.84**

Table A20: Best Found Base-stock Levels for PA Policy for Case B

Optimal Base-Stock Levels for PA for Case B					
	S0	S1	S2	S3	Objective Function
Scenario 1	11.03	27.03	27.03	27.03	**92.12**
Scenario 2	18.71	26.71	26.71	26.71	**98.84**
Scenario 3	19.83	35.83	35.83	35.83	**127.32**
Scenario 4	27.97	35.97	35.97	35.97	**135.88**
Scenario 5	14.23	30.24	30.24	30.24	**104.95**
Scenario 6	22.8	30.8	30.8	30.8	**115.2**
Scenario 7	11.37	27.37	27.37	27.37	**93.48**
Scenario 8	19.37	27.37	27.37	27.37	**101.48**
Scenario 9	15.41	31.41	31.41	31.41	**109.64**
Scenario 10	24.1	32.1	32.1	32.1	**120.4**
Scenario 11	22.17	46.17	46.17	46.17	**160.68**
Scenario 12	30.18	46.18	46.18	46.18	**168.72**

Table A21: Scenarios for Fixed Allocation (FA)

	Intermediate product, Demand 1	Final Product, Demand 2	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0	FQ 1 (fixed quantity for Intermediate prod	FQ 2 (fixed quantity for Final prod
Scenario 1	Norm(8,1)	Norm(4,1)	18	18	18	8	10	15
Scenario 2	Norm(4,1)	Norm(8,1)	18	18	18	15	5	15
Scenario 3	Norm(8,4)	Norm(4,4)	18	18	18	8	10	15
Scenario 4	Norm(4,4)	Norm(8,4)	18	18	18	15	5	15
Scenario 5	Norm(8,2)	Norm(4,2)	14	14	14	6	10	15
Scenario 6	Norm(4,2)	Norm(8,2)	14	14	14	12	5	15
Scenario 7	Norm(8,1)	Norm(4,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(8,3)	10	15
Scenario 8	Norm(4,1)	Norm(8,1)	Norm(18,3)	Norm(18,3)	Norm(18,3)	Norm(15,3)	5	15
Scenario 9	Norm(8,2)	Norm(4,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(6,2)	10	15
Scenario 10	Norm(4,2)	Norm(8,2)	Norm(14,2)	Norm(14,2)	Norm(14,2)	Norm(12,2)	5	15
Scenario 11	Norm(12,2)	Norm(8,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(12,2)	15	15
Scenario 12	Norm(8,2)	Norm(12,2)	Norm(24,2)	Norm(24,2)	Norm(24,2)	Norm(18,2)	10	20

Fixed Allocation (FA): In a fixed allocation a source of demand cannot receive more than fixed upper bound or fixed quantity. The last two columns in table A21 show the fixed quantity (subjective) for the intermediate product and the final product. The best found base-stock levels obtained using this inventory allocation policy using case A and case B described in chapter 8 is shown in table A22 and A23 respectively.

Table A22: Best Found Base-stock Levels for FA Policy for Case A

Optimal Base-Stock Levels for FA for Case A					
	S0	S1	S2	S3	Objective Function
Scenario 1	19.39	35.39	35.39	35.39	125.56
Scenario 2	23.71	31.71	31.71	31.71	118.84
Scenario 3	24.51	40.51	40.51	40.51	146.04
Scenario 4	30.78	39.23	39.23	39.23	148.47
Scenario 5	21.49	37.49	37.49	37.49	133.96
Scenario 6	26.64	34.64	34.64	34.64	130.56
Scenario 7	19.67	35.67	35.67	35.67	126.68
Scenario 8	18.11	34.12	34.12	34.12	120.47
Scenario 9	22.42	38.42	38.42	38.42	137.68
Scenario 10	27.61	35.62	35.62	35.62	134.47
Scenario 11	29.31	49.31	49.31	49.31	177.24
Scenario 12	34.43	50.43	50.43	50.43	185.72

Table A23: Best Found Base-stock Levels for FA Policy for Case B

Optimal Base-Stock Levels for FA Case B					
	S0	S1	S2	S3	Objective Function
Scenario 1	10.93	26.94	26.94	26.94	**91.75**
Scenario 2	18.93	26.94	26.94	26.94	**99.75**
Scenario 3	18.55	35.55	35.55	35.55	**125.2**
Scenario 4	27.06	35.06	35.06	35.06	**132.24**
Scenario 5	13.94	29.94	29.94	29.94	**103.76**
Scenario 6	22.36	30.36	30.36	30.36	**113.44**
Scenario 7	10.93	26.94	26.94	26.94	**91.75**
Scenario 8	18.61	26.99	26.61	26.61	**98.82**
Scenario 9	15.21	31.21	31.21	31.21	**108.84**
Scenario 10	23.85	31.85	31.85	31.85	**119.4**
Scenario 11	21.7	45.7	45.7	45.7	**158.8**
Scenario 12	29.78	45.78	45.78	45.78	**167.12**

A7. Update Equations for Five-echelon Allocation Policies

The update equations for model # 5 – 10 based of table 8.9 are described in sub-sections below, all the equations show only the modified equations with respect to the equations described in model #1:

A7.1 Update equations for model # 5

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, ratio\,3 * \left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \ldots - \xi_{n-1}^4\right)\right] \quad \text{(A.18)}$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{bmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, \left(1 - ratio\,3\right) * \\ \left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \ldots - \xi_{n-1}^4\right),\ \eta_n^2 \end{bmatrix} \quad \text{(A.19)}$$

$$Y_{n+1}^{2i} = Y_n^{2i} + \xi_n^2 - \min\left[Y_n^{2i} + \xi_n^2, s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^3 - \xi_{n-2}^4\right] \quad \text{(A.20)}$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, \\ s^2 - Y_{n-2}^2 - \xi_{n-1}^2 \ldots - \xi_{n-1}^4,\ \eta_n^1 \end{bmatrix} \quad \text{(A.21)}$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, ratio\,1 * \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right)\right] \quad \text{(A.22)}$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^4 - \min\begin{bmatrix} Y_n^0 + \xi_n^4, \left(1 - ratio\,1\right) * \\ \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right),\ \eta_n^0 \end{bmatrix} \quad \text{(A.23)}$$

A7.2 Update equations for model # 6

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^2 - \xi_{n-2}^3 - \xi_{n-2}^4\right] \quad \text{(A.24)}$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{bmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, \\ s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \ldots - \xi_{n-1}^4,\ \eta_n^2 \end{bmatrix} \quad \text{(A.25)}$$

$$Y_{n+1}^1 = Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, \\ s^2 - Y_{n-2}^2 - \xi_{n-1}^2 \ldots - \xi_{n-1}^4 - \xi_{n-2}^2,\ \eta_n^1 \end{bmatrix} \quad \text{(A.26)}$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, ratio\,1 * \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right)\right] \quad \text{(A.27)}$$

$$Y_{n+1}^{0} = Y_n^{0} + \xi_n^4 - \min\begin{bmatrix} Y_n^{0} + \xi_n^4, (1 - ratio\,1)* \\ \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right),\ \eta_n^0 \end{bmatrix} \tag{A.28}$$

A7.3 Update equations for model # 7

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, ratio\,1*\left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^4\right)\right] \tag{A.29}$$

$$Y_{n+1}^{0} = Y_n^{0} + \xi_n^4 - \min\left[Y_n^{0} + \xi_n^4, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4,\ \eta_n^0\right] \tag{A.30}$$

A7.4 Update equations for model # 8

$$Y_{n+1}^{2i} = Y_n^{2i} + \xi_n^2 - \min\left[Y_n^{2i} + \xi_n^2, s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^3 - \xi_{n-2}^4\right] \tag{A.31}$$

$$Y_{n+1}^{1} = Y_n^{1} + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, \\ s^2 - Y_{n-2}^2 - \xi_{n-1}^2 \ldots - \xi_{n-1}^4,\ \eta_n^1 \end{bmatrix} \tag{A.32}$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4 - \xi_{n-2}^4\right] \tag{A.33}$$

$$Y_{n+1}^{0} = Y_n^{0} + \xi_n^4 - \min\left[Y_n^{0} + \xi_n^4, s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4,\ \eta_n^0\right] \tag{A.34}$$

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, ratio\,3*\left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \ldots - \xi_{n-1}^4\right)\right] \tag{A.35}$$

A7.5 Update equations for model # 9

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - \ldots - \xi_{n-1}^4 - \xi_{n-2}^2 - \xi_{n-2}^3 - \xi_{n-2}^4\right] \tag{A.36}$$

$$Y_{n+1}^{2} = Y_n^{2} + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{bmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, \\ s^3 - Y_{n-2}^3 - \xi_{n-1}^1 \ldots - \xi_{n-1}^4,\ \eta_n^2 \end{bmatrix} \tag{A.37}$$

$$Y_{n+1}^{2i} = Y_n^{2i} + \xi_n^2 - \min\left[Y_n^{2i} + \xi_n^2, ratio\,2*\left(s^2 - Y_{n-2}^2 - \xi_{n-1}^2 - \ldots - \xi_{n-1}^4\right)\right] \tag{A.38}$$

$$Y_{n+1}^{1} = Y_n^{1} + \xi_n^3 + \xi_n^4 - DS_{n-1}^0 - \min\begin{bmatrix} Y_n^1 + \xi_n^3 + \xi_n^4 - DS_{n-1}^0, (1 - ratio\,2)* \\ \left(s^2 - Y_{n-2}^2 - \xi_{n-1}^2 \ldots - \xi_{n-1}^4\right),\ \eta_n^1 \end{bmatrix} \tag{A.39}$$

$$Y_{n+1}^{1i} = Y_n^{1i} + \xi_n^3 - \min\left[Y_n^{1i} + \xi_n^3, ratio\,1*\left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right)\right] \tag{A.40}$$

$$Y_{n+1}^0 = Y_n^0 + \xi_n^4 - \min\begin{bmatrix} Y_n^0 + \xi_n^4, (1 - ratio\,1)* \\ \left(s^1 - Y_{n-2}^1 - \xi_{n-1}^3 - \xi_{n-1}^4\right), \eta_n^0 \end{bmatrix} \tag{A.41}$$

A7.5 Update equations for model #10

$$Y_{n+1}^{3i} = Y_n^{3i} + \xi_n^1 - \min\left[Y_n^{3i} + \xi_n^1, ratio\,3*\left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 - ... - \xi_{n-1}^4\right)\right] \tag{A.42}$$

$$Y_{n+1}^2 = Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - \min\begin{bmatrix} Y_n^2 + \xi_n^2 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1, (1 - ratio\,3)* \\ \left(s^3 - Y_{n-2}^3 - \xi_{n-1}^1 ... - \xi_{n-1}^4\right), \eta_n^2 \end{bmatrix} \tag{A.43}$$

A8.Additional Computational Results for Five-Echelon Allocation Policies

Table A24: Demand and Capacity Values for Instance 1

Instance 1 : Normal Capacity			Normal Distribution with CV = 0.1							
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 1	8.2	8.8	8.6	3.5	40.4	38.9	44	29.1	16.7	4.8
Scenario 2	3.9	4.4	4	9.7	30.4	32.5	32	25.8	18.5	14.1
Scenario 3	4.4	4.4	9.1	8.8	38.4	36.4	38.6	33.6	26.5	12.9
Scenario 4	8.5	9.3	3.6	4	35.6	36.6	35.4	23.4	10.5	5.7
Scenario 5	5	8.6	9.6	3.7	40.6	40.8	37.6	30.5	18.1	5
Scenario 6	9	3.9	3.9	9.2	36.4	35.5	38.8	23.5	19.7	13.5

Table A25: Demand and Capacity Values for Instance 2

Instance 2: Tight Capacity			Normal Distribution with Variance CV = 0.1							
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 7	8.1	9.9	9.8	4.9	35.6	35.5	37.9	28.2	15.8	5.3
Scenario 8	3.8	4.6	4.5	10	26.2	25.7	25.8	20.7	15.9	11
Scenario 9	4.3	3.6	9.6	8.6	28.9	30.1	27.8	25.4	19.4	9.9
Scenario 10	9.3	8	4	3.7	27.7	28.4	27.2	16.8	9	4
Scenario 11	4.2	9.6	8.8	4.1	30.2	28.3	29.6	25.1	14.8	4.6
Scenario 12	9.2	3.9	4.2	8.4	29.8	29.2	27.7	17.5	14	8.9

Table A26: Demand and Capacity Values for Instance 3

Instance 3: Higher Demand Variance under Normal Capacity					Normal Distribution with CV = 0.3					
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 13	8.3	8.5	8.3	4.8	40.7	40.4	42.3	32.5	18.5	6.7
Scenario 14	4.5	4.9	3.9	9.8	35.1	32.7	34.9	25.8	19.1	14.9
Scenario 15	4.2	4.3	8.2	8.5	37.3	35.7	33.8	30.1	23.9	12.6
Scenario 16	9.3	9.4	4.6	3.6	37.7	36.8	39.5	25.2	11.8	4.9
Scenario 17	3.6	9.1	9	3.8	35.3	37.9	37.9	30.2	19.6	5.6
Scenario 18	8.9	4.7	4.9	9.7	42.2	41.1	42.4	29.3	20.9	14.8

Table A27: Demand and Capacity Values for Instance 4

Instance 4: Higher Demand Variance with Tighter Capacity (Worst Case Scenario)							Normal Distribution with CV = 0.3			
	Intermediate product, Demand 1	Intermediate product, Demand 2	Intermediate product, Demand 3	Final Product, Demand 4	Capacity At Node 5	Capacity At Node 4	Capacity At Node 3	Capacity At Node 2	Capacity At Node 1	Capacity At Node 0
Scenario 19	9.5	9.8	9.8	4.4	38	35.6	39.1	26.8	15.5	5
Scenario 20	3.7	5	4.1	10	24.3	26.1	25.7	20.5	16.2	11.5
Scenario 21	4.1	4.7	9	9.5	30.1	29.9	31.3	27.3	19.6	11.1
Scenario 22	8.7	8.8	4.2	3.9	27.1	29.8	29.4	17.8	9.4	4.4
Scenario 23	4.1	9	8.1	3.6	28.1	27.7	27.5	23	13.3	3.9
Scenario 24	9.9	4.6	4.9	9	32.1	32.8	30.8	19.9	16.2	10.1

Table A28: Best Found Base-stock Levels and Total Cost for Model #1

Optimal Base-Stock Levels for Model 1							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	9.85	27.06	44.84	61.24	61.06	61.24	**265.29**
Scenario 2	22.29	30.3	39.1	46.9	46.9	46.9	**232.39**
Scenario 3	20.8	39	47.61	56.41	56.6	56.41	**276.83**
Scenario 4	10.55	17.75	36.68	53.35	53.35	53.68	**225.36**
Scenario 5	10.29	29.49	46.69	56.69	56.69	56.69	**256.54**
Scenario 6	21.36	29.16	36.96	54.96	54.96	54.96	**252.36**
Scenario 7	14.81	34.42	54.22	71.26	71.26	71.26	**317.23**
Scenario 8	25.22	34.22	43.42	51.21	51.21	51.21	**256.49**
Scenario 9	24.71	43.91	51.11	59.71	59.71	59.71	**298.86**
Scenario 10	10.32	18.33	34.33	52.93	52.93	52.93	**221.77**
Scenario 11	14.47	32.07	51.27	59.67	59.67	59.67	**276.82**
Scenario 12	23.1	31.49	39.29	57.69	57.69	57.69	**266.95**
Scenario 13	28.93	45.53	62.53	78.78	79.13	79.13	**374.03**
Scenario 14	34.47	42.07	52.07	61.07	61.07	61.07	**311.82**
Scenario 15	36.32	53.72	61.32	69.72	69.72	69.72	**360.52**
Scenario 16	21.61	30.82	49.62	68.24	68.22	68.22	**306.73**
Scenario 17	24.73	42.73	60.93	68.13	68.13	68.13	**332.78**
Scenario 18	36.75	46.55	55.95	73.36	73.76	73.76	**360.13**
Scenario 19	40.45	60.05	79.65	98.65	98.65	98.65	**476.1**
Scenario 20	63.78	71.98	81.98	81.4	81.4	81.4	**461.94**
Scenario 21	63.49	81.49	91.21	99.47	99.47	99.47	**534.6**
Scenario 22	33.87	42.28	59.88	77.28	77.28	77.28	**367.87**
Scenario 23	36.78	52.98	70.98	79.18	79.18	79.18	**398.28**
Scenario 24	51.27	61.08	71.09	89.89	89.89	89.89	**453.11**

Table A29: Best Found Base-stock Levels and Total Cost for Model #2

Optimal Base-Stock Levels for Model 2							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	10.26	27.47	45.07	61.28	61.28	61.28	**266.64**
Scenario 2	22.02	30.02	38.82	46.62	46.62	46.62	**230.72**
Scenario 3	20.8	39	47.37	56.17	56.59	56.59	**276.52**
Scenario 4	10.85	18.05	36.65	53.65	53.65	53.65	**226.5**
Scenario 5	10.95	30.15	47.35	57.02	57.35	57.35	**260.17**
Scenario 6	21.37	29.18	36.98	54.98	54.98	54.98	**252.47**
Scenario 7	15.62	35.22	55.02	71.22	71.22	71.22	**319.52**
Scenario 8	24.51	33.52	42.72	50.32	50.32	50.32	**251.71**
Scenario 9	23.36	42.57	49.77	58.37	58.37	58.37	**290.81**
Scenario 10	10.24	18.24	32.24	52.4	52.84	52.84	**218.8**
Scenario 11	13.46	31.07	50.27	59.3	58.67	58.67	**271.44**
Scenario 12	21.17	29.58	37.38	55.78	55.78	55.78	**255.47**
Scenario 13	28.74	45.35	62.35	78.95	78.95	78.95	**373.29**
Scenario 14	33.2	41	50.8	59.51	59.8	59.8	**304.11**
Scenario 15	33.69	50.09	58.69	67.09	67.09	67.09	**343.74**
Scenario 16	20.27	29.28	48.28	66.88	66.88	66.88	**298.47**
Scenario 17	22.92	40.92	59.12	66.32	66.32	66.32	**321.92**
Scenario 18	35.57	45.37	54.77	72.57	72.57	72.57	**353.42**
Scenario 19	40.25	59.75	79.5	98.3	99.25	99.25	**476.3**
Scenario 20	60.31	68.51	78.51	85.91	85.91	85.91	**465.06**
Scenario 21	61.86	79.86	89.26	97.46	97.46	97.46	**523.36**
Scenario 22	32.24	40.64	58.24	75.64	75.64	75.64	**358.04**
Scenario 23	31.3	47.51	65.51	73.71	73.71	73.71	**365.45**
Scenario 24	46.67	56.48	65.68	85.48	85.48	85.48	**425.27**

Table A30: Best Found Base-stock Levels and Total Cost for Model #3

Optimal Base-Stock Levels for Model 3							
	S0	S1	S2	S3	S4	S5	Total Cost
Scenario 1	10.25	27.45	45.24	61.64	61.64	61.64	**267.86**
Scenario 2	22.97	30.97	39.77	47.57	47.57	47.86	**236.71**
Scenario 3	21.92	40.12	48.92	57.72	57.72	57.3	**283.7**
Scenario 4	10.95	17.29	36.27	53.27	53.27	53.27	**224.32**
Scenario 5	11.41	30.61	47.81	57.81	57.81	57.81	**263.26**
Scenario 6	21.82	29.63	37.43	55.43	55.43	55.43	**255.17**
Scenario 7	16.45	36.05	55.85	72.05	72.05	72.05	**324.5**
Scenario 8	27.11	36.11	45.31	52.95	52.91	52.91	**267.3**
Scenario 9	28.1	47.26	54.46	62.2	63.06	63.06	**318.14**
Scenario 10	10.41	18.41	34.41	52.05	53.01	53.01	**221.3**
Scenario 11	15.25	32.85	52.05	60.45	60.45	60.45	**281.5**
Scenario 12	24.1	32.49	40.29	58.69	58.69	58.69	**272.95**
Scenario 13	28.15	44.75	61.75	78.35	78.35	78.35	**369.7**
Scenario 14	38.44	42.25	54.05	63.05	63.05	63.05	**323.89**
Scenario 15	38.32	54.73	63.33	71.73	71.73	71.73	**371.57**
Scenario 16	19.6	28.8	47.6	65.81	66.8	66.8	**295.41**
Scenario 17	23.18	41.19	59.39	66.59	66.59	66.59	**323.53**
Scenario 18	36.1	45.89	55.29	73.09	73.09	73.09	**356.55**
Scenario 19	45.4	65.01	84.61	103.61	103.61	103.61	**505.85**
Scenario 20	63.57	71.77	81.77	89.17	89.17	89.17	**484.62**
Scenario 21	62.32	80.32	89.72	97.92	97.92	97.92	**526.12**
Scenario 22	32.81	41.22	58.82	76.22	76.22	76.22	**361.51**
Scenario 23	35.9	52.1	70.1	78.67	78.67	78.67	**394.11**
Scenario 24	51.45	61.25	70.45	90.25	90.25	90.25	**453.9**

Table A31: Best Found Base-stock Levels and Total Cost for Model #4

Optimal Base-Stock Levels for Model 4							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	12.13	29.33	46.31	63.71	63.71	63.71	**278.9**
Scenario 2	22.35	30.35	38.87	46.67	46.95	46.95	**232.14**
Scenario 3	20.67	38.87	47.67	56.47	56.47	56.47	**276.62**
Scenario 4	11	18.2	36.41	53.41	53.41	53.41	**225.84**
Scenario 5	10.32	29.52	46.96	56.96	56.72	56.72	**257.2**
Scenario 6	21.54	29.34	36.91	54.91	55.14	55.14	**252.98**
Scenario 7	15.88	35.48	55.28	71.48	71.48	71.48	**321.08**
Scenario 8	24.93	33.93	43.08	50.15	50.15	50.15	**252.39**
Scenario 9	26.1	45.31	52.51	59.97	61.11	61.11	**306.11**
Scenario 10	10.038	18.04	34.04	52.65	52.65	52.65	**220.068**
Scenario 11	15.73	33.35	52.55	60.95	60.95	61.33	**284.86**
Scenario 12	22.62	29.7	37.9	56.3	56.3	56.3	**259.12**
Scenario 13	27.59	44.19	61.19	77.79	77.79	77.79	**366.34**
Scenario 14	31.74	39.55	49.35	62.35	62.35	62.35	**307.69**
Scenario 15	34.06	50.47	59.07	67.47	67.47	67.47	**346.01**
Scenario 16	19.52	28.72	48.52	66.12	66.12	66.12	**295.12**
Scenario 17	21.84	39.84	58.04	65.24	65.24	65.24	**315.44**
Scenario 18	35	44.81	54.21	72.01	72.01	72.01	**350.05**
Scenario 19	43.76	63.35	82.95	101.95	101.95	101.95	**495.91**
Scenario 20	63.03	71.23	81.23	87.77	88.63	88.63	**480.52**
Scenario 21	64.21	82.21	91.61	99.81	99.81	99.81	**537.46**
Scenario 22	32.62	41.02	58.62	76.02	76.02	76.02	**360.32**
Scenario 23	33.72	49.92	67.92	76.12	76.12	76.12	**379.92**
Scenario 24	52.4	62.2	71.4	91.2	91.2	91.2	**459.6**

Table A32: Best Found Base-stock Levels and Total Cost for Model #5

Optimal Base-Stock Levels for Model 5							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	10.73	27.94	45.54	61.94	61.94	61.94	**270.03**
Scenario 2	22.9	30.9	39.7	47.5	47.5	47.5	**236**
Scenario 3	22.1	40.3	49.1	57.9	57.9	57.9	**285.2**
Scenario 4	10.58	17.79	36.39	53.39	53.39	53.39	**224.93**
Scenario 5	11.56	30.77	47.97	57.97	57.97	57.97	**264.21**
Scenario 6	21.9	29.71	37.51	55.51	55.51	55.51	**255.65**
Scenario 7	15.81	35.41	55.21	71.41	71.41	71.41	**320.66**
Scenario 8	26.42	35.42	44.62	51.84	52.22	52.22	**262.74**
Scenario 9	26.16	45.37	52.57	61.17	61.17	61.17	**307.61**
Scenario 10	10.12	18.12	34.12	52.72	52.72	52.72	**220.52**
Scenario 11	15.1	32.6	51.9	60.3	60.3	60.3	**280.5**
Scenario 12	22.79	31.2	39	57.4	57.4	57.4	**265.19**
Scenario 13	28.91	45.51	62.51	79.11	79.11	79.11	**374.26**
Scenario 14	37	44.8	54.6	63.6	63.6	63.6	**327.2**
Scenario 15	37.65	54.05	62.65	71.05	71.05	71.05	**367.5**
Scenario 16	19.43	28.64	47.44	66.04	66.04	66.04	**293.63**
Scenario 17	24.93	42.94	61.14	68.34	68.34	68.34	**334.03**
Scenario 18	36.16	45.97	55.37	73.17	73.17	73.17	**357.01**
Scenario 19	44.5	64.11	83.71	102.97	102.97	102.97	**501.23**
Scenario 20	65.31	73.52	83.52	90.92	90.92	90.92	**495.11**
Scenario 21	64.97	82.97	92.37	100.57	100.57	100.57	**542.02**
Scenario 22	35.5	43.9	61.5	78.9	78.9	78.9	**377.6**
Scenario 23	34.56	50.76	68.76	78.96	78.96	78.96	**390.96**
Scenario 24	51.92	61.73	70.93	90.73	90.73	90.73	**456.77**

Table A33: Best Found Base-stock Levels and Total Cost for Model #6

Optimal Base-Stock Levels for Model 6							
	S0	S1	S2	S3	S4	S5	Total Cost
Scenario 1	10.37	27.57	45.17	61.57	61.57	61.57	**267.82**
Scenario 2	22.19	30.19	38.99	46.79	46.79	46.79	**231.74**
Scenario 3	20.78	38.98	47.78	56.58	56.58	56.58	**277.28**
Scenario 4	10.81	18.01	36.61	53.61	53.61	53.61	**226.26**
Scenario 5	10.67	29.88	47.08	57.08	57.08	57.08	**258.87**
Scenario 6	21.49	29.29	37.09	55.09	55.09	55.09	**253.14**
Scenario 7	14.79	34.4	54.2	70.4	70.4	70.4	**314.59**
Scenario 8	25.08	34.08	43.28	50.88	50.88	50.88	**255.08**
Scenario 9	24.24	43.44	50.64	59.47	59.47	59.47	**296.73**
Scenario 10	10	18	34	52.6	52.6	52.6	**219.8**
Scenario 11	13.84	31.44	50.64	59.04	59.04	59.04	**273.04**
Scenario 12	21.6	30	37.8	56.2	56.2	56.2	**258**
Scenario 13	29.06	45.67	62.67	79.27	79.27	79.27	**375.21**
Scenario 14	34.42	42.23	52.03	61.03	61.03	61.03	**311.77**
Scenario 15	34.96	51.37	59.97	68.37	68.37	68.37	**351.41**
Scenario 16	19.95	29.16	47.96	66.56	66.56	66.56	**296.75**
Scenario 17	21.67	39.68	57.88	65.08	65.08	65.08	**314.47**
Scenario 18	35.49	45.29	54.69	72.49	72.49	72.49	**352.94**
Scenario 19	46.02	65.63	85.23	104.23	104.23	104.23	**509.57**
Scenario 20	60.94	69.15	79.15	86.55	86.55	86.55	**468.89**
Scenario 21	65.32	83.32	92.72	100.92	100.92	100.92	**544.12**
Scenario 22	33	41.4	59	76.4	76.4	76.4	**362.6**
Scenario 23	34.4	50.61	68.61	76.81	76.81	76.81	**384.05**
Scenario 24	52.03	61.84	71.04	90.64	90.84	90.84	**457.23**

Table A34: Best Found Base-stock Levels and Total Cost for Model #7

Optimal Base-Stock Levels for Model 7							
	S0	S1	S2	S3	S4	S5	Total Cost
Scenario 1	9.97	27.17	44.77	61.17	61.17	61.17	**265.42**
Scenario 2	22.61	30.61	39.41	47.21	47.21	47.21	**234.26**
Scenario 3	20.72	38.93	47.73	56.53	56.53	56.53	**276.97**
Scenario 4	10.58	17.79	36.39	53.39	53.39	53.39	**224.93**
Scenario 5	10.32	29.53	46.73	56.73	56.73	56.73	**256.77**
Scenario 6	21.51	29.31	37.11	55.11	55.11	55.11	**253.26**
Scenario 7	15.53	35.14	54.94	71.14	71.14	71.14	**319.03**
Scenario 8	25.58	34.59	43.79	51.39	51.39	51.39	**258.13**
Scenario 9	24.46	43.66	50.86	59.46	59.46	59.46	**297.36**
Scenario 10	10.053	18.05	34.05	52.65	52.65	52.65	**220.103**
Scenario 11	14.6	32.12	51.34	59.74	59.74	59.74	**277.28**
Scenario 12	22.28	30.69	32.49	56.89	56.89	56.89	**256.13**
Scenario 13	29.03	45.63	62.63	79.23	79.23	79.23	**374.98**
Scenario 14	34.1	41.8	51.5	60.5	60.5	60.5	**308.9**
Scenario 15	35.38	51.78	60.38	68.87	68.87	68.87	**354.15**
Scenario 16	21.98	31.19	50	68.6	68.6	68.6	**308.97**
Scenario 17	24.59	42.59	60.79	67.14	67.14	67.14	**329.39**
Scenario 18	35.51	45.31	54.71	72.71	72.71	72.71	**353.66**
Scenario 19	38.44	58.04	77.64	103.64	103.64	103.64	**485.04**
Scenario 20	62.57	70.78	80.78	80.18	80.18	80.18	**454.67**
Scenario 21	65.72	83.72	93.12	105.32	105.32	105.32	**558.52**
Scenario 22	35.76	44.17	61.77	79.17	79.17	79.17	**379.21**
Scenario 23	33.85	50.05	68.05	76.25	76.25	76.25	**380.7**
Scenario 24	53.43	63.23	72.23	92.23	92.23	92.23	**465.58**

Table A35: Best Found Base-stock Levels and Total Cost for Model #8

Optimal Base-Stock Levels for Model 8							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	9.85	27.05	44.65	61.05	61.05	61.05	**264.7**
Scenario 2	22.06	30.06	38.86	46.66	46.66	46.66	**230.96**
Scenario 3	20.63	38.83	47.63	56.43	56.43	56.43	**276.38**
Scenario 4	10.62	17.83	36.43	53.43	53.43	53.43	**225.17**
Scenario 5	10.7	29.9	47.11	57.11	57.11	57.11	**259.04**
Scenario 6	21.32	29.12	36.92	54.92	54.92	54.92	**252.12**
Scenario 7	15.64	35.25	55.05	71.25	71.25	71.25	**319.69**
Scenario 8	24.6	33.61	42.81	50.41	50.41	50.41	**252.25**
Scenario 9	24.53	43.75	50.95	59.55	59.55	59.55	**297.88**
Scenario 10	10.16	18.16	34.16	51.77	52.76	52.76	**219.77**
Scenario 11	14.27	31.87	51.07	59.47	59.47	59.47	**275.62**
Scenario 12	21.88	30.29	38.09	56.49	56.49	56.49	**259.73**
Scenario 13	29.4	46	63	79.6	79.6	79.6	**377.2**
Scenario 14	33.26	41.07	50.87	59.87	59.87	59.87	**304.81**
Scenario 15	33.61	50	58.6	67.01	67.01	67.01	**343.24**
Scenario 16	20.16	29.37	48.17	66.77	66.77	66.77	**298.01**
Scenario 17	22.94	40.95	59.15	66.35	66.35	66.35	**322.09**
Scenario 18	36.23	46.03	55.43	73.23	73.23	73.23	**357.38**
Scenario 19	37.2	56.82	76.42	103.42	103.42	103.42	**480.7**
Scenario 20	58.44	66.64	76.64	84.04	84.04	84.04	**453.84**
Scenario 21	62.61	80.61	90.01	98.21	98.21	98.21	**527.86**
Scenario 22	35.04	43.45	61.05	78.45	78.45	78.45	**374.89**
Scenario 23	35.84	52.05	70.05	77.25	77.25	77.25	**389.69**
Scenario 24	52.24	62.05	71.25	91.05	91.05	91.05	**458.69**

Table A36: Best Found Base-stock Levels and Total Cost for Model #9

Optimal Base-Stock Levels for Model 9							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	10.31	27.51	45.11	65.51	65.51	65.51	**279.46**
Scenario 2	22.19	30.19	38.99	46.79	46.79	46.79	**231.74**
Scenario 3	20.78	38.98	47.78	56.58	56.58	56.58	**277.28**
Scenario 4	10.81	18.01	36.61	53.61	53.61	53.61	**226.26**
Scenario 5	10.59	27.79	46.99	56.99	56.99	56.99	**256.34**
Scenario 6	21.76	29.56	37.36	55.36	55.36	55.36	**254.76**
Scenario 7	15.46	35.07	54.87	71.07	71.07	71.07	**318.61**
Scenario 8	24.6	33.26	42.46	49.64	50.06	50.06	**250.08**
Scenario 9	24.18	43.39	50.59	59.19	59.19	59.19	**295.73**
Scenario 10	10.22	18.22	34.22	52.59	52.82	52.82	**220.89**
Scenario 11	14.81	32.41	51.61	60.01	60.01	60.01	**278.86**
Scenario 12	21.3	29.7	37.5	55.9	55.9	55.9	**256.2**
Scenario 13	28.85	45.45	62.45	79.05	79.05	79.05	**373.9**
Scenario 14	32.98	40.78	50.58	59.58	59.58	59.58	**303.08**
Scenario 15	33.03	49.44	58.04	66.44	66.44	66.44	**339.83**
Scenario 16	20.27	29.48	48.28	66.88	66.88	66.88	**298.67**
Scenario 17	21.32	39.32	57.52	64.72	64.72	64.72	**312.32**
Scenario 18	35.14	44.94	54.34	72.14	72.14	72.14	**350.84**
Scenario 19	41.25	60.85	80.45	99.45	99.45	99.45	**480.9**
Scenario 20	60.46	68.67	78.67	86.07	86.07	86.07	**466.01**
Scenario 21	66.69	84.7	94.1	102.3	102.3	102.3	**552.39**
Scenario 22	34.48	42.89	60.49	77.89	77.89	77.89	**371.53**
Scenario 23	33.67	49.88	67.88	76.08	76.08	76.08	**379.67**
Scenario 24	51.99	61.79	70.79	90.79	90.79	90.79	**456.94**

Table A37: Best Found Base-stock Levels and Total Cost for Model #10

Optimal Base-Stock Levels for Model 10							
	S0	**S1**	**S2**	**S3**	**S4**	**S5**	**Total Cost**
Scenario 1	9.91	27.11	44.71	61.11	61.11	61.11	**265.06**
Scenario 2	22.29	30.03	38.83	46.63	46.63	46.63	**231.04**
Scenario 3	20.82	39.03	47.83	56.63	56.63	56.63	**277.57**
Scenario 4	10.78	17.98	36.58	53.58	53.58	53.58	**226.08**
Scenario 5	10.26	29.47	46.67	56.67	56.67	56.67	**256.41**
Scenario 6	21.52	29.33	37.13	55.13	55.13	55.13	**253.37**
Scenario 7	15.47	35.07	54.87	71.07	71.07	71.07	**318.62**
Scenario 8	25.32	34.33	43.53	51.13	51.13	51.13	**256.57**
Scenario 9	25.05	44.25	51.45	60.05	60.05	60.05	**300.9**
Scenario 10	10.37	18.37	34.37	52.97	52.97	52.97	**222.02**
Scenario 11	14.1	31.7	50.9	59.3	59.3	59.3	**274.6**
Scenario 12	22.52	30.93	38.73	57.13	57.13	57.13	**263.57**
Scenario 13	28.07	44.67	61.67	78.27	78.27	78.27	**369.22**
Scenario 14	33.85	41.65	51.45	60.45	60.45	60.45	**308.3**
Scenario 15	33.2	49.7	58.3	66.7	66.7	66.7	**341.3**
Scenario 16	19.52	28.72	47.52	66.12	66.12	66.12	**294.12**
Scenario 17	24.1	42.1	60.2	67.5	67.5	67.5	**328.9**
Scenario 18	35.58	45.38	54.78	72.58	72.58	72.58	**353.48**
Scenario 19	44.31	63.91	83.51	102.51	102.77	102.77	**499.78**
Scenario 20	62.7	70.93	80.93	88.33	88.33	88.33	**479.55**
Scenario 21	66.63	84.63	94.03	102.23	102.23	102.23	**551.98**
Scenario 22	35.1	43.5	61.1	78.5	78.5	78.5	**375.2**
Scenario 23	35.31	51.52	69.52	77.72	77.72	77.72	**389.51**
Scenario 24	52.96	62.76	71.96	91.76	91.76	91.76	**462.96**

A9.Additonal Information on Network 1 and Network 2

A9.1 Objective Function and Constraints for Network 1

Objective function:

$$\min_{s^i \geq 0} \sum_{i=0}^{7} c^i s^i \tag{A.44}$$

The constraints ensure that the desired service level is achieved at the node where external source of demand exists, in case of network 1 they are nodes 4, 3, 2 and 0.

$$\frac{1}{N}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{1} + \xi_{n-2}^{1} + \xi_{n-1}^{4} + \xi_{n-2}^{4} < s^4 - Y_{n-2}^{4} + DS_{n-1}^{1}\right\} = \alpha^4 \tag{A.45}$$

$$\frac{1}{N}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{2} + \xi_{n-2}^{2} + \xi_{n-1}^{4} + \xi_{n-2}^{4} < s^3 - Y_{n-2}^{3} + DS_{n-1}^{1}\right\} = \alpha^3 \tag{A.46}$$

$$\frac{1}{N}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{3} + \xi_{n-2}^{3} + \xi_{n-1}^{4} + \xi_{n-2}^{4} < s^2 - Y_{n-2}^{2} + DS_{n-1}^{1}\right\} = \alpha^2 \tag{A.47}$$

$$\frac{1}{N}\sum_{n=1}^{N} 1\left\{\xi_{n-1}^{4} + \xi_{n-2}^{4} < s^0 - Y_{n-2}^{0}\right\} = \alpha^0 \tag{A.48}$$

Where α^i represent the service-level for stage *i*. The constraints can also be written as:

$$N^{-1}\sum_{n=1}^{N} F_n\left(s^4 - Y_{n-2}^{4} + DS_{n-1}^{1}\right) - \alpha^4 \geq 0 \tag{A.49}$$

$$N^{-1}\sum_{n=1}^{N} F_n\left(s^3 - Y_{n-2}^{3} + DS_{n-1}^{1}\right) - \alpha^3 \geq 0 \tag{A.50}$$

$$N^{-1}\sum_{n=1}^{N} F_n\left(s^2 - Y_{n-2}^{2} + DS_{n-1}^{1}\right) - \alpha^2 \geq 0 \tag{A.51}$$

$$N^{-1}\sum_{n=1}^{N} F_n\left(s^0 - Y_{n-2}^{0}\right) - \alpha^0 \geq 0 \tag{A.52}$$

A9.2 Lagrange Function and Associated First-Order Equations for Network 1

The Lagrange function for network 1 is stated below:

$$L = \sum_{i=0}^{7} c^i s^i - \left\{ \begin{array}{l} u_4 \left[N^{-1} \sum_{n=1}^{N} F_n \left(s^4 - Y_{n-2}^4 + DS_{n-1}^1 \right) - \alpha^4 \right] \\ + u_3 \left[N^{-1} \sum_{n=1}^{N} F_n \left(s^3 - Y_{n-2}^3 + DS_{n-1}^1 \right) - \alpha^3 \right] \\ + u_2 \left[N^{-1} \sum_{n=1}^{N} F_n \left(s^2 - Y_{n-2}^2 + DS_{n-1}^1 \right) - \alpha^2 \right] \\ + u_0 \left[N^{-1} \sum_{n=1}^{N} F_n \left(s^0 - Y_{n-2}^0 \right) - \alpha^0 \right] \end{array} \right\} \tag{A.53}$$

The first order differentiation of the Lagrange function with respect to the base-stock levels and Lagrange multipliers is listed below:

$$\frac{dL}{ds^0} = c^0 - \frac{u_0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n \left(s^0 - Y_{n-2}^0 \right) \left[1 - \frac{dY_{n-2}^0}{ds^0} \right] \tag{A.54}$$

$$\frac{dL}{ds^1} = c^1 - \frac{u_0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n \left(s^0 - Y_{n-2}^0 \right) \left[- \frac{dY_{n-2}^0}{ds^1} \right] \tag{A.55}$$

$$\frac{dL}{ds^2} = \left\{ \begin{array}{l} c^2 - \frac{u_2}{N} \sum_{n=1}^{N} 1\left\{ s^2 - Y_{n-2}^2 - DS_{n-1}^1 > 0 \right\} f_n \left(s^2 - Y_{n-2}^2 - DS_{n-1}^1 \right) \left[1 - \frac{dY_{n-2}^2}{ds^2} \right] \\ - \frac{u_0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n \left(s^0 - Y_{n-2}^0 \right) \left[- \frac{dY_{n-2}^0}{ds^2} \right] \end{array} \right. \tag{A.56}$$

$$\frac{dL}{ds^3} = \left\{ \begin{array}{l} c^3 - \frac{u_3}{N} \sum_{n=1}^{N} 1\left\{ s^3 - Y_{n-2}^3 - DS_{n-1}^1 > 0 \right\} f_n \left(s^3 - Y_{n-2}^3 - DS_{n-1}^1 \right) \left[1 - \frac{dY_{n-2}^3}{ds^3} \right] \\ - \frac{u_0}{N} \sum_{n=1}^{N} 1\left\{ s^0 - Y_{n-2}^0 > 0 \right\} f_n \left(s^0 - Y_{n-2}^0 \right) \left[- \frac{dY_{n-2}^0}{ds^3} \right] \end{array} \right. \tag{A.57}$$

$$\frac{dL}{ds^4}=\begin{cases} c^4-\frac{u_4}{N}\sum_{n=1}^{N}1\left\{s^4-Y_{n-2}^4-DS_{n-1}^1>0\right\}f_n\left(s^4-Y_{n-2}^4-DS_{n-1}^1\right)\left[1-\frac{dY_{n-2}^4}{ds^4}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N}1\left\{s^0-Y_{n-2}^0>0\right\}f_n\left(s^0-Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^4}\right] \end{cases} \tag{A.58}$$

$$\frac{dL}{ds^5}=\begin{cases} c^5-\frac{u_2}{N}\sum_{n=1}^{N}1\left\{s^2-Y_{n-2}^2-DS_{n-1}^1>0\right\}f_n\left(s^2-Y_{n-2}^2-DS_{n-1}^1\right)\left[-\frac{dY_{n-2}^2}{ds^5}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N}1\left\{s^0-Y_{n-2}^0>0\right\}f_n\left(s^0-Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^5}\right] \end{cases} \tag{A.59}$$

$$\frac{dL}{ds^6}=\begin{cases} c^6-\frac{u_3}{N}\sum_{n=1}^{N}1\left\{s^3-Y_{n-2}^3-DS_{n-1}^1>0\right\}f_n\left(s^3-Y_{n-2}^3-DS_{n-1}^1\right)\left[-\frac{dY_{n-2}^3}{ds^6}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N}1\left\{s^0-Y_{n-2}^0>0\right\}f_n\left(s^0-Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^6}\right] \end{cases} \tag{A.60}$$

$$\frac{dL}{ds^7}=\begin{cases} c^7-\frac{u_4}{N}\sum_{n=1}^{N}1\left\{s^4-Y_{n-2}^4-DS_{n-1}^1>0\right\}f_n\left(s^4-Y_{n-2}^4-DS_{n-1}^1\right)\left[-\frac{dY_{n-2}^4}{ds^7}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N}1\left\{s^0-Y_{n-2}^0>0\right\}f_n\left(s^0-Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^7}\right] \end{cases} \tag{A.61}$$

$$\frac{dL}{du_0}=\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^0-Y_{n-2}^0\right)-\alpha^0\right] \tag{A.62}$$

$$\frac{dL}{du_2}=\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^2-Y_{n-2}^2+DS_{n-1}^1\right)-\alpha^2\right] \tag{A.63}$$

$$\frac{dL}{du_3}=\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^3-Y_{n-2}^3+DS_{n-1}^1\right)-\alpha^3\right] \tag{A.64}$$

$$\frac{dL}{du_4}=\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^4-Y_{n-2}^4+DS_{n-1}^1\right)-\alpha^4\right] \tag{A.65}$$

A9.3 First-Order Outstanding Order Equations for Network 1

The first order outstanding order equations for network 1are listed below

$$\frac{dY_{n+1}^{j}}{ds^{i}}=\begin{cases}0 & \text{if } Y_{n+1}^{j}=0\\ \dfrac{dY_{n}^{j}}{ds^{i}} & \text{otherwise}\end{cases} \quad \text{where } i\in\{0,....,7\};\ j\in\{5,6,7\} \tag{A.66}$$

$$\underset{\text{where } i\in\{0,....,7\}}{\frac{dY_{n+1}^{4}}{ds^{i}}}=\begin{cases}0 & \text{if } Y_{n+1}^{4}=0\\ \dfrac{dY_{n}^{4}}{ds^{i}}+\dfrac{dY_{n-2}^{7}}{ds^{i}}-1\left(\text{add -1only if } i\in\{7\}\right) & \text{if } Y_{n+1}^{4}=\begin{bmatrix}Y_{n}^{4}+\xi_{n}^{1}+\xi_{n}^{4}-DS_{n-1}^{1}\\ -s^{7}+Y_{n-2}^{7}+\xi_{n-1}^{1}+\xi_{n-1}^{4}\end{bmatrix}\\ \dfrac{dY_{n}^{4}}{ds^{i}} \qquad \text{where } i\in\{0,....,7\} & \text{otherwise}\end{cases}$$

(A.67)

$$\underset{\text{where } i\in\{0,...,7\}}{\frac{dY_{n+1}^{3}}{ds^{i}}}=\begin{cases}0 & \text{if } Y_{n+1}^{3}=0\\ \dfrac{dY_{n}^{3}}{ds^{i}}+\dfrac{dY_{n-2}^{6}}{ds^{i}}-1\left(\text{add -1only if } i\in\{6\}\right) & \text{if } Y_{n+1}^{3}=\begin{bmatrix}Y_{n}^{3}+\xi_{n}^{2}+\xi_{n}^{4}-DS_{n-1}^{1}\\ -s^{6}+Y_{n-2}^{6}+\xi_{n-1}^{2}+\xi_{n-1}^{4}\end{bmatrix}\\ \dfrac{dY_{n}^{3}}{ds^{i}} & \text{otherwise}\end{cases}$$

(A.68)

$$\underset{\text{where } i\in\{0,....,7\}}{\frac{dY_{n+1}^{2}}{ds^{i}}}=\begin{cases}0 & \text{if } Y_{n+1}^{2}=0\\ \dfrac{dY_{n}^{2}}{ds^{i}}+\dfrac{dY_{n-2}^{5}}{ds^{i}}-1\left(\text{add -1only if } i\in\{5\}\right) & \text{if } Y_{n+1}^{2}=\begin{bmatrix}Y_{n}^{2}+\xi_{n}^{3}+\xi_{n}^{4}-DS_{n-1}^{1}\\ -s^{5}+Y_{n-2}^{5}+\xi_{n-1}^{3}+\xi_{n-1}^{4}\end{bmatrix}\\ \dfrac{dY_{n}^{3}}{ds^{i}} & \text{otherwise}\end{cases}$$

(A.69)

$$\frac{dY_{n+1}^{1}}{ds^{i}} \underset{\text{where } i\in\{0,\dots,7\}}{=} \begin{cases} 0 & \text{if } Y_{n+1}^{1}=0 \\ \frac{dY_{n}^{1}}{ds^{i}}+\frac{dY_{n-2}^{4}}{ds^{i}}-1\left(\text{add -1 only if } i\in\{4\}\right) & \text{if } Y_{n+1}^{1}=\begin{bmatrix} Y_{n}^{1}+\xi_{n}^{4}-DS_{n-1}^{0}-s^{4} \\ +Y_{n-2}^{4}+\xi_{n-1}^{1}+\xi_{n-1}^{4}+\xi_{n-2}^{1} \end{bmatrix} \\ \frac{dY_{n}^{1}}{ds^{i}}+\frac{dY_{n-2}^{3}}{ds^{i}}-1\left(\text{add -1 only if } i\in\{3\}\right) & \text{if } Y_{n+1}^{1}=\begin{bmatrix} Y_{n}^{1}+\xi_{n}^{4}-DS_{n-1}^{0}-s^{3} \\ +Y_{n-2}^{3}+\xi_{n-1}^{2}+\xi_{n-1}^{4}+\xi_{n-2}^{2} \end{bmatrix} \\ \frac{dY_{n}^{1}}{ds^{i}}+\frac{dY_{n-2}^{2}}{ds^{i}}-1\left(\text{add -1 only if } i\in\{2\}\right) & \text{if } Y_{n+1}^{1}=\begin{bmatrix} Y_{n}^{1}+\xi_{n}^{4}-DS_{n-1}^{0}-s^{2} \\ +Y_{n-2}^{2}+\xi_{n-1}^{3}+\xi_{n-1}^{4}+\xi_{n-2}^{3} \end{bmatrix} \\ \frac{dY_{n}^{1}}{ds^{i}} & \text{otherwise} \end{cases} \tag{A.70}$$

$$\frac{dY_{n+1}^{0}}{ds^{i}} \underset{\text{where } i\in\{0,\dots,7\}}{=} \begin{cases} 0 & \text{if } Y_{n+1}^{0}=0 \\ \frac{dY_{n}^{0}}{ds^{i}}+\frac{dY_{n-2}^{1}}{ds^{i}}-1\left(\text{add -1 only if } i\in\{1\}\right) & \text{if } Y_{n+1}^{0}=\begin{bmatrix} Y_{n}^{0}+\xi_{n}^{4}-s^{1} \\ +Y_{n-2}^{1}+\xi_{n-1}^{4} \end{bmatrix} \\ \frac{dY_{n}^{0}}{ds^{i}} & \text{otherwise} \end{cases} \tag{A.71}$$

The first order on-hand inventory equations for network 1 are similar to the three-echelon assembly system.

A9.4 Objective Function and Constraints for Network 2

Objective function:

$$\min_{s^{i}\geq 0}\sum_{i=0}^{9}c^{i}s^{i} \tag{A.72}$$

The constraints ensure that the desired service level is achieved at the node where external source of demand exists, in case of network 1 they are nodes 0, 1, 2, and 3

$$\frac{1}{N}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{1}+\xi_{n-2}^{1}<s^{3}-Y_{n-2}^{3}\right\}=\alpha^{3} \tag{A.73}$$

$$\frac{1}{N}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{2}+\xi_{n-2}^{2}<s^{2}-Y_{n-2}^{2}\right\}=\alpha^{2} \tag{A.74}$$

$$\frac{1}{N}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{3}+\xi_{n-2}^{3}<s^{1}-Y_{n-2}^{1}\right\}=\alpha^{1} \tag{A.75}$$

$$\frac{1}{N}\sum_{n=1}^{N}1\left\{\xi_{n-1}^{4}+\xi_{n-2}^{4}<s^{0}-Y_{n-2}^{0}\right\}=\alpha^{0} \tag{A.76}$$

Where α^i represent the service-level for stage *i*. The constraints can also be written as:

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{3}-Y_{n-2}^{3}\right)-\alpha^{3}\geq 0 \tag{A.77}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{2}-Y_{n-2}^{2}\right)-\alpha^{2}\geq 0 \tag{A.78}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{1}-Y_{n-2}^{1}\right)-\alpha^{1}\geq 0 \tag{A.79}$$

$$N^{-1}\sum_{n=1}^{N}F_n\left(s^{0}-Y_{n-2}^{0}\right)-\alpha^{0}\geq 0 \tag{A.80}$$

A9.5 Lagrange Function and Associated First-Order Equations for Network 2

The Lagrange function for network 1 is stated below:

$$L=\sum_{i=0}^{9}c^{i}s^{i}-\left\{\begin{array}{l} u_3\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^{3}-Y_{n-2}^{3}\right)-\alpha^{3}\right]+u_2\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^{2}-Y_{n-2}^{2}\right)-\alpha^{2}\right] \\ +u_1\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^{1}-Y_{n-2}^{1}\right)-\alpha^{1}\right]+u_0\left[N^{-1}\sum_{n=1}^{N}F_n\left(s^{0}-Y_{n-2}^{0}\right)-\alpha^{0}\right]\end{array}\right\} \tag{A.81}$$

$$\frac{dL}{ds^0} = c^0 - \frac{u_0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left[1 - \frac{dY_{n-2}^0}{ds^0}\right] \tag{A.82}$$

$$\frac{dL}{ds^1} = c^1 - \frac{u_1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 > 0\right\} f_n\left(s^1 - Y_{n-2}^1\right)\left[1 - \frac{dY_{n-2}^1}{ds^1}\right] \tag{A.83}$$

$$\frac{dL}{ds^2} = c^2 - \frac{u_2}{N}\sum_{n=1}^{N} 1\left\{s^2 - Y_{n-2}^2 > 0\right\} f_n\left(s^2 - Y_{n-2}^2\right)\left[1 - \frac{dY_{n-2}^2}{ds^2}\right] \tag{A.84}$$

$$\frac{dL}{ds^3} = c^3 - \frac{u_3}{N}\sum_{n=1}^{N} 1\left\{s^3 - Y_{n-2}^3 > 0\right\} f_n\left(s^3 - Y_{n-2}^3\right)\left[1 - \frac{dY_{n-2}^3}{ds^3}\right] \tag{A.85}$$

$$\frac{dL}{ds^4} = \begin{cases} c^4 - \frac{u_1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 > 0\right\} f_n\left(s^1 - Y_{n-2}^1\right)\left[-\frac{dY_{n-2}^1}{ds^4}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^4}\right] \end{cases} \tag{A.86}$$

$$\frac{dL}{ds^5} = \begin{cases} c^5 - \frac{u_3}{N}\sum_{n=1}^{N} 1\left\{s^3 - Y_{n-2}^3 > 0\right\} f_n\left(s^3 - Y_{n-2}^3\right)\left[-\frac{dY_{n-2}^3}{ds^5}\right] \\ -\frac{u_2}{N}\sum_{n=1}^{N} 1\left\{s^2 - Y_{n-2}^2 > 0\right\} f_n\left(s^2 - Y_{n-2}^2\right)\left[-\frac{dY_{n-2}^2}{ds^5}\right] \end{cases} \tag{A.87}$$

$$\frac{dL}{ds^i} = \begin{cases} c^i - \frac{u_1}{N}\sum_{n=1}^{N} 1\left\{s^1 - Y_{n-2}^1 > 0\right\} f_n\left(s^1 - Y_{n-2}^1\right)\left[-\frac{dY_{n-2}^1}{ds^i}\right] \\ -\frac{u_0}{N}\sum_{n=1}^{N} 1\left\{s^0 - Y_{n-2}^0 > 0\right\} f_n\left(s^0 - Y_{n-2}^0\right)\left[-\frac{dY_{n-2}^0}{ds^i}\right] \\ -\frac{u_3}{N}\sum_{n=1}^{N} 1\left\{s^3 - Y_{n-2}^3 > 0\right\} f_n\left(s^3 - Y_{n-2}^3\right)\left[-\frac{dY_{n-2}^3}{ds^i}\right] \\ -\frac{u_2}{N}\sum_{n=1}^{N} 1\left\{s^2 - Y_{n-2}^2 > 0\right\} f_n\left(s^2 - Y_{n-2}^2\right)\left[-\frac{dY_{n-2}^2}{ds^i}\right] \end{cases} \quad where\, i \in \{6,7,8,9\} \tag{A.88}$$

A9.6 First-Order Outstanding Order Equations for Network 2

The first order outstanding order equations for network 1are listed below

$$\frac{dY_{n+1}^{j}}{ds^{i}} = \begin{cases} 0 & \text{if } Y_{n+1}^{j} = 0 \\ \dfrac{dY_{n}^{j}}{ds^{i}} & \text{otherwise} \end{cases} \quad \text{where } i \in \{0,....,9\};\ j \in \{9,8,7\} \tag{A.89}$$

$$\frac{dY_{n+1}^{6}}{ds^{i}} = \begin{cases} 0 & \text{if } Y_{n+1}^{6} = 0 \\ \dfrac{dY_{n}^{6}}{ds^{i}} + \dfrac{dY_{n-2}^{9}}{ds^{i}} - 1\left(\text{add -1 only if } i \in \{9\}\right) & \text{if } Y_{n+1}^{6} = \begin{bmatrix} Y_{n}^{6} + \sum_{j=1}^{4} \xi_{n}^{j} - DS_{n-1}^{5} - DS_{n-1}^{4} \\ - s^{9} + Y_{n-2}^{9} + \sum_{j=1}^{4} \xi_{n}^{j} \end{bmatrix} \\ \dfrac{dY_{n}^{6}}{ds^{i}} + \dfrac{dY_{n-2}^{8}}{ds^{i}} - 1\left(\text{add -1 only if } i \in \{8\}\right) & \text{if } Y_{n+1}^{6} = \begin{bmatrix} Y_{n}^{6} + \sum_{j=1}^{4} \xi_{n}^{j} - DS_{n-1}^{5} - DS_{n-1}^{4} \\ - s^{8} + Y_{n-2}^{8} + \sum_{j=1}^{4} \xi_{n}^{j} \end{bmatrix} \\ \dfrac{dY_{n}^{6}}{ds^{i}} + \dfrac{dY_{n-2}^{7}}{ds^{i}} - 1\left(\text{add -1 only if } i \in \{2\}\right) & \text{if } Y_{n+1}^{6} = \begin{bmatrix} Y_{n}^{6} + \sum_{j=1}^{4} \xi_{n}^{j} - DS_{n-1}^{5} - DS_{n-1}^{4} \\ - s^{2} + Y_{n-2}^{2} + \sum_{j=1}^{4} \xi_{n}^{j} \end{bmatrix} \\ \dfrac{dY_{n}^{6}}{ds^{i}} & \text{otherwise} \end{cases}$$

$$\text{where } i \in \{0,....,9\}$$

(A.90)

$$\frac{dY_{n+1}^{5}}{ds^{i}} = \begin{cases} 0 & \text{if } Y_{n+1}^{5} = 0 \\ \dfrac{dY_{n}^{5}}{ds^{i}} + \underbrace{\dfrac{dY_{n-2}^{6}}{ds^{i}} - 1}_{\left(\text{add -1 only if } i \in \{6\}\right)} & \text{if } Y_{n+1}^{5} = \begin{bmatrix} Y_{n}^{5} + \xi_{n}^{1} + \xi_{n}^{2} - DS_{n-1}^{3} - DS_{n-1}^{2} \\ - s^{6} + Y_{n-2}^{6} + \sum_{j=1}^{4} \xi_{n-1}^{j} \end{bmatrix} \\ \dfrac{dY_{n}^{5}}{ds^{i}} & \text{otherwise} \end{cases} \quad \text{where } i \in \{0,....,9\} \tag{A.91}$$

$$\underset{\text{where } i\in\{0,\ldots,9\}}{\frac{dY_{n+1}^4}{ds^i}} = \begin{cases} 0 & \text{if } Y_{n+1}^4 = 0 \\ \underset{(\text{add -1 only if } i\in\{6\})}{\frac{dY_n^4}{ds^i} + \frac{dY_{n-2}^6}{ds^i} - 1} & \text{if } Y_{n+1}^4 = \begin{bmatrix} Y_n^4 + \xi_n^3 + \xi_n^4 - DS_{n-1}^1 - DS_{n-1}^0 \\ -s^6 + Y_{n-2}^6 + \sum_{j=1}^{4} \xi_{n-1}^j + \xi_{n-2}^1 + \xi_{n-2}^2 \end{bmatrix} \\ \frac{dY_n^4}{ds^i} & \text{otherwise} \end{cases} \tag{A.91}$$

$$\underset{\text{where } i\in\{0,\ldots,9\}}{\frac{dY_{n+1}^3}{ds^i}} = \begin{cases} 0 & \text{if } Y_{n+1}^3 = 0 \\ \frac{dY_n^3}{ds^i} + \underset{(\text{add -1 only if } i\in\{5\})}{\left[\frac{dY_{n-2}^5}{ds^i} - 1\right]} * (ratio1) & \text{if } Y_{n+1}^3 = \begin{bmatrix} Y_n^3 + \xi_n^1 - \\ \left[s^5 - Y_{n-2}^5 - \xi_{n-1}^1 - \xi_{n-1}^2\right] \\ *(ratio1) \end{bmatrix} \\ \frac{dY_n^3}{ds^i} & \text{otherwise} \end{cases} \tag{A.92}$$

$$\underset{\text{where } i\in\{0,\ldots,9\}}{\frac{dY_{n+1}^2}{ds^i}} = \begin{cases} 0 & \text{if } Y_{n+1}^2 = 0 \\ \frac{dY_n^2}{ds^i} + \underset{(\text{add -1 only if } i\in\{5\})}{\left[\frac{dY_{n-2}^5}{ds^i} - 1\right]} * (1 - ratio1) & \text{if } Y_{n+1}^2 = \begin{bmatrix} Y_n^2 + \xi_n^2 - \\ \left[s^5 - Y_{n-2}^5 - \xi_{n-1}^1 - \xi_{n-1}^2\right] \\ *(1 - ratio1) \end{bmatrix} \\ \frac{dY_n^2}{ds^i} & \text{otherwise} \end{cases} \tag{A.93}$$

$$\underset{\text{where } i\in\{0,\ldots,9\}}{\frac{dY_{n+1}^1}{ds^i}} = \begin{cases} 0 & \text{if } Y_{n+1}^1 = 0 \\ \frac{dY_n^1}{ds^i} + \underset{(\text{add -1 only if } i\in\{4\})}{\left[\frac{dY_{n-2}^4}{ds^i} - 1\right]} * (ratio2) & \text{if } Y_{n+1}^1 = \begin{bmatrix} Y_n^1 + \xi_n^3 - \\ \left[s^4 - Y_{n-2}^4 - \xi_{n-1}^3 - \xi_{n-1}^4\right] \\ *(ratio2) \end{bmatrix} \\ \frac{dY_n^1}{ds^i} & \text{otherwise} \end{cases} \tag{A.94}$$

$$
\underset{\text{where } i\in\{0,\ldots,9\}}{\frac{dY_{n+1}^{0}}{ds^{i}}} = \begin{cases} 0 & \text{if } Y_{n+1}^{0}=0 \\ \dfrac{dY_{n}^{0}}{ds^{i}} + \underset{(\text{add -1 only if } i\in\{4\})}{\left[\dfrac{dY_{n-2}^{4}}{ds^{i}} - 1\right]} * \left(1 - ratio\,2\right) & \text{if } Y_{n+1}^{1} = \begin{bmatrix} Y_{n}^{0} + \xi_{n}^{4} - \\ \left[s^{4} - Y_{n-2}^{4} - \xi_{n-1}^{3} - \xi_{n-1}^{4}\right] \\ * \left(1 - ratio\,2\right) \end{bmatrix} \\ \dfrac{dY_{n}^{0}}{ds^{i}} & \text{otherwise} \end{cases} \qquad \text{(A.95)}
$$

A10. Additional Results for Network 1 and Network 2

Table A38: Base-stock Levels for Network 1

#	Base-Stock Levels for Network 1								
	S0	S1	S2	S3	S4	S5	S6	S7	Total
1	10.08	10.08	27.28	27.68	38.88	27.28	27.68	38.88	**207.84**
2	21.93	21.93	29.93	30.37	29.73	29.93	30.37	29.73	**223.92**
3	15.06	15.06	34.66	34.86	41.46	34.66	34.86	41.46	**252.08**
4	25.25	25.25	34.25	34.45	32.85	34.25	34.45	32.85	**253.6**
5	21.72	21.72	38.32	38.72	48.32	38.32	38.72	48.32	**294.16**
6	33.2	33.2	41.01	43.01	42.21	41.01	43.01	42.21	**318.86**
7	37	37	56.6	56.6	67.2	56.6	56.6	67.2	**434.8**
8	44.03	44.03	52.23	54.03	51.43	52.23	54.03	51.43	**403.44**

Table A39: Safety-stock Levels for Network 1

#	Safety-Stock Levels for Network 1								
	S0	S1	S2	S3	S4	S5	S6	S7	Total Safety-Stock
1	3.08	3.08	3.08	3.08	3.08	3.08	3.08	3.08	**24.64**
2	2.53	2.53	2.53	2.17	2.53	2.53	2.17	2.53	**19.52**
3	5.26	5.26	5.26	5.26	5.26	5.26	5.26	5.26	**42.08**
4	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	**42**
5	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	**96.96**
6	13.6	13.6	13.61	13.61	13.61	13.61	13.61	13.61	**108.86**
7	28.2	28.2	28.2	28.2	28.2	28.2	28.2	28.2	**225.6**
8	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	**192.24**

Table A40: Base-stock Levels for PT5 Inventory Allocation Under Network 2

#	Base-Stock Levels for Network 2 (Lexicographic Allocation with Priority to Node 5) PT5										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total
1	10	20.2	20.6	19.4	27.2	37	61.2	61.2	61.2	61.2	**379.2**
2	20.61	21.21	11.81	11.81	38.81	20.61	56.41	56.41	56.41	56.41	**350.5**
3	11	10.2	21.6	20	18.2	38.6	53.8	53.8	53.8	53.8	**334.8**
4	19.5	29.3	29.5	25.9	39.1	45.7	75.1	75.1	75.1	75.1	**489.4**
5	23.78	25.78	13.78	15.18	43	22.4	58.8	58.8	58.8	58.8	**379.12**
6	14.04	14.64	22.64	25.24	22.04	41.24	56.64	56.64	56.64	56.64	**366.4**
7	27.76	34.8	35.2	34.8	44.4	51.8	78	78	78	78	**540.76**
8	32.1	31.5	23.7	23.5	48.5	32.1	65.5	65.5	65.5	65.5	**453.4**
9	25.3	27.3	36.9	36.7	34.5	55.5	71.9	71.9	71.9	71.9	**503.8**
10	47.5	58.3	58.3	57.7	67.1	77.3	105.7	105.7	105.7	105.7	**789**
11	55	54	45.4	44.2	73	54.6	90.6	90.6	90.6	90.6	**688.6**
12	43.7	44.3	53.5	53.3	52.1	70.9	87.1	87.1	87.1	87.1	**666.2**

Table A41: Safety-stock Levels for PT5 Inventory Allocation Under Network 2

#	Safety-Stock Levels for Network 2 (Lexicographic Allocation with Priority to Node 5)										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total Safety-Stock
1	3	3	3	3	3	3	3	3	3	3	30
2	3.01	3.01	3.01	3.01	3.01	3.01	3.01	3.01	3.01	3.01	30.1
3	3	3	3	3	3	3	3	3	3	3	30
4	9.7	9.7	9.7	9.7	9.7	9.7	9.7	9.7	9.7	9.7	97
5	6.58	6.58	6.58	6.58	6.6	6.6	6.6	6.6	6.6	6.6	65.92
6	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	66.4
7	18.16	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	181.96
8	15.1	15.1	15.1	15.1	15.1	15.1	15.1	15.1	15.1	15.1	151
9	18.1	18.1	18.1	18.1	18.1	18.1	18.1	18.1	18.1	18.1	181
10	38.7	38.7	38.7	38.7	38.7	38.7	38.7	38.7	38.7	38.7	387
11	36	36	36	36	36	37	36	36	36	36	361
12	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	359

Table A42: Base-stock Levels for PT4 Inventory Allocation Under Network 2

#	Base-Stock Levels for Network 2 PT4										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total
1	9.9	19.6	20.5	19.3	27	36.7	61	61	61	61	377
2	20.8	21.1	12	12	39	20.8	56.3	56.3	56.3	56.3	350.9
3	10.8	9.7	21.4	19.8	18	38.4	53.6	53.6	53.6	53.6	332.5
4	19.2	29	29.2	25.6	38.8	45.4	74.8	74.8	74.8	74.8	486.4
5	25.89	27.89	15.89	17.29	45.09	24.49	60.89	60.89	60.89	60.89	400.1
6	14.04	14.64	22.64	25.24	22.04	41.24	56.64	56.64	56.64	56.64	366.4
7	27.54	34.54	34.94	34.54	44.14	51.54	77.74	77.74	77.74	77.74	538.2
8	34	33	25.6	25.4	50.4	34	67.37	67.37	67.37	67.37	471.88
9	25.17	27.55	36.77	36.57	34.4	55.4	71.8	71.8	71.8	71.8	503.06
10	48	58.8	58.8	58.2	67.6	77.8	106.15	106.15	106.15	106.15	793.8
11	55.1	54.1	45.5	44.3	73.1	53.7	90.7	90.7	90.7	90.7	688.6
12	43.7	44.3	53.5	53.3	52.1	70.9	87.1	87.1	87.1	87.1	666.2

Table A43: Safety-stock Levels for PT4 Inventory Allocation Under Network 2

#	Safety-Stock Levels for Network 2 (Lexicographic Allocation with Priority to Node 4)										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total Safety-Stock
1	2.9	2.4	2.9	2.9	2.8	2.7	2.8	2.8	2.8	2.8	27.8
2	3.2	2.9	3.2	3.2	3.2	3.2	2.9	2.9	2.9	2.9	30.5
3	2.8	2.5	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	27.7
4	9.4	9.4	9.4	9.4	9.4	9.4	9.4	9.4	9.4	9.4	94
5	8.69	8.69	8.69	8.69	8.69	8.69	8.69	8.69	8.69	8.69	86.9
6	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	6.64	66.4
7	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	179.4
8	17	16.6	17	17	17	17	16.97	16.97	16.97	16.97	169.48
9	17.97	18.35	17.97	17.97	18	18	18	18	18	18	180.26
10	39.2	39.2	39.2	39.2	39.2	39.2	39.15	39.15	39.15	39.15	391.8
11	36.1	36.1	36.1	36.1	36.1	36.1	36.1	36.1	36.1	36.1	361
12	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	359

Table A44: Base-stock Levels for PA Inventory Allocation Under Network 2

Scenario #	Base-Stock Levels for Network 2 (PA Policy)										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total
1	13.04	22.86	23.64	22.44	30.24	40	64.24	64.24	64.24	64.24	**409.18**
2	20.66	21.26	11.86	11.86	38.86	20.66	56.46	56.46	56.46	56.46	**351**
3	11	10.2	21.6	20	18.2	38.6	53.8	53.8	53.8	53.8	**334.8**
4	20	29.8	30	26.4	39.6	41.2	75.6	75.6	75.6	75.6	**489.4**
5	24.78	26.78	14.78	16.18	43.98	23.38	59.8	59.8	59.8	59.8	**389.08**
6	14.15	14.75	22.75	25.35	22.15	41.35	56.75	56.75	56.75	56.75	**367.5**
7	27.21	34.21	34.61	34.21	43.81	51.21	77.41	77.41	77.41	77.41	**534.9**
8	33.55	32.95	25.15	24.95	49.95	33.55	66.95	66.95	66.95	66.95	**467.9**
9	22.8	24.8	34.4	34.2	32	53	69.4	69.4	69.4	69.4	**478.8**
10	47.47	58.27	58.27	57.67	67.07	77.27	105.67	105.67	105.67	105.67	**788.7**
11	55.14	55.14	45.54	44.34	73.14	53.74	90.74	90.74	90.74	90.74	**690**
12	44.74	45.34	54.54	54.34	53.14	71.94	88.14	88.14	88.14	88.14	**676.6**

Table A43a: Safety-stock Levels for PA Inventory Allocation Under Network 2

Scenario #	Safety-Stock Levels for Network 2 (PA Policy)										
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total Safety-Stock
1	6.04	5.66	6.04	6.04	6.04	6	6.04	6.04	6.04	6.04	**59.98**
2	3.06	3.06	3.06	3.06	3.06	3.06	3.06	3.06	3.06	3.06	**30.6**
3	3	3	3	3	3	3	3	3	3	3	**30**
4	10.2	10.2	10.2	10.2	10.2	5.2	10.2	10.2	10.2	10.2	**97**
5	7.58	7.58	7.58	7.58	7.58	7.58	7.6	7.6	7.6	7.6	**75.88**
6	6.75	6.75	6.75	6.75	6.75	6.75	6.75	6.75	6.75	6.75	**67.5**
7	17.61	17.61	17.61	17.61	17.61	17.61	17.61	17.61	17.61	17.61	**176.1**
8	16.55	16.55	16.55	16.55	16.55	16.55	16.55	16.55	16.55	16.55	**165.5**
9	15.6	15.6	15.6	15.6	15.6	15.6	15.6	15.6	15.6	15.6	**156**
10	38.67	38.67	38.67	38.67	38.67	38.67	38.67	38.67	38.67	38.67	**386.7**
11	36.14	37.14	36.14	36.14	36.14	36.14	36.14	36.14	36.14	36.14	**362.4**
12	36.94	36.94	36.94	36.94	36.94	36.94	36.94	36.94	36.94	36.94	**369.4**

A11. Additional Results Heuristic Starting Points

Table A44a: Starting Points for Three-Echelon Assembly System Instance 1

	Starting Points							
	Traditional				Rule Based Approach			
	Node 0	Node 1	Node 2	Node 3	Node 0	Node 1	Node 2	Node 3
Case 1	9.32	26.51	26.51	26.51	9.9	29.21	27.62	27.62
Case 2	8.57	26.5	26.5	26.5	9.1	29.2	27.61	27.61
Case 3	6.9	19.45	19.45	19.45	8.58	24.97	23.81	23.81
Case 4	24.34	51.68	51.68	51.68	25.36	56.43	52.82	52.82
Case 5	10.32	30.82	30.82	30.82	12.63	39.26	37.11	37.11
Case 6	14.18	51.88	51.88	51.88	17.36	66.1	62.46	62.46
Case 7	35.88	80.52	80.52	80.52	36.67	86.32	80.68	80.68
Case 8	21.45	78	78	78	21.92	83.62	78.16	78.16
Case 9	23.82	68.67	68.67	68.67	28.68	86.11	81.31	81.31

Table A45: Best Found Base-stock Levels for Three-Echelon Assembly System under Instance 1 with Traditional Approach

	Optimal Base-stock Levels Using Traditional Approach					
	S0	S1	S2	S3	Total	Time(min)
Case 1	10.28	27.48	27.48	27.48	**92.72**	0.84
Case 2	9.53	27.47	27.47	27.47	**91.94**	0.58
Case 3	16.52	29.08	29.08	29.08	**103.76**	11.85
Case 4	25.3	52.65	52.65	52.65	**183.25**	0.54
Case 5	15.98	36.48	36.48	36.48	**125.42**	4.96
Case 6	24.08	61.25	61.25	61.25	**207.83**	14.26
Case 7	36.84	81.49	81.49	81.49	**281.31**	0.64
Case 8	22.48	79	79	79	**259.48**	0.56
Case 9	34.44	79.3	79.3	79.3	**272.34**	14.52

Table A46: Target Base-stock Levels for Three-Echelon Assembly System under Instance 1 with Rule Based Approach

Optimal Base-stock Levels Using Rule Based Approach					
S0	S1	S2	S3	Total	Time (Min)
9.9	29.21	27.62	27.62	**94.35**	0.10
9.1	29.2	27.61	27.61	**93.52**	0.06
14.73	31.86	30.89	30.89	**108.37**	8.4
25.36	56.43	52.82	52.82	**187.43**	0.06
12.63	39.26	37.11	37.11	**126.11**	0.06
17.36	66.1	62.46	62.46	**208.38**	0.05
36.9	86.56	80.92	80.92	**285.3**	0.11
22.15	83.86	78.4	78.4	**262.81**	0.1
28.68	86.11	81.31	81.31	**277.41**	0.06

Table A47: Starting Points for Three-Echelon Assembly System under Instance 1 with Decomposition Approach

	Starting Points			
	Decomposition Approach			
	Node 0	Node 1	Node 2	Node 3
Case 4	26.84	56.4	56.4	56.4
Case 5	13.48	39.14	39.1	39.81
Case 6	18.25	65.86	65.86	65

Table A48: Target Base-stock Levels for Three-Echelon Assembly System under Instance 1 with Decomposition Approach

	Target Base-Stock Levels					
	Decomposition Approach					
	S0	S1	S2	S3	Total	Time (min)
Case 4	26.84	56.4	56.4	56.4	196.04	0.1
Case 5	13.48	39.14	39.1	39.81	131.53	0.07
Case 6	19	66.68	66.68	66.68	219.04	0.5

Table A49: Starting Points for Three-Echelon Assembly System Instance 2

	Starting Points							
	Traditional Approach				Rule Based Approach			
	Node 0	Node 1	Node 2	Node 3	Node 0	Node 1	Node 2	Node 3
Case 1	9.32	26.51	26.51	26.51	9.9	29.21	27.62	27.62
Case 2	8.57	26.5	26.5	26.5	9.1	29.2	27.61	27.61
Case 3	6.9	19.45	19.45	19.45	8.58	24.97	23.81	23.81
Case 4	24.34	51.68	51.68	51.68	25.36	56.43	52.82	52.82
Case 5	10.32	30.82	30.82	30.82	12.63	39.26	37.11	37.11
Case 6	14.18	51.88	51.88	51.88	17.36	66.1	62.46	62.46
Case 7	35.88	80.52	80.52	80.52	36.67	86.32	80.68	80.68
Case 8	21.45	78	78	78	21.92	83.62	78.16	78.16
Case 9	23.82	68.67	68.67	68.67	28.68	86.11	81.31	81.31

Table A50: Best Found Base-stock Levels for Three-Echelon Assembly System under Instance 2 with Traditional Approach

	Optimal Base-stock Levels Using Traditional Approach					
	S0	S1	S2	S3	Total	Time(min)
Case 1	10.28	27.48	27.48	27.48	**92.72**	0.84
Case 2	9.53	27.47	27.47	27.47	**91.94**	0.58
Case 3	16.52	29.08	29.08	29.08	**103.76**	11.85
Case 4	24.8	52.1	52.1	52.1	**181.1**	0.54
Case 5	15.98	36.48	36.48	36.48	**125.42**	4.96
Case 6	24.08	61.25	61.25	61.25	**207.83**	14.26
Case 7	36.84	81.49	81.49	81.49	**281.31**	0.64
Case 8	22.48	79	79	79	**259.48**	0.56
Case 9	34.44	79.3	79.3	79.3	**272.34**	14.52

Table A51: Target Base-stock Levels for Three-Echelon Assembly System under Instance 2 with Rule Based Approach

Target Base-stock Levels Using Rule Based Approach					
S0	S1	S2	S3	Total	Time (Min)
9.9	29.21	27.62	27.62	**94.35**	0.10
9.1	29.2	27.61	27.61	**93.52**	0.06
14.84	31.23	30.07	30.07	**106.21**	8.83
25.36	56.43	52.82	52.82	**187.43**	0.06
13.19	39.82	37.67	37.67	**128.35**	0.4
17.34	66.08	62.44	62.44	**208.3**	0.4
36.67	86.32	80.68	80.68	**284.35**	0.11
22.15	83.86	78.4	78.4	**262.81**	0.11
28.68	86.11	81.31	81.31	**277.41**	0.06

Table A52: Starting Points for Three-Echelon Assembly System under Instance 2 with Decomposition Approach

	Starting Points			
	Decomposition Approach			
	Node 0	Node 1	Node 2	Node 3
Case 4	24.57	51.95	51.95	51.95
Case 5	11.28	34.82	34.82	34.82
Case 6	15.14	55.88	55.88	55.88

Table A53: Target Base-stock Levels for Three-Echelon Assembly System under Instance 2 with Decomposition Approach

	Target Base-Stock Levels					
	Decomposition Approach					
	S0	S1	S2	S3	Total	Time (min)
Case 4	24.8	52.19	52.19	52.19	**181.37**	**0.1**
Case 5	14.6	38.14	38.14	38.14	**129.02**	**5.4**
Case 6	22.79	63.53	63.53	63.53	**213.38**	**6.67**

Table A54: Starting Points for Three-Echelon Assembly System Instance 3

	Starting Points							
	Traditional Approach				Rule Based Approach			
	Node 0	Node 1	Node 2	Node 3	Node 0	Node 1	Node 2	Node 3
Case 1	9.32	26.51	26.51	26.51	10.38	30.59	29	29
Case 2	8.57	26.5	26.5	26.5	9.55	30.58	28.99	28.99
Case 3	6.9	19.45	19.45	19.45	10.2	29.53	28.36	28.36
Case 4	24.34	51.68	51.68	51.68	26.63	59.12	55.5	55.5
Case 5	10.32	30.82	30.82	30.82	15.05	46.48	44.32	44.32
Case 6	14.18	51.88	51.88	51.88	20.67	78.24	74.6	74.6
Case 7	35.88	80.52	80.52	80.52	36.38	85.67	80.04	80.04
Case 8	21.45	78	78	78	21.75	82.99	77.53	77.53
Case 9	23.82	68.67	68.67	68.67	24.15	73.06	68.26	68.26

Table A55: Best Found Base-stock Levels for Three-Echelon Assembly System under Instance 3 with Traditional Approach

	Optimal Base-stock Levels Using Traditional Approach					
	S0	S1	S2	S3	Total	Time(min)
Case 1	12.5	29.69	29.69	29.69	**101.57**	1.59
Case 2	11.75	29.68	29.68	29.68	**100.79**	1.45
Case 3	19.8	32.44	32.44	32.44	**117.12**	24.77
Case 4	28.17	55.52	55.52	55.52	**194.73**	4.25
Case 5	20.03	40.48	40.48	40.48	**141.47**	6.59
Case 6	31.4	69.15	69.15	69.15	**238.85**	26.85
Case 7	36.84	81.49	81.49	81.49	**281.31**	0.6
Case 8	22.41	79	79	79	**259.41**	0.56
Case 9	24.78	69.64	69.64	69.64	**233.7**	0.56

Table A56: Target Base-stock Levels for Three-Echelon Assembly System under Instance 3 with Rule Based Approach

Target Base-stock Levels Using Rule Based Approach					
S0	S1	S2	S3	Total	Time (Min)
11.27	31.49	29.9	29.9	**102.56**	0.42
10.44	31.48	29.89	29.89	**101.7**	0.4
12.59	35.83	34.67	34.67	**117.76**	4.6
27.52	60.02	56.4	56.4	**200.34**	0.46
15.05	46.48	44.32	44.32	**150.17**	0.06
20.34	78	74.36	74.36	**247.06**	0.15
36.38	85.67	80.04	80.04	**282.13**	0.06
21.75	82.99	77.53	77.53	**259.8**	0.06
24.15	73.06	68.26	68.26	**233.73**	0.06

Table A57: Starting Points for Three-Echelon Assembly System under Instance 3 with Decomposition Approach

	Starting Points			
	Decomposition Approach			
	Node 0	Node 1	Node 2	Node 3
Case 4	26.7	55.85	56.04	56.67
Case 5	13.92	40.42	39.82	41.53
Case 6	19.18	69.34	68.08	68.59

Table A58: Target Base-stock Levels for Three-Echelon Assembly System under Instance 3 with Decomposition Approach

	Target Base-Stock Levels					
	Decomposition Approach					
	S0	S1	S2	S3	Total	Time (min)
Case 4	26.7	55.85	56.04	56.67	**195.26**	0.05
Case 5	15.95	42.45	41.85	41.85	**142.1**	3.83
Case 6	22.76	72.93	71.67	71.67	**239.03**	9.23

Table A59: Starting Points for Five-Echelon Assembly System Under Instance 3 with Traditional Approach

	Starting Points					
	Traditional Approach					
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5
Case 10	18.44	44.2	71.88	107.82	107.82	107.82
Case 11	16.12	32.44	60.48	87.58	87.58	87.58
Case 12	12.4	30.68	51.84	83.46	83.46	83.46

Table A60: Starting Points for Five-Echelon Assembly System Under Instance 3 with Rule Based Approach

	Starting Points					
	Rule Based Approach					
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5
Case 10	20.08	49.68	80.07	119.03	115.26	115.26
Case 11	17.55	36.46	67.37	96.69	93.62	93.62
Case 12	16.47	41.82	70.15	112.1	109.18	109.18

Table A61: Starting Points for Five-Echelon Assembly System Under Instance 3 with Decomposition Approach

	Starting Points for Decomposition Approach					
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5
Case 10	20.21	48.2	78.88	117.58	117.58	117.58
Case 11	17.89	35.44	66.08	95.78	95.78	95.78
Case 12	16.4	44.68	71.84	110.43	110.43	110.43

Table A62: Best Found Base-stock Levels for Five-Echelon Assembly System under Instance 3 with Traditional Approach

	Optimal Base-stock Levels for Traditional Approach							
	S0	S1	S2	S3	S4	S5	Total	Time(min)
Case 10	23.65	49.41	77	113	113	113	**489.06**	6.02
Case 11	20.52	36.57	64.61	91.71	91.71	91.71	**396.83**	12.12
Case 12	32.49	50.77	71.93	103.55	103.55	103.55	**465.84**	65

Table A63: Target Base-stock Levels for Five-Echelon Assembly System under Instance 3 with Rule Based Approach

	Target Base-stock Levels for Rule Based Approach							
	S0	S1	S2	S3	S4	S5	Total	Time (Min)
Case 10	20.08	49.68	80.07	119.03	115.26	115.26	**499.38**	0.06
Case 11	17.78	36.7	67.61	96.93	93.86	93.86	**406.74**	0.14
Case 12	17.44	42.8	71.13	113.45	110.53	110.53	**465.88**	4.49

Table A64: Target Base-stock Levels for Five-Echelon Assembly System under Instance 3 with Decomposition Approach

	Target Base-stock Levels for Decomposition Approach							
	S0	S1	S2	S3	S4	S5	Total	Time (min)
Case 10	20.44	48.44	79.12	117.82	117.82	117.82	**501.46**	0.1
Case 11	17.89	35.44	66.08	95.78	95.78	95.78	**406.75**	0.06
Case 12	17.91	46.19	73.35	111.94	111.94	111.94	**473.27**	1.67

Table A65: Starting Points for Network 1 Under Instance 3 with Traditional Approach

	Starting Points for Traditional Approach							
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
Case 1	13.34	13.34	35	36.36	38.94	35	36.36	38.94
Case 2	13.26	13.26	33.72	37.92	30.92	33.72	37.92	30.92
Case 3	18.94	18.94	44.88	43.56	41.14	44.88	43.56	41.14

Table A66: Starting Points for Network 1 Under Instance 3 with Rule Based Approach

	Starting Points for Rule Based Approach							
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
Case 13	14.46	14.39	38.64	40.14	40.09	37.42	38.87	41.63
Case 14	14.37	14.51	37.23	41.86	42	36.05	40.54	33.05
Case 15	29.2	29.39	70.08	68.02	68.07	68.51	66.5	62.8

Table A67: Starting Points for Network 1 Under Instance 3 with Decomposition Approach

	Starting Points for Decomposition Approach							
	S0	S1	S2	S3	S4	S5	S6	S7
Case 13	15.1	15.1	38.41	39.78	43.14	38.41	**39.78**	43.14
Case 14	15	15	37.24	42.13	34.33	37.24	**42.13**	34.33
Case 15	26.35	26.35	58.8	58.53	53.82	58.8	**58.53**	53.82

Table A68: Best Found Base-stock Levels for Network 1 under Instance 3 with Traditional Approach

	Optimal Base-stock Level for Traditional Approach									
	S0	S1	S2	S3	S4	S5	S6	S7	Total	Time(min)
Case 13	16.47	16.48	38.22	39.22	42.16	38.22	39.58	41.95	**272.3**	2.25
Case 14	16.2	16.2	36.7	40.9	33.9	36.7	40.9	33.9	**255.4**	2.8
Case 15	38	38	63.6	62.3	60.4	63.6	62.3	60.4	**448.6**	39

Table A69: Target Base-stock Levels for Network 1 under Instance 3 with Rule Based Approach

	Target Base-stock Level for Rule Based Approach									
	S0	S1	S2	S3	S4	S5	S6	S7	Total	Time (Min)
Case 13	16.16	16.09	40.34	41.84	41.79	39.12	40.57	43.33	**279.24**	0.80
Case 14	15.51	15.67	38.4	43	43.1	37.2	41.7	33.8	**268.38**	0.5
Case 15	29.61	29.81	70.5	68.5	68.5	69	67	64	**466.92**	1.06

Table A70: Target Base-stock Levels for Network 1 under Instance 3 with Decomposition Approach

	Target Base-stock Level for Decomposition Approach									
	S0	S1	S2	S3	S4	S5	S6	S7	Total	Time(min)
Case 13	15.34	15.34	38.65	40.02	43.38	38.65	43.38	43.38	278.14	0.12
Case 14	15.94	15.94	38.18	43.07	35.27	38.18	35.27	35.27	257.12	0.47
Case 15	33.7	33.7	66.15	65.88	61.17	66.15	61.17	61.17	449.09	21.1

Table A71: Starting Points for Network 2 Under Instance 3 with Traditional Approach

	Starting Points for Traditional Approach									
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	Node 8	Node 9
Case 16	13.98	21.32	17.94	16.9	35.3	34.84	70.14	70.14	70.14	70.14
Case 17	11.22	22.12	19.04	18.26	33.34	37.3	70.64	70.64	70.64	70.64
Case 18	19.44	20.24	16.28	22.74	39.68	39.02	78.7	78.7	78.7	78.7

Table A72: Starting Points for Network 2 Under Instance 3 with Rule Based Approach

	Starting Points for Rule Based Approach									
	Node 0	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	Node 8	Node 9
Case 16	15.15	23.11	19.45	18.32	38.09	37.24	75.33	74.98	74.98	74.98
Case 17	12.16	23.98	20.64	19.79	35.97	39.87	75.87	75.51	75.51	75.51
Case 18	28.15	29.31	23.57	32.93	57.26	55.92	113.17	112.78	112.78	112.78

Table A73: Starting Points for Network 2 Under Instance 3 with Decomposition Approach

	Starting Points for Decomposition Approach									
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Case 16	15.74	23.32	19.71	18.67	38.71	38.25	76.46	76.46	76.46	76.46
Case 17	12.27	24.12	21.04	20.14	36.75	40.94	76.96	76.96	76.96	76.96
Case 18	26.08	27.06	22.69	30.67	51.81	51.15	104.58	104.58	104.58	104.58

Table A74: Best Found Base-stock Levels for Network 2 under Instance 3 with Traditional Approach

	Optimal Base-stock Level for Traditional Approach											
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total	Time(min)
Case 16	18.1	25.1	22.1	21.1	39.1	39.3	74.1	74.1	74.1	74.1	**461.2**	7.8
Case 17	16.1	27.31	24.61	23.8	38.9	42.3	76.21	76.2	76.2	76.2	**477.83**	7.4
Case 18	37.44	38.24	34.28	40.74	57.68	57.02	96.7	96.7	96.7	96.7	**652.2**	71.8

Table A75: Target Base-stock Levels for Network 2 under Instance 3 with Rule Based Approach

	Target Base-stock Level for Rule Based Approach											
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total	Time (Min)
Case 16	16.11	24.04	20.41	19.28	39	38.2	76.3	76.3	76.3	76.3	**462.2**	1.03
Case 17	15.1	26.1	22.8	21.9	38.11	42.01	78.01	78.01	78.01	78.01	**478.1**	3
Case 18	28.15	29.31	23.57	32.93	57.26	55.92	113.17	112.78	112.8	112.8	**678.7**	0.06

Table A75: Target Base-stock Levels for Network 2 under Instance 3 with Decomposition Approach

Target Base-stock Level for Decomposition Approach												
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total	Time(min)
Case 16	15.98	23.58	19.9	18.9	38.9	38.5	76.7	76.7	76.7	76.7	462.6	0.06
Case 17	14	25.84	22.76	21.86	38.47	42.66	78.68	78.68	78.68	78.68	480.3	0.12
Case 18	28.86	29.84	25.47	33.45	54.59	53.93	107.4	107.36	107.36	107.36	655.6	34.32