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A Study of Nonlinear Combustion Instability

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To the Graduate Council:

I am submitting herewith a dissertation written by Eric J. Jacob entitled "A Study of Nonlinear Combustion Instability." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Aerospace Engineering.

Gary A. Flandro, Major Professor

We have read this dissertation and recommend its acceptance:

John S. Steinhoff, Trevor M. Moeller, Christian G. Parigger, Boris A. Kupershmidt

Accepted for the Council:

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A Study of Nonlinear Combustion Instability

A Dissertation Presented for
the Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Eric J. Jacob

December 2009

Dedicated to my loving family.

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I would first like to acknowledge my advisor Dr. Flandro for inviting me into his research group and providing me with enlightening classes and stimulating research topics. Furthermore, Joel Davenport and Keith Walker have continuously been a tremendous help in all of my experimental endeavors and I am indebted to them for their assistance. I'd like to thank my fellow colleagues Lutz Blaette who I have worked with, side by side, through many stubborn experiments and Tony Saad for his reservoir of knowledge including that in mathematics and all things regarding computers. Also, the members of my committee have been a tremendous help and inspiration. Additionally, I'd like to thank Paul Gloyer and all of GTL for involving me in their research with NASA and ATK. The experience has been thrilling. Finally, I would like to thank my Family for their undying support.

Problems worthy

of attack,

prove their worth

by hitting back.

-Piet Hein

Abstract

Combustion instability (CI) has been persistent in all forms of propulsion since their inception. CI is characterized by pressure oscillations within the propulsion system. If even a small fraction of the dense energy within the system is converted to acoustic oscillations the system vibrations can be devastating. The coupling of combustion and fluid dynamic phenomena in a nonlinear system poses CI as a significant engineering challenge.

Drawing from previous analysis, second order acoustic energy models are taken to third order. Second order analysis predicts exponential growth. The addition of the third order terms capture the nonlinear acoustic phenomena (such as wave steepening) observed in experiments. The analytical framework is derived such that the energy sources and sinks are properly accounted for. The resulting third order solution is compared against a newly performed simplified acoustic closed tube experiment. This experiment provides the interesting result that in a forced system, as the 2nd harmonic is driven, no energy is transferred back into the 1st mode. The subsequent steepened waveform is a summation of 2nd mode harmonics (2, 4, 6, 8...) where all odd modes are nonexistent. The current third order acoustic model recreates the physics as seen in the experiment.

Numerical experiments show the sensitivity of the pressure wave limit cycle amplitude to the second order growth rate, highlighting the importance of correctly calculating the growth rates. The sensitivity of the solution to the third order parameter is shown as well. Exponential growth is found if the third order parameter is removed, and increased nonlinear behavior is found if it retained and as it is increased. The solutions sensitivity to this term highlights its importance and shows the need for continued analysis via increasing the models

generality by including neglected effects. In addition, the affect of a time varying second order growth rate is shown. This effect shows the importance of modeling the system in time because of the time lag between changes in the growth rate to a change in the limit cycle amplitude.

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Chapter 1

Introduction

Combustion instability (CI) is a major concern in all propulsion and power generation devices. It is characterized by vibrations within the combustion system, generally measured as an oscillating pressure. CI has been a problem in propulsion devices since their inception. Combustion Instability occurs when dynamic phenomena within the propulsion system drive oscillations. Typically these oscillations arise from a driving phenomena, a random noise or a sudden pulse which then couples with the driving phenomena, drawing energy from the system internally, and resonates with the acoustic oscillations in the system. The resulting large oscillation is then responsible for design failures. Figure 1.1 depicts a typical example of combustion instability in a solid rocket motor.

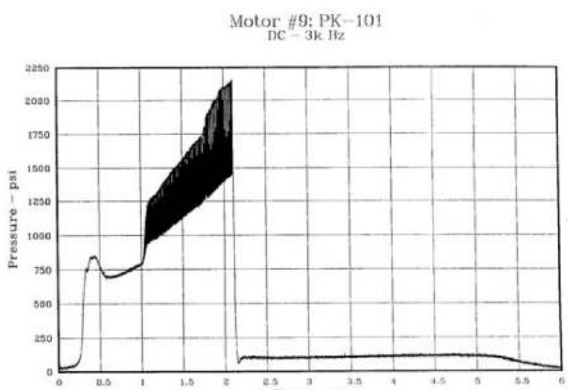


Figure 1.1: Example Pressure Plot from a Solid Rocket Motor (1)

Combustion Instability is a serious difficulty while designing propulsion devices because of their large internal energy density. If even a small amount of the contained energy is converted into organized pressure oscillations the results are disastrous. Mechanical failure, vibration loads, and increased heat transfer are examples of the destructive nature of unstable systems.

The name combustion instability itself is a misnomer. In many cases the oscillations have little or nothing to do with combustion. In a simple system, such as a flute, oscillations occur naturally with no combustion involved. The same phenomena are at work in a large solid rocket booster.

1.1 Forced Oscillatory Behavior

Undoubtedly, the simplest oscillatory system is the mass-spring system. This type of system is referred to as a forced system because the driving mechanism is an outside forcing function. In its simplest form we have a mass attached to a spring riding on a frictionless platform as shown in Figure 1.2. The mass may or may not have an externally applied force and damping. For small amplitudes the dynamics of this system are defined by the differential equation,

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = F(t). \tag{1.1}$$

Where ω_0 is the undamped natural frequency of the oscillator, ζ is the damping ratio and $F(t)$ is the driving force. For a mass on a spring having a spring constant k and a damping

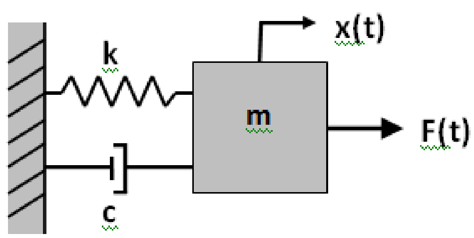


Figure 1.2: Damped Mass Spring System

coefficient c, $\omega_0 = \sqrt{k/m}$ and $\zeta = c/2m\omega_0$. Solving this equation for the unforced, undamped solution we find,

$$x(t) = A\sin(2\pi f t + \phi) \tag{1.2}$$

This solution demonstrates the natural sinusoidal nature of the oscillations. In later chapters we will extend this thinking to fluid dynamics via acoustics. Assuming a sinusoidal driving force $F(t) = F_0 \sin(\omega t)$, after stead state is achieved, we find for the damped forced solution,

$$x(t) = \frac{F_0}{Z_m \omega}\sin(\omega t + \phi) \tag{1.3}$$

Where, $Z_m = \sqrt{(2\omega_0\zeta)^2 + \frac{1}{\omega^2}\left(\omega_0^2 - \omega^2\right)^2}$ and $\phi = \arctan\left(\frac{2\omega\omega_0\zeta}{\omega^2 - \omega_0^2}\right)$. This solution shows the origins of resonance. When the system is driven near to is natural frequency the amplitude of the oscillations is increased. Figure 1.3 depicts the growth transient of a forced-damped oscillator. The internal mechanisms in propulsion systems drive at varying frequencies and amplitudes. This affects the resulting observed oscillations making them a mix of different phenomena. We will find later that a combined forced-self excited system is a more appropriate model, but the simple mass spring system demonstrates the resonance principle nicely.

Additionally, the damping in a system can be positive or negative. Positive damping decreases the amplitude of oscillation in time, and a negative damping coefficient will increase it. In the field of combustion instability this concept of damping is referred to as alpha, α, or the 'linear' growth rate. The use of the term 'linear' is due to the fact that α arrises from linearized differential equations. In essence, α is the internally driven damping coefficient just as in $e^{\alpha t}$. However, the notation is reversed from that of damping, μ. If it is negative, oscillations decrease in time, if it is positive, they increase.

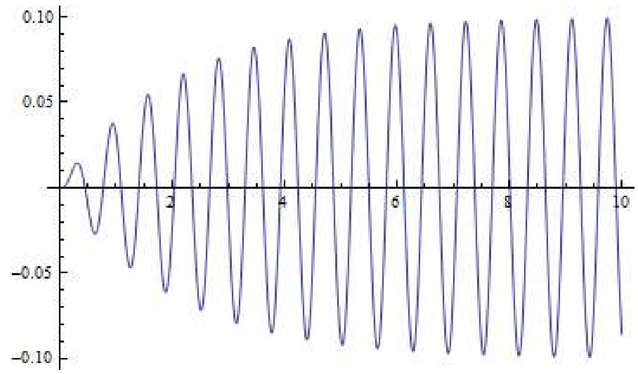


Figure 1.3: Example Waveform from a Driven Damped Harmonic Oscillator

1.2 Motor Stability

A system, within the context of combustion instability, is defined as stable when the net effect on the organized oscillations is to dampen them. Mathematically, this is achieved when, $\alpha < 0$. In an unstable system the oscillation amplitudes grow. This is contrary to the industry's standard practice where a low amplitude pressure oscillation (below 10% of the mean pressure) would be considered 'stable.' In this case the system is clearly unstable as seen by the presence of organized oscillations. A truly stable motor dampens out all oscillations if pulsed and runs smoothly (excluding turbulent noise). In the end, knowing if a motor is stable or unstable is not sufficient. The pressure amplitude at which the system oscillates differentiates varying degrees of instability. The system designers need to know the actual operating conditions and these design parameters are functions of a complex nonlinear system with many energy sources and interactions.

1.3 Self-Excited Systems

It was mentioned previously that a forced system alone is not an appropriate way to model combustion instability. The mechanisms which drive internal oscillations within a rocket motor are typically dependent on the oscillations within the motor and not solely an outside force. For example, the propellant burning rate is dependent on the pressure which, in turn, is dependent on the burning rate of the propellant. This reinforcing nature shows that this is primarily a self excited system. Oscillations within the rocket motor arise from chaotic turbulent behavior as well as organized flow instabilities such as vortex shedding. These oscillations are then either reinforced or damped internally by the system.

Van Der Pol Equation

The Van Der Pol equation is a classic example of a nonlinear self excited system. It arises from the study of electrical circuits including vacuum tubes. It is given by,

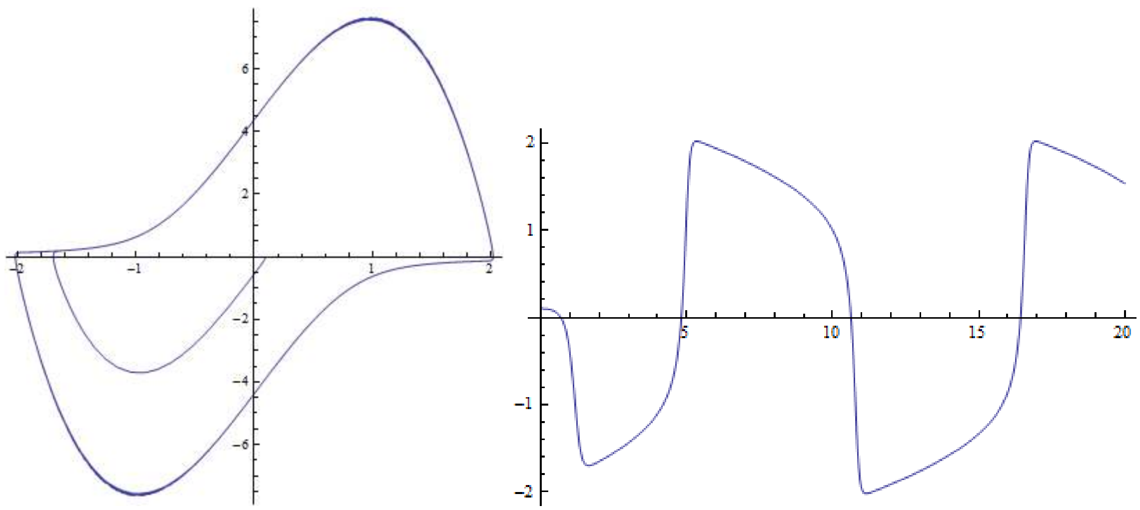
$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \tag{1.4}$$

Where μ is the strength of the damping. When the system has an initial condition of $x = .1$ and $x' = 0$ with a damping coefficient of $\mu = 5$ the system exhibits nonlinear behavior unlike the sinusoidal nature of the linear differential system. Figure 1.4(b) depicts this nonlinear waveform. Figure 1.4(a) shows the phase plane plot for the same case. The system starts at $(.1, 0)$ and quickly grows to a larger amplitude where the system reaches its limit.

As μ approaches zero the system becomes a linear oscillator, $\frac{d^2x}{dt^2} + x = 0$. Thus, when a smaller damping value of $\mu = .1$ is used the system exhibits nearly sinusoidal behavior seen in the nearly circular phase plane plot in Figure 1.5(a). The low level of driving allows for a nice example of the growth transient shown in Figure 1.5(b).

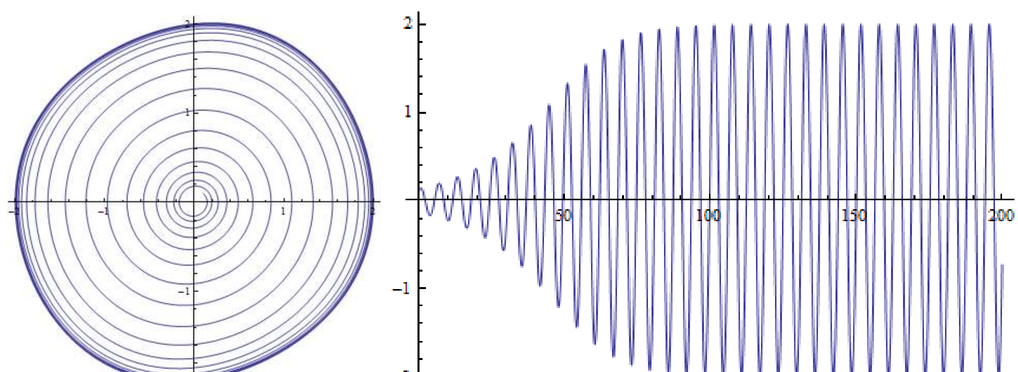
1.4 Forced vs. Self-Excited Systems

The difference in the two systems is seen in their waveforms. First, in Figure 1.6(a), we see a driven damped oscillator. The concave down exponential growth, proportional to $1 - e^{-\beta t}$, is a solution to a linear differential equation which is referred to henceforth as 'linear' behavior. Additionally, it is linear on a time scale $t \ll 1/\beta$. However, in Figure 1.6(b), the system starts with exponential, $e^{\alpha t}$ growth, which transitions to nonlinear limiting. Both cases arrive at an equilibrium, or a limit cycle amplitude, but they do so by different means. In reality, the complex system within a rocket motor is a combination of driven and self excited



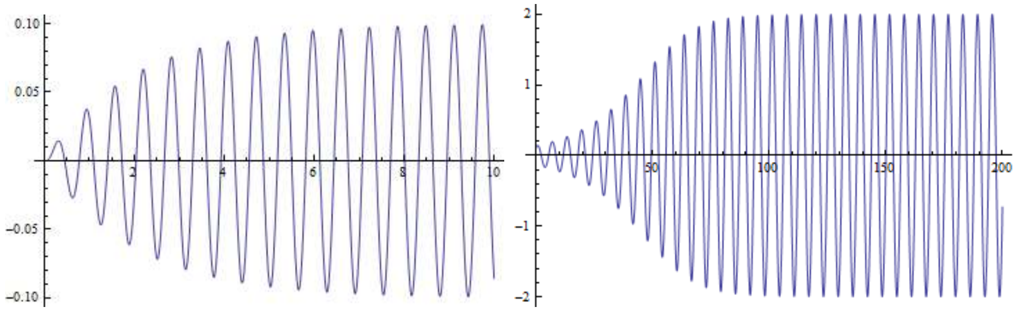
(a) Phase Plane (b) Waveform

Figure 1.4: Van Der Pol Equation $\mu = 5$



(a) Phase Plane (b) Waveform

Figure 1.5: Van Der Pol Equation $\mu = 0.1$



(a) Forced (b) Self-Excited

Figure 1.6: Comparing Forced vs. Self Excited Systems

mechanisms. Additionally, because of the random nature of the measured pressure signals and their rapid time dependence it is difficult to make any firm conclusions based on the experimental data.

1.5 Acoustics

Before we move onto nonlinear dynamics we will introduce the equations for acoustics. A detailed derivation of this equation is found is Appendix A.2. More information on acoustics can be found in (9) or (8). By combining the continuity and momentum equations we arrive at,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0 \tag{1.5}$$

Several assumptions have been made to bring us to this point. These assumptions play a critical role in future models. In the acoustic solution we assume that the system is isentropic and thermally perfect. Later, when evaluating CI, there is motivation to avoid these same assumptions. For example, the isentropic assumption is not appropriate when distributed combustion is present in the flow field.

The solution to the acoustic equation depends on the geometry involved. For closed tube, which represents a choked combustion chamber, in the 1-D axial case we arrive at the classical sines and cosines solution. The first four longitudinal mode shapes for a closed-closed acoustic tube are shown in Figure 1.7.

In a real system many other modes are prevalent in a closed tube. For example, tangential and radial acoustic modes form Bessel functions. These solutions represent a linear acoustic solution similar in nature to the mass spring problem. However, in real fluids the oscillations behave in a nonlinear fashion, thus, we need to account for the deviation from the linear model.

1.6 Nonlinear Behavior

Nonlinear systems have interesting consequences which are not seen in linear systems. These nonlinear effects are critical in understanding combustion instability. Unstable pressure traces are well documented in the literature. Several key characteristics are prevalent in the data. These are:

- Limit Cycle Amplitudes
- Wave Steepening
- 'DC' or Mean Pressure Shift

1.6.1 Limit Cycle Amplitudes

In a simplified case we can imagine a negatively damped spring-mass oscillator. The mass would begin oscillating at a given amplitude and in time it would increase in amplitude. If the damping value were dependent upon the oscillation amplitude, we could image a system

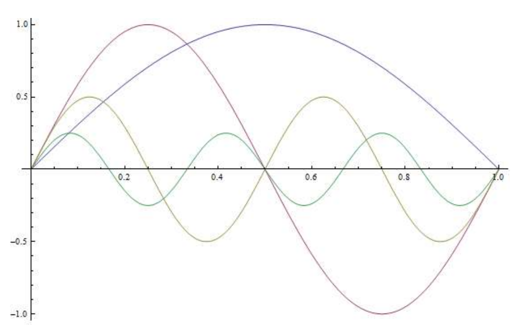


Figure 1.7: Longitudinal Mode Shapes, First Four Modes

where the damping coefficient would increase as the oscillation amplitude rises. Thus, as the system increases in amplitude the damping coefficient becomes neutral and we reach a balanced oscillation amplitude. This is referred to as the limit cycle amplitude. In dynamical systems the limit cycle is an orbit in phase space on which trajectories converge. This orbit may be stable or unstable and at least one trajectory must spiral into it. This phenomena can be seen clearly in Figure 1.6 as both systems reach a limit amplitude. More information on dynamical systems and limit cycles can be found in "Nonlinear Dynamics and Chaos" by Steven Strogatz (69) and "Deterministic Chaos" by Heinz Schuster (70).

In a real fluid dynamic system this 'damping coefficient' is comprised of many mechanisms and thereby many variables. In a linear system it is known as the growth rate, or α, as in $e^{\alpha t}$. Some mechanisms dampen oscillations, some drive them. The large number of unique mechanisms all working simultaneously in a complex nonlinear system poses combustion instability as one of the more difficult engineering challenges.

1.6.2 Wave Steepening

When an acoustic wave travels through a medium it tends to steepen. This is a natural process due to localized changes in the speed of sound. As a result the waveform will converge to a steepened waveform depicted in Figure 1.8. This shock-like structure is the result of the nonlinear nature of fluids and is well documented in acoustic literature (8). This is nature's method of limiting the infinite exponential growth predicted with linear resonance. The exponential growth is only valid in the system initially and predicts an eventual unphysical infinite pressure rise. In the real nonlinear system energy cascades from the high amplitude low modes to higher modes. This process limits the wave growth as the higher modes require more energy to sustain.

1.6.3 Mean Pressure Shift

When steepened waves are present in a closed chamber an interesting phenomena occurs. The mean pressure in the chamber increases. This can be see clearly in Figure 1.1. This

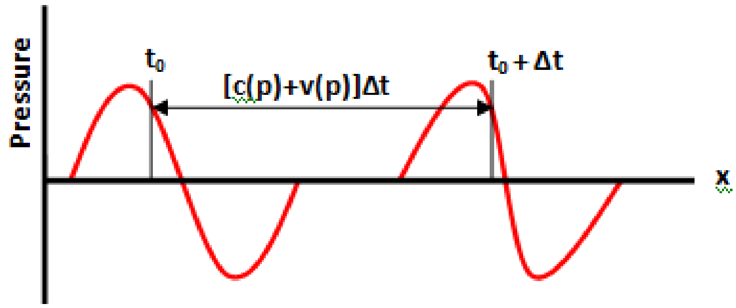


Figure 1.8: Pierce's description of wave steepening (8)

effect is due to the work done on the fluid by the waveform at the surface. If it is not considered in the design, this mean pressure increase, or 'DC' shift, as it is called in the industry, combined with shock-like waves can result in catastrophic failure of a pressurized chamber.

The phrase 'DC shift' comes from the analogy between alternating current (AC) and direct current (DC), where the AC is analogous to the oscillating pressure and the DC represents the mean pressure. This notation likely originated from the fact that the pressure is measured with an electrical sensor where the oscillating pressure response would be observed as AC and the mean pressure as DC.

1.6.4 Nonlinear Systems

There are many interesting characteristics which are prevalent in unstable systems. Many of these are the result of non-linear processes. Figure 1.9 shows a collection of mechanisms which all play a role in the transfer of energy within the system. These mechanisms either add energy into the organized oscillations (propellant pressure response, acoustic boundary layer pumping, vortex shedding, etc) or remove energy (viscous damping, nozzle damping, particle damping, etc). This balance, looked at from a nonlinear perspective, determines the stability of the system and if it is unstable it will define the character of the pressure trace. More information on the individual mechanisms can be found in Culicks book (28).

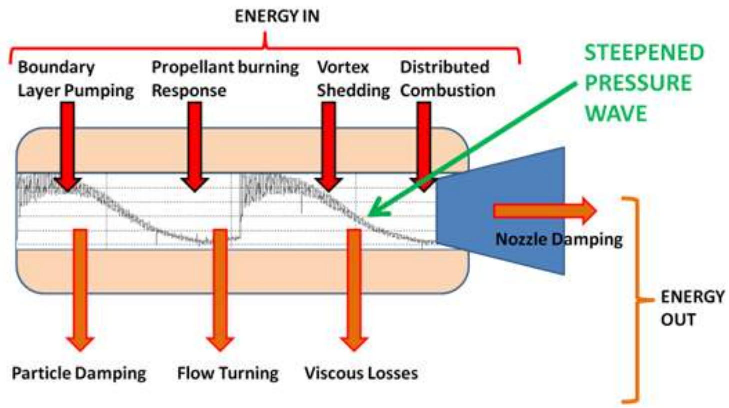


Figure 1.9: Example System Energy Gains and Losses

1.6.5 Damping as a Function of Time

In a rocket system the net damping coefficient, or α, is changing rapidly in time. This effect makes it difficult to visualize the waveform and phase diagram. Additionally, there is a transient between a change in α and arriving at a limit amplitude. When α is changing in time before the steady state is achieved the true limit amplitude may never be reached. This fact shows the need for detailed analysis of the individual mechanisms and a time dependant solution of the entire system. As an example, the Van Der Pol equation is modified so that μ is a variable. It has a mean value of 0.1 with sinusoidal and random additions. This results in a random appearing, or noisy, phase plane plot shown in Figure 1.10 with crossing lines and no clear limit cycle. An actual data signal is bound to have many more mechanisms varying the driving α, along with more noise in the system, causing the phase plane and waveform plots to appear disorganized.

$$y\text{''}(x) - \left(.1 + 2(\sin(.5x))^2 + .2\sin(10x) + \text{RandomReal}(0,1)\right)\left(1 - y(x)^2\right) y'[x] + y(x) = 0 \quad (1.6)$$

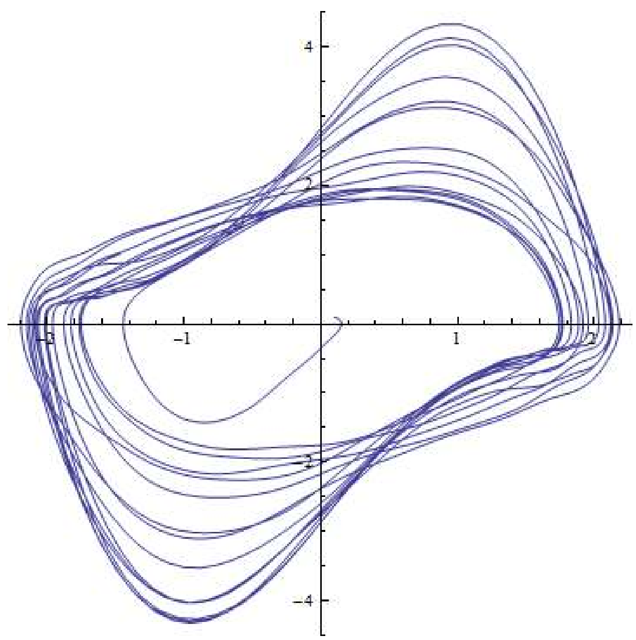


Figure 1.10: Van Der Pol with variable μ

1.7 Experimental and Historical Examples

Since the inception of rocketry, combustion instability has been a serious problem. Several real world examples will highlight the various characteristics of combustion instability.

1.7.1 Rocketdyne F-1 Engine

Producing over 1.5 million lbf thrust, five of these motors were used in the first stage of the Saturn V. The F-1 engine, shown in Figure 1.11, had serious combustion instability issues during its development (11). Over 2700 full scale tests were performed to analyze its performance. During these tests 'bombs' (acting as a pressure pulses) were detonated within the combustion chamber to test for stability. Pressure pulses such as these occur naturally in real systems sparking the growth of instabilities. For example, in a solid rocket motor, if a piece of propellant or igniter breaks off within the motor it will travel through the nozzle instantaneously changing the throat area. This sudden change will cause a immediate rise in chamber pressure, acting like a pressure pulse or 'bomb'. For practical testing reasons, since instabilities can grow from noise within the system, the pulse allows for an immediate testing of instability without waiting for the natural transient to grow.

An example of the pressure trace produced by the F-1 is shown in Figure 1.12. High amplitude steepened waves are seen on the order of 500psi. This is roughly the same order as the chamber pressure given at 965 psia. The intense pressure waves cause extreme localized heating causing damage to the injector surface.

An injector design was arrived at involving baffles, shown in Figure 1.13, presumably to break up the coherence of the tangential oscillations in the chamber and reduce their interaction with the injection process (11). This design reduced the oscillations to below an acceptable amplitude of 10% of the mean pressure. In modern systems this is considered a 'stable rocket,' but as mentioned in the introduction this is still an unstable system because the system is sustaining organized oscillations.

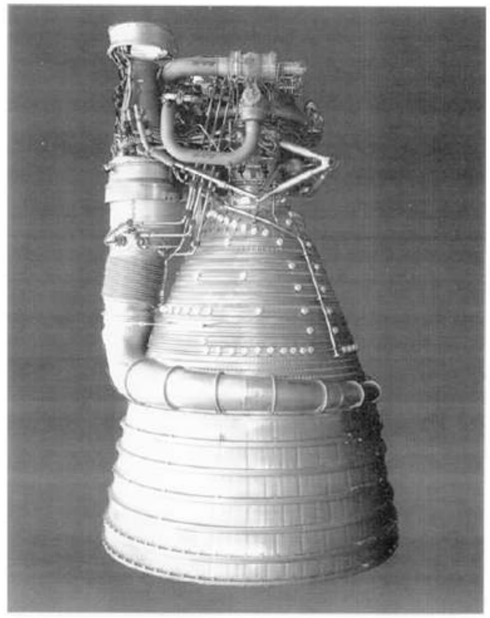


Figure 1.11: F-1 Liquid Rocket Engine (11)

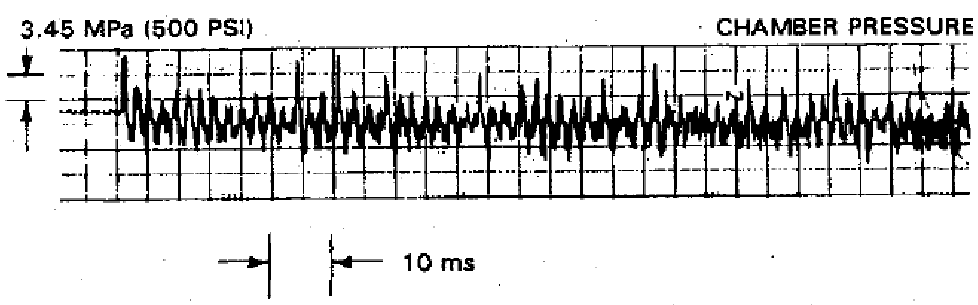


Figure 1.12: F-1 example pressure trace (11)

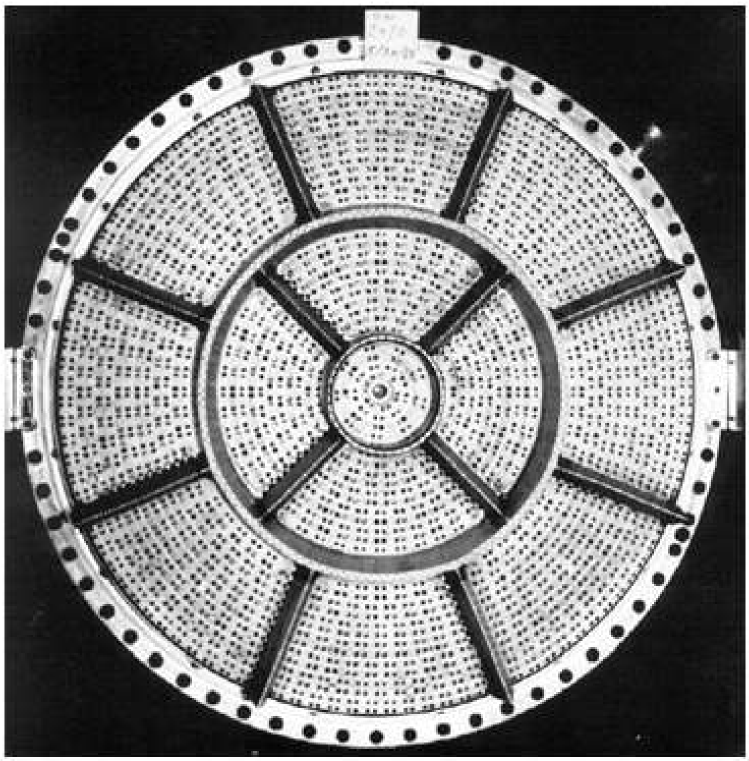


Figure 1.13: F-1 injector plate showing the presence of baffles (11)

1.7.2 Brownlee-Marble Solid Rocket Motor Data

Tests performed by W.G. Brownlee and F.E. Marble at the Caltech Jet Propulsion Laboratory in 1959 (12) (13) (14) were one of the first controlled and fully instrumented experimental studies of solid rocket motors. Figure 1.14 shows the experimental setup of a 5in diameter solid rocket motor. Nonlinear features such as limit cycle amplitude and mean pressure increase were observed in a repeatable manner in over 400 tests. Carefully observed stability boundaries were measured, shown in Figure 1.15. In a paper by Flandro, et all (50) the data was recently analyzed using current techniques and, as shown in Figure 1.16, it is found that the current theoretical basis captures the true physics seen in the experimental data. The key features of these tests are illuminated with improvements in data analysis of E.W. Perry's T-Burner data (50).

1.7.3 Minuteman III, 3rd Stage

The Minuteman III, 3rd stage solid rocket motor is a clear example of vortex shedding induced combustion instability (60). The internal ballistics cause large shear layers to form as the flow passes protrusions in the propellant grain. This sheared flow, depicted in Figure 1.17, is unstable and results in the generation of vortices. These vortices travel downstream

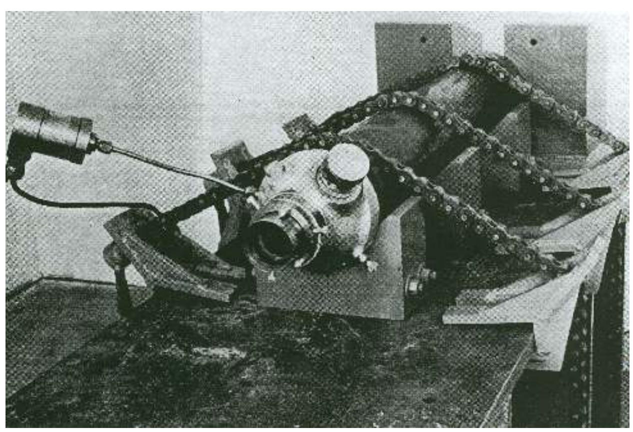


Figure 1.14: Experimental Setup of 5in Motor (13)

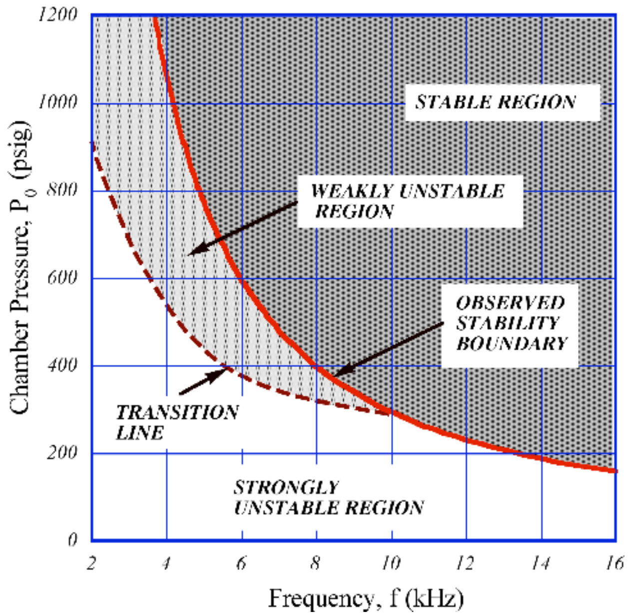


Figure 1.15: Experimental Setup of 5in Motor (50)

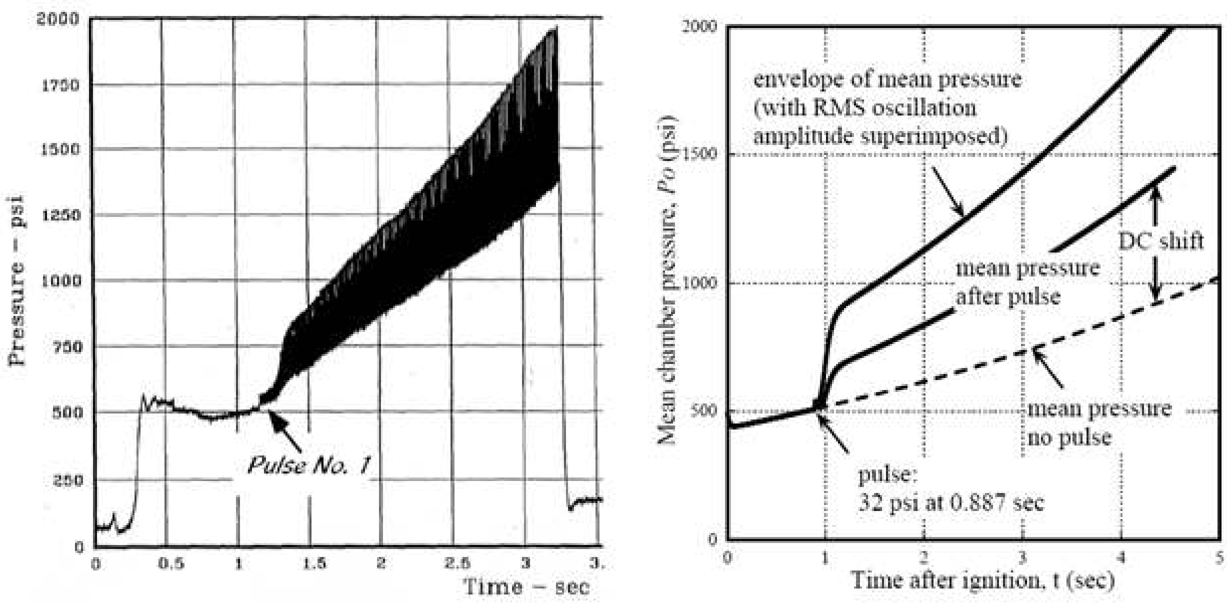


Figure 1.16: Experimental Data vs. Simulation Data (50)

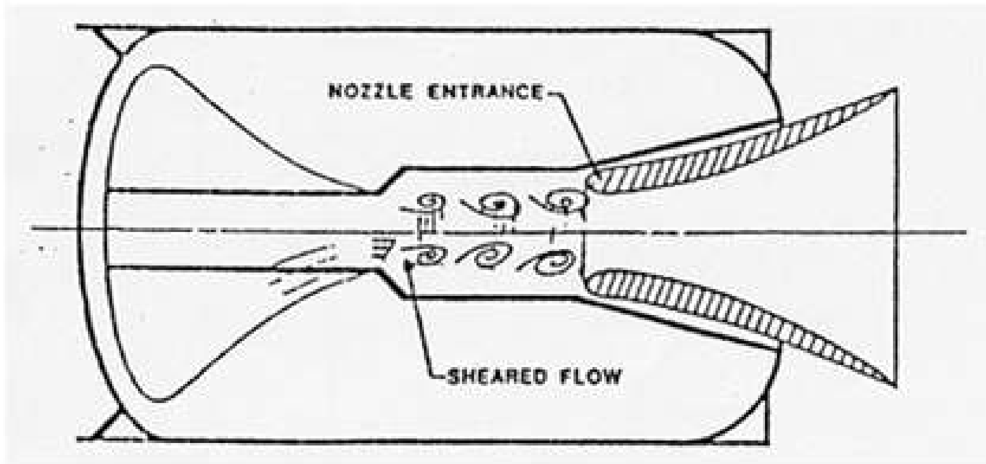


Figure 1.17: Minuteman III Vortex Shedding (60)

and impinge on the interior nozzle surface. The instability of the shear layer and the vortex impingement are both sources of sound and drive the pressure oscillations.

The internal geometry changes as the propellant grain burns back in time which in turn changes both the frequency at which the vortices are shed and the acoustic natural frequency. These shed vortices prefer to fit into the geometrical gap between the shedding protrusion and the nozzle surface in integer values. As a result, the frequency of the vortex shedding can shift throughout the burn. Oscillation amplitudes are highest when the vortex impulse driving frequency resonates with the chamber acoustics. Similarly, chamber pressure oscillations alter the shedding of the vortices. Thus we find a coupling between the vortex shedding frequency and the chamber acoustics. Figure 1.18 shows this 'locking in' phenomenon characterized by the sudden shifts in oscillatory frequency as the vortex shedding frequency shifts to match the acoustic frequency.

1.7.4 Experimental Verification of Mean Pressure Increase and Wave Steepening

This nonlinear phenomenon has been characterized in previous work by Saenger (17) and Jacob (18). Both phenomena are easily seen in a simple acoustics experiment such as a

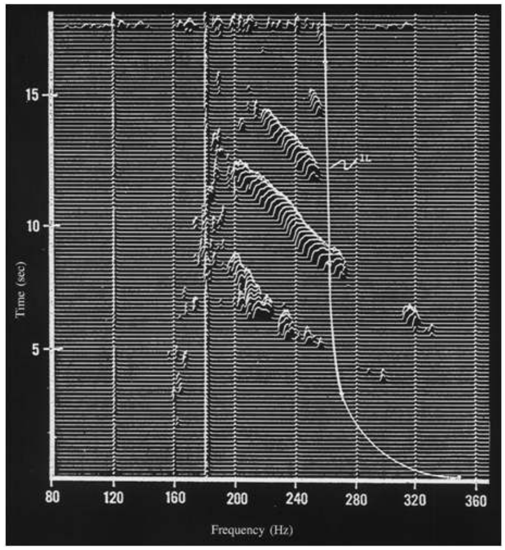


Figure 1.18: Typical Pressure Amplitude Waterfall for Minuteman III, Third Stage (61)

closed tube with a piston driven on one end and a pressure sensor on the other. Figure 1.19 shows the experimental setup by Jacob (18).

When the system is driven off the fundamental resonant frequency, the system displays traditional sinusoidal acoustic behavior as shown in Figure 1.20(a). Yet, when driven near the natural frequency, the system quickly transitions to a steepened waveform shown in Figure 1.20(b).

The spectrum of the steepened waveform is shown in Figure 1.21. As the system is driven near to the fundamental acoustic mode, the waveform steepens. This steepening is characterized by the energy flowing from the first mode to higher modes.

Additionally, as the system is driven near to the fundamental mode, an increase in mean pressure is measured. This is a result of the work done on the fluid by the piston (52). Figure 1.22 shows the increase in mean pressure as a function of the forcing frequency. In a rocket system this same phenomena is observed. Driving mechanisms such as the burning rate pressure response act as the piston. Both steepened waves and mean pressure increase are measure in liquid and solid rocket motors.

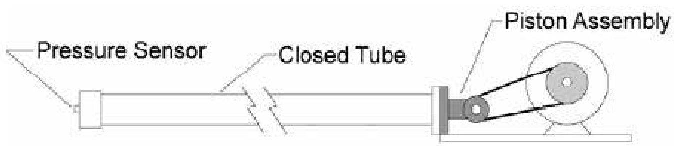
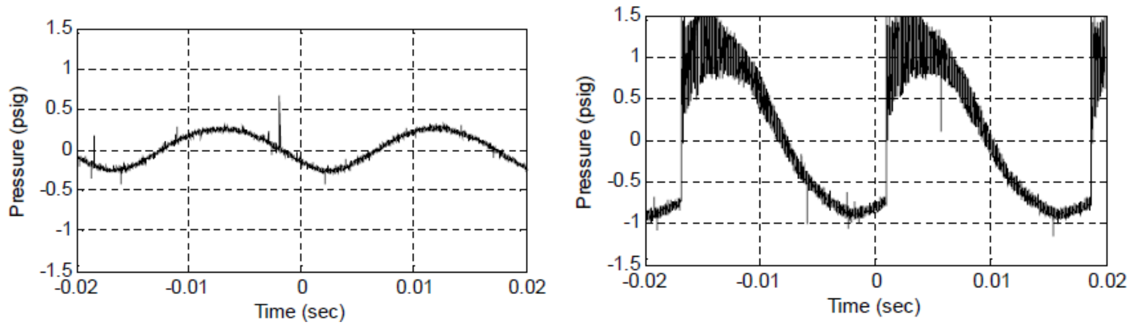


Figure 1.19: Shock Tube Experiment Setup (18)



(a) Off Resonance

(b) On Resonance

Figure 1.20: Shock Tube Pressure Plots (18)

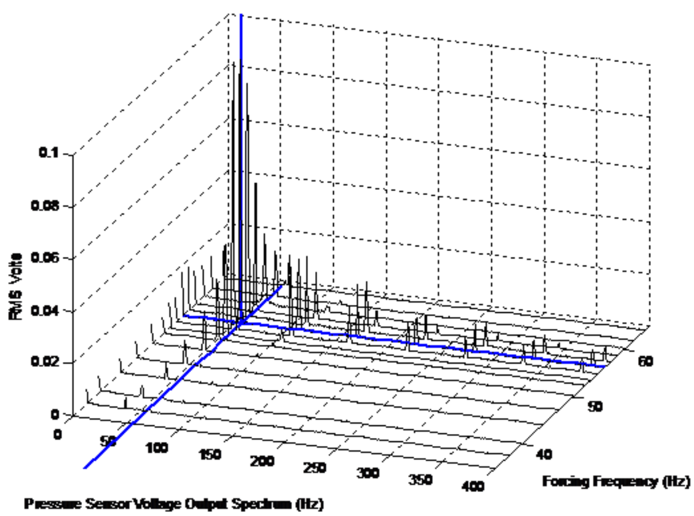


Figure 1.21: Waterfall Plot from Shock Tube Experiment (18)

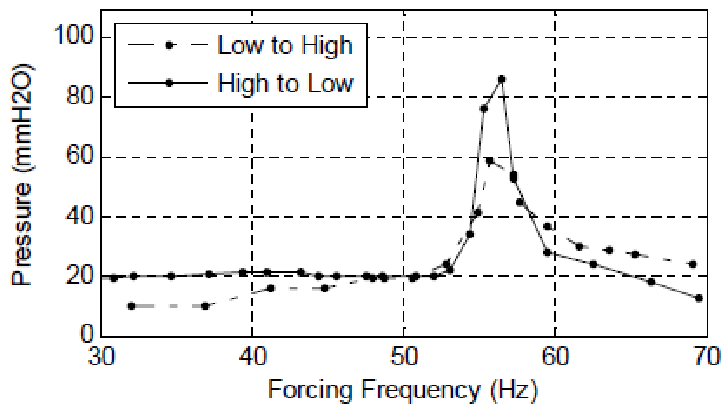


Figure 1.22: Mean Pressure Increase by Jacob (18)

Second Harmonic Driving

In further previously undocumented work, when the second harmonic is driven by the piston the higher modes which are divisible by 2 are driven. Very little energy transfers back into the 1st mode. Figure 1.23 shows the spectrum of the pressure measurement (taken by a dynamic pressure transducer) when the tube is driven near to the second harmonic at 110 Hz. These spectra were taken with an FFT analysis of the pressure measurement at the end of the closed tube. In future analysis it may be appropriate to use wavelet analysis to capture time dependence. Figure 1.24 depicts the same data highlighting the first three modes showing the high amplitude at 110 Hz and no observable frequency components at the first or third modes. A small frequency component is seen at 60Hz due to wall voltage noise.

Tangential Steepening

Tangential waves do not steepen in the same manner as the longitudinal waves (57) and are often mistaken as detonation waves. Additionally, the mean pressure increase may not be present. The steepened tangential waveform is shown in Figure 1.25.

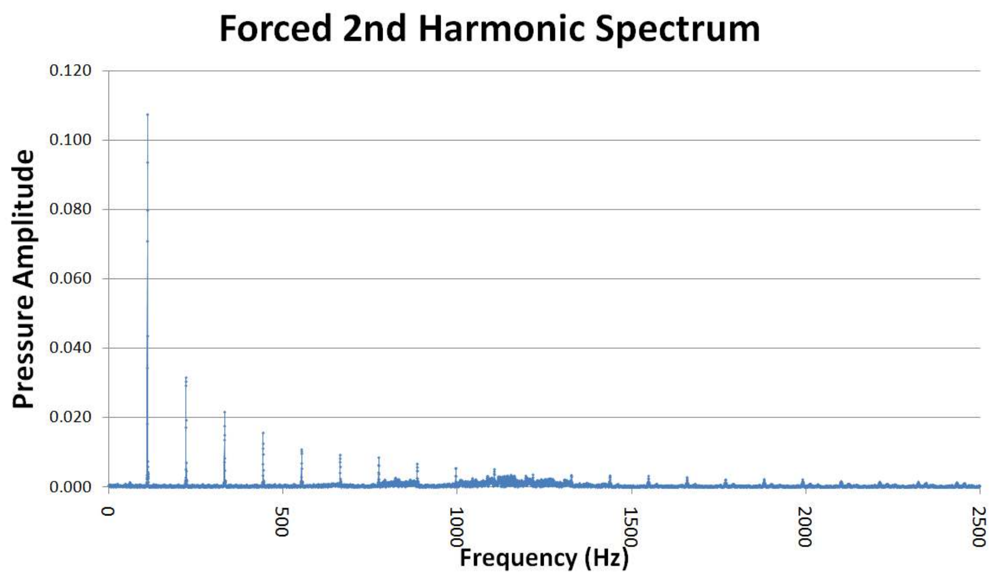


Figure 1.23: Forced Second Harmonic Spectrum

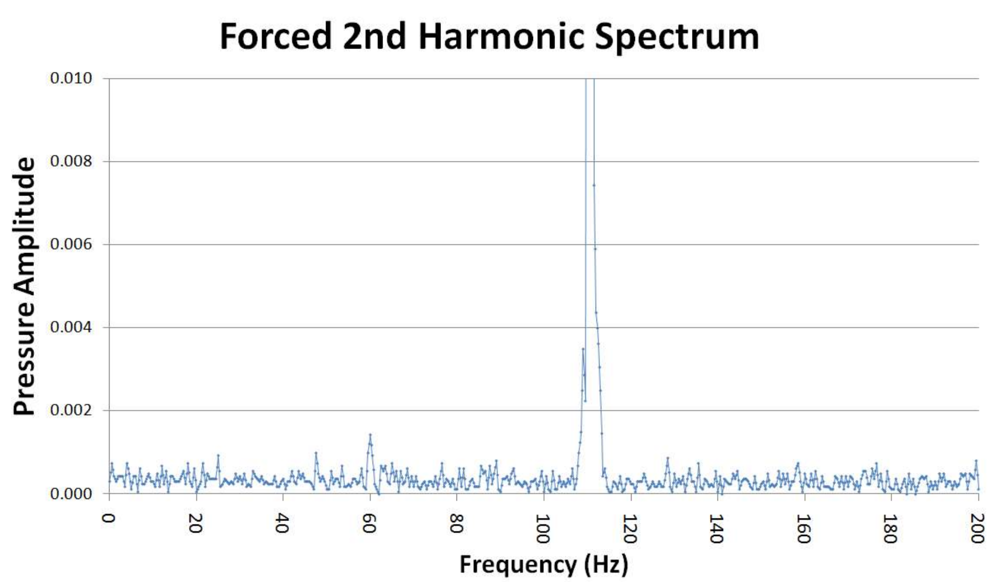


Figure 1.24: Forced Second Harmonic Spectrum Zoomed in on Missing First Mode

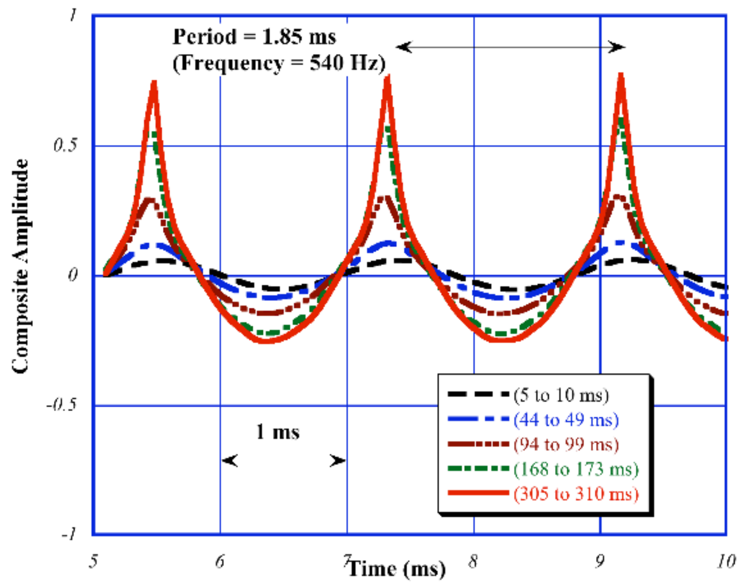


Figure 1.25: Steepened Tangential Waveform (57)

1.7.5 JPL Data

Tangential oscillations have been studied extensively in a test motor developed at JPL (88). This motor was highly instrumented with multiple pressure taps. Figure 1.26 shows the motor setup. Figure 1.27 shows a series of pressure readings from pressure sensors located throughout the chamber length. This data shows the variation in the pressure amplitude across the axial length. The steepened waveform exhibits a high amplitude at the injection surface highlighting the importance of the injection interaction. Figure 1.28 is a detailed plot of a pressure signal. The waveform is similar to the steepened tangential waves seen in Figure 1.25.

1.7.6 Space Shuttle Solid Rocket Booster (SRB)

Modern motivation comes from recent additions to the NASA space fleet in the ARES rockets. The space shuttle solid rocket booster (SRB) has been adapted for future use in both the ARES I and ARES V flight vehicles. In the ARES I vehicle, oscillations within the boosters resonate with structural harmonics. This causes a dangerous level of oscillatory

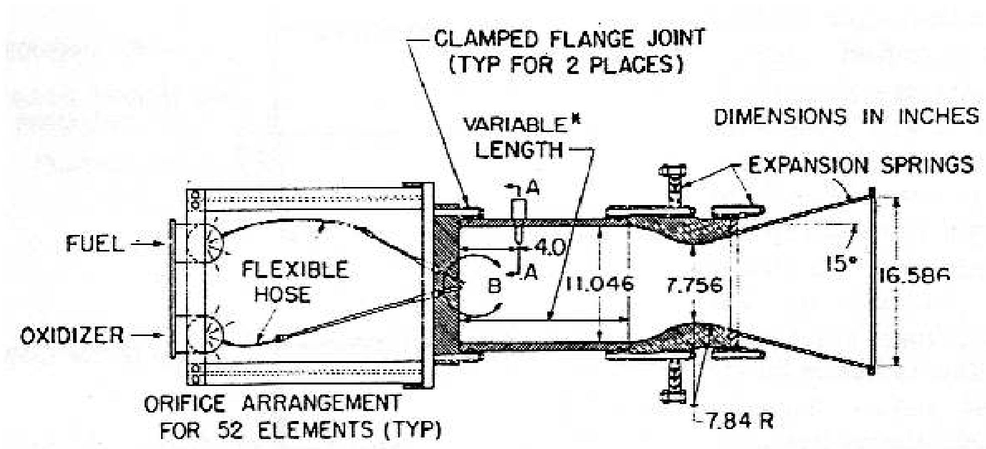


Figure 1.26: Layout of JPL Test Engine (88)

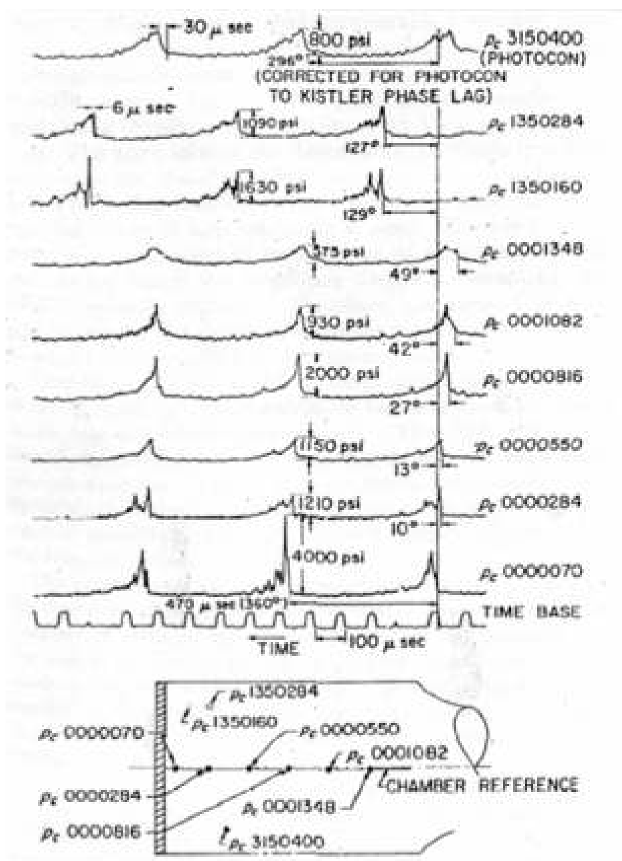


Figure 1.27: Multiple Simultaneous Pressure Traces from JPL Engine (88)

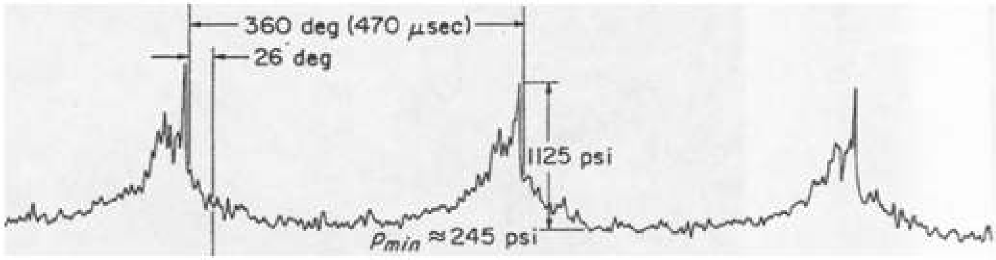


Figure 1.28: Pressure Trace from JPL Engine (89)

acceleration placed on the astronauts and the rocket (72). The ARES I oscillations involve a modified SRB, RSRMV, which will be discussed later. These modern oscillations originate in the space shuttle SRB design. Data from the space shuttle flights and ground tests confirm this. Figure 1.29 depicts a collection of cases of SRB combustion instability. A small pressure oscillation amplitude, considering that the mean pressure is on the order of 600-800 psia, is measured at one to four psi. This is certainly a much smaller oscillation than seen in the F-1 which was deemed stable. However, due to the large size of the motors this translates to a large thrust oscillation.

1.7.7 ARES I RSRMV

The modified space shuttle solid rocket boosters (RSRM) are lengthened versions of the SRB. An fifth section has been added to the length of the motor. The increase in length decreases the fundamental harmonic to approximately 12 Hz. Additional changes were made to the grain geometry shape. The increase in total surface area along with other instability mechanisms such as vortex shedding from protrusions into the mean flow yield an initial prediction that the RSRM will oscillate at a higher amplitude than the SRB.

Unlike the space shuttle, the ARES I vehiclehas a similar structural harmonic near to the 12 Hz created by the motor. This coupling causes large acceleration oscillations in the rocket. This vibration threatens the astronauts lives and the integrity of the rocket itself. Because of these issues increased study has gone into the solution to the CI problem in the RSRM and other large solid rocket motors.

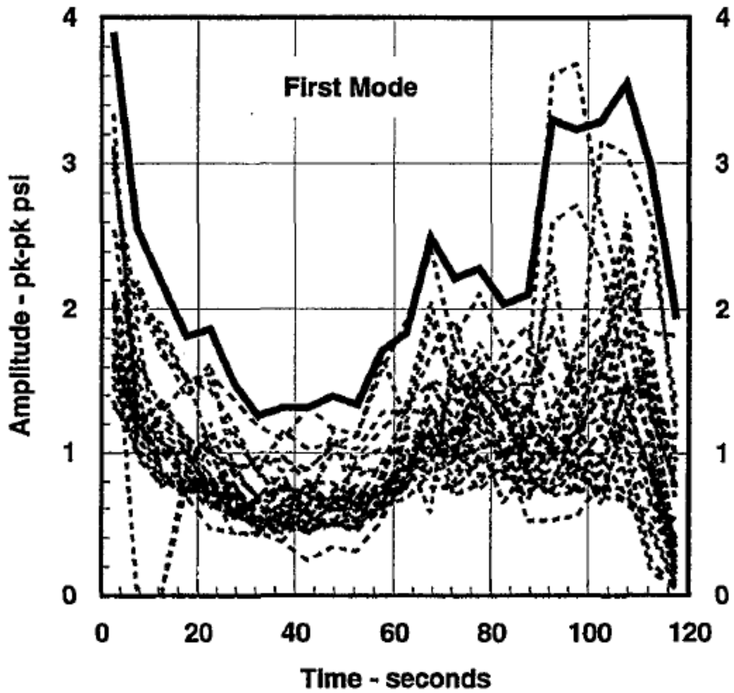


Figure 1.29: SRB First Mode Pressure Oscillations (4)

1.7.8 Non-Rocket Examples: Ramjets, Scramjets, Afterburners, etc.

Rocketry is not the only example of a system which will exhibit combustion instability, though, it is the most documented. Any system which contains large amounts of energy will find ways to oscillate. If only a small fraction of that energy couples with the natural system frequencies, harsh oscillatory amplitudes are produced. As a result all propulsion devices will likely exhibit combustion instability. These alternative systems exhibit the same phenomena demonstrated in the previous rocketry examples. Vortices are shed and couple with acoustic oscillations, injection processes couple with flow conditions, and steepened waves and mean pressure shift are general nonlinear phenomena that are applicable to all propulsion systems. Unfortunately, data for these systems is scarce because the majority of the data is either proprietary or classified. However, it is beneficial to keep these alternative systems in mind when developing combustion instability theories and codes so that they will be applicable to all systems. For example, no limitations will be placed on the mean flow Mach number so that the theory is applicable to afterburners and scramjets. The most current derivations fit these requirements and are applicable to all systems. This will maintain the solution techniques validity for all systems from solid rocket motors to scramjets. More information is available in Culick's report, "Unsteady Motions in Combustion Chambers for Propulsion Systems," (28).

1.8 Summary of Applicable Theoretical

Figure 1.30 shows the progression of relevant combustion instability research in a chronological flow chart. The center column shows the direct lineage leading to the current analysis. Due to the large variety of theories, the following section will briefly discuss the applicable prior theoretical models relevance to current models. Other valuable information on combustion instability testing and research, shown in the side columns, is still relevant and portions of their work are applicable in modern analysis and will be referenced when needed. This list is by no means exhaustive. The field of propulsion research dates back hundreds of years. However, the current combustion instability analysis relies primarily on work by Kirchoff (20), Cantrell and Hart (21), Morfey (31), Myers (42), Culick (28), Yang (11) and Flandro (57).

1.8.1 Kirchoff

Kirchoff's paper (20) provides us with the principle of energy conservation in the acoustic system. The change in the oscillatory energy must equal the work done on the system in addition to the energy sources. This is the foundation of combustion instability research. The change in the pressure oscillations in a system are caused by specific phenomena, or mechanisms, which either increase the oscillatory energy or decrease it. From Kirchoff,

$$\frac{\partial E}{\partial t} + \nabla \cdot W = D \tag{1.7}$$

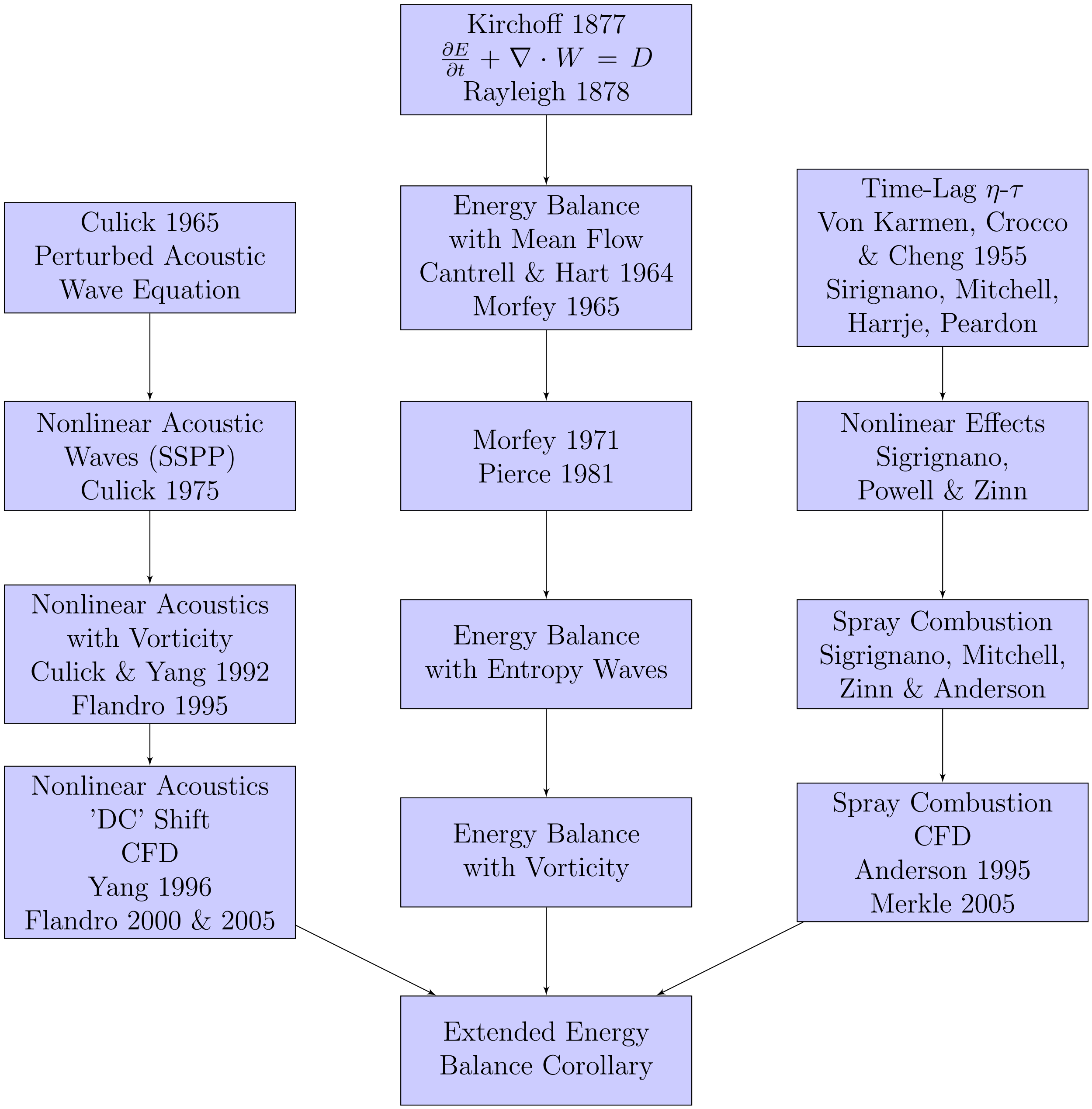


Figure 1.30: Theoretical History Flow Chart

Where E is the system energy, W is the total work done on the system and D is the summation of the energy sources.

1.8.2 Cantrell and Hart

Cantrell and Hart wrote their paper "Interaction between Sound and Flow in Acoustic Cavities: Mass, Momentum, and Energy Considerations" (21) in 1964. Their use of the energy method is a key motivation in current work. They start with the mass, momentum and energy fluid dynamic equations,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{m}) = 0 \tag{1.8}$$

$$0 = \frac{\partial \mathbf{v}}{\partial t} + \nabla(v^2/2) + (1/\rho)\nabla p = \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[(v^2/2) + h\right] \tag{1.9}$$

$$\frac{\partial}{\partial t}\left\{\rho\left[C_p T + (v^2/2)\right]\right\} \tag{1.10}$$

Where $\mathbf{m} = \rho\mathbf{v}$ the product of the density,ρ, and the velocity, $\mathbf{v}$. p is the pressure, T is the temperature, $h = C_p T = C_v T + p/\rho$ is the specific enthalpy. Cantrell and Hart argue that acoustic stability is determined by calculating the average rate of work done on the surface of the flow field.

$$\left\langle \int_S p_1 \mathbf{v}_1 \cdot dS \right\rangle = 0 \tag{1.11}$$

Where the notation $\langle\rangle$ denotes the time average. The variables are expanded by inserting an expansion of the field variables. Then each order is separated into its respective part.

$$\mathbf{m}_0 = \rho_0 \mathbf{v}_0 \tag{1.12}$$

$$\mathbf{m}_1 = \rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0 \tag{1.13}$$

$$\mathbf{m}_2 = \rho_0 \mathbf{v}_2 + \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_0 \tag{1.14}$$

$$h_1 = c_0^2 \frac{\rho_1}{\rho_0} = \frac{p_1}{\rho_0} \tag{1.15}$$

$$h_2 = \frac{c_0^2}{\rho_0} \left[\rho_2 + \frac{\gamma - 2}{2} \frac{\rho_1^2}{\rho_0} \right] \tag{1.16}$$

$$\left(\frac{v^2}{2} \right)_1 = \mathbf{v}_0 \cdot \mathbf{v}_1 \tag{1.17}$$

$$\left(\frac{v^2}{2} \right)_2 = \mathbf{v}_0 \cdot \mathbf{v}_2 + \frac{v_1^2}{2} \tag{1.18}$$

These similar expansion philosophies will be used in later work as well. Using the energy growth rate equation, $\frac{dE}{dt}/(E - E_0)$ and applying the previous relations to an acoustic cavity yields,

$$2\alpha = -\frac{\left\langle \int_S dS \cdot \left[p_1 v_1 + \frac{p_1^2 v_0}{\rho_0 c_0^2} + \rho_0 (v_0 \cdot v_1) v_1 + \frac{p_1}{c_0^2} (v_0 \cdot v_1) v_0 \right] \right\rangle}{\left\langle \int_v dv \left[\frac{\rho_0 v_1^2}{2} + \frac{p_1^2}{2 \rho_0 c_0^2} + \frac{p_1 (v_0 \cdot v_1)}{c_0^2} \right] \right\rangle} \tag{1.19}$$

This equation relates the first order acoustic field and the mean flow variables to the energy growth rate, α. The growth rate can then be applied to a rocket system, yielding,

$$\alpha = -\frac{\rho_0 c_0^2}{L} \left\{ \mathrm{Re}[Y_b] - \frac{v_{0b}}{\gamma p_0} + \frac{2L}{a} \mathrm{Re}[Y_w] \right\} \tag{1.20}$$

Where Y_b is the specific acoustic admittance at the propellant surface, Y_w is the specific acoustic admittance of the walls, v_{0b} is the mean speed of the burned gas leaving the burning surface, L is the chamber length, and a is the chamber radius. In the paper, “Acoustic Energy in Non-uniform Flows," Morfey (31) extends the concept of acoustic energy to nonuniform flows.

1.8.3 Myers

Continuing from Cantrell and Harts and Morfeys work, Myers (42) finds the solution of the acoustic energy in a non-uniform flow. The derivation of this work will be highlighted in detail in the dissertation. Continuing with the energy methodology, using the relation,

$$\frac{\partial E}{\partial t} + \nabla \cdot W = D, \tag{1.21}$$

he finds that for the general case including entropy fluctuations,

$$E = \rho\left[H - H_0 - T_0\left(s - s_0\right)\right] - \mathbf{m}_0 \cdot \left(\mathbf{u} - \mathbf{u}_0\right) - \left(p - p_0\right) \tag{1.22}$$

$$\begin{aligned} W = {} & \left(\mathbf{m} - \mathbf{m}_0\right)\left[H - H_0 + T_0\left(s - s_0\right)\right] + \mathbf{m}_0\left(T - T_0\right)\left(s - s_0\right) \\ & - \left(m_j - m_{0_j}\right)\left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0}\right) + \left(T - T_0\right)\left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0}\right) \end{aligned} \tag{1.23}$$

$$\begin{aligned} D = {} & \mathbf{m} \cdot \zeta_0 + \mathbf{m}_0 \zeta - \left(s - s_0\right)\mathbf{m} \cdot \nabla T_0 + \left(s - s_0\right)\mathbf{m}_0 \cdot \nabla T \\ & - \left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0}\right)\frac{\partial}{\partial x_i}\left(m_j - m_{0_j}\right) + \left(m_j - m_{0_j}\right)\left(\frac{1 P_{ij}}{\rho^2}\frac{\partial \rho}{\partial x_i} - \frac{P_{0_{ij}}}{\rho_0^2}\frac{\partial \rho_0}{\partial x_i}\right) \\ & + \left(T - T_0\right)\left(\frac{\phi}{T} - \frac{\phi_0}{T_0}\right) + \left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0}\right) \cdot \nabla\left(T - T_0\right) \\ & - \left(T - T_0\right)\left(\frac{\mathbf{q} \cdot \nabla T}{T^2} - \frac{\mathbf{q_0} \cdot \nabla T_0}{T_0^2}\right) \end{aligned} \tag{1.24}$$

These relations play a key role in the oscillations seen in rocket motors. These equations will be expanded upon and applied to the rocket motor system showing the result that the essential physics of combustion instability can be reproduced.

1.8.4 Flandro

Current work by Flandro (45) (46) applies Myers energy balance (42) to combustion instability. His work combines previous combustion instability analysis with the modern acoustic energy method. This analysis yields a more complete picture of the system, while making minimal assumptions. In addition to his continuation of the energy method, Dr. Flandro has championed modern analysis by leading the understanding on interior motor dynamics and their impact on combustion instability.

Chapter 2

Introductory Equations

The foundation of the generalized combustion instability is in the thermodynamic and fluid dynamic equations. Every attempt is made to minimize assumptions in the initial formulations. This ensures that the solution is applicable in a wide variety of situations. Several assumptions will be made. These assumptions are used to simplify the mathematics allowing for an analytical solution. The effect of these assumptions and the possibility of removing them will be discussed in the document. Restrictive conditions may be applied in later analysis to allow for computation, however, the derivation will remain complete and allow for future analysis of neglected variables.

Once the governing equations are settled, each field variable is split into slowly changing and oscillatory parts. These algebraic expansions are applied to the governing equations. After these expansions are applied the thermodynamic relations are used to relate the oscillatory field variables to one another. This is accomplished by using a Taylor series expansion of each variable where the thermodynamics are used to evaluate the partial derivatives.

In later Chapters these expansions will be applied to the energy equation, and the subsequent relations will be used in the solution of the systems oscillatory energy. Then using Galerkin spectral decomposition and acoustics, the oscillatory pressure amplitude is solved for numerically.

2.1 Thermodynamics

We begin with the thermodynamic relations for a single species closed system. The combustion zone in a solid rocket motor is very small and located near to the burning surface. Therefore it is reasonable to assume that the bulk of the system acts as a single species with no chemical reactions. This is an assumption, and extending the presented analysis to multiple species including combustion would make for a very interesting future study. Three of these equations define the whole thermodynamic system, several more have been stated for use in later algebra. More information on the thermodynamic equations can be found in Liepmann and Roshko's "Gas Dynamics" (64).

$$de = T ds + \frac{p}{\rho^2} d\rho \tag{2.1a}$$

$$dh = T ds + \frac{1}{\rho} dp \tag{2.1b}$$

$$dh = \gamma T ds + \frac{a^2}{\rho} d\rho \tag{2.1c}$$

$$dp = \frac{a^2 \rho}{c_p} ds + a^2 d\rho \tag{2.1d}$$

$$dT = \frac{T}{c_p} ds + \frac{1}{\rho c_p} dp \tag{2.1e}$$

$$dT = \frac{1}{c_p}\left(\gamma T ds + \frac{a^2}{\rho} d\rho\right) \tag{2.1f}$$

Where p is the pressure, ρ is the density, T is the temperature, s is the entropy, e is the internal energy, h is the enthalpy, a is the speed of sound, γ is the specific heat ratio, and C_p is the constant pressure specific heat. A thermally perfect gas is assumed. The state equation for a thermally perfect gas is given by $p = \rho Rt$. The speed of sound, a, is given by $\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho}\right)_T}$ and if thermally perfect, $a = \sqrt{\gamma RT}$. In order to preserve the entropy oscillations the isentropic assumption is avoided at all times.

2.2 Fluid Dynamics

Three governing equations are required to fully define the fluid dynamics. Continuity (C), Momentum (L), Entropy (S) are chosen. These three equations correspond to the three types of waves: pressure, vorticity and entropy. The entropy equation is chosen over the energy equation for this purpose. Since we are primarily concerned with the net energy balance, the energy equation will be used later to pull all the pieces together. The derivation of the fluid mechanic equations is shown clearly in “Fluid Mechanics” by Landau and Lifshitz (66) and “Theory of Nonlinear Acoustics in Fluids” by Enflo and Hedberg (65).

Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{m} = 0 \tag{2.2}$$

Momentum:

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + \nabla H - T\nabla s = \psi \tag{2.3}$$

Entropy:

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot (\mathbf{m}s) = Q \tag{2.4}$$

Energy:

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla \cdot (\mathbf{m}H) - \mathbf{m} \cdot \psi - TQ = 0 \tag{2.5}$$

Definitions:

$$\text{Enthalpy:} \qquad h = e + \frac{p}{\rho} \tag{2.6}$$

$$\text{Total Enthalpy:} \qquad H = h + \frac{1}{2}\mathbf{u}^2 \tag{2.7}$$

$$\text{Mass Flow Rate:} \qquad \mathbf{m} = \rho \mathbf{u} \tag{2.8}$$

$$\text{Vorticity:} \qquad \omega = \nabla \times \mathbf{u} \tag{2.9}$$

$$\text{Lamda Vector:} \qquad \zeta = \omega \times \mathbf{u} \tag{2.10}$$

$$\text{Heat Release:} \qquad Q = \frac{\Phi - \nabla \mathbf{q} + \mathcal{H}}{T} \tag{2.11}$$

Where $\mathcal{H}$ is the distributed combustion heat release, $\mathbf{q}$ is the heat transfer, and ψ, the viscous stress, is given in vector form as:

$$\psi = \frac{1}{\rho}\left[-\mu\nabla\times\nabla\times\mathbf{u} + \left(\eta + \frac{4}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + \mathbf{F}\right] \tag{2.12}$$

Where η is the bulk viscosity, μ is the dynamic viscosity and $\mathbf{F}$ is an external body force. The viscous dissipation term cannot be written in purely vector form. It is instead written in tensor form.

$$\Phi = P_{ij}\frac{\partial u_j}{\partial x_i} \tag{2.13}$$

Where,

$$P_{ij} = {\sigma'}_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \eta\delta_{ik}\frac{\partial v_l}{\partial x_l} \tag{2.14}$$

2.3 Variable Expansions

The idea of separating the mean parameter from the fluctuating variable is taken from the study of acoustics. There, the fluctuating pressure is separated from the mean pressure, which is generally assumed to be constant. This principle can be extended to all variables, were all variables are algebraically expanded into the form,

$$q(\mathbf{x},t) = q_0(\mathbf{x}) + \sum_{n=1}^{\infty}\varepsilon^n q_n(\mathbf{x},t) \tag{2.15}$$

Where q_0 is the slowly changing mean flow term steady state value, q_1 is the first order oscillatory value, and q_2 is the second order perturbation of order $(q_1)^2$. ϵ^n is a small parameter. This expansion is useful given the fact that in general the pressure oscillations are much smaller than the mean pressure. Thus for the pressure expansion the small parameter $\epsilon = p_1/P_0$. This term is always less than one. Therefore, $p_1 > p_2 > p_3 > etc.$ The same logic follows for the other variables.

$$
\begin{aligned}
\mathbf{m} &= \mathbf{m}_0 + \varepsilon \mathbf{m}_1 + \varepsilon^2 \mathbf{m}_2 + \ldots \\
\mathbf{u} &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots \\
p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \\
\rho &= \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \ldots \\
T &= T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots \\
s &= s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \ldots
\end{aligned}
\tag{2.16}
$$

These expansions are inserted into the set of governing equations. Each order is equated on an individual basis. Note that the time dependency of the mean flow parameters are considered small and therefore neglected. However, spatial dependence is retained.

Mean Flow:

$$\nabla \cdot \mathbf{m}_0 = 0 \tag{2.17a}$$

$$\zeta_0 + \nabla H_0 - T_0 \nabla s_0 = \psi_0 \tag{2.17b}$$

$$\nabla \cdot (\mathbf{m}_0 s_0) = Q_0 \tag{2.17c}$$

First Order:

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \mathbf{m}_1 = 0 \tag{2.18a}$$

$$\frac{\partial \mathbf{u}_1}{\partial t} + \zeta_1 + \nabla H_1 - T_0 \nabla s_1 - T_1 \nabla s_0 = \psi_1 \tag{2.18b}$$

$$\frac{\partial (\rho_0 s_1 + \rho_1 s_0)}{\partial t} + \nabla \cdot (\mathbf{m}_0 s_1 + \mathbf{m}_1 s_0) = Q_1 \tag{2.18c}$$

Second Order:

$$\frac{\partial \rho_2}{\partial t} + \nabla \cdot \mathbf{m}_2 = 0 \tag{2.19a}$$

$$\frac{\partial \mathbf{u_2}}{\partial t} + \zeta_2 + \nabla H_2 - T_0 \nabla s_2 - T_1 \nabla s_1 - T_2 \nabla s_0 = \psi_2 \tag{2.19b}$$

$$\frac{\partial(\rho_0 s_2 + \rho_1 s_1 + \rho_2 s_0)}{\partial t} + \nabla \cdot (\mathbf{m}_0 s_2 + \mathbf{m}_1 s_1 + \mathbf{m}_2 s_0) = Q_2 \tag{2.19c}$$

Third Order:

$$\frac{\partial \rho_3}{\partial t} + \nabla \cdot \mathbf{m}_3 = 0 \tag{2.20a}$$

$$\frac{\partial \mathbf{u_3}}{\partial t} + \zeta_3 + \nabla H_3 - T_0 \nabla s_3 - T_1 \nabla s_2 - T_2 \nabla s_1 - T_3 \nabla s_0 = \psi_3 \tag{2.20b}$$

$$\frac{\partial(\rho_0 s_3 + \rho_1 s_2 + \rho_2 s_1 + \rho_3 s_0)}{\partial t} + \nabla \cdot (\mathbf{m}_0 s_3 + \mathbf{m}_1 s_2 + \mathbf{m}_2 s_1 + \mathbf{m}_3 s_0) = Q_3 \tag{2.20c}$$

Note that analysis goes beyond that done by Myers' (42) calculations. Most notably, the term $\mathcal{H}$ is included as the distributed heat release. To save time in further analysis the following notation may be used,

$$C_n = continuity\_order\_n$$

$$L_n = linear\_momentum\_order\_n$$

$$S_n = entroy\_order\_n$$

Then the three fluid dynamic governing equations can be written as:

$$C = 0$$

$$L - \psi = 0$$

$$S - Q = 0$$

2.4 Thermodynamic Power Series

To complete the system of equations shown in Section 2.3 the expansion of the thermodynamic quantities h, T, p and e are required. Each variable is expanded as a power series in s, ρ, or p about the respective stagnate values, s_0, ρ_0, or p_0. All thermodynamic variables can be written as a function of two other thermodynamic variables; thus, every variable is expanded with a two dimensional Taylor series (64). These expansions will provide the relationships between the oscillatory field variables and will be used later in the expansion of the governing equations, including the energy equation.

2.4.1 Expansion of ρe

The Taylor series expansion of ρe about the point (ρ_0, s_0) is,

$$\begin{aligned}\rho e = \rho_0 e_0 &+ \left.\frac{\partial(\rho e)}{\partial \rho}\right|_0 (\rho - \rho_0) + \left.\frac{\partial(\rho e)}{\partial s}\right|_0 (s - s_0) \\ &+ \left.\frac{\partial^2(\rho e)}{\partial \rho^2}\right|_0 \frac{(\rho-\rho_0)^2}{2!} + \left.\frac{\partial^2(\rho e)}{\partial \rho \partial s}\right|_0 (\rho - \rho_0)(s - s_0) + \left.\frac{\partial^2(\rho e)}{\partial s^2}\right|_0 \frac{(s-s_0)^2}{2!} \\ &+ \left.\frac{\partial^3(\rho e)}{\partial \rho^3}\right|_0 \frac{(\rho-\rho_0)^3}{3!} + \left.\frac{\partial^3(\rho e)}{\partial \rho^2 \partial s}\right|_0 \frac{(\rho-\rho_0)^2 (s - s_0)}{2!} \\ &+ \left.\frac{\partial^3(\rho e)}{\partial \rho \partial s^2}\right|_0 \frac{(\rho-\rho_0)(s-s_0)^2}{2!} + \left.\frac{\partial^3(\rho e)}{\partial s^3}\right|_0 \frac{(s-s_0)^3}{3!} + \ldots \end{aligned} \quad (2.21)$$

Each individual partial derivative is evaluated. The fundamental thermodynamic relations (2.1) are used heavily in these algebraic manipulations. For example completing the derivatives with respect to ρ, remembering that entropy, s, is held constant yields,

$$\left.\frac{\partial(\rho e)}{\partial \rho}\right|_s = \rho \left.\frac{\partial e}{\partial \rho}\right|_s + e \left.\frac{\partial \rho}{\partial \rho}\right|_s = \rho \left.\frac{\partial e}{\partial \rho}\right|_s + e \quad (2.22)$$

And from (2.1a),

$$\left.\frac{\partial e}{\partial \rho}\right|_s = \frac{p}{\rho^2}. \quad (2.23)$$

Therefore,

$$\rho \left.\frac{\partial e}{\partial \rho}\right|_s + e = \rho \frac{p}{\rho^2} + e. = \frac{p}{\rho} + e = h \tag{2.24}$$

Similarly, continuing through the rest of the partial derivatives.

$$\left.\frac{\partial^2(\rho e)}{\partial \rho^2}\right|_s = \left.\frac{\partial(h)}{\partial \rho}\right|_s = \left.\frac{\partial h}{\partial p}\right|_s \left.\frac{\partial p}{\partial \rho}\right|_s = \frac{1}{\rho}a^2 \tag{2.25a}$$

$$\begin{aligned}\left.\frac{\partial^3(\rho e)}{\partial \rho^3}\right|_s &= \frac{\partial\left(\frac{a^2}{\rho}\right)_s}{\partial \rho} = \frac{\rho \left.\frac{\partial a^2}{\partial \rho}\right|_s - a^2}{\rho^2} \\ &= \frac{\rho\left(\gamma - 1\right)\frac{a^2}{\rho} - a^2}{\rho^2} = \left(\gamma - 2\right)\frac{a^2}{\rho^2}\end{aligned} \tag{2.25b}$$

Furthermore, the derivatives with respect to entropy,s, are completed,

$$\left.\frac{\partial(\rho e)}{\partial s}\right|_\rho = \rho \left.\frac{\partial e}{\partial s}\right|_\rho = \rho T \tag{2.26a}$$

$$\left.\frac{\partial^2(\rho e)}{\partial s^2}\right|_\rho = \left.\frac{\partial(\rho T)}{\partial s}\right|_\rho = \rho \left.\frac{\partial T}{\partial s}\right|_\rho = \frac{\rho \gamma T}{c_p} \tag{2.26b}$$

$$\left.\frac{\partial^3(\rho e)}{\partial s^3}\right|_\rho = \frac{\partial\left(\frac{\rho\gamma T}{c_p}\right)_\rho}{\partial s} = \frac{\rho\gamma}{c_p}\left.\frac{\partial T}{\partial s}\right|_\rho = \frac{\rho\gamma}{c_p}\frac{\gamma T}{c_p} = \frac{\rho T}{{c_v}^2} \tag{2.26c}$$

And the mixed derivatives yield,

$$\frac{\partial^2(\rho e)}{\partial \rho \partial s} = \left.\frac{\partial h}{\partial s}\right|_\rho = \gamma T \tag{2.27a}$$

$$\frac{\partial^3(\rho e)}{\partial \rho^2 \partial s} = \left.\frac{\partial\left(\gamma T\right)}{\partial \rho}\right|_s = \frac{\gamma a^2}{c_p \rho} = \frac{a^2}{c_v \rho} \tag{2.27b}$$

$$\frac{\partial^3(\rho e)}{\partial \rho \partial s^2} = \left.\frac{\partial\left(\gamma T\right)}{\partial s}\right|_\rho = \frac{\gamma^2 T}{c_p} \tag{2.27c}$$

In addition to the partial derivatives, the perturbation expansions of the differential ρ and s terms are needed in order to complete the expansion of (2.21). The expansions of density, entropy, pressure and mixed terms are shown through third order as they are all used in later analysis.

$$(\rho-\rho_0)=\varepsilon\rho_1+\varepsilon^2\rho_2+\varepsilon^3\rho_3+\dots \tag{2.28a}$$

$$(\rho-\rho_0)^2=\varepsilon^2{\rho_1}^2+\varepsilon^3\left(2\rho_1\rho_2\right)+\dots \tag{2.28b}$$

$$(\rho-\rho_0)^3=\varepsilon^3{\rho_1}^3+\dots \tag{2.28c}$$

$$(s-s_0)=\varepsilon s_1+\varepsilon^2 s_2+\varepsilon^3 s_3+\dots \tag{2.29a}$$

$$(s-s_0)^2=\varepsilon^2{s_1}^2+\varepsilon^3\left(2s_1s_2\right)+\dots \tag{2.29b}$$

$$(s-s_0)^3=\varepsilon^3{s_1}^3+\dots \tag{2.29c}$$

$$(p-p_0)=\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+\dots \tag{2.30a}$$

$$(p-p_0)^2=\varepsilon^2{p_1}^2+\varepsilon^3\left(2p_1p_2\right)+\dots \tag{2.30b}$$

$$(p-p_0)^3=\varepsilon^3{p_1}^3+\dots \tag{2.30c}$$

$$(\rho-\rho_0)(s-s_0)=\left(\varepsilon\rho_1+\varepsilon^2\rho_2+\varepsilon^3\rho_3\right)\left(\varepsilon s_1+\varepsilon^2 s_2+\varepsilon^3 s_3\right) \tag{2.31a}$$

$$=\varepsilon^2\rho_1 s_1+\varepsilon^3\left(\rho_1 s_2+\rho_2 s_1\right)+\dots. \tag{2.31b}$$

$$(\rho-\rho_0)^2(s-s_0)=\varepsilon^3{\rho_1}^2 s_1+\dots \tag{2.31c}$$

$$(\rho-\rho_0)(s-s_0)^2=\varepsilon^3\rho_1{s_1}^2+\dots \tag{2.31d}$$

$$
\begin{aligned}
(p - p_0)(s - s_0) &= \left(\varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3\right)\left(\varepsilon s_1 + \varepsilon^2 s_2 + \varepsilon^3 s_3\right) &\text{(2.32a)}\\
&= \varepsilon^2 p_1 s_1 + \varepsilon^3 \left(p_1 s_2 + p_2 s_1\right) + \dots . &\text{(2.32b)}\\
(p - p_0)^2 (s - s_0) &= \varepsilon^3 {p_1}^2 s_1 + \dots &\text{(2.32c)}\\
(p - p_0)(s - s_0)^2 &= \varepsilon^3 p_1 {s_1}^2 + \dots &\text{(2.32d)}
\end{aligned}
$$

Now insert the partial derivatives into 2.1a,

$$
\begin{aligned}
\rho e = \rho_0 e_0 &+ h_0(\rho - \rho_0) + \rho_0 T_0 (s - s_0) \\
&+ \frac{{a_0}^2}{2\rho_0}(\rho - \rho_0)^2 + \gamma T_0 (\rho - \rho_0)(s - s_0) + \frac{\rho_0 \gamma T_0}{2c_p}(s - s_0)^2 \\
&+ (\gamma - 2)\frac{{a_0}^2}{6{\rho_0}^2}(\rho - \rho_0)^3 + \frac{\gamma {a_0}^2}{2c_p \rho_0}(\rho - \rho_0)^2 (s - s_0) \\
&+ \frac{\gamma^2 T_0}{2c_p}(\rho - \rho_0)(s - s_0)^2 + \frac{\gamma^2 \rho_0 T_0}{6{c_p}^2}(s - s_0)^3 + \dots \qquad (2.33)
\end{aligned}
$$

And finally insert the perturbation expansions,

$$
\begin{aligned}
\rho e = \rho_0 e_0 &+ h_0(\varepsilon \rho_1 + \varepsilon^2 \rho_2 + \varepsilon^3 \rho_3) + \rho_0 T_0 (\varepsilon s_1 + \varepsilon^2 s_2 + \varepsilon^3 s_3) \\
&+ \frac{{a_0}^2}{2\rho_0}(\varepsilon^2 {\rho_1}^2 + \varepsilon^3 (2\rho_1 \rho_2)) + \gamma T_0 \left(\varepsilon^2 \rho_1 s_1 + \varepsilon^3 (\rho_1 s_2 + \rho_2 s_1)\right) \\
&+ \frac{\rho_0 \gamma T_0}{2c_p}(\varepsilon^2 {s_1}^2 + \varepsilon^3 (2 s_1 s_2)) + (\gamma - 2)\frac{{a_0}^2}{6{\rho_0}^2}(\varepsilon^3 {\rho_1}^3) \\
&+ \frac{\gamma {a_0}^2}{2c_p \rho_0}(\varepsilon^3 {\rho_1}^2 s_1) + \frac{\gamma^2 T_0}{2c_p}(\varepsilon^3 \rho_1 {s_1}^2) + \frac{\gamma^2 \rho_0 T_0}{6{c_p}^2}(\varepsilon^3 {s_1}^3) + O(\varepsilon^4) \qquad (2.34)
\end{aligned}
$$

The individual terms are gathered into their respective orders,

$$\rho e = \rho_0 e_0 + \varepsilon \left\{ h_0 \rho_1 + \rho_0 T_0 s_1 \right\} + \varepsilon^2 \left\{ h_0 \rho_2 + \rho_0 T s_2 + \frac{{a_0}^2 {\rho_1}^2}{2\rho_0} + \gamma T_0 \rho_1 s_1 + \frac{\rho_0 \gamma T_0}{2c_p} {s_1}^2 \right\} + \varepsilon^3 \left\{ \begin{gathered} h_0 \rho_3 + \rho_0 T_0 s_3 + \frac{{a_0}^2 \rho_1 \rho_2}{\rho_0} + \gamma T_0 \left(\rho_1 s_2 + \rho_2 s_1 \right) \\ + \frac{\rho_0 \gamma T_0}{c_p} s_1 s_2 + (\gamma - 2) \frac{{a_0}^2 {\rho_1}^3}{6{\rho_0}^2} \\ + \frac{\gamma {a_0}^2 {\rho_1}^2 s_1}{2 c_p \rho_0} + \frac{\gamma^2 T_0 \rho_1 {s_1}^2}{2c_p} + \frac{\gamma^2 \rho_0 T_0 {s_1}^3}{6{c_p}^2} \end{gathered} \right\} + O(\varepsilon^4) \quad (2.35)$$

Using the notation which will be implemented later in the expansion of the energy equation the terms are separated.

$$(\rho e)_0 = \rho_0 e_0 \quad (2.36a)$$

$$(\rho e)_1 = \left\{ h_0 \rho_1 + \rho_0 T_0 s_1 \right\} \quad (2.36b)$$

$$(\rho e)_2 = \left\{ h_0 \rho_2 + \rho_0 T s_2 + \frac{{a_0}^2 {\rho_1}^2}{2\rho_0} + \gamma T_0 \rho_1 s_1 + \frac{\rho_0 \gamma T_0}{2c_p} {s_1}^2 \right\} \quad (2.36c)$$

$$(\rho e)_3 = \left\{ \begin{gathered} h_0 \rho_3 + \rho_0 T_0 s_3 + \frac{{a_0}^2 \rho_1 \rho_2}{\rho_0} + \gamma T_0 \left(\rho_1 s_2 + \rho_2 s_1 \right) \\ + \frac{\rho_0 \gamma T_0}{c_p} s_1 s_2 + (\gamma - 2) \frac{{a_0}^2 {\rho_1}^3}{6{\rho_0}^2} \\ + \frac{\gamma {a_0}^2 {\rho_1}^2 s_1}{2 c_p \rho_0} + \frac{\gamma^2 T_0 \rho_1 {s_1}^2}{2c_p} + \frac{\gamma^2 \rho_0 T_0 {s_1}^3}{6{c_p}^2} \end{gathered} \right\} \quad (2.36d)$$

In later analysis we will prefer the use of p_1 over ρ_1 terms. In order to make these modifications additional expansions of h, p, ρ are required. Starting with the multivariable Taylor series expansion for each term.

$$h = h_0 + \left.\frac{\partial h}{\partial \rho}\right|_0 (\rho - \rho_0) + \left.\frac{\partial h}{\partial s}\right|_0 (s - s_0) + \left.\frac{\partial^2 h}{\partial \rho^2}\right|_0 \frac{(\rho - \rho_0)^2}{2} + \left.\frac{\partial^2 h}{\partial s \partial \rho}\right|_0 (\rho - \rho_0)(s - s_0) + \left.\frac{\partial^2 h}{\partial s^2}\right|_0 \frac{(s - s_0)^2}{2} + \dots \quad (2.37)$$

$$p = p_0 + \left.\frac{\partial p}{\partial \rho}\right|_0 (\rho - \rho_0) + \left.\frac{\partial p}{\partial s}\right|_0 (s - s_0) + \left.\frac{\partial^2 p}{\partial \rho^2}\right|_0 \frac{(\rho - \rho_0)^2}{2} + \left.\frac{\partial^2 p}{\partial s \partial \rho}\right|_0 (\rho - \rho_0)(s - s_0) + \left.\frac{\partial^2 p}{\partial s^2}\right|_0 \frac{(s - s_0)^2}{2} + \ldots \quad (2.38)$$

$$\rho = \rho_0 + \left.\frac{\partial \rho}{\partial p}\right|_0 (p - p_0) + \left.\frac{\partial \rho}{\partial s}\right|_0 (s - s_0) + \left.\frac{\partial^2 \rho}{\partial p^2}\right|_0 \frac{(p - p_0)^2}{2} + \left.\frac{\partial^2 \rho}{\partial s \partial p}\right|_0 (p - p_0)(s - s_0) + \left.\frac{\partial^2 \rho}{\partial s^2}\right|_0 \frac{(s - s_0)^2}{2} + \ldots \quad (2.39)$$

Just as in the expansion of ρe, each of the individual partial derivatives need to be evaluated. Again, this is done using the thermodynamic relations. First, the derivatives of a^2 are evaluated since they are used throughout the other derivatives.

$$\left.\frac{\partial a^2}{\partial \rho}\right|_s = \left.\frac{\partial (\gamma R T)}{\partial \rho}\right|_s = \gamma \frac{\partial \left(\frac{p}{\rho}\right)}{\partial \rho} = \gamma \left(\frac{\rho \left.\frac{\partial p}{\partial \rho}\right|_s - p \left.\frac{\partial \rho}{\partial \rho}\right|_s}{\rho^2} \right) = \gamma \left(\frac{\rho a^2 - p}{\rho^2} \right) = \frac{\gamma a^2}{\rho} - \frac{\gamma p}{\rho^2} = \frac{\gamma a^2}{\rho} - \frac{\gamma \rho R T}{\rho^2} = \frac{\gamma a^2}{\rho} - \frac{a^2}{\rho} = (\gamma - 1) \frac{a^2}{\rho} \quad (2.40a)$$

$$\left.\frac{\partial a^2}{\partial s}\right|_\rho = \left.\frac{\partial (\gamma R T)}{\partial s}\right|_\rho = \gamma R \left.\frac{\partial T}{\partial s}\right|_\rho = \gamma R \left(\frac{\gamma T}{c_p} \right) = \frac{\gamma a^2}{c_p} = \frac{a^2}{c_v} \quad (2.40b)$$

$$\left.\frac{\partial a^2}{\partial p}\right|_s = \left.\frac{\partial a^2}{\partial \rho}\right|_s \left.\frac{\partial \rho}{\partial p}\right|_s = (\gamma - 1) \frac{a^2}{\rho} \frac{1}{a^2} = \frac{(\gamma - 1)}{\rho} \quad (2.40c)$$

$$\left.\frac{\partial a^2}{\partial s}\right|_p = \left.\frac{\partial (\gamma R T)}{\partial s}\right|_p = \gamma R \left.\frac{\partial T}{\partial s}\right|_p = \gamma R \frac{T}{c_p} \quad (2.40d)$$

Now the enthalpy derivative terms,

$$\left.\frac{\partial h}{\partial \rho}\right|_s = \left.\frac{\partial h}{\partial p}\right|_s \left.\frac{\partial p}{\partial \rho}\right|_s = \frac{a^2}{\rho} \tag{2.41a}$$

$$\left.\frac{\partial h}{\partial s}\right|_\rho = \gamma T \tag{2.41b}$$

$$\begin{aligned}\left.\frac{\partial^2 h}{\partial \rho^2}\right|_s &= \left.\frac{\partial\left(a^2/\rho\right)}{\partial \rho}\right|_s = \frac{1}{\rho^2}\left[\rho\left.\frac{\partial a^2}{\partial \rho}\right|_s - a^2\left.\frac{\partial \rho}{\partial \rho}\right|_s\right] \\ &= \frac{1}{\rho^2}\left[\rho\left(\gamma-1\right)\frac{a^2}{\rho} - a^2\right] = \left(\gamma-2\right)\frac{a^2}{\rho^2}\end{aligned} \tag{2.41c}$$

$$\left.\frac{\partial^2 h}{\partial s^2}\right|_\rho = \gamma\left.\frac{\partial T}{\partial s}\right|_\rho = \gamma\left(\frac{\gamma T}{c_p}\right) = \frac{\gamma T}{c_v} = \frac{a^2}{R c_v} \tag{2.41d}$$

$$\frac{\partial^2 h}{\partial \rho \partial s} = \left.\frac{\partial}{\partial s}\right|_\rho \left(\left.\frac{\partial h}{\partial \rho}\right|_s\right) = \frac{\partial}{\partial s}\left(\frac{a^2}{\rho}\right)_\rho = \frac{1}{\rho}\left(\left.\frac{\partial a^2}{\partial s}\right|_\rho\right) = \frac{a^2}{\rho c_v} \tag{2.41e}$$

And the pressure derivative terms,

$$\left.\frac{\partial p}{\partial \rho}\right|_s = a^2 \tag{2.42a}$$

$$\left.\frac{\partial p}{\partial s}\right|_\rho = \frac{a^2 \rho}{c_p} \tag{2.42b}$$

$$\left.\frac{\partial^2 p}{\partial \rho^2}\right|_s = \left.\frac{\partial a^2}{\partial \rho}\right|_s = \left(\gamma-1\right)\frac{a^2}{\rho} \tag{2.42c}$$

$$\left.\frac{\partial^2 p}{\partial s^2}\right|_\rho = \frac{1}{c_p}\left.\frac{\partial\left(a^2\rho\right)}{\partial s}\right|_\rho = \frac{1}{c_p}\left[\rho\left.\frac{\partial a^2}{\partial s}\right|_\rho + a^2\left.\frac{\partial \rho}{\partial s}\right|_\rho\right] = \frac{1}{c_p}\rho\frac{a^2}{c_v} = \frac{\gamma\rho a^2}{{c_p}^2} \tag{2.42d}$$

$$\frac{\partial^2 p}{\partial s \partial \rho} = \left.\frac{\partial\left(a^2\right)}{\partial s}\right|_\rho = \frac{a^2}{c_v} = \frac{\gamma a^2}{c_p} \tag{2.42e}$$

And finally the density derivative terms,

$$\left.\frac{\partial \rho}{\partial p}\right|_s = \frac{1}{a^2} \tag{2.43a}$$

$$\left.\frac{\partial \rho}{\partial s}\right|_p = -\frac{\rho}{c_p} \tag{2.43b}$$

$$\left.\frac{\partial^2 \rho}{\partial p^2}\right|_s = \left.\frac{\partial\left(1/a^2\right)}{\partial p}\right|_s = \frac{1}{a^4}\left[a^2 \left.\frac{\partial 1}{\partial p}\right|_s - \left.\frac{\partial a^2}{\partial p}\right|_s\right] = -\frac{(\gamma-1)}{a^4\rho} \tag{2.43c}$$

$$\left.\frac{\partial^2 \rho}{\partial s^2}\right|_p = -\frac{1}{c_p}\left.\frac{\partial \rho}{\partial s}\right|_p = -\frac{1}{c_p}\left(-\frac{\rho}{c_p}\right) = \frac{\rho}{{c_p}^2} \tag{2.43d}$$

$$\frac{\partial^2 \rho}{\partial s \partial p} = -\frac{1}{c_p}\left.\frac{\partial \rho}{\partial p}\right|_s = -\frac{1}{c_p a^2} \tag{2.43e}$$

$$\frac{\partial^2 \rho}{\partial s \partial p} = \left.\frac{\partial\left(1/a^2\right)}{\partial s}\right|_p = \frac{1}{a^4}\left[-\left.\frac{\partial a^2}{\partial s}\right|_p\right] = -\frac{1}{a^4}\gamma R\frac{T}{c_p} = -\frac{1}{a^4}\frac{a^2}{c_p} = -\frac{1}{c_p a^2} \tag{2.43f}$$

Using these completed partial derivatives, the Taylor series expansions are completed. The partial derivatives are applied which are followed by the perturbation expansions. Then the terms are separated into their respective orders. First enthalpy, inserting Equation's 2.41 into Eqn 2.37.

$$h = h_0 + \frac{{a_0}^2}{\rho_0}(\rho-\rho_0) + \gamma T_0(s-s_0) + (\gamma-2)\frac{{a_0}^2}{{\rho_0}^2}\frac{(\rho-\rho_0)^2}{2} + \frac{{a_0}^2}{\rho_0 c_v}(\rho-\rho_0)(s-s_0) + \frac{{a_0}^2}{R c_v}\frac{(s-s_0)^2}{2} + ... \tag{2.44}$$

Then using the perturbation expansions, Eqn's 2.28, 2.29, and 2.31.

$$h = h_0 + \frac{{a_0}^2}{\rho_0}(\varepsilon\rho_1 + \varepsilon^2\rho_2) + \gamma T_0(\varepsilon s_1 + \varepsilon^2 s_2) + (\gamma-2)\frac{{a_0}^2}{{\rho_0}^2}\frac{\varepsilon^2{\rho_1}^2}{2} + \frac{{a_0}^2}{\rho_0 c_v}\varepsilon^2\rho_1 s_1 + \frac{{a_0}^2}{R c_v}\frac{\varepsilon^2{s_1}^2}{2} + ... \tag{2.45}$$

And arranging the terms into their respective orders,

$$h = h_0 + \varepsilon \left\{ \frac{{a_0}^2}{\rho_0} \rho_1 + \gamma T_0 s_1 \right\} + \varepsilon^2 \left\{ \frac{{a_0}^2 \rho_2}{\rho_0} + \gamma T_0 s_2 + (\gamma - 2) \frac{{a_0}^2 {\rho_1}^2}{2{\rho_0}^2} + \frac{{a_0}^2 \rho_1 s_1}{\rho_0 c_v} + \frac{{a_0}^2 {s_1}^2}{2R c_v} \right\} + \ldots \quad (2.46)$$

Second, pressure, inserting Equation's 2.42 into Eqn 2.38.

$$p = p_0 + {a_0}^2 (\rho - \rho_0) + \frac{{a_0}^2 \rho_0}{c_p} (s - s_0) + (\gamma - 1) \frac{{a_0}^2}{\rho_0} \frac{(\rho - \rho_0)^2}{2} + \frac{\gamma {a_0}^2}{c_p} (\rho - \rho_0)(s - s_0) + \frac{\gamma \rho_0 {a_0}^2}{{c_p}^2} \frac{(s - s_0)^2}{2} + \ldots \quad (2.47)$$

Then using the perturbation expansions, Eqn's 2.28, 2.29, and 2.31.

$$p = p_0 + {a_0}^2 (\varepsilon \rho_1 + \varepsilon^2 \rho_2) + \frac{{a_0}^2 \rho_0}{c_p} (\varepsilon s_1 + \varepsilon^2 s_2) + (\gamma - 1) \frac{{a_0}^2}{\rho_0} \frac{\varepsilon^2 {\rho_1}^2}{2} + \frac{\gamma {a_0}^2}{c_p} \varepsilon^2 \rho_1 s_1 + \frac{\gamma \rho_0 {a_0}^2}{{c_p}^2} \frac{\varepsilon^2 {s_1}^2}{2} + \ldots \quad (2.48)$$

And arranging the terms into their respective orders,

$$p = p_0 + \varepsilon \left\{ {a_0}^2 \rho_1 + \frac{{a_0}^2 \rho_0 s_1}{c_p} \right\} + \varepsilon^2 \left\{ \begin{array}{c} {a_0}^2 \rho_2 + \frac{{a_0}^2 \rho_0 s_2}{c_p} + (\gamma - 1) \frac{{a_0}^2 {\rho_1}^2}{2\rho_0} \\ + \frac{\gamma {a_0}^2 \rho_1 s_1}{c_p} + \frac{\gamma \rho_0 {a_0}^2 {s_1}^2}{2{c_p}^2} \end{array} \right\} + \ldots \quad (2.49)$$

Finally density, inserting Equation's 2.43 into Eqn 2.39.

$$\rho = \rho_0 + \frac{1}{{a_0}^2} (p - p_0) - \frac{\rho_0}{c_p} (s - s_0) - \frac{(\gamma - 1)}{{a_0}^4 \rho_0} \frac{(p - p_0)^2}{2} - \frac{1}{c_p {a_0}^2} (p - p_0)(s - s_0) + \frac{\rho_0}{{c_p}^2} \frac{(s - s_0)^2}{2} + \ldots \quad (2.50)$$

Then using the perturbation expansions, Eqn's 2.30, 2.29, and 2.31.

$$\rho = \rho_0 + \frac{1}{{a_0}^2}(\varepsilon p_1 + \varepsilon^2 p_2) - \frac{\rho_0}{c_p}(\varepsilon s_1 + \varepsilon^2 s_2) \\ - \frac{(\gamma - 1)}{{a_0}^4 \rho_0}\frac{\varepsilon^2 {p_1}^2}{2} - \frac{1}{c_p {a_0}^2}\varepsilon^2 p_1 s_1 + \frac{\rho_0}{{c_p}^2}\frac{\varepsilon^2 {s_1}^2}{2} + ... \quad (2.51)$$

And arranging the terms into their respective orders,

$$\rho = \rho_0 + \varepsilon \left\{ \frac{p_1}{{a_0}^2} - \frac{\rho_0 s_1}{c_p} \right\} + \varepsilon^2 \left\{ \frac{p_2}{{a_0}^2} - \frac{\rho_0 s_2}{c_p} - \frac{(\gamma - 1)\, {p_1}^2}{2{a_0}^4 \rho_0} - \frac{p_1 s_1}{c_p {a_0}^2} + \frac{\rho_0 {s_1}^2}{2{c_p}^2} \right\} + ... \quad (2.52)$$

Drawing from these relations, we find that,

$$h_1 = \frac{{a_0}^2}{\rho_0}\rho_1 + \gamma T_0 s_1 = \frac{p_1}{\rho_0} + T_0 s_1 \quad (2.53)$$

$$p_1 = {a_0}^2 \rho_1 + \frac{{a_0}^2 \rho_0}{c_p} s_1 \quad (2.54)$$

$$\rho_1 = \frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p} s_1 \quad (2.55)$$

We can see that they are self consistent, such that ρ_1 can be derived from p_1. One reason for expanding ρ was to ensure that this is correct. Additionally we see a clear similarity between these equations and the thermodynamic relations for the set (h, ρ, s). For instance, the thermodynamic relation,

$$dp = {a_0}^2 d\rho + \frac{{a_0}^2 \rho_0}{c_p} ds, \quad (2.56)$$

corresponds directly with Eqn 2.54. The second order relations are similarly extracted and are shown to be,

$$h_2 = \frac{{a_0}^2 \rho_2}{\rho_0} + \gamma T_0 s_2 + (\gamma - 2)\frac{{a_0}^2 {\rho_1}^2}{2{\rho_0}^2} + \frac{{a_0}^2 \rho_1 s_1}{\rho_0 c_v} + \frac{{a_0}^2 {s_1}^2}{2Rc_v} \quad (2.57)$$

$$p_2 = {a_0}^2 \rho_2 + \frac{{a_0}^2 \rho_0 s_2}{c_p} + (\gamma - 1)\frac{{a_0}^2 {\rho_1}^2}{2\rho_0} + \frac{\gamma {a_0}^2 \rho_1 s_1}{c_p} + \frac{\gamma \rho_0 {a_0}^2 {s_1}^2}{2{c_p}^2} \quad (2.58)$$

$$\rho_2 = \frac{p_2}{{a_0}^2} - \frac{\rho_0 s_2}{c_p} - \frac{(\gamma - 1)\, {p_1}^2}{2{a_0}^4 \rho_0} - \frac{p_1 s_1}{c_p {a_0}^2} + \frac{\rho_0 {s_1}^2}{2{c_p}^2} \quad (2.59)$$

These equations are also self consistent.

2.4.2 Manipulating ρe terms

The previously derived expansions for ρe are manipulated using the relationships between the field variables. This allows ρe terms to be put into a form which prefers pressure over density fluctuations. The details of these manipulations are shown in Appendix A.3.

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \tag{2.60}$$

$$\begin{aligned}(\rho e)_3 = h_0\rho_3 + \rho_0 T_0 s_3 + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2 {a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{p_2 s_1}{\gamma R} \\ + \frac{p_1 s_2}{\gamma R} - \frac{{p_1}^2 s_1}{2c_p\rho_0 {a_0}^2} - \frac{p_1 {s_1}^2}{2c_p\gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{T_0\rho_0 {s_1}^3}{6{c_p}^2}\end{aligned} \tag{2.61}$$

Chapter 3

Energy Corollary

When using the three full governing equations (continuity, momentum and entropy) the energy equation becomes redundant. Thus any solution which satisfies the previous set of governing equations automatically satisfies the energy equation. Thereby, it is interesting to explore the consequences of introducing the previous field variable expansions into the energy equation. Care is taken to reduce the energy equation down to an form related to the acoustic energy (42). In this chapter the expansions shown previously are used to reduce the energy equation. The energy equation given in vector form is,

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla \cdot (\mathbf{m}H) - \mathbf{m} \cdot \psi - TQ = 0 \tag{3.1}$$

The energy equation is expanded and shown through second order by splitting the field variables as shown in Chapter 2.3. Beginning with the base (or zeroth) order,

$$\nabla \cdot (\mathbf{m}_0 H_0) - \mathbf{m}_0 \cdot \psi_0 - T_0 Q_0 = 0 \tag{3.2}$$

And first order,

$$\frac{\partial}{\partial t}(\rho H - p)_1 + \nabla \cdot (\mathbf{m}_0 H_1 + \mathbf{m}_1 H_0) - \mathbf{m}_0 \cdot \psi_1 - \mathbf{m}_1 \cdot \psi_0 - T_0 Q_1 - T_1 Q_0 = 0 \tag{3.3}$$

And second order,

$$\frac{\partial}{\partial t}(\rho H - p)_2 + \nabla \cdot (\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) \\ - \mathbf{m}_0 \cdot \psi_2 - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad (3.4)$$

3.1 Base Order or Zeroth Order Energy

Each order is analyzed individually, beginning with the base order energy.

$$\nabla \cdot (\mathbf{m}_0 H_0) - \mathbf{m}_0 \cdot \psi_0 - T_0 Q_0 = 0 \quad (3.5)$$

Expanding the divergence term and rearranging.

$$H_0 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot (\nabla H_0 - \psi_0) - T_0 Q_0 = 0 \quad (3.6)$$

Recalling the base order relations (2.17) and insert them into the zeroth order energy. Use the $\mathbf{m}_0 \cdot L_0$ expansion for the $\nabla \mathrm{H}_0$ term.

$$\mathbf{m}_0 \cdot L_0 = \mathbf{m}_0 \cdot (\zeta_0 + \nabla H_0 - T_0 \nabla s_0) \quad (3.7)$$

Rearrange the terms to solve for $\mathbf{m}_0 \cdot \nabla H_0$,

$$\mathbf{m}_0 \cdot \nabla H_0 = \mathbf{m}_0 \cdot L_0 - \mathbf{m}_0 \cdot \zeta_0 + \mathbf{m}_0 \cdot T_0 \nabla s_0 \quad (3.8)$$

Insert into (3.6).

$$H_0 \nabla \cdot \mathbf{m}_0 - \mathbf{m}_0 \cdot \psi_0 - T_0 Q_0 + \mathbf{m}_0 \cdot L_0 - \mathbf{m}_0 \cdot \zeta_0 + \mathbf{m}_0 \cdot T_0 \nabla s_0 = 0 \quad (3.9)$$

Using the vector definition for ζ and expanding we find the identity,

$$\mathbf{m}_0 \cdot \zeta_0 = \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_0) = 0 \tag{3.10}$$

As a result, there are no vorticity contributions to the base order energy balance. Again, rearranging the terms in (3.9),

$$H_0 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot (L_0 - \psi_0) - T_0 Q_0 + \mathbf{m}_0 \cdot T_0 \nabla s_0 = 0 \tag{3.11}$$

Expand and insert the definition of the base order entropy equation, S_o, and then multiply by T_o.

$$\begin{aligned} S_0 = \nabla \cdot (\mathbf{m}_0 s_0) &= s_0 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot \nabla s_0 \\ \mathbf{m}_0 \cdot \nabla s_0 &= S_0 - s_0 \nabla \cdot \mathbf{m}_0 \\ T_0 \mathbf{m}_0 \cdot \nabla s_0 &= T_0 S_0 - T_0 s_0 \nabla \cdot \mathbf{m}_0 \end{aligned} \tag{3.12}$$

Insert back into (3.11).

$$H_0 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot (L_0 - \psi_0) - T_0 Q_0 + T_0 S_0 - T_0 s_0 \nabla \cdot \mathbf{m}_0 = 0 \tag{3.13}$$

Collect the terms.

$$(H_0 - T_0 s_0) \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot (L_0 - \psi_0) + T_0 (S_0 - Q_0) = 0 \tag{3.14}$$

Insert the base order continuity equation, $\nabla \cdot \mathbf{m}_0 = C_0$

$$(H_0 - T_0 s_0) C_0 + \mathbf{m}_0 \cdot (L_0 - \psi_0) + T_0 (S_0 - Q_0) = 0 \tag{3.15}$$

Since $C_0 = L_0 - \psi_0 = S_0 - Q_0 = 0$, Equation (3.15) shows that based on prior knowledge $0 = 0$. Therefore the base order energy reveals no new information.

3.2 First Order Energy

The same process is performed on the first order equations, only with more algebra this time.

$$\frac{\partial}{\partial t}(\rho H - p)_1 + \nabla\cdot(\mathbf{m}_0 H_1 + \mathbf{m}_1 H_0) - \mathbf{m}_0\cdot\psi_1 - \mathbf{m}_1\cdot\psi_0 - T_0 Q_1 - T_1 Q_0 = 0 \tag{3.16}$$

To start, the expansion of the first term is needed. Inserting the definition of the total enthalpy and expanding,

$$\begin{aligned}
(\rho H - p)_1 &= \left(\rho\left(h + \frac{1}{2}\mathbf{u}^2\right) - p\right)_1 \\
&= \left(\rho\left(e + \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2\right) - p\right)_1 \\
&= \left(\rho e + \frac{1}{2}\rho\mathbf{u}^2\right)_1 \\
&= (\rho e)_1 + \frac{1}{2}\rho_1 {u_0}^2 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_1 \\
&= h_0\rho_1 + \rho_0 T_0 s_1 + \frac{1}{2}\rho_1 {u_0}^2 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_1 \\
(\rho H - p)_1 &= \rho_1 H_0 + \rho_0 T_0 s_1 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_1
\end{aligned} \tag{3.17}$$

The results of Equation (3.17) are placed back into the first order energy equation.

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho_1 H_0 + \rho_0 T_0 s_1 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_1) + \nabla\cdot(\mathbf{m}_0 H_1 + \mathbf{m}_1 H_0) \\
&\qquad - \mathbf{m}_0\cdot\psi_1 - \mathbf{m}_1\cdot\psi_0 - T_0 Q_1 - T_1 Q_0 = 0
\end{aligned} \tag{3.18}$$

Expand the time derivative term remembering that mean flow variables time dependency is neglected. Then all terms are algebraically reduced and the first order governing equations are used to reduce the first order energy balance. The details of this process are shown in Appendix A.4. The primary concern is the reduction of the terms containing vorticity.

Remembering that,

$$\begin{aligned} \mathbf{m}_1 &= \rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0 \\ \zeta_1 &= \omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_0 \end{aligned} \tag{3.19}$$

Apply these definitions to the vorticity terms,

$$\begin{aligned} \mathbf{m}_1 \cdot \zeta_0 + \mathbf{m}_0 \cdot \zeta_1 &= (\rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0) \cdot (\omega_0 \times \mathbf{u}_0) \\ &\quad + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_0) \\ &= \rho_0 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_0) \\ &\quad + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) + \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_0) \end{aligned} \tag{3.20}$$

$$\begin{aligned} \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_0) &= 0 \\ \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_0) &= 0 \end{aligned} \tag{3.21}$$

Recalling scalar triple product rules we find that there is no vorticity contribution to the first order energy balance.

$$\mathbf{m}_1 \cdot \zeta_0 + \mathbf{m}_0 \cdot \zeta_1 = \rho_0 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) = 0 \tag{3.22}$$

So as before in the base order energy equation, regardless of entropy or vorticity fluctuations, the energy equation for the first order reduces to zero.

$$(H_0 - T_0 s_0)\, C_1 + \mathrm{T}_0(\mathrm{S}_1 - Q_1) + \mathbf{m}_0 \cdot (L_1 - \psi_1) = 0 \tag{3.23}$$

3.3 Second Order Energy Expansions

The second order process follows exactly as the first order process did. Each term is expanded with the thermodynamic expansions and then simplified.

$$\frac{\partial}{\partial t}(\rho H - p)_2 + \nabla \cdot (\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) - \mathbf{m}_0 \cdot \psi_2 - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad (3.24)$$

The expansion of the first terms is found. The details of the algebra in this Section is shown in Appendix A.5.

$$(\rho H - p)_2 = H_0 \rho_2 + T_0(\rho_1 s_1 + \rho_0 s_2) + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 + \mathbf{m}_0 \cdot \mathbf{u}_2 \quad (3.25)$$

The expanded form of the time derivative term is inserted into Eqn. 3.24,

$$\frac{\partial}{\partial t}\begin{pmatrix} H_0 \rho_2 + T_0(\rho_1 s_1 + \rho_0 s_2) + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \\ + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 + \mathbf{m}_0 \cdot \mathbf{u}_2 \end{pmatrix} + \nabla \cdot (\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) - \mathbf{m}_0 \cdot \psi_2 - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad (3.26)$$

To assist in the bookkeeping the following terms in the time derivative are defined as,

$$E_2 = \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 \quad (3.27)$$

After applying the field variable expansions and the second order governing equations the following equation is found,

$$\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + \mathbf{m}_0 \cdot (L_2 - \psi_2) + T_0\,(S_2 - Q_2) \\ - T_0 \nabla \cdot (\mathbf{m}_1 s_1) - \mathbf{m}_0 \cdot \zeta_2 + T_1 \nabla \cdot (\mathbf{m}_0 s_1) + \nabla \cdot (\mathbf{m}_1 H_1) \\ - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \zeta_0 - T_1 Q_1 = 0 \quad (3.28)$$

Now the vorticity terms are dealt with similar to the first order energy balance, remembering that, $\zeta_2 = \omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_0$ and $\mathbf{m}_2 = \rho_0 \mathbf{u}_2 + \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_0$

$$\mathbf{m}_0 \cdot \zeta_2 + \mathbf{m}_2 \cdot \zeta_0 = \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_0) \\ + (\rho_0 \mathbf{u}_2 + \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_0) \cdot (\omega_0 \times \mathbf{u}_0) \\ = \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) + \rho_1 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) \quad (3.29)$$

In the second order case, all of the terms in the energy equation do not fall out. Thus we get some more information from it. Redundant terms are removed and the vorticity terms are simplified.Terms are arranged into three parts: the time derivative energy terms, the work or $\nabla\cdot$ terms, and the remaining source terms. This collection is motivated by recalling the energy balance form of $\frac{\partial E}{\partial t} + \nabla \cdot W = D$.

$$\frac{\partial E_2}{\partial t} + \nabla \cdot [\mathbf{m}_1 (H_1 - T_0 s_1) + \mathbf{m}_0 T_1 s_1] + \mathbf{m}_0 s_1 \cdot \nabla T_0 - \mathbf{m}_0 s_1 \cdot \nabla T_1 \\ - \mathbf{m}_1 \cdot \psi_1 - T_1 Q_1 - \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) - \rho_1 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) = 0 \quad (3.30)$$

Now the terms $\mathbf{m}_1 \cdot \psi_1$ and $T_1 Q_1$ are expanded upon. These terms are in tensor form and cannot be written in purely vector form. Thus, it is required to split them and leave them in tensor form,

$$\mathbf{m}_1 \cdot \psi_1 = m_{1_j} \left(\frac{1}{\rho} \frac{\partial P_i j}{\partial x_i} \right)_1 \tag{3.31}$$

$$= \frac{\partial}{\partial x_i} \left[m_{1_j} \left(\frac{P_i j}{\rho} \right)_1 \right] - \left(\frac{P_i j}{\rho^2} \right)_1 \frac{\partial m_{1_j}}{\partial x_i} + m_{1j} \left(\frac{P_i j}{\rho} \frac{\partial \rho}{\partial x_i} \right)_1 \tag{3.32}$$

$$T_1 Q_1 = T_1 \left(\frac{\phi - \nabla \cdot q}{T} \right)_1 \tag{3.33}$$

$$= T_1 \left(\frac{\phi}{T} \right)_1 - \nabla \cdot \left[T_1 \left(\frac{q}{T} \right)_1 \right] + \left(\frac{q}{T} \right)_1 \cdot \nabla T_1 - T_1 \left(\frac{q \cdot \nabla T}{T^2} \right)_1 \tag{3.34}$$

3.3.1 Final Second Order (Energy, Work and Sources)

The base order and first order expanded energy equation yielded no new information. However, the second order equation produced a relation between the first order oscillatory field variables. Using the form of Kirchoff's equation (20), we have the form,

$$\frac{\partial E_2}{\partial t} + \nabla \cdot W_2 = D_2 \tag{3.35}$$

Where,

$$E_2 = \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 \tag{3.36a}$$

$$W_2 = \mathbf{m}_1 \left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 - T_0 s_1 \right) + \mathbf{m}_0 T_1 s_1 + T_1 \left(\frac{q}{T} \right)_1 - m_{1j} \left(\frac{P_{ij}}{\rho} \right)_1 \tag{3.36b}$$

$$\begin{aligned} D_2 = & -\mathbf{m}_1 s_1 \cdot \nabla T_0 + \mathbf{m}_0 s_1 \cdot \nabla T_1 - \rho_0 \mathbf{u}_0 \cdot (\mathbf{u}_1 \times \omega_1) - \rho_1 \mathbf{u}_1 \cdot (\mathbf{u}_0 \times \omega_0) \\ & - \left(\frac{P_{ij}}{\rho} \right)_1 \frac{\partial m_{1j}}{\partial x_i} + m_{1j} \left(\frac{P_{ij}}{\rho^2} \frac{\partial \rho}{\partial x_i} \right)_1 \\ & + T_1 \left(\frac{\phi}{T} \right)_1 + \left(\frac{q}{T} \right)_1 \cdot \nabla T_1 - T_1 \left(\frac{q \cdot \nabla T}{T^2} \right)_1 \end{aligned} \tag{3.36c}$$

These equations represent the second order change in energy in a closed system. This analysis can then be applied to a rocket system where the volume integral of the total energy equation is evaluated yielding the total system change in oscillatory energy. This total change in energy is then numerically solved. This second order energy change reduces to a linear differential equation where the solutions are exponential functions. As a result, the third order energy balance is needed to capture nonlinear effects. Details on the application of the second order total energy balance are shown in chapter 5.

3.4 General Energy Corollary

Instead of algebraically expanding each order as shown earlier in chapter 3, it is possible to construct a general case which can later be expanded to any order. This method is preferred as it is algebraically simpler and it will be used to derive the third order energy balance. Following work by Myers (42),
Start with the energy equation,

$$\frac{\partial}{\partial t}\left[\rho H - p\right] + \nabla \cdot (\mathbf{m} H) - \mathbf{m} \cdot \psi - TQ = 0 \tag{3.37}$$

Subtract the general relations,

$$(H_0 - T_0 s_0) = 0 \tag{3.38a}$$

$$T_0\left(S - Q\right) = 0 \tag{3.38b}$$

$$\mathbf{m}_0 \cdot (L - \psi) = 0 \tag{3.38c}$$

Remembering that,

$$C = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{m} = 0 \tag{3.39a}$$

$$L = \frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + \nabla H - T \nabla s = \psi \tag{3.39b}$$

$$S = \frac{\partial \rho s}{\partial t} + \nabla \cdot (\mathbf{m} s) = Q \tag{3.39c}$$

Using the base order equations and algebraic expansions the energy equation is manipulated. The details are shown in Appendix A.6. After some work the following expansion is found,

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\rho\left[H-H_0-T_0\left(s-s_0\right)\right]-\mathbf{m}_0\cdot\left(\mathbf{u}-\mathbf{u}_0\right)-\left(p-p_0\right)\right\}\\+\nabla\cdot\left[\left(\mathbf{m}-\mathbf{m}_0\right)\left[H-H_0+T_0\left(s-s_0\right)\right]+\mathbf{m}_0\left(T-T_0\right)\left(s-s_0\right)\right]\\-\left(\mathbf{m}-\mathbf{m}_0\right)\cdot\left(\psi-\psi_0\right)-\left(T-T_0\right)\left(Q-Q_0\right)\\-\mathbf{m}\cdot\zeta_0-\mathbf{m}_0\zeta+\left(s-s_0\right)\mathbf{m}\cdot\nabla T_0-\left(s-s_0\right)\mathbf{m}_0\cdot\nabla T=0\end{aligned} \quad (3.40)$$

The heat transfer and viscous terms have to be split into work and source terms. Beginning with the stress terms, we expand them with their tensor form.

$$\begin{aligned}\left(\mathbf{m}-\mathbf{m}_0\right)\cdot\left(\psi-\psi_0\right)&=\left(m_j-m_{0_j}\right)\left(\frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}-\frac{1}{\rho_0}\frac{\partial P_{0ij}}{\partial x_i}\right)\\&=\frac{\partial}{\partial x_i}\left[\left(m_j-m_{0_j}\right)\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\right]-\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\frac{\partial}{\partial x_i}\left(m_j-m_{0_j}\right)\\&\quad+\left(m_j-m_{0_j}\right)\left(\frac{1P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}-\frac{P_{0_{ij}}}{\rho_0^2}\frac{\partial\rho_0}{\partial x_i}\right)\end{aligned} \quad (3.41)$$

$$\begin{aligned}(T-T_0)(Q-Q_0)&=\left(T-T_0\right)\left(\frac{\phi}{T}-\frac{\phi_0}{T_0}-\frac{\nabla\cdot\mathbf{q}}{T}+\frac{\nabla\cdot\mathbf{q_0}}{T_0}\right)\\&=\left(T-T_0\right)\left(\frac{\phi}{T}-\frac{\phi_0}{T_0}\right)-\nabla\cdot\left[\left(T-T_0\right)\left(\frac{\mathbf{q}}{T}-\frac{\mathbf{q_0}}{T_0}\right)\right]\\&\quad+\left(\frac{\mathbf{q}}{T}-\frac{\mathbf{q_0}}{T_0}\right)\cdot\nabla(T-T_0)-\left(T-T_0\right)\left(\frac{\mathbf{q}\cdot\nabla T}{T^2}-\frac{\mathbf{q_0}\cdot\nabla T_0}{T_0^2}\right)\end{aligned} \quad (3.42)$$

These relations are separated into work and source terms. Inserting them back into Eqn. 3.40 and separating Energy, Work and Source terms with the form, $\frac{\partial E}{\partial t}+\nabla\cdot W=D$ it is found that,

$$E = \rho \left[H - H_0 - T_0 \left(s - s_0 \right) \right] - \mathbf{m}_0 \cdot \left(\mathbf{u} - \mathbf{u}_0 \right) - \left(p - p_0 \right) \tag{3.43}$$

$$\begin{aligned} W = &\left(\mathbf{m} - \mathbf{m}_0 \right) \left[H - H_0 + T_0 \left(s - s_0 \right) \right] + \mathbf{m}_0 \left(T - T_0 \right) \left(s - s_0 \right) \\ &- \left(m_j - m_{0_j} \right) \left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0} \right) + \left(T - T_0 \right) \left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0} \right) \end{aligned} \tag{3.44}$$

$$\begin{aligned} D = &\mathbf{m} \cdot \zeta_0 + \mathbf{m}_0 \zeta - \left(s - s_0 \right) \mathbf{m} \cdot \nabla T_0 + \left(s - s_0 \right) \mathbf{m}_0 \cdot \nabla T \\ &- \left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0} \right) \frac{\partial}{\partial x_i} (m_j - m_{0_j}) + \left(m_j - m_{0_j} \right) \left(\frac{1 P_{ij}}{\rho^2} \frac{\partial \rho}{\partial x_i} - \frac{P_{0_{ij}}}{\rho_0^2} \frac{\partial \rho_0}{\partial x_i} \right) \\ &+ \left(T - T_0 \right) \left(\frac{\phi}{T} - \frac{\phi_0}{T_0} \right) + \left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0} \right) \cdot \nabla (T - T_0) \\ &- \left(T - T_0 \right) \left(\frac{\mathbf{q} \cdot \nabla T}{T^2} - \frac{\mathbf{q_0} \cdot \nabla T_0}{T_0^2} \right) \end{aligned} \tag{3.45}$$

This method is applied to the second order expansion shown in Chapter 3 and the results are recreated in Appendix B.

Chapter 4

Third Order Expansion

Section 3.4 showed that it is possible to use the general energy corollary to quickly and more simply (and therefore less prone to error) arrive at the second order energy balance. In the base, first and second orders it was possible, without too much trouble, to arrive at the energy balance by expanding the governing equations and inserting them into the energy equation. However, due to the increasing amount of algebraic manipulation required the third order will be derived solely using the exact energy corollary.

Typically, previous combustion instability theories only extend to second order. This is justifiable as higher orders are increasingly negligible. This expansion will allow us to quantify the third order terms and judge their true value numerically. In this chapter the algebraic steps are shown in detail.

4.1 Energy

Starting with the general energy equation given from the general energy corollary from Chapter 3,

$$E = \rho \left[(H - H_0) - T_0 (s - s_0) \right] - \mathbf{m}_0 \cdot (\mathbf{u} - \mathbf{u}_0) - (p - p_0) \tag{4.1}$$

Each field variable is algebraically expanded upon,

$$E = (\rho_0 + \rho_1 + \rho_2 + \rho_3)\left[H_1 + H_2 + H_3 - T_0\left(s_1 + s_2 + s_3\right)\right] \\ - \mathbf{m}_0 \cdot (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) - (p_1 + p_2 + p_3) \quad (4.2)$$

The third order terms are extracted from the algebraic expansion of the general energy corollary,

$$E_3 = \rho_0 H_3 + \rho_1 H_2 + \rho_2 H_1 - \rho_0 T_0 s_3 - \rho_1 T_0 s_2 - \rho_2 T_0 s_1 - \mathbf{m}_0 \cdot \mathbf{u}_3 - p_3 \quad (4.3)$$

Recalling the relations,

$$\begin{cases} H_1 = h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 \\ H_2 = h_2 + \dfrac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 \\ H_3 = h_3 + \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_0 \cdot \mathbf{u}_3 \end{cases} \quad (4.4)$$

These expansions are used within Eqn. 4.3,

$$E_3 = \rho_0\left(h_3 + \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_0 \cdot \mathbf{u}_3\right) + \rho_1\left(h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2\right) + \rho_2\left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1\right) \\ - \rho_0 T_0 s_3 - \rho_1 T_0 s_2 - \rho_2 T_0 s_1 - \mathbf{m}_0 \cdot \mathbf{u}_3 - p_3 \quad (4.5)$$

Further algebraic expansion of Eqn. 4.5 yields,

$$E_3 = \rho_0 h_3 + \rho_0 \mathbf{u}_1 \cdot \mathbf{u}_2 + \rho_0 \mathbf{u}_0 \cdot \mathbf{u}_3 + \rho_1 h_2 + \frac{1}{2}\rho_1 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_2 \\ + \rho_2 h_1 + \rho_2 \mathbf{u}_0 \cdot \mathbf{u}_1 - \rho_0 T_0 s_3 - \rho_1 T_0 s_2 - \rho_2 T_0 s_1 - \mathbf{m}_0 \cdot \mathbf{u}_3 - p_3 \quad (4.6)$$

Equal terms are canceled and the relations below are applied,

$$\begin{aligned} \rho_0 h_3 + \rho_1 h_2 + \rho_2 h_1 + \rho_3 h_0 &= (\rho e)_3 + p_3 \\ \rho_0 h_3 + \rho_1 h_2 + \rho_2 h_1 - p_3 &= (\rho e)_3 - \rho_3 h_0 \end{aligned} \quad (4.7)$$

Also, recalling the expansion for ρe_3,

$$(\rho e)_3 = h_0\rho_3 + \rho_0 T_0 s_3 + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2{a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{p_2 s_1}{\gamma R} + \frac{p_1 s_2}{\gamma R} - \frac{{p_1}^2 s_1}{2c_p\rho_0{a_0}^2} - \frac{p_1{s_1}^2}{2c_p\gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{T_0\rho_0{s_1}^3}{6{c_p}^2} \quad (4.8)$$

We arrive at,

$$E_3 = \rho_0\mathbf{u}_1\cdot\mathbf{u}_2 + \frac{1}{2}\rho_1{\mathbf{u}_1}^2 + \rho_1\mathbf{u}_0\cdot\mathbf{u}_2 + \rho_2\mathbf{u}_0\cdot\mathbf{u}_1 - \rho_0T_0s_3 - \rho_1T_0s_2 - \rho_2T_0s_1 + \rho_0T_0s_3 + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2{a_0}^4} + \frac{p_1p_2}{\rho_0{a_0}^2} + \frac{p_2s_1}{\gamma R} + \frac{p_1s_2}{\gamma R} - \frac{{p_1}^2s_1}{2c_p\rho_0{a_0}^2} - \frac{p_1{s_1}^2}{2c_p\gamma R} - \frac{\rho_0T_0s_1s_2}{c_p} + \frac{T_0\rho_0{s_1}^3}{6{c_p}^2} \quad (4.9)$$

Canceling equal terms,

$$E_3 = \rho_0\mathbf{u}_1\cdot\mathbf{u}_2 + \frac{1}{2}\rho_1{\mathbf{u}_1}^2 + \rho_1\mathbf{u}_0\cdot\mathbf{u}_2 + \rho_2\mathbf{u}_0\cdot\mathbf{u}_1 - \rho_1T_0s_2 - \rho_2T_0s_1 + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2{a_0}^4} + \frac{p_1p_2}{\rho_0{a_0}^2} + \frac{p_2s_1}{\gamma R} + \frac{p_1s_2}{\gamma R} - \frac{{p_1}^2s_1}{2c_p\rho_0{a_0}^2} - \frac{p_1{s_1}^2}{2c_p\gamma R} - \frac{\rho_0T_0s_1s_2}{c_p} + \frac{T_0\rho_0{s_1}^3}{6{c_p}^2} \quad (4.10)$$

Recalling,

$$\rho_1 = \frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1$$
$$\rho_2 = \frac{p_2}{{a_0}^2} - \frac{\rho_0 s_2}{c_p} - \frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0} - \frac{p_1s_1}{c_p{a_0}^2} + \frac{\rho_0{s_1}^2}{2{c_p}^2} \quad (4.11)$$

The following term is expanded in order to convert the ρ_1 term into a p_1 term,

$$\rho_1T_0s_2 = \left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right)T_0s_2 = \frac{p_1T_0s_2}{{a_0}^2} - \frac{\rho_0T_0s_2s_1}{c_p} = \frac{p_1s_2}{\gamma R} - \frac{\rho_0T_0s_2s_1}{c_p} \quad (4.12)$$

$$\rho_2 T_0 s_1 = \left(\frac{p_2}{{a_0}^2} - \frac{\rho_0 s_2}{c_p} - \frac{(\gamma - 1)\, {p_1}^2}{2{a_0}^4 \rho_0} - \frac{p_1 s_1}{c_p {a_0}^2} + \frac{\rho_0 {s_1}^2}{2{c_p}^2} \right) T_0 s_1 \tag{4.13}$$

$$= \frac{p_2 T_0 s_1}{{a_0}^2} - \frac{\rho_0 T_0 s_1 s_2}{c_p} - \frac{(\gamma - 1)\, T_0 s_1 {p_1}^2}{2{a_0}^4 \rho_0} - \frac{p_1 T_0 s_1 s_1}{c_p {a_0}^2} + \frac{\rho_0 T_0 s_1 {s_1}^2}{2{c_p}^2} \tag{4.14}$$

$$= \frac{p_2 s_1}{\gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} - \frac{(\gamma - 1)\, s_1 {p_1}^2}{2\gamma R {a_0}^2 \rho_0} - \frac{p_1 {s_1}^2}{c_p \gamma R} + \frac{\rho_0 T_0 {s_1}^3}{2{c_p}^2} \tag{4.15}$$

Using this expansion,

$$\begin{aligned} E_3 = {} & \rho_0 \mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{1}{2}\rho_1 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_2 + \rho_2 \mathbf{u}_0 \cdot \mathbf{u}_1 - \frac{p_1 s_2}{\gamma R} + \frac{\rho_0 T_0 s_2 s_1}{c_p} \\ & - \left(\frac{p_2 s_1}{\gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} - \frac{(\gamma - 1)\, s_1 {p_1}^2}{2\gamma R {a_0}^2 \rho_0} - \frac{p_1 {s_1}^2}{c_p \gamma R} + \frac{\rho_0 T_0 {s_1}^3}{2{c_p}^2} \right) \\ & + \frac{(1 - 2\gamma)\, {p_1}^3}{6{\rho_0}^2 {a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{p_2 s_1}{\gamma R} + \frac{p_1 s_2}{\gamma R} - \frac{{p_1}^2 s_1}{2 c_p \rho_0 {a_0}^2} \\ & - \frac{p_1 {s_1}^2}{2 c_p \gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{T_0 \rho_0 {s_1}^3}{6{c_p}^2} \end{aligned} \tag{4.16}$$

Now several terms cancel,

$$\begin{aligned} E_3 = {} & \rho_0 \mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{1}{2}\rho_1 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_2 + \rho_2 \mathbf{u}_0 \cdot \mathbf{u}_1 \\ & + \frac{\rho_0 T_0 s_2 s_1}{c_p} + \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{(\gamma - 1)\, s_1 {p_1}^2}{2\gamma R {a_0}^2 \rho_0} + \frac{p_1 {s_1}^2}{c_p \gamma R} - \frac{\rho_0 T_0 {s_1}^3}{2{c_p}^2} \\ & + \frac{(1 - 2\gamma)\, {p_1}^3}{6{\rho_0}^2 {a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} - \frac{{p_1}^2 s_1}{2 c_p \rho_0 {a_0}^2} - \frac{p_1 {s_1}^2}{2 c_p \gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{T_0 \rho_0 {s_1}^3}{6{c_p}^2} \end{aligned} \tag{4.17}$$

Individual sets of terms are reduced,

$$\frac{\rho_0 T_0 s_2 s_1}{c_p} + \frac{\rho_0 T_0 s_1 s_2}{c_p} - \frac{\rho_0 T_0 s_1 s_2}{c_p} = \frac{\rho_0 T_0 s_1 s_2}{c_p} \tag{4.18a}$$

$$\frac{T_0 \rho_0 {s_1}^3}{6{c_p}^2} - \frac{\rho_0 T_0 {s_1}^3}{2{c_p}^2} = -\frac{\rho_0 T_0 {s_1}^3}{3{c_p}^2} \tag{4.18b}$$

$$\frac{p_1 {s_1}^2}{c_p \gamma R} - \frac{p_1 {s_1}^2}{2 c_p \gamma R} = \frac{p_1 {s_1}^2}{2 c_p \gamma R} \tag{4.18c}$$

$$\frac{(\gamma - 1)\, s_1 {p_1}^2}{2\gamma R {a_0}^2 \rho_0} - \frac{{p_1}^2 s_1}{2 c_p \rho_0 {a_0}^2} = 0 \tag{4.18d}$$

And a small form of E_3 is captured. Again, as in the derivation of E_2 this equation is ambiguous. We could have solved it with terms including ρ_1. The form shown is simply the path chosen due to the familiarity of the pressure oscillations.

$$E_3 = \rho_0 \mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{1}{2}\rho_1 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_2 + \rho_2 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{(1 - 2\gamma)\, {p_1}^3}{6{\rho_0}^2 {a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{\rho_0 T_0 s_2 s_1}{c_p} + \frac{p_1 {s_1}^2}{2 c_p \gamma R} - \frac{\rho_0 T_0 {s_1}^3}{3{c_p}^2} \tag{4.19}$$

4.2 Work

The work term is reasonably simple. Beginning with the work term given from the general energy corollary,

$$\nabla \cdot W = \nabla \cdot \{(\mathbf{m} - \mathbf{m}_0)\,[H - H_0 - T_0\,(s - s_0)] + \mathbf{m}_0\,(T - T_0)\,(s - s_0)\} \tag{4.20}$$

The equation for W is extracted,

$$W = (\mathbf{m} - \mathbf{m}_0)\,[H - H_0 - T_0\,(s - s_0)] + \mathbf{m}_0\,(T - T_0)\,(s - s_0) \tag{4.21}$$

Now, the algebraic expansions of the field variables are inserted,

$$W = (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3)\left[H_1 + H_2 + H_3 - T_0\left(s_1 + s_2 + s_3\right)\right] \\ + \mathbf{m}_0\left(T_1 + T_2 + T_3\right)\left(s_1 + s_2 + s_3\right) \quad (4.22)$$

And the third order terms are gathered,

$$W_3 = \mathbf{m}_1 H_2 - \mathbf{m}_1 T_0 s_2 + \mathbf{m}_2 H_1 - \mathbf{m}_2 T_0 s_1 + \mathbf{m}_0 T_1 s_2 + \mathbf{m}_0 T_2 s_1 \quad (4.23)$$

Recalling the relations for total enthalpy,

$$\begin{aligned} H_1 &= h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 \\ H_2 &= h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 \\ H_3 &= h_3 + \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_0 \cdot \mathbf{u}_3 \end{aligned} \quad (4.24)$$

We arrive at,

$$W_3 = \mathbf{m}_1\left(h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2\right) - \mathbf{m}_1 T_0 s_2 \\ + \mathbf{m}_2\left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1\right) - \mathbf{m}_2 T_0 s_1 + \mathbf{m}_0 T_1 s_2 + \mathbf{m}_0 T_2 s_1 \quad (4.25)$$

Finally, collecting terms conveniently, we arrive at the third order work equation.

$$W_3 = \mathbf{m}_0\left(T_1 s_2 + T_2 s_1\right) + \mathbf{m}_1\left(h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 - T_0 s_2\right) \\ + \mathbf{m}_2\left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 - T_0 s_1\right) \quad (4.26)$$

4.3 Sources

Beginning with the general energy corollary source term,

$$D = (\mathbf{m} - \mathbf{m}_0) \cdot [\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 + (s - s_0) \nabla T_0] \\ - (s - s_0) \mathbf{m}_0 \cdot \nabla (T - T_0) + viscous + h.t. \quad (4.27)$$

Expanding the individual terms, remembering that the viscous and heat transfer terms are kept separate,

$$D = (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3) \cdot [\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 + (s_1 + s_2 + s_3) \nabla T_0] \\ - (s_1 + s_2 + s_3) \mathbf{m}_0 \cdot \nabla (T_1 + T_2 + T_3) \quad (4.28)$$

Expanding the vorticity terms separately,

$$\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 = (\omega_0 + \omega_1 + \omega_2) \times (\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2) - \omega_0 \times \mathbf{u}_0 \quad (4.29)$$

$$= \left\{ \begin{array}{c} \omega_0 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_0 + \omega_2 \times \mathbf{u}_0 \\ +\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_1 \\ +\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_2 + \omega_2 \times \mathbf{u}_2 \end{array} \right\} - \omega_0 \times \mathbf{u}_0 \quad (4.30)$$

$$= \left\{ \begin{array}{c} \omega_1 \times \mathbf{u}_0 + \omega_2 \times \mathbf{u}_0 \\ +\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_1 \\ +\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_2 + \omega_2 \times \mathbf{u}_2 \end{array} \right\} \quad (4.31)$$

$$= \left\{ \begin{array}{c} \omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1 \\ +\omega_2 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_1 + \omega_0 \times \mathbf{u}_2 \\ +\omega_2 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_2 \\ +\omega_2 \times \mathbf{u}_2 \end{array} \right\} \quad (4.32)$$

Using these expansions and extracting the third order terms,

$$D_3 = \begin{cases} \mathbf{m}_1 \cdot (\omega_2 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_1 + \omega_0 \times \mathbf{u}_2) + \mathbf{m}_2 \cdot (\omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1) \\ +\mathbf{m}_1 \cdot s_2 \nabla T_0 + \mathbf{m}_2 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_2 - s_2 \mathbf{m}_0 \cdot \nabla T_1 \end{cases} \tag{4.33}$$

Remembering that $A \cdot (A \times B) = 0$ for all vectors B, the vorticity terms are manipulated,

$$\begin{aligned} &\mathbf{m}_1 \cdot (\omega_2 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_1 + \omega_0 \times \mathbf{u}_2) \\ &\qquad = (\rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0) \cdot (\omega_2 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_1 + \omega_0 \times \mathbf{u}_2) \\ &\qquad = \rho_0 \mathbf{u}_1 \cdot \omega_2 \times \mathbf{u}_0 + \rho_0 \mathbf{u}_1 \cdot \omega_1 \times \mathbf{u}_1 + \rho_0 \mathbf{u}_1 \cdot \omega_0 \times \mathbf{u}_2 \\ &\qquad + \rho_1 \mathbf{u}_0 \cdot \omega_2 \times \mathbf{u}_0 + \rho_1 \mathbf{u}_0 \cdot \omega_1 \times \mathbf{u}_1 + \rho_1 \mathbf{u}_0 \cdot \omega_0 \times \mathbf{u}_2 \\ &\qquad = \rho_0 \mathbf{u}_1 \cdot \omega_2 \times \mathbf{u}_0 + \rho_0 \mathbf{u}_1 \cdot \omega_0 \times \mathbf{u}_2 + \rho_1 \mathbf{u}_0 \cdot \omega_1 \times \mathbf{u}_1 + \rho_1 \mathbf{u}_0 \cdot \omega_0 \times \mathbf{u}_2 \\ &\mathbf{m}_2 \cdot (\omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1) = (\rho_0 \mathbf{u}_2 + \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_0) \cdot (\omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1) \\ &\qquad = \rho_0 \mathbf{u}_2 \cdot \omega_1 \times \mathbf{u}_0 + \rho_0 \mathbf{u}_2 \cdot \omega_0 \times \mathbf{u}_1 + \rho_1 \mathbf{u}_1 \cdot \omega_1 \times \mathbf{u}_0 \\ &\qquad + \rho_1 \mathbf{u}_1 \cdot \omega_0 \times \mathbf{u}_1 + \rho_2 \mathbf{u}_0 \cdot \omega_1 \times \mathbf{u}_0 + \rho_2 \mathbf{u}_0 \cdot \omega_0 \times \mathbf{u}_1 \\ &\qquad = \rho_0 \mathbf{u}_2 \cdot \omega_1 \times \mathbf{u}_0 + \rho_0 \mathbf{u}_2 \cdot \omega_0 \times \mathbf{u}_1 + \rho_1 \mathbf{u}_1 \cdot \omega_1 \times \mathbf{u}_0 + \rho_2 \mathbf{u}_0 \cdot \omega_0 \times \mathbf{u}_1 \end{aligned} \tag{4.34}$$

Applying these manipulations,

$$D_3 = \begin{cases} \rho_0 \mathbf{u}_1 \cdot (\omega_2 \times \mathbf{u}_0) + \rho_0 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_2) + \rho_1 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2) \\ +\rho_0 \mathbf{u}_2 \cdot (\omega_1 \times \mathbf{u}_0) + \rho_0 \mathbf{u}_2 \cdot (\omega_0 \times \mathbf{u}_1) + \rho_1 \mathbf{u}_1 \cdot (\omega_1 \times \mathbf{u}_0) + \rho_2 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) \\ +\mathbf{m}_1 \cdot s_2 \nabla T_0 + \mathbf{m}_2 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_2 - s_2 \mathbf{m}_0 \cdot \nabla T_1 \end{cases} \tag{4.35}$$

Separating the terms into orders of ρ

$$D_3 = \begin{cases} \rho_0 \left[\mathbf{u}_1 \cdot (\omega_2 \times \mathbf{u}_0) + \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_2) + \mathbf{u}_2 \cdot (\omega_1 \times \mathbf{u}_0) + \mathbf{u}_2 \cdot (\omega_0 \times \mathbf{u}_1) \right] \\ +\rho_1 \left[\mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) + \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2) + \mathbf{u}_1 \cdot (\omega_1 \times \mathbf{u}_0) \right] \\ +\rho_2 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) + \mathbf{m}_1 \cdot s_2 \nabla T_0 + \mathbf{m}_2 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_2 - s_2 \mathbf{m}_0 \cdot \nabla T_1 \end{cases} \tag{4.36}$$

Applying triple product rules, $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) = -C \cdot (B \times A)$

$$D_3 = \begin{cases} \rho_0 \left[\mathbf{u}_1 \cdot (\omega_2 \times \mathbf{u}_0) + \mathbf{u}_2 \cdot (\omega_1 \times \mathbf{u}_0)\right] + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2) + \rho_2 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) \\ +\mathbf{m}_1 \cdot s_2 \nabla T_0 + \mathbf{m}_2 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_2 - s_2 \mathbf{m}_0 \cdot \nabla T_1 \end{cases} \tag{4.37}$$

4.4 Third Order Viscous and Heat Transfer Terms

The viscous and heat transfer terms are treated separately. After expansion they may be converted into a tensor form and separated into work and source terms. This is unnecessary because both the work and source terms contribute to the change in the oscillatory energy density. Therefore, separating them and then recombining them is fruitless.

$$(\mathbf{m} - \mathbf{m}_0) \cdot (\psi - \psi_0) + (T - T_0)(Q - Q_0) = (\mathbf{m}_1 + \mathbf{m}_2) \cdot (\psi_1 + \psi_2) + (T_1 + T_2)(Q_1 + Q_2) \tag{4.38}$$

Upon expansion, these terms are separated into second order terms, $VHT_2 = \mathbf{m}_1 \cdot \psi_1 + T_1 Q_1$ and third order terms,

$$VHT_3 = \mathbf{m}_1 \cdot \psi_2 + \mathbf{m}_2 \cdot \psi_1 + T_1 Q_2 + T_2 Q_1 \tag{4.39}$$

4.5 Third Order Summary

In summary, for the third order the energy corollary is:

$$\frac{\partial E_3}{\partial t} + \nabla \cdot W_3 = VHT_3 - D_3 \tag{4.40}$$

$$E_3 = \begin{cases} \rho_0 \mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{1}{2}\rho_1 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_2 + \rho_2 \mathbf{u}_0 \cdot \mathbf{u}_1 \\ + \frac{(1-2\gamma)\, {p_1}^3}{6{\rho_0}^2 {a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{\rho_0 T_0 s_2 s_1}{c_p} + \frac{p_1 {s_1}^2}{2 c_p \gamma R} - \frac{\rho_0 T_0 {s_1}^3}{3 {c_p}^2} \end{cases} \tag{4.41}$$

$$W_3 = \mathbf{m}_0 \left(T_1 s_2 + T_2 s_1\right) + \mathbf{m}_1 \left(h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 - T_0 s_2\right) + \mathbf{m}_2 \left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 - T_0 s_1\right) \quad (4.42)$$

$$D_3 = \begin{cases} \rho_0 \left[\mathbf{u}_1 \cdot (\omega_2 \times \mathbf{u}_0) + \mathbf{u}_2 \cdot (\omega_1 \times \mathbf{u}_0)\right] + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2) + \rho_2 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) \\ +\mathbf{m}_1 \cdot s_2 \nabla T_0 + \mathbf{m}_2 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_2 - s_2 \mathbf{m}_0 \cdot \nabla T_1 \end{cases} \quad (4.43)$$

$$VHT_3 = \mathbf{m}_1 \cdot \psi_2 + \mathbf{m}_2 \cdot \psi_1 + T_1 Q_2 + T_2 Q \quad (4.44)$$

These equations can then be added to the second order relations to generate an improved system of equations. In order to use these equations, assumptions on the pressure and velocity field will be employed. This analysis is shown in the following Chapter.

4.6 Second Order Field Variables

Is is to be noted that in the final 3rd order equation there exists field variables of 2nd order. These variables, known as secondary flow phenomena, are notoriously difficult to calculate. Because of that fact it is tempting to neglect them all together. Also, it may be tempting to attempt to reduce these terms to a product of first order terms. This method cannot be implemented due to the fact that the second order thermodynamic relations, which are displayed in Chapter 2.4, show that all the second order variables are functions of two other second order variables. Some fundamental physical problems have been worked to second order, these examples are noteworthy for further investigation of the importance of these terms in the stability analysis. For example, see Morse and Ingard's "Theoretical Acoustics" pages 863 to 874 (9). In the current analysis these terms are neglected. However, their influence on nonlinear behavior is a recommended subject of further work.

Chapter 5

Results of the Energy Growth Model

5.1 Outline

The process of analyzing the energy growth consists of four parts. First, the fluctuating energy, E_2 or E_3, is expanded using Galerkin spectral decomposition (68). Second, a volume integral is performed on the total energy equation, yielding the net system energy balance. Third, the mode shapes and subsequent orthogonality is evaluated. And finally, a time average is performed yielding the change in the pressure amplitudes. This final equation can then be numerically analyzed yielding time dependant oscillation amplitudes.

5.2 Second Order Energy Growth

The process outlined is applied to the second order oscillatory energy growth.

$$\frac{\partial E_2}{\partial t} + \nabla \cdot W_2 = D_2 \tag{5.1}$$

If we assume the flow to be isentropic ($s_1 = 0$) the second order equations reduce down to,

$$E_2 = \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 \tag{5.2a}$$

$$W_2 = \mathbf{m}_1 \left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1\right) + T_1 \left(\frac{q}{T}\right)_1 - m_{1j} \left(\frac{P_{ij}}{\rho}\right)_1 \tag{5.2b}$$

$$\begin{aligned} D_2 = &-\rho_0 \mathbf{u}_0 \cdot (\mathbf{u}_1 \times \omega_1) - \rho_1 \mathbf{u}_1 \cdot (\mathbf{u}_0 \times \omega_0) \\ &- \left(\frac{P_{ij}}{\rho}\right)_1 \frac{\partial m_{1j}}{\partial x_i} + m_{1j} \left(\frac{P_{ij}}{\rho^2}\frac{\partial \rho}{\partial x_i}\right)_1 \\ &+ T_1 \left(\frac{\phi}{T}\right)_1 + \left(\frac{q}{T}\right)_1 \cdot \nabla T_1 - T_1 \left(\frac{q \cdot \nabla T}{T^2}\right)_1 \end{aligned} \tag{5.2c}$$

5.2.1 Volume Integral

A volume integral is performed over the entire volume in order to solve for the total system energy flux. This produces,

$$\int_V \frac{\partial E_2}{\partial t} dV = \int_V D_2 dV - \int_V \nabla \cdot W_2 dV \tag{5.3}$$

The work integral is transformed into a surface integral,

$$\int_V \frac{\partial E_2}{\partial t} dV = \int_V D_2 dV - \int_S \hat{n} \cdot W_2 dS \tag{5.4}$$

Where the left hand side represents the change in the fluctuating energy, and the right hand side represents the volumetric energy production and surface energy flux. The right hand side yields the definition of α, the linear growth rate, or the change in oscillatory energy in time.

5.2.2 Galerkin Spectral Decomposition

The left hand side energy fluctuation is analyzed via a Galerkin expansion (68). This assumes that the oscillatory pressure is a superposition of all harmonics with varying amplitude as motivated by experimental evidence. The amplitudes are scaled by P_0.

$$p_1(\mathbf{r},t) = P_0(t)\sum_{m=1}^{\infty}\eta_m(t)\,\psi_m(\mathbf{r}) \tag{5.5}$$

Beginning with pressure summation the irrotational velocity is derived from the linear acoustics. Velocity is split into irrotational and rotational parts. Vorticity is the curl of the rotational velocity. The derivation below shows the relationship between p_1 and irrotational u_1, or $\hat{\mathbf{u}}_1$

From the linearized momentum equation,

$$\rho_0\frac{\partial\hat{\mathbf{u}}_1}{\partial t} + \nabla p_1 = 0. \tag{5.6}$$

Inserting the pressure summation,

$$\rho_0\frac{\partial\hat{\mathbf{u}}_1}{\partial t} + \nabla\left[P_0(t)\sum_{m=1}^{\infty}\eta_m(t)\,\psi_m(\mathbf{r})\right] = 0. \tag{5.7}$$

Pulling the gradient inside the summation,

$$\frac{\partial\hat{\mathbf{u}}_1}{\partial t} = -\frac{P_0(t)}{\rho_0}\sum_{m=1}^{\infty}\eta_m(t)\,\nabla\psi_m(\mathbf{r}). \tag{5.8}$$

Remembering that,

$$\frac{{a_0}^2}{\gamma} = \frac{\gamma R T_0}{\gamma} = \frac{P_0}{\rho_0} \tag{5.9}$$

$$\frac{\partial\hat{\mathbf{u}}_1}{\partial t} = -\frac{{a_0}^2}{\gamma}\sum_{m=1}^{\infty}\eta_m(t)\nabla\psi_m(\mathbf{r}) = -\sum_{m=1}^{\infty}\frac{{a_0}^2\eta_m(t)}{\gamma}\nabla\psi_m(\mathbf{r}) \tag{5.10}$$

Then, with $k_m = \frac{w_m}{a_0}$

$$\frac{\partial\hat{\mathbf{u}}_1}{\partial t} = -\sum_{m=1}^{\infty}\frac{{w_m}^2\eta_m(t)}{\gamma{k_m}^2}\nabla\psi_m(\mathbf{r}) \tag{5.11}$$

Then since, $\eta_m(t) = R_m(t)\sin(w_m t)$ and R_m is a slow function in time,

$$\dot{\eta}_m(t) = R_m(t)\, w_m \cos(w_m t) \tag{5.12}$$

$$\ddot{\eta}_m(t) = -R_m(t)\, {w_m}^2 \sin(w_m t) = -{w_m}^2 \eta_m(t) \tag{5.13}$$

$$\frac{\partial \hat{\mathbf{u}}_1}{\partial t} = \sum_{m=1}^{\infty} \frac{\ddot{\eta}_m(t)}{\gamma {k_m}^2} \nabla \psi_m(\mathbf{r}) \tag{5.14}$$

$$\frac{\partial \hat{\mathbf{u}}_1}{\partial t} = \sum_{m=1}^{\infty} \frac{\ddot{\eta}_m(t)}{\gamma {k_m}^2} \nabla \psi_m(\mathbf{r}) = \frac{\partial}{\partial t}\left[\sum_{m=1}^{\infty} \frac{\dot{\eta}_m(t)}{\gamma {k_m}^2} \nabla \psi_m(\mathbf{r})\right] \tag{5.15}$$

Thus, solving for the irrotational velocity, $\hat{\mathbf{u}}_1$,

$$\hat{\mathbf{u}}_1(\mathbf{r}, t) = \sum_{m=1}^{\infty} \frac{\dot{\eta}_m(t)}{\gamma {k_m}^2} \nabla \psi_m(\mathbf{r}) \tag{5.16}$$

5.2.3 Algebraic Expansion

The product of the decomposed field variables are needed in the expansion of E_2 and they are shown below,

$$\begin{aligned} {p_1}^2 &= {P_0}^2(t) \sum_{m=1}^{\infty} \eta_m(t)\, \psi_m(\mathbf{r}) \sum_{n=1}^{\infty} \eta_n(t)\, \psi_n(\mathbf{r}) = {P_0}^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \eta_m \eta_n \psi_m \psi_n \\ &= {P_0}^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_m \sin(w_m t)\, R_n \sin(w_n t) \psi_m \psi_n \end{aligned} \tag{5.17}$$

$$
\begin{aligned}
\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_1 &= \left[\sum_{m=1}^{\infty} \frac{\dot{\eta}_m(t)}{\gamma {k_m}^2} \nabla \psi_m(\mathbf{r})\right] \cdot \left[\sum_{n=1}^{\infty} \frac{\dot{\eta}_n(t)}{\gamma {k_n}^2} \nabla \psi_n(\mathbf{r})\right] \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\dot{\eta}_m \dot{\eta}_n}{\gamma^2 {k_m}^2 {k_n}^2} \nabla \psi_m \cdot \nabla \psi_n \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{w_m w_n}{\gamma^2 {k_m}^2 {k_n}^2} \left[R_m \cos(w_m t)\right] \left[R_n \cos(w_n t)\right] \nabla \psi_m \cdot \nabla \psi_n \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{{a_0}^2}{\gamma^2 k_m k_n} \left[R_m \cos(w_m t)\right] \left[R_n \cos(w_n t)\right] \nabla \psi_m \cdot \nabla \psi_n
\end{aligned}
\tag{5.18}
$$

$$
\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_1 = \frac{P_0}{\gamma \rho_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k_m k_n} \left[R_m \cos(w_m t)\right] \left[R_n \cos(w_n t)\right] \nabla \psi_m \cdot \nabla \psi_n \tag{5.19}
$$

5.2.4 E_2 Expanded

If $s_1 = 0$ then, $\rho_1 = p_1 / {a_0}^2$,

$$
E_2 = \frac{{p_1}^2}{2 \rho_0 {a_0}^2} + \frac{p_1}{{a_0}^2} \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2} \rho_0 {u_1}^2 \tag{5.20}
$$

Each term is treated separately, beginning with the pressure squared term,

$$
\frac{{p_1}^2}{2 \rho_0 {a_0}^2} = \frac{{P_0}^2}{2 \rho_0 {a_0}^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} R_m \sin(w_m t)\, R_n \sin(w_n t) \psi_m \psi_n \tag{5.21}
$$

Additionally, the velocity squared term,

$$
\frac{1}{2} \rho_0 {u_1}^2 = \frac{1}{2} \rho_0 \frac{P_0}{\gamma \rho_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k_m k_n} \left[R_m \cos(w_m t)\right] \left[R_n \cos(w_n t)\right] \nabla \psi_m \cdot \nabla \psi_n \tag{5.22}
$$

The remaining term $(p_1/{a_0}^2)\, \mathbf{u}_0 \cdot \mathbf{u}_1$ is left as it is because it will drop out in the time averaging analysis.

5.2.5 Orthogonality

The volume integral can be brought inside the summations since the remaining terms are all time dependant and not spatially dependant. Thus, the first and third terms are evaluated as,

$$\int_V \frac{{p_1}^2}{2\rho_0 {a_0}^2} dV = \frac{{P_0}^2}{2\rho_0 {a_0}^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} R_m \sin(w_m t)\, R_n \sin(w_n t) \int_V \psi_m \psi_n dV \tag{5.23}$$

$$\int_V \frac{1}{2} \rho_0 {u_1}^2 dV = \frac{P_0}{2\gamma} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k_m k_n} \left[R_m \cos(w_m t)\right] \left[R_n \cos(w_n t)\right] \int_V \nabla\psi_m \cdot \nabla\psi_n dV \tag{5.24}$$

The two volume integrals are evaluated knowing that $\psi_m = \cos(k_m z)$ and $k_m = m\pi/L$,

$$\int_V \psi_m \psi_n dV = A \int_0^L \psi_m \psi_n dz = A \int_0^L \cos(k_m z) \cos(k_n z)\, dz = \begin{cases} \dfrac{AL}{2} \text{ for } (m = n) \\ 0 \text{ for } (m \neq n) \end{cases} \tag{5.25}$$

$$\begin{aligned} \int_V \nabla\psi_m \cdot \nabla\psi_n dV &= A \int_0^L \nabla\psi_m \cdot \nabla\psi_n dz \\ &= A \int_0^L k_m k_n \sin(k_m z) \sin(k_n z)\, dV = \begin{cases} k_m^2 \dfrac{AL}{2} \text{ for } (m = n) \\ 0 \text{ for } (m \neq n) \end{cases} \end{aligned} \tag{5.26}$$

Where $A = \pi R^2$, is the cross sectional chamber area and R is the chamber radius.

5.2.6 Time Average

The definition of the time average is,

$$\langle f(t) \rangle = \frac{1}{\tau} \int_0^{\tau} f(t) dt \tag{5.27}$$

Where in the longitudinal case, $\omega_m = m\pi a_0/L$ and $\tau = 2\pi/\omega_1$ thus,

$$\langle f(t)\rangle = \frac{\omega_1}{2\pi}\int_0^{2\pi/\omega_1} f(t)dt \tag{5.28}$$

Applying this definition to the term $(p_1/{a_0}^2)\,\mathbf{u}_0 \cdot \mathbf{u}_1$ yields,

$$\frac{\mathbf{u}_0\omega_1\psi_m\nabla\psi_n}{2\pi a_0^2}\int_0^{2\pi/\omega_1}\frac{d}{dt}\left[\sin\left(\omega_m t\right)\cos\left(\omega_n t\right)\right]dt = 0 \tag{5.29}$$

And therefore the term disappears. The remaining terms are algebraically manipulated. Now that $m = n$, because of orthogonality, and $\frac{{P_0}^2}{2\rho_0 {a_0}^2} = \frac{P_0}{2\gamma}$, the two remaining terms are combined.

$$\begin{aligned}\int_V \frac{d}{dt}\left[\frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{1}{2}\rho_0 {u_1}^2\right]dV &= \frac{P_0}{2\gamma}\sum_{m=1}^{\infty}\frac{d}{dt}\left[R_m^2\left(\sin^2\left(w_m t\right) + \cos^2\left(w_m t\right)\right)\frac{L}{2}\right]\\ &= \frac{LP_0}{4\gamma}\sum_{m=1}^{\infty}\frac{d}{dt}\left[R_m^2\right]\end{aligned} \tag{5.30}$$

The time average is not needed since the term R_m changes slowly in time and is nearly constant over one period. Putting these results back into the total energy equation,

$$\int_V \frac{\partial E_2}{\partial t}dV = \frac{LP_0}{4\gamma}\sum_{m=1}^{\infty}R_m^2 = \int_V D_2 dV - \int_S \hat{n}\cdot W_2 dS \tag{5.31}$$

5.2.7 Linear Alpha

The terms which make up alpha are individually complicated in of themselves, each requiring their own analysis. For demonstration of the linear differential alpha and its relation we expand upon the surface work term, assuming no heat wall heat transfer, isentropic and inviscid flow,

$$W_2 = \mathbf{m}_1\left(\frac{p_1}{\rho_0} + \mathbf{u}_0\cdot\mathbf{u}_1\right) = \mathbf{u}_1 p_1 + \rho_0\mathbf{u}_1(\mathbf{u}_0\cdot\mathbf{u}_1) + \frac{\mathbf{u}_0 p_1^2}{\rho_0 a_0^2} + \rho_1\mathbf{u}_0(\mathbf{u}_0\cdot\mathbf{u}_1) \tag{5.32}$$

Each algebraic group yields an individual mechanism. These terms are evaluated at the surface. For instance the first term, $\mathbf{u}_1 p_1$ is evaluated over the volume. The volume integral of $\nabla \cdot W_2$ reduces down to a surface integral. This surface integral is evaluated over the entire surface. The surface of the rocket is split into three categories: inert surfaces, burning surfaces and nozzle entrance planes. For example, for the first term,

$$\int_S \hat{n} \cdot \mathbf{u}_1 p_1 dS = \int_{Sb} \hat{n} \cdot \mathbf{u}_1 p_1 dS + \int_S N\hat{n} \cdot \mathbf{u}_1 p_1 dS + \int_S i\hat{n} \cdot \mathbf{u}_1 p_1 dS \tag{5.33}$$

Where on inert surfaces, $\hat{n} \cdot \hat{\mathbf{u}}_1 = 0$ and on the burning surface,

$$\hat{n} \cdot \hat{\mathbf{u}}_1 = -a_0 M_b A_b \frac{p'}{\gamma P_0} \tag{5.34}$$

And on the nozzle entrance plane,

$$\hat{n} \cdot \hat{\mathbf{u}}_1 = -a_0 M_N A_N \frac{p'}{\gamma P_0} \tag{5.35}$$

Where A_b and A_N are the burning and nozzle acoustic admittance which are typically found experimentally or computationally. M_b and M_N are the Mach numbers of the flow at the burning surface and at the nozzle entrance plane. Each term in W_2 is evaluated on each portion of the surface each yielding a different α. The summation of these α's is the total system growth rate. Where, $\alpha_T = \alpha_1 + \alpha_2 + ...$ for each mode.
As an example, for the first term on the burning propellant, the solution expanded yields,

$$\int_{Sb} \hat{n} \cdot \mathbf{u}_1 p_1 dS = \int_{Sb} -a_0 M_b A_b \frac{p_1}{\gamma P_0} p_1 dS \tag{5.36}$$

This solution is then expanded upon as before,

$$\begin{aligned} \int_{S_b} -a_0 M_b A_b \frac{p_1}{\gamma P_0} p_1 dS &= -\frac{a_0}{\gamma P_0} \int_{Sb} M_b A_b p_1^2 dS \\ &= -\frac{a_0 P_0}{\gamma} \int_{Sb} M_b A_b R_m R_n \sin(w_m t) \sin(w_n t) \cos(k_m z) \cos(k_n z) dS \end{aligned} \tag{5.37}$$

Once time averaged, $m = n$ and $\langle \sin(w_m t) \sin(w_n t) \rangle = \frac{1}{2}$,

$$-\frac{a_0 P_0}{\gamma} \int_{Sb} M_b A_b R_m R_n \sin(w_m t) \sin(w_n t) \cos(k_m z) \cos(k_n z) dS$$
$$= -\frac{a_0 P_0 R_m^2}{2\gamma} \int_{Sb} M_b A_b \cos^2(k_m z) dS = -\frac{a_0 P_0 R_m^2}{2\gamma} \alpha_{1,m} \quad (5.38)$$

In previous analysis α is defined in the form,

$$\alpha_{past} = \frac{a_0}{2E_m^2} \int_{Sb} M_b A_b \psi_m^2 dS \quad (5.39)$$

Where $E_m^2 = \pi R^2 L/2$ for a cylindrical chamber. Comparing past notation verses the current derivation yields.

$$\alpha_m = \alpha_{past} \frac{AL}{a_0} \quad (5.40)$$

The past notation is preferred in order to compare to previous analysis.

5.2.8 Second Order Wave Growth Equations

The results from the second order equations are summarized in that for an individual mode,

$$\frac{d}{dt} \left[R_m^2 \right] = 2 R_m^2 \alpha_m \quad (5.41)$$

Where all the length and area parameters cancel each other out. By expanding the time derivative and reducing a simple linear differential equation is found.

$$\frac{d}{dt} \left[R_m \right] = R_m \alpha_{T,m} \quad (5.42)$$

This equation describes the growth as a linear differential equation with an exponential solution. The result is that amplitudes will grow to infinity which is clearly not physical. In order to capture the wave steepening process, third order effects must be retained. Figure 5.1 shows the exponential growth of the first mode given a positive α_1 and negative α's of

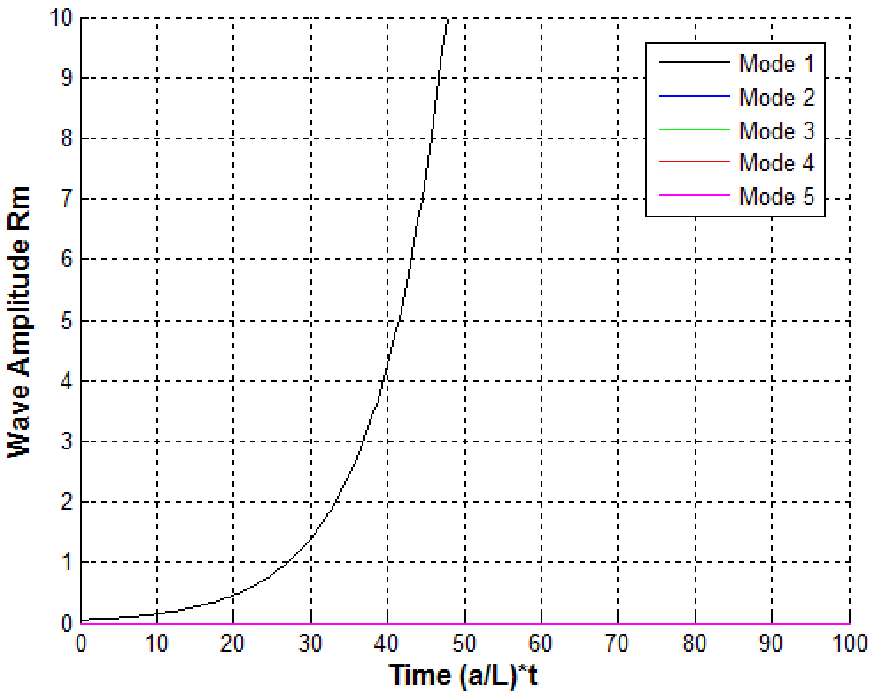


Figure 5.1: Second Order Exponential Growth

higher order. The first mode grows exponentially where the higher modes stay at zero. No energy cascading is seen, and the exponential growth is unphysical.

5.3 Third Order Energy Growth

A similar process is used in the analysis of the third order energy, E_3. The terms W_3 and D_3 lead to third order alpha terms. If we still assume the flow is isentropic and now that all secondary flow effects are small we yield,

$$E_3 = \frac{1}{2}\rho_1 \mathbf{u}_1{}^2 + \frac{(1-2\gamma)\, p_1{}^3}{6\rho_0{}^2 a_0{}^4} \tag{5.43}$$

These terms provide the nonlinear relationship between the modes most importantly seen as wave steepening.

5.3.1 E_3 Expanded

If $s_1 = 0$ then, $\rho_1 = p_1/{a_0}^2$,

$$\frac{dE_3}{dt} = \frac{d}{dt}\left[\frac{p_1}{2a_0^2}{\mathbf{u}_1}^2 + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2{a_0}^4}\right] \tag{5.44}$$

$$\frac{dE_3}{dt} = \frac{1}{2a_0^2}\frac{dp_1}{dt}{\mathbf{u}_1}^2 + \frac{p_1\mathbf{u}_1}{a_0^2}\frac{d\mathbf{u}_1}{dt} + \frac{(1-2\gamma)\,{p_1}^2}{2{\rho_0}^2{a_0}^4}\frac{dp_1}{dt} \tag{5.45}$$

Each term is individually expanded,

$$\begin{aligned}&\frac{1}{2a_0^2}\frac{dp_1}{dt}{\mathbf{u}_1}^2\\ &\quad= \frac{P_0}{2a_0^2}\sum_{l=1}^{\infty} w_l R_l \cos{(w_l t)}\psi_l \frac{P_0}{\gamma\rho_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_m k_n}R_m\cos{(w_m t)}R_n\cos{(w_n t)}\nabla\psi_m\cdot\nabla\psi_n\\ &\qquad= \frac{P_0}{2\gamma^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{w_l}{k_m k_n}R_l\cos{(w_l t)}R_m\cos{(w_m t)}R_n\cos{(w_n t)}\psi_l\nabla\psi_m\cdot\nabla\psi_n\end{aligned} \tag{5.46}$$

$$\begin{aligned}\frac{p_1}{a_0^2}\mathbf{u}_1\frac{d\mathbf{u}_1}{dt} &= -\frac{P_0}{a_0^2}\sum_{l=1}^{\infty}R_l\sin{(w_l t)}\psi_l\sum_{m=1}^{\infty}\frac{R_m w_m\cos{(w_m t)}}{\gamma {k_m}^2}\nabla\psi_m\sum_{n=1}^{\infty}\frac{R_n a_0^2\sin{(w_n t)}}{\gamma}\nabla\psi_n\\ &= -\frac{P_0}{a_0^2\gamma^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}R_l R_m R_n\frac{w_m w_n^2}{{k_m}^2{k_n}^2}\sin{(w_l t)}\cos{(w_m t)}\sin{(w_n t)}\,\psi_l\nabla\psi_m\nabla\psi_n\end{aligned} \tag{5.47}$$

$$\begin{aligned}&\frac{(1-2\gamma)\,{p_1}^2}{2{\rho_0}^2{a_0}^4}\frac{dp_1}{dt}\\ &\qquad= \frac{(1-2\gamma)P_0}{2\gamma^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}w_n R_l R_m R_n\sin{(w_l t)}\sin{(w_m t)}\cos{(w_n t)}\psi_m\psi_n\psi_l\end{aligned} \tag{5.48}$$

5.3.2 Orthogonality

The third order equations are evaluated over the volume just as in the second order. This produces a more complicated solution showing the interconnectedness of the wave amplitudes. Since all other terms are not functions of space, the volume integral can be brought inside the summations and evaluated on the mode shapes, ψ.

$$A\int_0^L \psi_m\psi_n\psi_l dz = A\int_0^L \cos\left(k_m z\right)\cos\left(k_n z\right)\cos\left(k_l z\right)dz$$

$$= \frac{AL}{4}A_{m,n,l} = \frac{AL}{4}\begin{cases} \delta(l-m-n) \\ +\delta(l+m-n) \\ +\delta(l-m+n) \end{cases} \quad (5.49)$$

Where, δ is the Kronecker Delta.

$$A\int_0^L \psi_l\nabla\psi_m\cdot\nabla\psi_n dz = A\int_0^L k_m k_n \sin\left(k_m z\right)\sin\left(k_n z\right)\cos\left(k_l z\right)dz$$

$$= k_m k_n\frac{AL}{4}B_{m,n,l} = k_m k_n\frac{AL}{4}\begin{cases} -\delta(l-m-n) \\ +\delta(l+m-n) \\ +\delta(l-m+n) \end{cases} \quad (5.50)$$

5.3.3 Time Average

Several time averages are needed. The following list will be used in the time averaging the the above three terms and combined with the volume integral of the spatial terms.

$$\langle\cos(w_l t)\cos(w_m t)\cos(w_n t)\rangle = C_{m,n,l} = \begin{cases} \delta(l-m-n) \\ +\delta(l+m-n) \\ +\delta(l-m+n) \end{cases} \quad (5.51)$$

$$\langle\cos(w_n t)\sin(w_l t)\sin(w_m t)\rangle = D_{m,n,l} = \begin{cases} \delta(l-m-n) \\ -\delta(l+m-n) \\ +\delta(l-m+n) \end{cases} \quad (5.52)$$

$$\langle\cos(w_m t)\sin(w_l t)\sin(w_n t)\rangle = E_{m,n,l} = \begin{cases} \delta(l-m-n) \\ +\delta(l+m-n) \\ -\delta(l-m+n) \end{cases} \tag{5.53}$$

$$\langle\cos(w_l t)\sin(w_n t)\sin(w_m t)\rangle = F_{m,n,l} = \begin{cases} -\delta(l-m-n) \\ +\delta(l+m-n) \\ +\delta(l-m+n) \end{cases} \tag{5.54}$$

For completeness it is noted that,

$$\langle\sin(w_l t)\cos(w_m t)\cos(w_n t)\rangle = 0 \tag{5.55}$$

$$\langle\sin(w_l t)\sin(w_m t)\sin(w_n t)\rangle = 0 \tag{5.56}$$

5.3.4 Combined Time Average and Orthogonality

The results from orthogonality and time averaging are joined for each term. The indices used above are not in any way important, that is, m can be interchanged for n and so on, as long as it is done consistently. It is important the the correct time average is used in conjunction with the volume integral. For instance in the third term, w_n came from the differentiation of $\sin(w_n t)$ which yields $w_n\cos(w_n t)$, therefore it is important to maintain the consistent notation between the two terms.

In the first term the wave numbers are eliminated by the volume integral and the remainder is $LB_{mnl}/4$. The time average is given by B_{mnl} as well, since $F_{mnl} = B_{mnl}$. Because of the properties of delta functions, $B_{mnl} * B_{mnl} = B_{mnl}$. Therefore the first term reduces down to,

$$\frac{1}{2a_0^2}\frac{dp_1}{dt}\mathbf{u}_1{}^2 = \frac{AP_0}{2\gamma^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_l R_l R_m R_n \frac{L}{4} B_{mnl} \tag{5.57}$$

The second term has an unusual character, but it reduces down nicely as well. The volume integral yields $k_m k_n \frac{L}{4} B_{m,n,l}$ and the time average yields $E_m nl$

$$\frac{p_1}{a_0^2}\mathbf{u}_1\frac{d\mathbf{u}_1}{dt} = -\frac{AP_0}{\gamma^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_l R_m R_n w_n E_{mnl}\frac{L}{4}F_{mnl} \tag{5.58}$$

The third term reduces simply. The time average yields D_{mnl} and the volume integral is $\frac{L}{4}A_{mnl}$ and then because $A_{mnl} * D_{mnl} = D_{mnl}$,

$$\frac{(1-2\gamma)\,{p_1}^2}{2{\rho_0}^2{a_0}^4}\frac{dp_1}{dt} = \frac{(1-2\gamma)AP_0}{2\gamma^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} w_n R_l R_m R_n D_{mnl}\frac{L}{4} \tag{5.59}$$

Indices Rotation

The indices of each equation are now rotated to allow for easier manipulation. For the first term rotate about n (switch l with m,)

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_l R_l R_m R_n B_{mnl} \rightarrow \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_m R_l R_m R_n E_{mnl} \tag{5.60}$$

In the second equation m is switched with n,

$$-\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_l R_m R_n w_n E_{mnl} F_{mnl} \rightarrow -\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_l R_m R_n w_m D_{mnl} F_{mnl} \tag{5.61}$$

And knowing that $D_{mnl} * F_{mnl} = -E_{mnl}$ we have,

$$-\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_l R_m R_n w_m D_{mnl} F_{mnl} = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_l R_m R_n w_m E_{mnl} \tag{5.62}$$

The third equation is rotated about l as well (switch m and n,)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} w_n R_l R_m R_n D_{mnl} \rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} w_m R_l R_m R_n E_{mnl} \tag{5.63}$$

And now all terms have the same combined delta functions, $w_m E_{mnl}$ and all three equations can be combined.

Summation of Summations

The summation of the three terms yields,

$$\begin{aligned}\frac{dE_3}{dt} &= \frac{AP_0L}{4}\left(\frac{1}{2\gamma^2}+\frac{1}{\gamma^2}+\frac{(1-2\gamma)}{2\gamma^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} w_m R_l R_m R_n E_{mnl} \\ &= \frac{AP_0L}{4}\left(\frac{2-\gamma}{\gamma^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} w_m R_l R_m R_n E_{mnl}\end{aligned} \tag{5.64}$$

5.3.5 Third Order Alpha

The third order alpha is not used at this point. It is easily derived from the third order relations. At the present moment the second order growth rate is still being determined and it is likely that the third order corrects will be small.

5.3.6 Nonlinear Wave Growth Equations

Adding the third order corrections to the second order terms and moving to the right hand side yields,

$$\frac{d}{dt}\left[R_m\right] = R_m \alpha_{T,m} - \left(\frac{2-\gamma}{2\gamma}\right) w_m \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} R_l R_n E_{mnl} \tag{5.65}$$

This equation can be numerically evaluated using fourth order Runge-Kutta. The advantage of this whole method over a CFD simulation is that the complicated interactions between modes and mechanisms is reduced down to a simple code which takes only minutes, instead of days for a CFD simulation, to evaluate while retaining the full physics. The origin of individual effects are preserved for analysis instead of being clumped together in one numerical solution. As we will see, a simple numerical solution is still capable of capturing the important physics of the problem.

5.4 First Mode Driving and Wave Steepening

The first mode is driven with $\alpha_1 = 200$ with negative values for higher modes. These values for α are on the same order as those seen in practice. An initial first mode amplitude of 0.05 is used, all other modes start at zero. 15 modes were used. The results are shown in Figure 5.2.

Just as in the experimental data the first mode amplitude increases to a limit cycle. As the wave amplitude grows energy cascades down to the lower modes. Summing the modes generates the pressure waveform which is shown in Figure 5.3.

The waveform starts out as a sinusoidal wave given by the initial conditions shown in Figure 5.4(a). By time the limit cycle is reached the waveform has steepened into a shock like waveform shown in Figure 5.4(b). The phenomena of wave steepening is captured just as in the physical experiment shown in Figure 1.20(b).

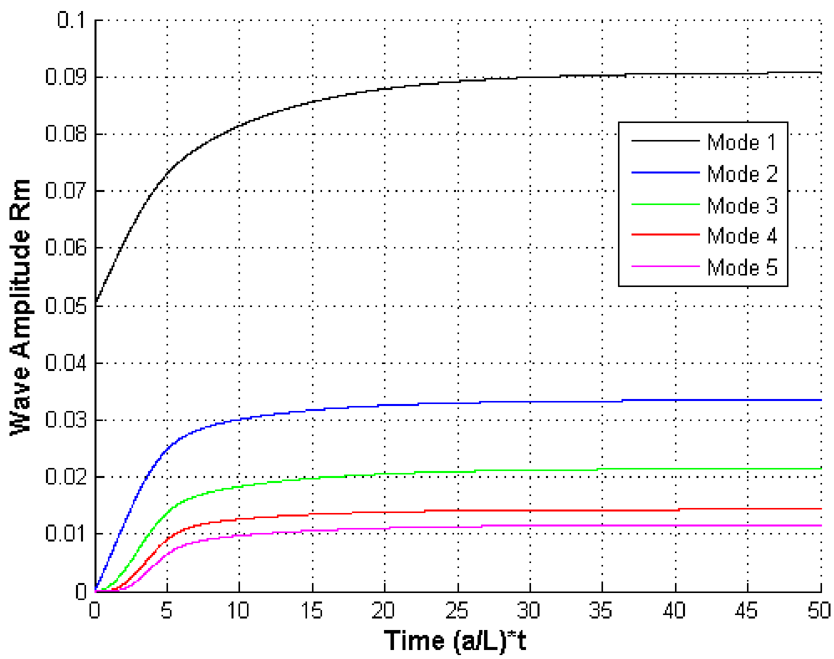


Figure 5.2: First Mode Driving Wave Amplitudes

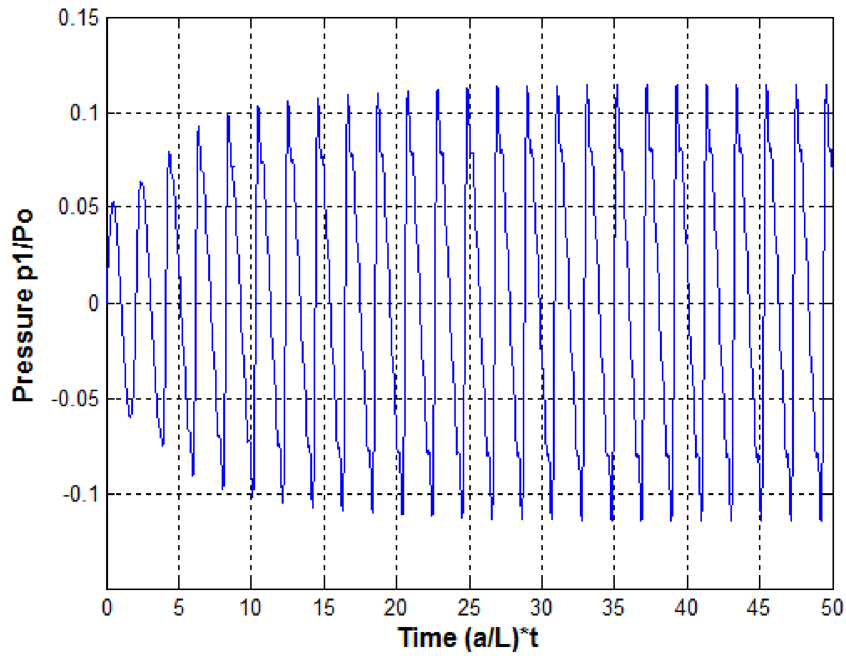
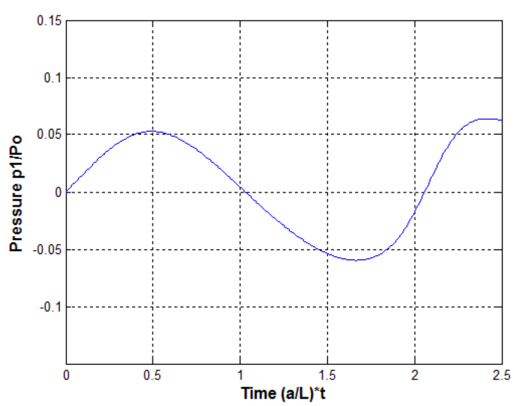
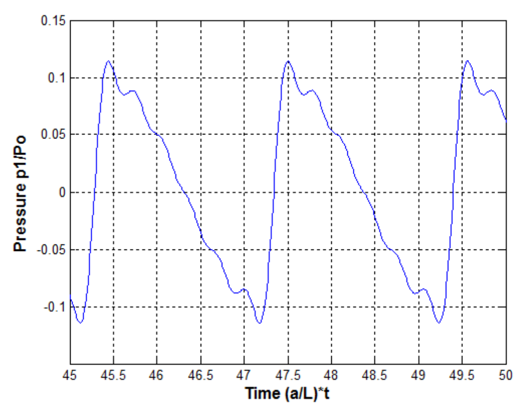


Figure 5.3: First Mode Driving Waveform



(a) First Mode Driving Waveform Beginning



(b) First Mode Driving Waveform Steepened

Figure 5.4: Steepening of a Longitudinal Wave

5.5 Second and Third Mode Driving

When the second mode is driven a similar process occurs as in the first mode driving and model correctly captures the physics as shown in the shock tube experiments as shown in section 1.7.4. The second mode is driven with $\alpha_2 = 200$ and all other modes are given negative values. A initial condition of 0.05 for the second mode is used, all other modes start at zero. 15 modes were used. The results are shown in Figure 5.5

The second mode amplitude increases and converges to a limit cycle and just as in the shock tube experiments, only the second, fourth, sixth, etc. modes are driven just as in Figure 1.23. No energy is drawn back into the first mode. Also, given the same value for alpha the limit cycle is much smaller for the second mode. This is expected as it is harder to drive higher harmonics. The resulting second harmonic steepened waveform is shown in Figure 5.6.

To further show the fact that no energy goes back into the first mode, Figure 5.7 shows an initial first mode amplitude of 0.01 started alongside the initial 0.05 second mode amplitude. The second mode is driven and reaches a limit cycle, and because no energy is given to the first mode it quickly reduces back to zero.

5.6 Initial Condition Sensitivity

The initial conditions are varied, keeping all other things equal. The effect is shown in Figure 5.8. 15 modes were used. The initial conditions have very little effect on the limit cycle. Interestingly there is a small change when the initial amplitude is near to the limit cycle, however this variation is very small and beyond the expectations of error in an engineering solution.

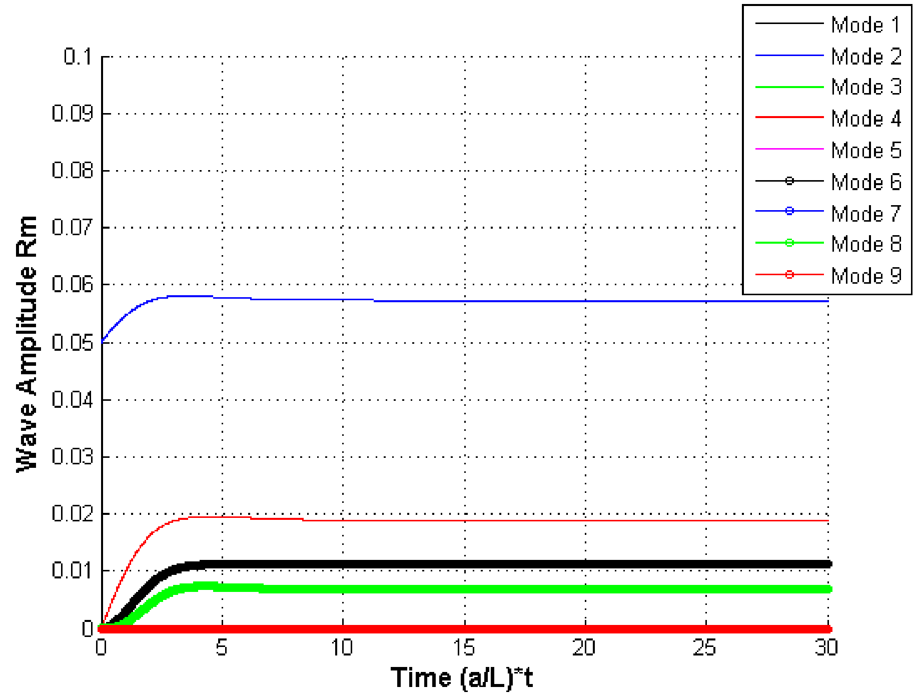


Figure 5.5: Second Mode Driving Wave Amplitudes

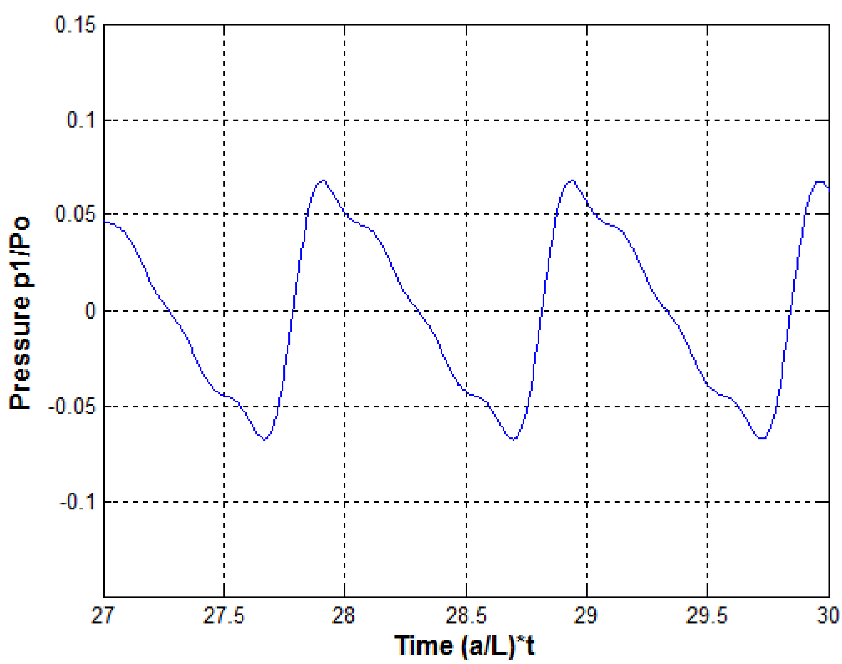


Figure 5.6: Second Mode Driving Steepened Waveform

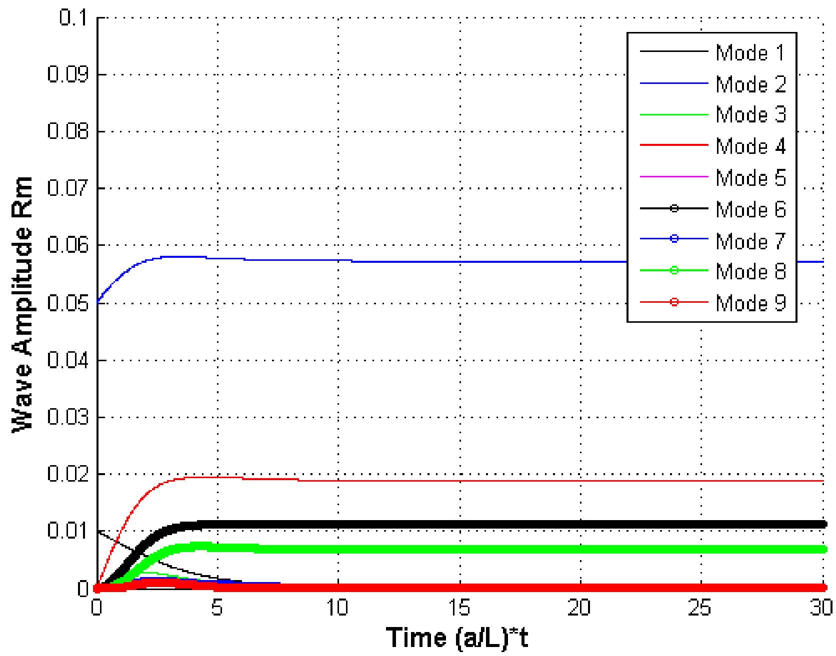


Figure 5.7: Second Mode Driving Wave Amplitudes

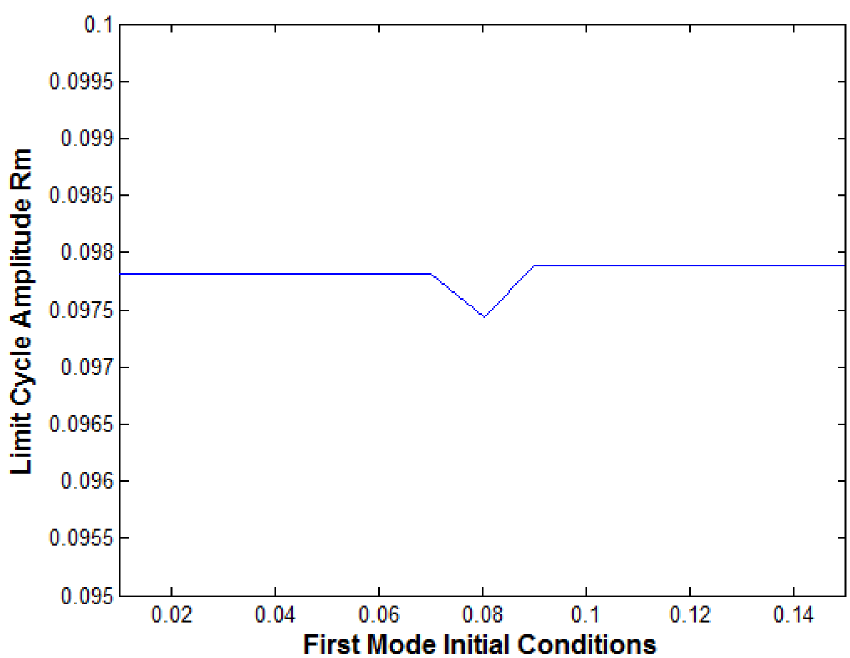


Figure 5.8: Effect of the Initial Conditions on the Limit Cycle Amplitude

5.7 Limit Cycle Alpha Sensitivity

The sensitivity of the limit cycle to alpha is tested. The first mode is driven with a positive value shown in Figure 5.9, all other modes are held at constant negative values, and the resulting limit cycle amplitude is shown.

The limit cycle varies significantly with the value of the driving alpha. This is well known; the harder the mode is driven, the harder it will oscillate. It is interesting to note how sensitive the solution is to alpha. Small changes can make large differences. Because alpha is an intrinsically hard value to calculate and there are multiple alpha's all adding and subtracting, the total error can become quite large as compared to the value of alpha. This in turn can drastically change the value of the limit cycle. Therefore it is important to accurately calculate alpha and know the total error in alpha so as to know the possibly large variation in pressure amplitudes.

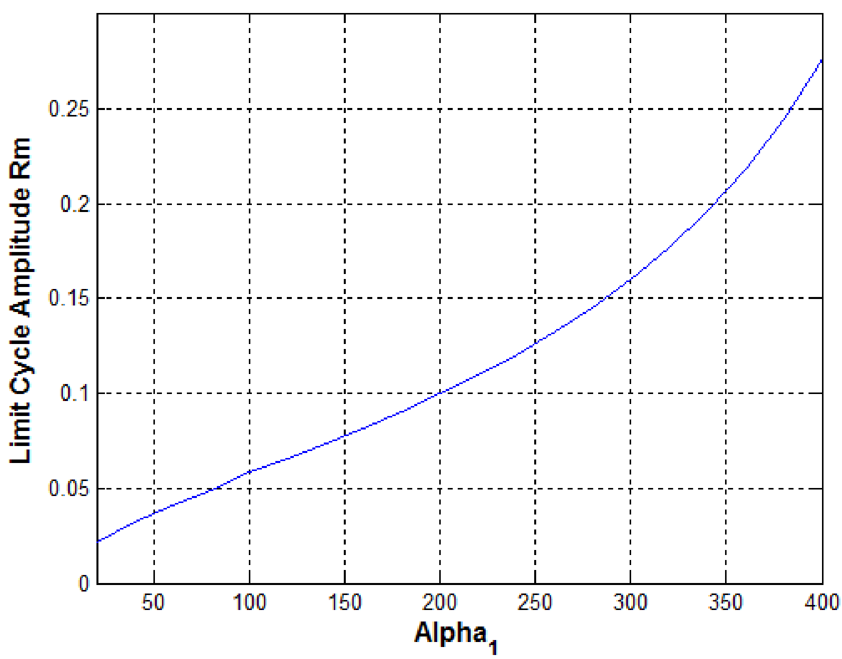


Figure 5.9: Effect of α_1 on the Limit Cycle Amplitude

5.8 Nonlinear Correction Sensitivity

In order to judge the sensitivity of the nonlinear term $(2-\gamma)/(2\gamma)$, it was multiplied by a factor, NF, varying from 0.1 to 2.0. The code was ran for each case capturing the first mode limit cycle amplitude. $\alpha_1 = 200$ and the other terms are set to negative values just as in the first mode driving case. The results are shown in Figure 5.10.

This plot shows the sensitivity of the solution to the nonlinear factor. As the nonlinear factor goes to zero the limit cycle will go to infinity just as the linear model predicts. The limit cycle is approximately inversely proportional to the nonlinear factor, or $A \sim \frac{A_0}{NF}$ where A is the limit cycle, A_0 is the original limit cycle and NF is a factor multiplying the nonlinear correction. The maximum pressure amplitude was also calculated for each case. This value comes from the maximum of the pressure waveform which is found by summing of all the harmonics with their respective amplitudes. The results are shown in Figure 5.11. The conclusion is similar to the first mode limit cycle amplitude. The energy is swept from the first mode to higher modes, and there the energy is dissipated thereby decreasing the amplitude of the pressure oscillations.

5.9 Mode Number Sensitivity

The number of modes used in the numerical simulation is important to the creation of a limit cycle. The value of the first mode limit cycle as compared to the number of modes used is shown in Figure 5.12.

Before five modes the solution is very unstable and the solution does not always converge. After that point the solution stabilizes to a slightly decreasing limit cycle amplitude. As more modes are introduced, slightly more energy is drawn from the system which decreases the limit amplitude of the first mode. The total amplitude, which is the summation of the modes may then go nearly unchanged. The instability when using only a few modes can be minimized by picking appropriate initial conditions and using a small time step. However,

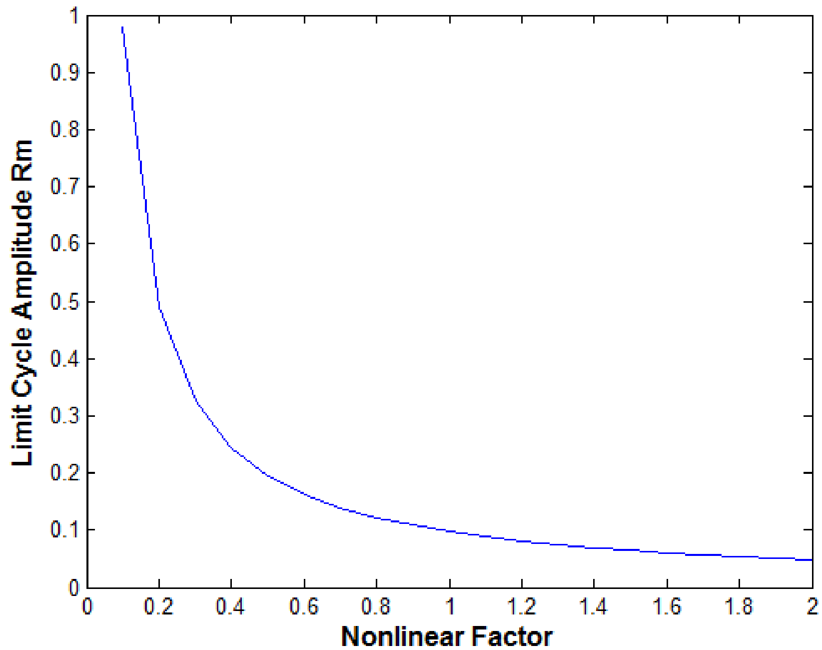


Figure 5.10: Effect of the Nonlinear Factor on the Limit Cycle

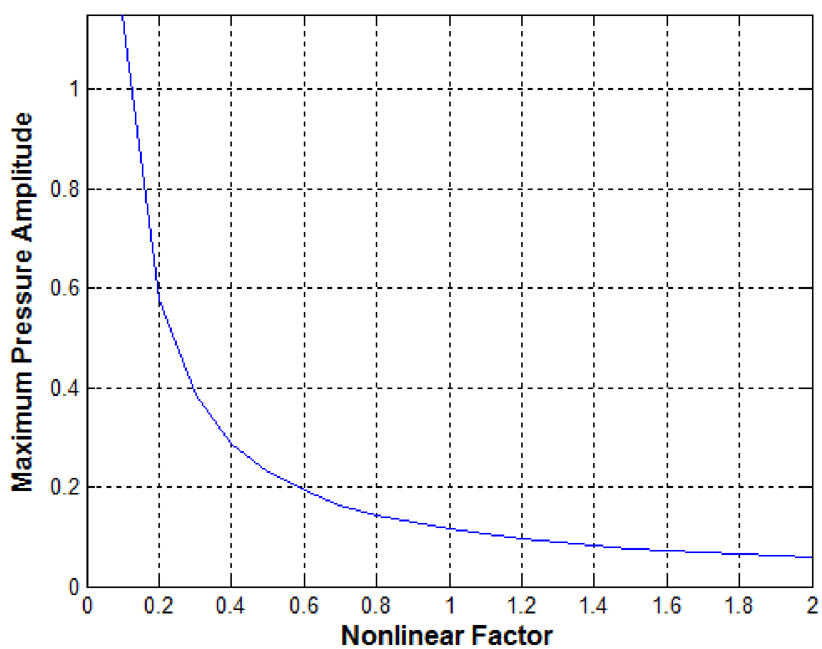


Figure 5.11: Effect of the Nonlinear Factor on the Normalized Maximum Pressure Amplitude

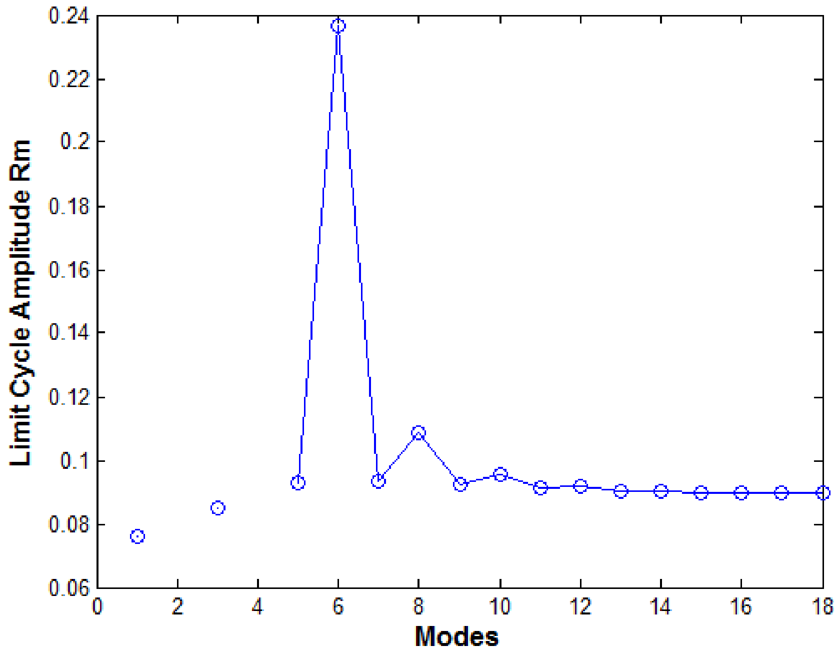


Figure 5.12: Effect of the Number of Modes Used in Simulation

because the solution is very compact, it is simply easier to run with greater than 10 modes. In previous typical motor simulations up to 30 modes have been used.

5.10 Time Dependant Alpha

When a change in alpha is observed the oscillations change to meet the new limit cycle. This process takes some length of time, therefore, if alpha is changing faster than the wave amplitude can change, the system might never be at it's limit cycle. This idea is tested using a time variable alpha. First, if $\alpha_1 = 200\cos(\pi * t/\tau)$ where t is the nondimensional time, $t = time * a_0/L$, and τ is α's oscillatory period. Figure 5.13, 5.14, and 5.15 show the change in the individual modes amplitudes with a changing alpha with a varying period of $\tau = 50, 100, \text{and} 200$ respectively. These figures show the delay in arriving at the limit amplitude. Also, since α is changing from negative to positive values the oscillations are driven to zero and then increase back to a temporary maximum.

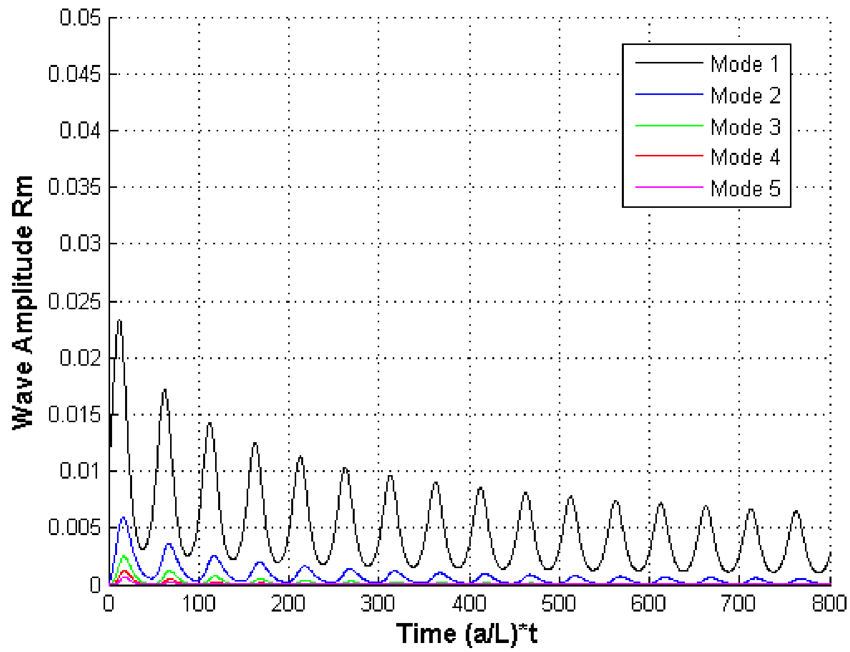


Figure 5.13: Time Dependant Alpha, $\tau = 50$

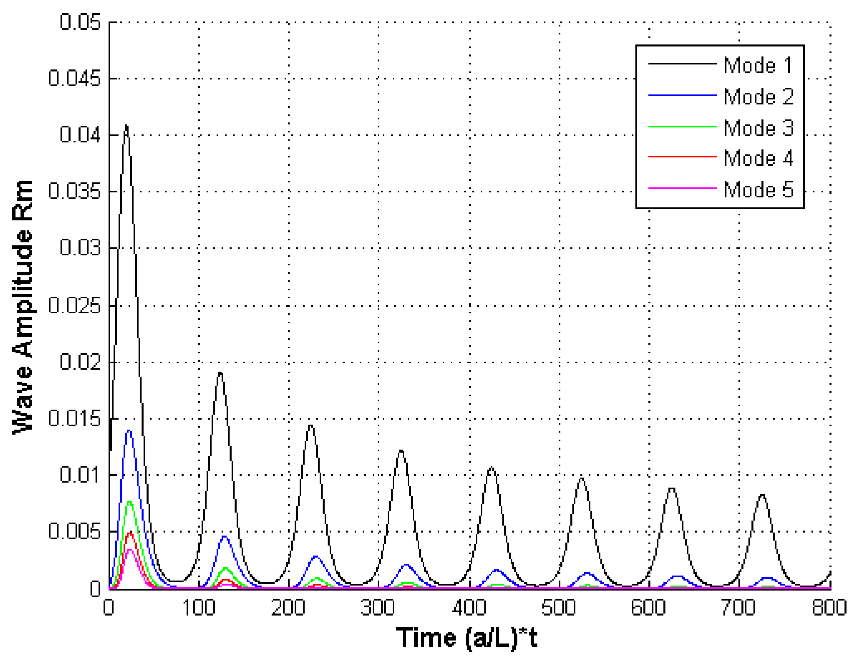


Figure 5.14: Time Dependant Alpha, $\tau = 100$

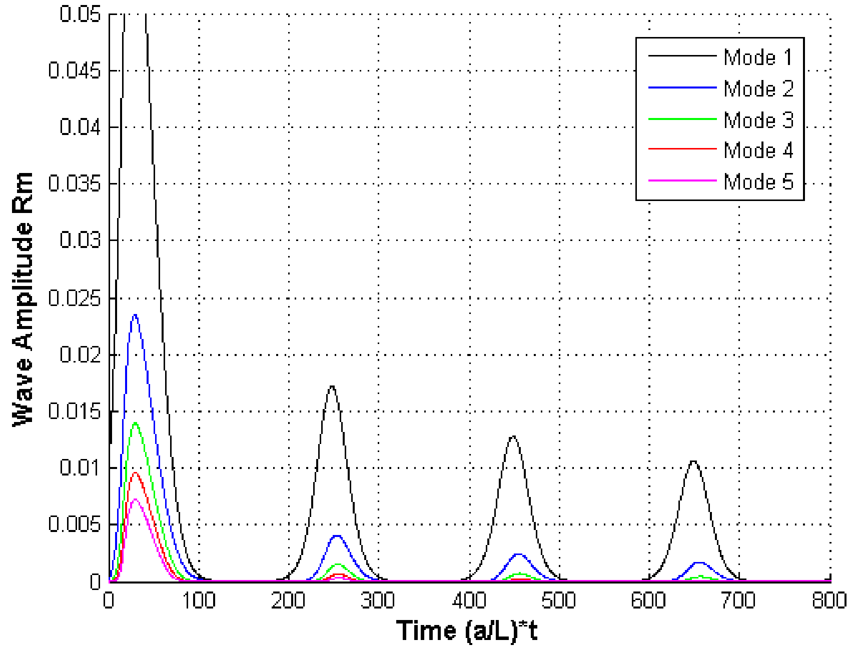


Figure 5.15: Time Dependant Alpha, $\tau = 200$

In Figure 5.16, 5.17, and 5.18 $\alpha = 100 + 100 * \cos(\pi * t/\tau)$ where now $\tau = 200, 50, \text{and} 10$ respectively. In these cases the wave grows to a quasi steady state with a mean first mode amplitude of slightly greater that 0.05. The maximum amplitude is diminished by the rapidly varying alpha, where in the highest frequency case the maximum amplitude is far less than in the lower frequency case. These facts reinforce the premise that the time dependant solution needs to be solved. Also, in the reduction of data, it is important to note that, if these phenomena were lost in the noise, the limit cycle amplitude, and therefore the estimated total alpha, may be underestimated if α is oscillatory. Fortunately, the changing α shown in these numerical tests were changing much more rapidly than seen in tests, so it is not a significant concern.

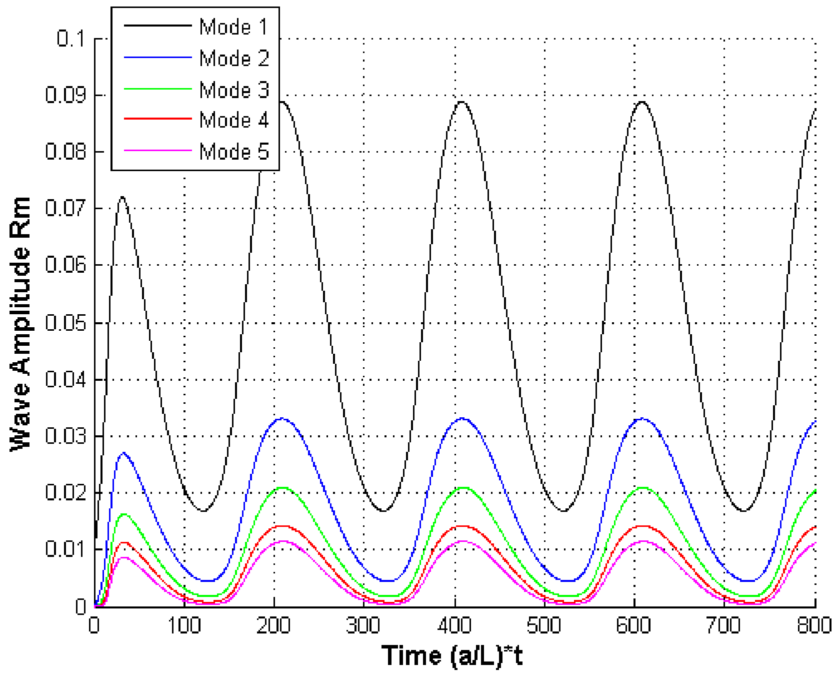


Figure 5.16: Time Dependant Alpha, $\tau = 200$

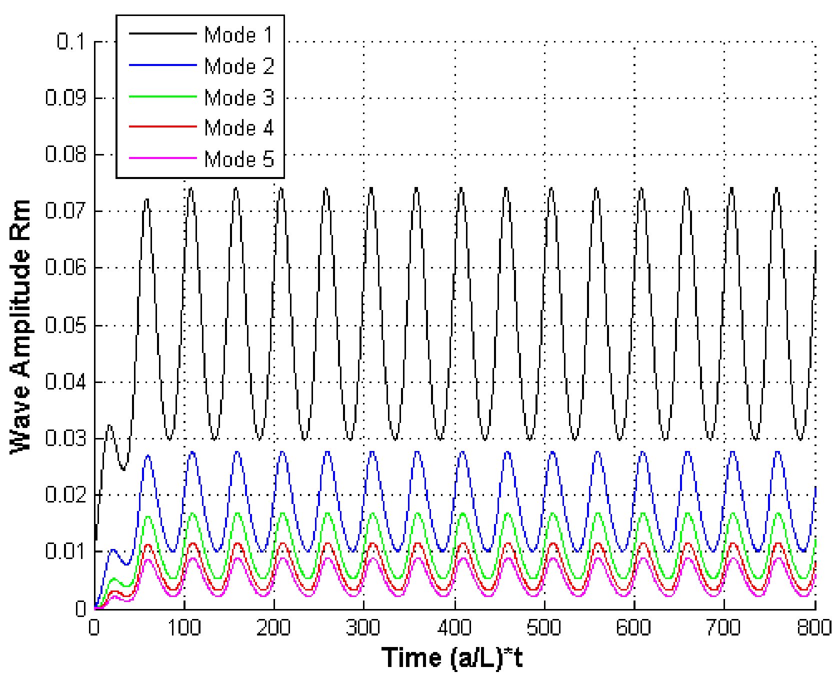


Figure 5.17: Time Dependant Alpha, $\tau = 50$

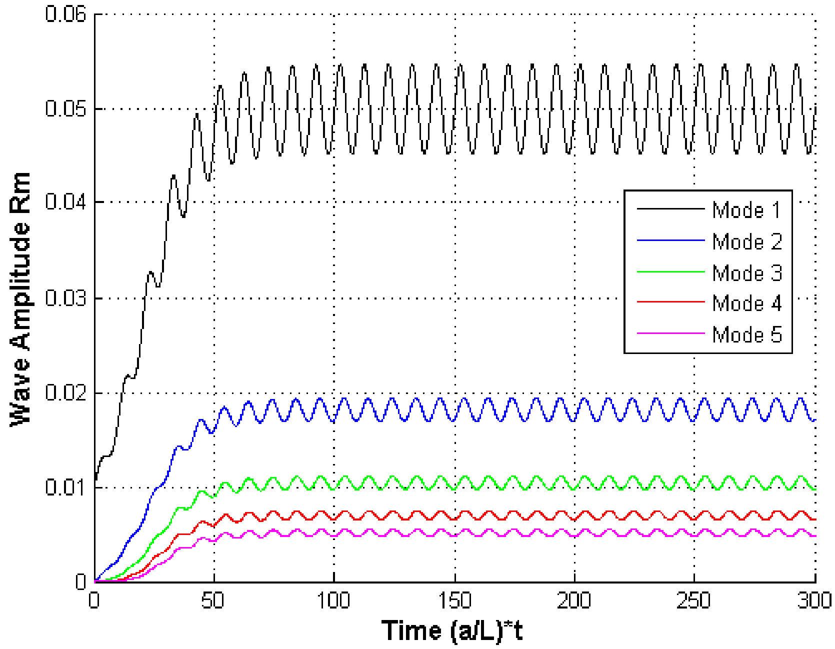


Figure 5.18: Time Dependant Alpha, $\tau = 10$

5.11 Tangential Steepening

A similar process as done with the longitudinal modes can be performed with tangential modes. The difference lies in the volume integral performed on the modes. As a result for tangential modes the matrix $A_m nl$ which previously multiplied the matrix $E_m nl$ is changed to,

$$A_{mnl} = \frac{\int_V J_m(k_m r) J_n(k_n r) J_l(k_l r) \cos(m\theta)\cos(n\theta)\cos(l\theta) r dr d\theta dz}{J_m^2(k_m n)\left(1 - \frac{m^2}{k_m n^2}\right)}(-\delta_1 + \delta_2 + \delta_3) \quad (5.66)$$

Then, using the modified $A_m nl$ matrix we arrive at a similar physical process shown in Figure 5.19. The wave amplitude grows, Figure 5.20, and it is limited as energy transfers to higher modes. Figure 5.21 shows the transition of the waveform from sinusoidal to a steeped tangential wave. This result is compared to Figure 1.28, and the model is shown to accurately model the physics seen in actual motor tests.

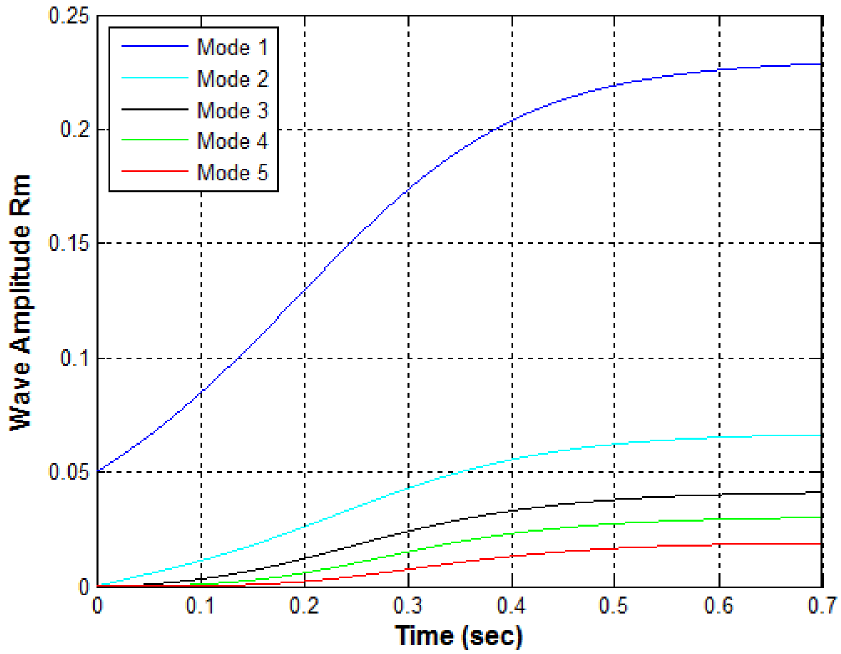


Figure 5.19: Limit Amplitudes with Tangential Modes

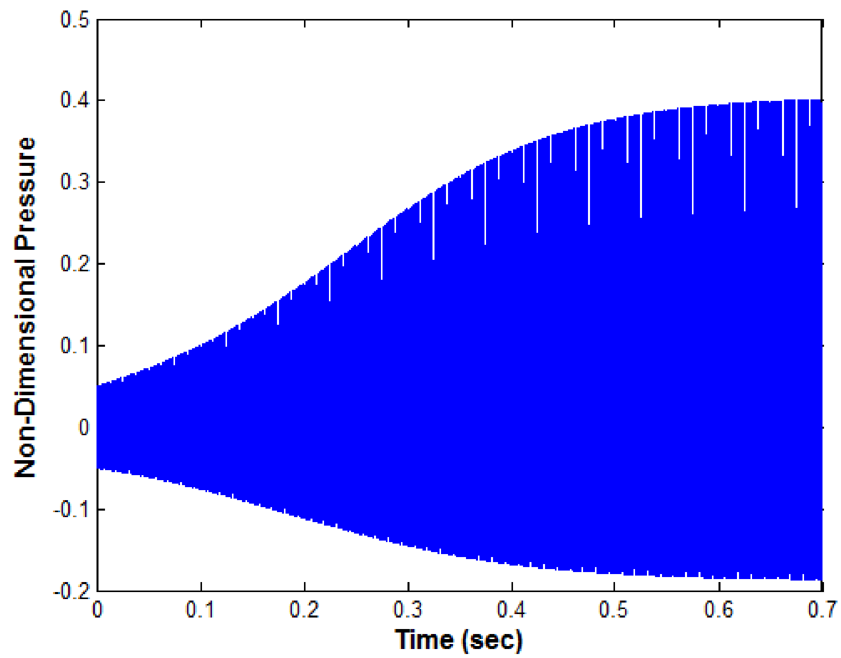
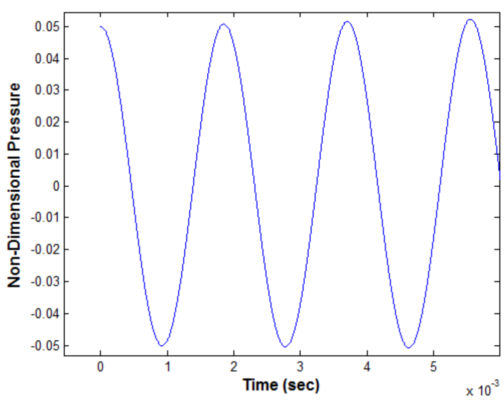
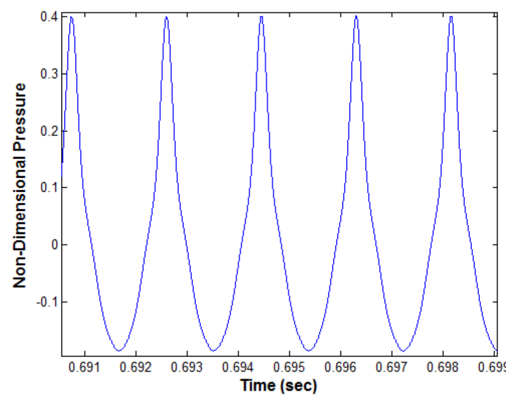


Figure 5.20: Limit Amplitudes with Tangential Modes



(a) Initial Tangential Waveform



(b) Steepened Tangential Waveform

Figure 5.21: Steepening of a Tangential Wave

Chapter 6

Summary

In summary, this dissertation shows the application of third order nonlinear modeling of acoustic energy applied to the problem of combustion instability. Second order results produce unphysical exponential growth, while the inclusion of the third order reproduces nonlinear behavior seen in recently performed experiments. This nonlinear behavior is characterized as the steepening of the waveform as energy cascades from low modes to high modes. The model also accurately predicts the interesting experimental result that when the second mode is driven in an acoustic system the fluctuating energy cascades only to subsequent even harmonics and as in the experiment the waveform still maintains a shock like character.

The solution depends on multiple parameters. Firstly, the number of modes used in the numerical solution is important. A minimum number of modes is needed to correctly model the system. This is generally not a problem because the numerical solutions are arrived at quickly. Running a numerical solution with up to 30 modes does not present a problem. Secondly, the final limit cycle amplitude is not affected by the initial conditions, as one would expect. The system simply grows until it finds an equilibrium and that equilibrium is not dependant on the initial conditions.

The solution is affected greatly by the use of different values of α, the second order growth rate. These values were assumed in the numerical experiments; however, in actual motor simulations the value of α is calculated using complicated experimental and numerical means. Given the error in mechanism's α calculation, the value for the total α has a significant total error. This variation in α leads to a significant error in the limit cycle amplitude. This clearly shows the need for rigorous alpha calculations and error estimations.

The solution is also significantly affected by the third order nonlinear parameter, found to be $(2-\gamma)/(2\gamma)$. This value is larger than previous estimations. A numerical test was performed varying the value of this nonlinear parameter and the limit cycle amplitude is shown to be nearly inversely proportional to its value. This sensitivity shows the need for further analysis on the nonlinear terms. In this case, the solution was assumed to be isentropic and irrotational, the removal of these assumptions may modify the nonlinear parameter to a significant degree.

6.1 Recommendations

In the process of performing this task many topics for future investigation are uncovered. The importance of the third order nonlinear components are demonstrated and shown to capture the physics seen in combustion instability. However, the numerical solution is shown to be very sensitive to this nonlinear parameter as the limit cycle amplitude is inversely proportional to its value. In this analysis the flow field was assumed isentropic and irrotational. Generally, these additions are small, but given the large sensitivity of the solution to the nonlinear parameter, their inclusion in future combustion instability models is likely important.

The application of other methods may be applied to this problem as well. FFT analysis is already performed on experimental data. A more modern approach would be to use wavelet analysis to capture time dependence as well. Additionally, other numerical techniques will be investigated to potentially allow for the removal of time averaging.

In the third order analysis the second order field variables were neglected. They are generally small compared to first order effects and require complex mathematical calculations. A study of their impact on the nonlinear solution is important in order to justify their removal. This task would be a complex, yet important future task.

Finally, given the sensitivity of the solution to the linear growth rate and the desire to accurately predict these oscillations, it is imperative that more work is performed to more accurately calculate the second order growth rate. This future work will take the form of additional analytical, numerical, and most importantly, experimental efforts. Further experiments are currently being performed and more are planned to quantify all individual alpha's to improve analytical and numerical efforts.

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Appendices

Appendix A

The appendix contains details of the derivations not shown in chapters two though four.

A.1 Common Vector Identities:

Several vector identities are useful in the manipulation of vector equations.

$$
\begin{aligned}
\nabla \times (\nabla \mathrm{s}) &= 0 \\
\nabla \cdot (\nabla \times \mathbf{v}) &= 0 \\
\nabla \cdot (\nabla s) &= \nabla^2 s \\
\nabla \times \nabla \times \mathbf{v} &= \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \\
\nabla \cdot (\mathbf{u} + \mathbf{v}) &= \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{v} \\
\nabla \times (\mathbf{u} + \mathbf{v}) &= \nabla \times \mathbf{u} + \nabla \times \mathbf{v} \\
\nabla (\mathbf{u} \cdot \mathbf{v}) &= (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \\
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v} \\
\nabla \cdot (s\mathbf{v}) &= \mathbf{v} \cdot \nabla s + s \nabla \cdot \mathbf{v} \\
\nabla \times (s\mathbf{v}) &= s \nabla \times \mathbf{v} - \mathbf{v} \times \nabla s \\
\nabla (sa) &= s \nabla a + a \nabla s \\
\frac{1}{2} \nabla \left(\mathbf{v}^2\right) &= \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{v}
\end{aligned}
\tag{A.1}
$$

A.2 Acoustic Wave Equation Derivation

Shown below is a derivation of the acoustic wave equation. Beginning with the continuity equation with mass sources,

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = q\left(\mathbf{r}, t\right) \tag{A.2}$$

And the momentum equation with body forces,

$$\frac{\partial\left(\rho\mathbf{u}\right)}{\partial t} + \left(\rho\mathbf{u}\right)\cdot\left(\nabla\mathbf{u}\right) + \mathbf{u}\nabla\cdot\left(\rho\mathbf{u}\right) = -\nabla p - \mu\nabla\times\omega + \left(\eta + \tfrac{4}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + \rho\mathbf{b} + \mathbf{F} \tag{A.3}$$

Linearize equations,

$$\frac{1}{\rho_0}\frac{\partial\rho'}{\partial t} + \nabla\cdot\mathbf{u}' = q\left(\mathbf{r}, t\right) \tag{A.4}$$

$$\rho_0\frac{d\mathbf{u}'}{dt} + \nabla p' = \mathbf{F} \tag{A.5}$$

Applying the isentropic relation, $\rho' = \frac{p'}{a^2}$, to A.4,

$$\frac{1}{\rho_0 a^2}\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u}' = q\left(\mathbf{r}, t\right) \tag{A.6}$$

Take the divergence of A.5

$$\nabla\cdot\left\{\rho_0\frac{d\mathbf{u}'}{dt} + \nabla p' = \mathbf{F}\right\} = \rho_0\frac{d\left(\nabla\cdot\mathbf{u}'\right)}{dt} + \nabla^2 p' = \nabla\cdot\mathbf{F} \tag{A.7}$$

Multiply A.5 by $\rho_0\frac{\partial}{\partial t}$,

$$\rho_0\frac{\partial}{\partial t}\left\{\frac{1}{\rho_0 a^2}\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u}' = q\left(\mathbf{r}, t\right)\right\} = \frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} + \rho_0\frac{\partial\left(\nabla\cdot\mathbf{u}'\right)}{\partial t} = \rho_0 q'\left(\mathbf{r}, t\right) \tag{A.8}$$

Subtracting A.8 from A.7 to arrive that the acoustic wave equation including sources,

$$\nabla^2 p' - \frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} = \nabla\cdot\mathbf{F} - \rho_0 q'\left(\mathbf{r}, t\right) \tag{A.9}$$

Remembering that this equation is valid for small amplitudes and is isentropic.

A.3 Manipulating ρe terms

Shown below are the details used in the manipulation of the thermodynamic expansions in section 2.4.2. With the relation for ρ_1^2, we insert it into the expansion for $(\rho e)_2$.

$$\rho_1{}^2 = \left(\frac{p_1}{a_0{}^2} - \frac{\rho_0}{c_p}s_1\right)^2 = \frac{p_1{}^2}{a_0{}^4} - 2\frac{p_1\rho_0 s_1}{a_0{}^2 c_p} + \frac{\rho_0{}^2}{c_p{}^2}s_1{}^2 \tag{A.10}$$

A.3.1 Second Order ρe

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \left(\frac{p_1{}^2}{2\rho_0 a_0{}^2} - \frac{p_1 s_1}{c_p} + \frac{a_0{}^2\rho_0}{2c_p{}^2}s_1{}^2\right) + \gamma T_0\rho_1 s_1 + \frac{\gamma\rho_0 T_0 s_1{}^2}{2c_p} \tag{A.11}$$

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{p_1{}^2}{2\rho_0 a_0{}^2} - \frac{\left(a_0{}^2\rho_1 + \frac{a_0{}^2\rho_0}{c_p}s_1\right)s_1}{c_p} + \gamma T_0\rho_1 s_1 + \frac{a_0{}^2\rho_0}{2c_p{}^2}s_1{}^2 + \frac{\gamma\rho_0 T_0 s_1{}^2}{2c_p} \tag{A.12}$$

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{p_1{}^2}{2\rho_0 a_0{}^2} - \frac{a_0{}^2\rho_1 s_1}{c_p} - \frac{a_0{}^2\rho_0}{c_p{}^2}s_1{}^2 + \gamma T_0\rho_1 s_1 + \frac{a_0{}^2\rho_0}{2c_p{}^2}s_1{}^2 + \frac{\gamma\rho_0 T_0 s_1{}^2}{2c_p} \tag{A.13}$$

Reducing the terms including $\rho_1 s_1$,

$$
\begin{aligned}
-\frac{{a_0}^2\rho_1 s_1}{c_p} + \gamma T_0\rho_1 s_1 &= \rho_1 s_1\left(-\frac{{a_0}^2}{c_p} + \gamma T_0\right)\\
&= \rho_1 s_1\left(-\frac{\gamma R T_0}{c_p} + \gamma T_0\right)\\
&= \rho_1 s_1 T_0\left(-\left(\gamma - 1\right) + \gamma\right)\\
&= \rho_1 s_1 T_0
\end{aligned}
\tag{A.14}
$$

Reducing the terms including s_2,

$$
\begin{aligned}
\frac{{a_0}^2\rho_0}{2{c_p}^2}{s_1}^2 + \frac{\gamma\rho_0 T_0 {s_1}^2}{2c_p} - \frac{{a_0}^2\rho_0}{{c_p}^2}{s_1}^2 &= \rho_0 T_0 {s_1}^2\left(\frac{\gamma R}{2{c_p}^2} + \frac{\gamma}{2c_p} - \frac{\gamma R}{{c_p}^2}\right)\\
&= \frac{\rho_0 T_0 {s_1}^2}{2c_p}\left(\gamma - \frac{\gamma R}{c_p}\right)\\
&= \frac{\rho_0 T_0 {s_1}^2}{2c_p}
\end{aligned}
\tag{A.15}
$$

Insert these back into the previous equation and we end up with the form shown below which is the same as Myers (42). This puts the terms back into a more classical form which can be compared to traditional acoustic analysis.

$$
\left(\rho e\right)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \tag{A.16}
$$

A.3.2 Third Order

To start, the ρ_1^2 and the ρ_1^3 cube terms are manipulated. First, some algebra,

$$
{\rho_1}^2 = \left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right)^2 = \frac{{p_1}^2}{{a_0}^4} - 2\frac{p_1\rho_0 s_1}{{a_0}^2 c_p} + \frac{{\rho_0}^2}{{c_p}^2}{s_1}^2 \tag{A.17}
$$

$$
\begin{aligned}
{\rho_1}^3 &= \left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right)^3 \\
&= \left(\frac{p_1}{{a_0}^2}\right)^3 - 3\left(\frac{p_1}{{a_0}^2}\right)^2\frac{\rho_0}{c_p}s_1 + 3\left(\frac{\rho_0}{c_p}s_1\right)^2\frac{p_1}{{a_0}^2} - \left(\frac{\rho_0}{c_p}s_1\right)^3 \\
&= \frac{{p_1}^3}{{a_0}^6} - 3\frac{\rho_0{p_1}^2s_1}{{a_0}^4c_p} + 3\frac{{\rho_0}^2p_1{s_1}^2}{{a_0}^2{c_p}^2} - \frac{{\rho_0}^3{s_1}^3}{{c_p}^3}
\end{aligned} \tag{A.18}
$$

Recalling 2.36c,

$$
\rho_2 = \frac{p_2}{{a_0}^2} - \frac{\rho_0 s_2}{c_p} - \frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0} - \frac{p_1s_1}{c_p{a_0}^2} + \frac{\rho_0{s_1}^2}{2{c_p}^2} \tag{A.19}
$$

Now, taking Equations A.17, A.17, 2.36b, and 2.36c into Eqn. 2.36d,

$$
\begin{aligned}
(\rho e)_3 = {} & h_0\rho_3 + \rho_0T_0s_3 \\
& + \frac{{a_0}^2}{\rho_0}\left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right)\left(\frac{p_2}{{a_0}^2} - \frac{\rho_0s_2}{c_p} - \frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0} - \frac{p_1s_1}{c_p{a_0}^2} + \frac{\rho_0{s_1}^2}{2{c_p}^2}\right) \\
& + \gamma T_0\left[\begin{array}{c} \left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right)s_2 \\ + \left(\frac{p_2}{{a_0}^2} - \frac{\rho_0s_2}{c_p} - \frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0} - \frac{p_1s_1}{c_p{a_0}^2} + \frac{\rho_0{s_1}^2}{2{c_p}^2}\right)s_1 \end{array}\right] \\
& + \frac{\rho_0\gamma T_0}{c_p}s_1s_2 + \frac{(\gamma-2)}{6{\rho_0}^2}{a_0}^2\left(\frac{{p_1}^3}{{a_0}^6} - 3\frac{\rho_0{p_1}^2s_1}{{a_0}^4c_p} + 3\frac{{\rho_0}^2p_1{s_1}^2}{{a_0}^2{c_p}^2} - \frac{{\rho_0}^3{s_1}^3}{{c_p}^3}\right) \\
& + \frac{\gamma{a_0}^2}{2c_p\rho_0}\left(\frac{{p_1}^2}{{a_0}^4} - 2\frac{p_1\rho_0s_1}{{a_0}^2c_p} + \frac{{\rho_0}^2}{{c_p}^2}{s_1}^2\right)s_1 \\
& + \frac{\gamma^2T_0}{2c_p}\left(\frac{p_1}{{a_0}^2} - \frac{\rho_0}{c_p}s_1\right){s_1}^2 + \frac{\gamma^2\rho_0T_0{s_1}^3}{6{c_p}^2}
\end{aligned} \tag{A.20}
$$

Several terms are unaffected by this insertion,

$$
h_0\rho_3 + \rho_0T_0s_3 + \frac{\rho_0\gamma T_0}{c_p}s_1s_2 + \frac{\gamma^2\rho_0T_0{s_1}^3}{6{c_p}^2} \tag{A.21}
$$

Expanding the affected terms,

$$
\begin{aligned}
&\frac{{a_0}^2}{\rho_0}\left\{\begin{matrix}
\frac{p_1}{{a_0}^2}\frac{p_2}{{a_0}^2}-\frac{p_1}{{a_0}^2}\frac{\rho_0 s_2}{c_p}-\frac{p_1}{{a_0}^2}\frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0}\\
-\frac{p_1}{{a_0}^2}\frac{p_1 s_1}{c_p {a_0}^2}+\frac{p_1}{{a_0}^2}\frac{\rho_0 {s_1}^2}{2{c_p}^2}-\frac{\rho_0}{c_p}s_1\frac{p_2}{{a_0}^2}+\frac{\rho_0}{c_p}s_1\frac{\rho_0 s_2}{c_p}\\
+\frac{\rho_0}{c_p}s_1\frac{(\gamma-1)\,{p_1}^2}{2{a_0}^4\rho_0}+\frac{\rho_0}{c_p}s_1\frac{p_1 s_1}{c_p {a_0}^2}-\frac{\rho_0}{c_p}s_1\frac{\rho_0 {s_1}^2}{2{c_p}^2}
\end{matrix}\right\}\\
&+\gamma T_0\left\{\begin{matrix}
\frac{p_1 s_2}{{a_0}^2}-\frac{\rho_0}{c_p}s_1 s_2+\frac{p_2 s_1}{{a_0}^2}-\frac{\rho_0 s_2 s_1}{c_p}\\
-\frac{(\gamma-1)\,{p_1}^2 s_1}{2{a_0}^4\rho_0}-\frac{p_1 {s_1}^2}{c_p {a_0}^2}+\frac{\rho_0 {s_1}^3}{2{c_p}^2}
\end{matrix}\right\}\\
&+\frac{(\gamma-2)\,{a_0}^2}{6{\rho_0}^2}\left(\frac{{p_1}^3}{{a_0}^6}-3\frac{\rho_0 {p_1}^2 s_1}{{a_0}^4 c_p}+3\frac{{\rho_0}^2 p_1 {s_1}^2}{{a_0}^2{c_p}^2}-\frac{{\rho_0}^3{s_1}^3}{{c_p}^3}\right)\\
&+\frac{\gamma {a_0}^2}{2c_p\rho_0}\left(\frac{{p_1}^2 s_1}{{a_0}^4}-2\frac{p_1\rho_0 {s_1}^2}{{a_0}^2 c_p}+\frac{{\rho_0}^2}{{c_p}^2}{s_1}^3\right)\\
&+\frac{\gamma^2 T_0}{2c_p}\left(\frac{p_1 {s_1}^2}{{a_0}^2}-\frac{\rho_0}{c_p}{s_1}^3\right)
\end{aligned}
\tag{A.22}
$$

$$
\begin{aligned}
(\rho e)_3 &= h_0\rho_3+\rho_0 T_0 s_3+\frac{\rho_0\gamma T_0}{c_p}s_1 s_2+\frac{\gamma^2\rho_0 T_0 {s_1}^3}{6{c_p}^2}\\
&+\left\{\begin{matrix}
\frac{p_1 p_2}{\rho_0 {a_0}^2}-\frac{p_1 s_2}{c_p}-\frac{(\gamma-1)\,{p_1}^3}{2{a_0}^4{\rho_0}^2}-\frac{{p_1}^2 s_1}{\rho_0 c_p {a_0}^2}+\frac{p_1 {s_1}^2}{2{c_p}^2}\\
-\frac{p_2 s_1}{c_p}+\frac{\rho_0 {a_0}^2 s_1 s_2}{{c_p}^2}+\frac{(\gamma-1)\,{p_1}^2 s_1}{2c_p\rho_0 {a_0}^2}+\frac{p_1 {s_1}^2}{{c_p}^2}-\frac{{a_0}^2\rho_0 {s_1}^3}{2{c_p}^3}
\end{matrix}\right\}\\
&+\left\{\begin{matrix}
\gamma T_0\frac{p_1 s_2}{{a_0}^2}-\gamma T_0\frac{\rho_0}{c_p}s_1 s_2+\gamma T_0\frac{p_2 s_1}{{a_0}^2}-\gamma T_0\frac{\rho_0 s_2 s_1}{c_p}\\
-\gamma T_0\frac{(\gamma-1)\,{p_1}^2 s_1}{2{a_0}^4\rho_0}-\gamma T_0\frac{p_1 {s_1}^2}{c_p {a_0}^2}+\gamma T_0\frac{\rho_0 {s_1}^3}{2{c_p}^2}
\end{matrix}\right\}\\
&+\left\{\frac{(\gamma-2)}{6{\rho_0}^2}\frac{{p_1}^3}{{a_0}^4}-\frac{(\gamma-2)}{2\rho_0}\frac{{p_1}^2 s_1}{{a_0}^2 c_p}+\frac{(\gamma-2)}{2}\frac{p_1 {s_1}^2}{{c_p}^2}-\frac{(\gamma-2)\,{a_0}^2}{6}\frac{\rho_0 {s_1}^3}{{c_p}^3}\right\}\\
&+\left\{\frac{\gamma}{2c_p\rho_0}\frac{{p_1}^2 s_1}{{a_0}^2}-\frac{\gamma p_1 {s_1}^2}{{c_p}^2}+\frac{\gamma {a_0}^2}{2}\frac{\rho_0}{{c_p}^3}{s_1}^3\right\}\\
&+\left\{\frac{\gamma^2 T_0}{2c_p}\frac{p_1 {s_1}^2}{{a_0}^2}-\frac{\gamma^2 T_0}{2}\frac{\rho_0}{{c_p}^2}{s_1}^3\right\}
\end{aligned}
\tag{A.23}
$$

Now gathering the similar terms, preparing to simplify them individually,

$$
\begin{aligned}
(\rho e)_3 = {} & h_0\rho_3 + \rho_0 T_0 s_3 \\
& + \left\{ -\frac{(\gamma-1)\,{p_1}^3}{2{a_0}^4{\rho_0}^2} + \frac{(\gamma-2)}{6{\rho_0}^2}\frac{{p_1}^3}{{a_0}^4} \right\} \\
& + \left\{ \frac{p_1 p_2}{\rho_0 {a_0}^2} \right\} + \left\{ \gamma T_0 \frac{p_2 s_1}{{a_0}^2} - \frac{p_2 s_1}{c_p} \right\} + \left\{ -\frac{p_1 s_2}{c_p} + \gamma T_0 \frac{p_1 s_2}{{a_0}^2} \right\} \\
& + \left\{ \begin{array}{c} -\dfrac{{p_1}^2 s_1}{\rho_0 c_p {a_0}^2} + \dfrac{(\gamma-1)\,{p_1}^2 s_1}{2c_p\rho_0{a_0}^2} - \gamma T_0 \dfrac{(\gamma-1)\,{p_1}^2 s_1}{2{a_0}^4\rho_0} \\ -\dfrac{(\gamma-2)}{2\rho_0}\dfrac{{p_1}^2 s_1}{{a_0}^2 c_p} + \dfrac{\gamma}{2c_p\rho_0}\dfrac{{p_1}^2 s_1}{{a_0}^2} \end{array} \right\} \\
& + \left\{ \begin{array}{c} \dfrac{p_1 {s_1}^2}{2{c_p}^2} + \dfrac{p_1 {s_1}^2}{{c_p}^2} - \gamma T_0 \dfrac{p_1 {s_1}^2}{c_p {a_0}^2} \\ +\dfrac{(\gamma-2)}{2}\dfrac{p_1 {s_1}^2}{{c_p}^2} - \dfrac{\gamma p_1 {s_1}^2}{{c_p}^2} + \dfrac{\gamma^2 T_0}{2c_p}\dfrac{p_1 {s_1}^2}{{a_0}^2} \end{array} \right\} \\
& + \left\{ \frac{\rho_0\gamma T_0}{c_p} s_1 s_2 + \frac{\rho_0 {a_0}^2 s_1 s_2}{{c_p}^2} - \gamma T_0 \frac{\rho_0 s_2 s_1}{c_p} - \gamma T_0 \frac{\rho_0}{c_p} s_1 s_2 \right\} \\
& + \left\{ \begin{array}{c} \dfrac{\gamma^2 \rho_0 T_0 {s_1}^3}{6{c_p}^2} - \dfrac{{a_0}^2 \rho_0 {s_1}^3}{2{c_p}^3} + \gamma T_0 \dfrac{\rho_0 {s_1}^3}{2{c_p}^2} \\ -\dfrac{(\gamma-2)\,{a_0}^2}{6}\dfrac{\rho_0 {s_1}^3}{{c_p}^3} + \dfrac{\gamma {a_0}^2}{2}\dfrac{\rho_0}{{c_p}^3}{s_1}^3 - \dfrac{\gamma^2 T_0}{2}\dfrac{\rho_0}{{c_p}^2}{s_1}^3 \end{array} \right\}
\end{aligned} \tag{A.24}
$$

Reducing the terms remembering that $\gamma = C_p/C_v$ and $R = C_p - C_v$,

$$
\begin{aligned}
(\rho e)_3 = {} & h_0\rho_3 + \rho_0 T_0 s_3 \\
& + \frac{(1-2\gamma)\,{p_1}^3}{6{\rho_0}^2{a_0}^4} + \frac{p_1 p_2}{\rho_0 {a_0}^2} + \frac{p_2 s_1}{\gamma R} + \frac{p_1 s_2}{\gamma R} - \frac{{p_1}^2 s_1}{2c_p\rho_0{a_0}^2} \\
& - \frac{p_1 {s_1}^2}{2c_p\gamma R} - \frac{\rho_0 T_0 s_1 s_2}{c_p} + \frac{T_0\rho_0 {s_1}^3}{6{c_p}^2}
\end{aligned} \tag{A.25}
$$

This is a simplified version of ρe_3. The final form is flexible since at any point one of the many first, second or third order relations could be reinserted which would change the look of the equation. This form is preferred over others since it opts to use p_1 and p_2 terms instead of ρ_1 and ρ_2, because in general the pressure fluctuations are the only fluctuations measured in a real system.

A.4 First Order Energy

Shown below are details not shown in Section 3.2. First order energy equation,

$$\frac{\partial}{\partial t}(\rho H - p)_1 + \nabla \cdot (\mathbf{m}_0 H_1 + \mathbf{m}_1 H_0) - \mathbf{m}_0 \cdot \psi_1 - \mathbf{m}_1 \cdot \psi_0 - T_0 Q_1 - T_1 Q_0 = 0 \tag{A.26}$$

To start, the expansion of the first term is needed. Inserting the definition of the total enthalpy and expanding,

$$\begin{aligned}
(\rho H - p)_1 &= \left(\rho\left(h + \frac{1}{2}\mathbf{u}^2\right) - p\right)_1 \\
&= \left(\rho\left(e + \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2\right) - p\right)_1 \\
&= \left(\rho e + \frac{1}{2}\rho\mathbf{u}^2\right)_1 \\
&= (\rho e)_1 + \frac{1}{2}\rho_1 {u_0}^2 + \rho_0 \mathbf{u}_0 \cdot \mathbf{u}_1 \\
&= h_0 \rho_1 + \rho_0 T_0 s_1 + \frac{1}{2}\rho_1 {u_0}^2 + \rho_0 \mathbf{u}_0 \cdot \mathbf{u}_1 \\
(\rho H - p)_1 &= \rho_1 H_0 + \rho_0 T_0 s_1 + \rho_0 \mathbf{u}_0 \cdot \mathbf{u}_1
\end{aligned} \tag{A.27}$$

The results of Equation (A.27) are placed back into the first order energy equation.

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_1 H_0 + \rho_0 T_0 s_1 + \rho_0 \mathbf{u}_0 \cdot \mathbf{u}_1\right) + \nabla \cdot (\mathbf{m}_0 H_1 + \mathbf{m}_1 H_0) \\
&\qquad - \mathbf{m}_0 \cdot \psi_1 - \mathbf{m}_1 \cdot \psi_0 - T_0 Q_1 - T_1 Q_0 = 0
\end{aligned} \tag{A.28}$$

Expand the time derivative term remembering that mean flow variables time dependency is neglected.

$$\begin{aligned}
&H_0 \frac{\partial \rho_1}{\partial t} + \rho_0 T_0 \frac{\partial s_1}{\partial t} + \rho_0 \mathbf{u}_0 \cdot \frac{\partial \mathbf{u}_1}{\partial t} + H_1 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot \nabla H_1 \\
&\qquad + H_0 \nabla \cdot \mathbf{m}_1 + \mathbf{m}_1 \cdot \nabla H_0 - \mathbf{m}_0 \cdot \psi_1 - \mathbf{m}_1 \cdot \psi_0 - T_0 Q_1 - T_1 Q_0 = 0
\end{aligned} \tag{A.29}$$

At this point a similar process is performed as before. Terms are manipulated to include C_0,L_0,andS_0. Manipulating L_0,

$$\nabla H_0 = L_0 - \zeta_0 + T_0 \nabla s_0 = \psi_0 - \zeta_0 + T_0 \nabla s_0 \tag{A.30}$$

Insert into (A.29) and collect terms.

$$H_0 \left(\frac{\partial \rho_1}{\partial t} + \nabla \cdot \mathbf{m}_1 \right) + \rho_0 T_0 \frac{\partial s_1}{\partial t} + \mathbf{m}_0 \cdot \left(\frac{\partial \mathbf{u}_1}{\partial t} + \nabla H_1 - \psi_1 \right) \\ + \mathbf{m}_1 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_1 - T_1 Q_0 = 0 \tag{A.31}$$

C_1 is then identified multiplying H_0 in the first term on the left,

$$H_0 C_1 + \rho_0 T_0 \frac{\partial s_1}{\partial t} + \mathbf{m}_0 \cdot \left(\frac{\partial \mathbf{u}_1}{\partial t} + \nabla H_1 - \psi_1 \right) \\ + \mathbf{m}_1 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_1 - T_1 Q_0 = 0 \tag{A.32}$$

Remembering that,

$$L_1 = \frac{\partial \mathbf{u}_1}{\partial t} + \zeta_1 + \nabla H_1 - T_0 \nabla s_1 - T_1 \nabla s_0 \tag{A.33}$$

Thus,

$$\frac{\partial \mathbf{u}_1}{\partial t} + \nabla H_1 - \psi_1 = L_1 - \zeta_1 + T_0 \nabla s_1 + T_1 \nabla s_0 - \psi_1 \tag{A.34}$$

And then,

$$H_0 C_1 + \rho_0 T_0 \frac{\partial s_1}{\partial t} + \mathbf{m}_0 \cdot (L_1 - \zeta_1 + T_0 \nabla s_1 + T_1 \nabla s_0 - \psi_1) \\ + \mathbf{m}_1 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_1 - T_1 Q_0 = 0 \tag{A.35}$$

Now add,

$$T_0 s_0 C_1 = T_0 s_0 \left(\frac{\partial \rho_1}{\partial t} + \nabla \cdot \mathbf{m}_1 \right) = 0 \tag{A.36}$$

and expand the time derivative terms remembering that the time dependency of the mean flow values is neglected. This allows us to take,

$$T_0 s_0 \frac{\partial \rho_1}{\partial t} = T_0 \frac{\partial s_0 \rho_1}{\partial t}, \tag{A.37}$$

for example.

$$\begin{aligned}(H_0 - T_0 s_0)\, C_1 + T_0 \frac{\partial}{\partial t}\left(s_0 \rho_1 + s_1 \rho_0\right) + T_0 s_0 \nabla \cdot \mathbf{m}_1 \\ + \mathbf{m}_0 \cdot \left(L_1 - \zeta_1 + T_0 \nabla s_1 + T_1 \nabla s_0 - \psi_1\right) \\ + \mathbf{m}_1 \cdot \left(T_0 \nabla s_0 - \zeta_0\right) - T_0 Q_1 - T_1 Q_0 = 0\end{aligned} \tag{A.38}$$

$$\begin{aligned}T_0 S_1 &= T_0 \frac{\partial(\rho_0 s_1 + \rho_1 s_0)}{\partial t} + T_0 \nabla \cdot \left(\mathbf{m}_0 s_1 + \mathbf{m}_1 s_0\right) \\ &= \mathrm{T}_0 \frac{\partial(\rho_0 s_1 + \rho_1 s_0)}{\partial t} + T_0 s_1 \nabla \cdot \mathbf{m}_0 \\ &\quad + T_0 \mathbf{m}_0 \cdot \nabla s_1 + T_0 s_0 \nabla \cdot \mathbf{m}_1 + T_0 \mathbf{m}_1 \cdot \nabla s_0\end{aligned} \tag{A.39}$$

Insert,

$$\begin{aligned}(H_0 - T_0 s_0)\, C_1 + \mathrm{T}_0 \mathrm{S}_1 - T_0 s_1 \nabla \cdot \mathbf{m}_0 + \mathbf{m}_0 \cdot \left(L_1 - \zeta_1 + T_1 \nabla s_0 - \psi_1\right) \\ - \mathbf{m}_1 \cdot \zeta_0 - T_0 Q_1 - T_1 Q_0 = 0\end{aligned} \tag{A.40}$$

And,

$$T_1 S_0 = T_1 \mathbf{m}_0 \cdot \nabla s_0 + T_1 s_0 \nabla \cdot \mathbf{m}_0 = T_1 \mathbf{m}_0 \cdot \nabla s_0 \tag{A.41}$$

$$(H_0 - T_0 s_0)\, C_1 + \mathrm{T}_0 \mathrm{S}_1 + \mathbf{m}_0 \cdot \left(L_1 - \zeta_1 - \psi_1\right) - \mathbf{m}_1 \cdot \zeta_0 - T_0 Q_1 + T_1 (S_0 - Q_0) = 0 \tag{A.42}$$

Rearrange and separate out vorticity terms,

$$(H_0 - T_0 s_0)\, C_1 + \mathrm{T}_0(\mathrm{S}_1 - Q_1) + \mathbf{m}_0 \cdot \left(L_1 - \psi_1\right) - \mathbf{m}_1 \cdot \zeta_0 - \mathbf{m}_0 \cdot \zeta_1 = 0 \tag{A.43}$$

Now the vorticity terms need to be dealt with. Remembering that,

$$\begin{aligned} \mathbf{m}_1 &= \rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0 \\ \zeta_1 &= \omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_0 \end{aligned} \tag{A.44}$$

Apply these definitions to the vorticity terms,

$$\begin{aligned} \mathbf{m}_1 \cdot \zeta_0 + \mathbf{m}_0 \cdot \zeta_1 &= (\rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0) \cdot (\omega_0 \times \mathbf{u}_0) \\ &\quad + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_0) \\ &= \rho_0 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_0) \\ &\quad + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) + \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_0) \end{aligned} \tag{A.45}$$

$$\begin{aligned} \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_0) &= 0 \\ \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_0) &= 0 \end{aligned} \tag{A.46}$$

Recalling scalar triple product rules we find that there is no vorticity contribution to the first order energy balance.

$$\mathbf{m}_1 \cdot \zeta_0 + \mathbf{m}_0 \cdot \zeta_1 = \rho_0 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) + \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) = 0 \tag{A.47}$$

So as before in the base order energy equation, regardless of entropy or vorticity fluctuations, the energy equation for the first order reduces to zero.

$$(H_0 - T_0 s_0)\, C_1 + \mathrm{T}_0(\mathrm{S}_1 - Q_1) + \mathbf{m}_0 \cdot (L_1 - \psi_1) = 0 \tag{A.48}$$

A.5 Second Order Energy Expansions

Below are details not shown in Section 3.3. The second order process follows exactly as the first order process did. Each term is expanded with the thermodynamic expansions and then simplified.

$$\frac{\partial}{\partial t}(\rho H - p)_2 + \nabla\cdot(\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) - \mathbf{m}_0\cdot\psi_2 - \mathbf{m}_1\cdot\psi_1 - \mathbf{m}_2\cdot\psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.49)}$$

We need an additional expansion:

$$\begin{aligned}(\rho H - p)_2 &= \left(\rho e + \frac{1}{2}\rho\mathbf{u}^2\right)_2 \\ &= (\rho e)_2 + \frac{1}{2}\rho_2 {u_0}^2 + \rho_1\mathbf{u}_0\cdot\mathbf{u}_1 + \frac{1}{2}\rho_0\mathbf{u}_1\cdot\mathbf{u}_1 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_2\end{aligned} \quad \text{(A.50)}$$

Recalling,

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \quad \text{(A.51)}$$

It follows that,

$$(\rho H - p)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \frac{1}{2}\rho_2 {u_0}^2 + \rho_1\mathbf{u}_0\cdot\mathbf{u}_1 + \frac{1}{2}\rho_0\mathbf{u}_1\cdot\mathbf{u}_1 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_2 \quad \text{(A.52)}$$

Using, $H_0 = h_0 + \frac{1}{2}{u_0}^2$,

$$(\rho H - p)_2 = H_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1\mathbf{u}_0\cdot\mathbf{u}_1 + \frac{1}{2}\rho_0\mathbf{u}_1\cdot\mathbf{u}_1 + \rho_0\mathbf{u}_0\cdot\mathbf{u}_2 \quad \text{(A.53)}$$

Collecting terms,

$$(\rho H - p)_2 = H_0\rho_2 + T_0(\rho_1 s_1 + \rho_0 s_2) + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1\mathbf{u}_0\cdot\mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 + \mathbf{m}_0\cdot\mathbf{u}_2 \quad \text{(A.54)}$$

The expanded form of the time derivative term is inserted into Eqn. (A.49),

$$\frac{\partial}{\partial t}\left(\begin{array}{c} H_0\rho_2 + T_0(\rho_1 s_1 + \rho_0 s_2) + \dfrac{{p_1}^2}{2\rho_0 {a_0}^2} + \dfrac{\rho_0 T_0 {s_1}^2}{2c_p} \\ +\rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \dfrac{1}{2}\rho_0 {u_1}^2 + \mathbf{m}_0 \cdot \mathbf{u}_2 \end{array}\right) + \nabla \cdot (\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) - \mathbf{m}_0 \cdot \psi_2 - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.55)}$$

To assist in the bookkeeping the following terms in the time derivative are defined as,

$$E_2 = \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{1}{2}\rho_0 {u_1}^2 \quad \text{(A.56)}$$

We will see the motivation for this definition as the mathematics progresses.

$$\frac{\partial}{\partial t}\left(H_0\rho_2 + T_0(\rho_1 s_1 + \rho_0 s_2) + E_2 + \mathbf{m}_0 \cdot \mathbf{u}_2\right) + + \nabla \cdot (\mathbf{m}_0 H_2 + \mathbf{m}_1 H_1 + \mathbf{m}_2 H_0) - \mathbf{m}_0 \cdot \psi_2 - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \psi_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.57)}$$

Terms are collected into subsets which correspond to the three governing equations,

$$\frac{\partial E_2}{\partial t} + H_0\left(\frac{\partial \rho_2}{\partial t} + \nabla \cdot \mathbf{m}_2\right) + T_0\frac{\partial}{\partial t}\left(\rho_1 s_1 + \rho_0 s_2\right) + \mathbf{m}_0 \cdot \left(\frac{\partial \mathbf{u}_2}{\partial t} + \nabla H_2 - \psi_2\right) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 + \mathbf{m}_2 \cdot (\nabla H_0 - \psi_0) - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.58)}$$

Recalling the base order momentum equation,

$$\zeta_0 + \nabla H_0 - T_0 \nabla s_0 = \psi_0 \quad \text{(A.59a)}$$

$$\nabla H_0 - \psi_0 = T_0 \nabla s_0 - \zeta_0 \quad \text{(A.59b)}$$

The base order momentum equation is used,

$$
\begin{aligned}
\frac{\partial E_2}{\partial t} + H_0 \left(\frac{\partial \rho_2}{\partial t} + \nabla \cdot \mathbf{m}_2 \right) + T_0 \frac{\partial}{\partial t} \left(\rho_1 s_1 + \rho_0 s_2 \right) \\
+ \mathbf{m}_0 \cdot \left(\frac{\partial \mathbf{u}_2}{\partial t} + \nabla H_2 - \psi_2 \right) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 \\
+ \mathbf{m}_2 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.60)}
\end{aligned}
$$

Recall the second order governing equations,

$$
C_2 = \frac{\partial \rho_2}{\partial t} + \nabla \cdot \mathbf{m}_2 \quad \text{(A.61a)}
$$

$$
L_2 = \frac{\partial \mathbf{u}_2}{\partial t} + \zeta_2 + \nabla H_2 - T_0 \nabla s_2 - T_1 \nabla s_1 - T_2 \nabla s_0 = \psi_2 \quad \text{(A.61b)}
$$

$$
S_2 = \frac{\partial (\rho_0 s_2 + \rho_1 s_1 + \rho_2 s_0)}{\partial t} + \nabla \cdot (\mathbf{m}_0 s_2 + \mathbf{m}_1 s_1 + \mathbf{m}_2 s_0) = Q_2 \quad \text{(A.61c)}
$$

Manipulating the second order momentum equation,

$$
\frac{\partial \mathbf{u}_2}{\partial t} + \nabla H_2 = L_2 - \zeta_2 + T_0 \nabla s_2 + T_1 \nabla s_1 + T_2 \nabla s_0 \quad \text{(A.62)}
$$

Identify C_2 and L_2,

$$
\begin{aligned}
\frac{\partial E_2}{\partial t} + H_0 C_2 + T_0 \frac{\partial}{\partial t} \left(\rho_1 s_1 + \rho_0 s_2 \right) \\
+ \mathbf{m}_0 \cdot (L_2 - \zeta_2 + T_0 \nabla s_2 + T_1 \nabla s_1 + T_2 \nabla s_0 - \psi_2) + \nabla \cdot (\mathbf{m}_1 H_1) \\
- \mathbf{m}_1 \cdot \psi_1 + \mathbf{m}_2 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.63)}
\end{aligned}
$$

Subtract $T_0 s_0$ time the second order continuity equation,

$$
T_0 s_0 C_2 = T_0 s_0 \frac{\partial \rho_2}{\partial t} + T_0 s_0 \nabla \cdot \mathbf{m}_2 \quad \text{(A.64)}
$$

To arrive at,

$$\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + T_0 \frac{\partial}{\partial t} (\rho_1 s_1 + \rho_0 s_2 + s_0 \rho_2) + T_0 s_0 \nabla \cdot \mathbf{m}_2 + \mathbf{m}_0 \cdot (L_2 - \zeta_2 + T_0 \nabla s_2 + T_1 \nabla s_1 + T_2 \nabla s_0 - \psi_2) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 + \mathbf{m}_2 \cdot (T_0 \nabla s_0 - \zeta_0) - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.65)}$$

Now identify the second order entropy equation,

$$T_0 S_2 = T_0 \frac{\partial(\rho_0 s_2 + \rho_1 s_1 + \rho_2 s_0)}{\partial t} + T_0 \nabla \cdot (\mathbf{m}_0 s_2 + \mathbf{m}_1 s_1 + \mathbf{m}_2 s_0) \quad \text{(A.66)}$$

Rearrange and expand the divergence term,

$$T_0 S_2 - T_0 (\mathbf{m}_0 \cdot \nabla s_2 + \nabla \cdot (\mathbf{m}_1 s_1) + s_0 \nabla \cdot \mathbf{m}_2 + \mathbf{m}_2 \cdot \nabla s_0) = T_0 \frac{\partial(\rho_0 s_2 + \rho_1 s_1 + \rho_2 s_0)}{\partial t} \quad \text{(A.67)}$$

And insert,

$$\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + T_0 (S_2 - Q_2) - T_0 (\mathbf{m}_0 \cdot \nabla s_2 + \nabla \cdot (\mathbf{m}_1 s_1) + s_0 \nabla \cdot \mathbf{m}_2 + \mathbf{m}_2 \cdot \nabla s_0) + T_0 s_0 \nabla \cdot \mathbf{m}_2 + \mathbf{m}_0 \cdot (L_2 - \zeta_2 + T_0 \nabla s_2 + T_1 \nabla s_1 + T_2 \nabla s_0 - \psi_2) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 + \mathbf{m}_2 \cdot (T_0 \nabla s_0 - \zeta_0) - T_1 Q_1 - T_2 Q_0 = 0 \quad \text{(A.68)}$$

Cancel equal terms,

$$
\begin{aligned}
\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + T_0\,(S_2 - Q_2) - T_0 \nabla \cdot (\mathbf{m}_1 s_1) \\
+ \mathbf{m}_0 \cdot (L_2 - \zeta_2 + T_1 \nabla s_1 + T_2 \nabla s_0 - \psi_2) \\
+ \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \zeta_0 - T_1 Q_1 - T_2 Q_0 = 0
\end{aligned} \quad \text{(A.69)}
$$

Eliminate $T_2 S_0$,

$$
\begin{aligned}
\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + \mathbf{m}_0 \cdot (L_2 - \psi_2) + T_0\,(S_2 - Q_2) \\
- T_0 \nabla \cdot (\mathbf{m}_1 s_1) - \mathbf{m}_0 \cdot \zeta_2 + T_1 \nabla \cdot (\mathbf{m}_0 s_1) + \nabla \cdot (\mathbf{m}_1 H_1) \\
- \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \zeta_0 - T_1 Q_1 = 0
\end{aligned} \quad \text{(A.70)}
$$

Now the vorticity terms are dealt with similar to the first order energy balance, remembering that, $\zeta_2 = \omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_0$ and $\mathbf{m}_2 = \rho_0 \mathbf{u}_2 + \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_0$

$$
\begin{aligned}
\mathbf{m}_0 \cdot \zeta_2 + \mathbf{m}_2 \cdot \zeta_0 &= \rho_0 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_0) \\
&\quad + (\rho_0 \mathbf{u}_2 + \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_0) \cdot (\omega_0 \times \mathbf{u}_0) \\
&= \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) + \rho_1 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0)
\end{aligned} \quad \text{(A.71)}
$$

In the second order case, all of the terms in the energy equation do not fall out. Thus we get some more information from it. By removing the redundant information and simplifying the vorticity terms we arrive at,

$$
\begin{aligned}
\frac{\partial E_2}{\partial t} + (H_0 - T_0 s_0)\, C_2 + \mathbf{m}_0 \cdot (L_2 - \psi_2) + T_0\,(S_2 - Q_2) \\
- T_0 \nabla \cdot (\mathbf{m}_1 s_1) - \mathbf{m}_0 \cdot \zeta_2 + T_1 \nabla \cdot (\mathbf{m}_0 s_1) \\
+ \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \zeta_0 - T_1 Q_1 = 0
\end{aligned} \quad \text{(A.72)}
$$

Now removing the second order relations which equal zero,

$$\frac{\partial E_2}{\partial t} - T_0 \nabla \cdot (\mathbf{m}_1 s_1) - \mathbf{m}_0 \cdot \zeta_2 + T_1 \nabla \cdot (\mathbf{m}_0 s_1) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 - \mathbf{m}_2 \cdot \zeta_0 - T_1 Q_1 = 0 \quad \text{(A.73)}$$

Apply the vector expansion for the vorticity terms,

$$\frac{\partial E_2}{\partial t} - T_0 \nabla \cdot (\mathbf{m}_1 s_1) + T_1 \nabla \cdot (\mathbf{m}_0 s_1) + \nabla \cdot (\mathbf{m}_1 H_1) - \mathbf{m}_1 \cdot \psi_1 - T_1 Q_1 - \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) - \rho_1 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) = 0 \quad \text{(A.74)}$$

Terms are arranged into three parts: the time derivative energy terms, the work or $\nabla\cdot$ terms, and the remaining source terms. This collection is motivated by recalling the energy balance form of $\frac{\partial E}{\partial t} + \nabla \cdot W = D$.

$$\frac{\partial E_2}{\partial t} + \nabla \cdot \left[\mathbf{m}_1 \left(H_1 - T_0 s_1\right) + \mathbf{m}_0 T_1 s_1\right] + \mathbf{m}_0 s_1 \cdot \nabla T_0 - \mathbf{m}_0 s_1 \cdot \nabla T_1 - \mathbf{m}_1 \cdot \psi_1 - T_1 Q_1 - \rho_0 \mathbf{u}_0 \cdot (\omega_1 \times \mathbf{u}_1) - \rho_1 \mathbf{u}_1 \cdot (\omega_0 \times \mathbf{u}_0) = 0 \quad \text{(A.75)}$$

Now the terms $\mathbf{m}_1 \cdot \psi_1$ and $T_1 Q_1$ are expanded upon. These terms are in tensor form and cannot be written in purely vector form. Thus, it is required to split them and leave them in tensor form,

$$\mathbf{m}_1 \cdot \psi_1 = m_{1_j} \left(\frac{1}{\rho} \frac{\partial P_i j}{\partial x_i}\right)_1 \quad \text{(A.76)}$$

$$= \frac{\partial}{\partial x_i} \left[m_{1_j} \left(\frac{P_i j}{\rho}\right)_1\right] - \left(\frac{P_i j}{\rho^2}\right)_1 \frac{\partial m_{1_j}}{\partial x_i} + m_{1j} \left(\frac{P_i j}{\rho} \frac{\partial \rho}{\partial x_i}\right)_1 \quad \text{(A.77)}$$

$$T_1 Q_1 = T_1 \left(\frac{\phi - \nabla \cdot q}{T}\right)_1 \quad \text{(A.78)}$$

$$= T_1 \left(\frac{\phi}{T}\right)_1 - \nabla \cdot \left[T_1 \left(\frac{q}{T}\right)_1\right] + \left(\frac{q}{T}\right)_1 \cdot \nabla T_1 - T_1 \left(\frac{q \cdot \nabla T}{T^2}\right)_1 \quad \text{(A.79)}$$

A.6 General Energy Corollary

Shown below are the details of the derivation of the general energy corollary shown in Section 3.4. Starting with the energy equation,

$$\frac{\partial}{\partial t}\left[\rho H - p\right] + \nabla\cdot(\mathbf{m}H) - \mathbf{m}\cdot\psi - TQ = 0, \tag{A.80}$$

subtract the general relations,

$$(H_0 - T_0 s_0) = 0 \tag{A.81a}$$

$$T_0\,(S - Q) = 0 \tag{A.81b}$$

$$\mathbf{m}_0\cdot(L - \psi) = 0 \tag{A.81c}$$

Remembering that,

$$C = \frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{m} = 0 \tag{A.82a}$$

$$L = \frac{\partial \mathbf{u}}{\partial t} + \omega\times\mathbf{u} + \nabla H - T\nabla s = \psi \tag{A.82b}$$

$$S = \frac{\partial \rho s}{\partial t} + \nabla\cdot(\mathbf{m}s) = Q \tag{A.82c}$$

Using the base order equations we rearrange into,

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho H - p - \rho\,(H_0 - T_0 s_0) - T_0\rho s - \mathbf{m}_0\cdot\mathbf{u}\right] + \nabla\cdot(\mathbf{m}H)&\\ - \mathbf{m}\cdot\psi - TQ - (H_0 - T_0 s_0)\,\nabla\cdot\mathbf{m} - T_0\nabla\cdot(\mathbf{m}s) + T_0 Q&\\ - \mathbf{m}_0\cdot\left[\zeta + \nabla H - T\nabla s - \psi\right] = 0&\end{aligned} \tag{A.83}$$

Rearrange the time derivative term,

$$\frac{\partial}{\partial t}\left[\rho\left(H-H_0\right)-T_0\rho\left(s-s_0\right)-\mathbf{m}_0\cdot\mathbf{u}-p\right]+\nabla\cdot\left(\mathbf{m}H\right)$$
$$-\mathbf{m}\cdot\psi-TQ-\left(H_0-T_0s_0\right)\nabla\cdot\mathbf{m}-T_0\nabla\cdot\left(\mathbf{m}s\right)+T_0Q$$
$$-\mathbf{m}_0\cdot\left[\zeta+\nabla H-T\nabla s-\psi\right]=0 \quad \text{(A.84)}$$

Effort is placed into rearranging the divergence terms,

$$\nabla\cdot\left[\mathbf{m}\left(H_0-T_0s_0\right)\right]=\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)+\left(H_0-T_0s_0\right)\nabla\cdot\mathbf{m}$$
$$-\left(H_0-T_0s_0\right)\nabla\cdot\mathbf{m}$$
$$=\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)-\nabla\cdot\left[\mathbf{m}\left(H_0-T_0s_0\right)\right] \quad \text{(A.85)}$$

inserting back,

$$\frac{\partial}{\partial t}\left[\rho\left(H-H_0\right)-T_0\rho\left(s-s_0\right)-\mathbf{m}_0\cdot\mathbf{u}-p\right]+\nabla\cdot\left(\mathbf{m}H\right)$$
$$-\mathbf{m}\cdot\psi-TQ+\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)-\nabla\cdot\left[\mathbf{m}\left(H_0-T_0s_0\right)\right]$$
$$-T_0\nabla\cdot\left(\mathbf{m}s\right)+T_0Q-\mathbf{m}_0\cdot\left[\zeta+\nabla H-T\nabla s-\psi\right]=0 \quad \text{(A.86)}$$

expand this term,

$$\nabla\cdot\left[\mathbf{m}sT_0\right]=T_0\nabla\cdot\left(\mathbf{m}s\right)+\mathbf{m}s\cdot\nabla T_0-T_0\nabla\cdot\left(\mathbf{m}s\right)$$
$$=\mathbf{m}s\cdot\nabla T_0-\nabla\cdot\left[\mathbf{m}sT_0\right] \quad \text{(A.87)}$$

$$\frac{\partial}{\partial t}\left[\rho\left(H-H_0\right)-T_0\rho\left(s-s_0\right)-\mathbf{m}_0\cdot\mathbf{u}-p\right]+\nabla\cdot\left(\mathbf{m}H\right)-\mathbf{m}\cdot\psi$$
$$-TQ+\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)-\nabla\cdot\left[\mathbf{m}\left(H_0-T_0s_0\right)\right]+\mathbf{m}s\cdot\nabla T_0$$
$$-\nabla\cdot\left[\mathbf{m}sT_0\right]+T_0Q-\mathbf{m}_0\cdot\left[\zeta+\nabla H-T\nabla s-\psi\right]=0 \quad \text{(A.88)}$$

chain rule and base order continuity,

$$\nabla \cdot [\mathbf{m}_0 H] = \mathbf{m}_0 \cdot \nabla H + H \nabla \cdot \mathbf{m}_0 = \mathbf{m}_0 \cdot \nabla H \tag{A.89}$$

back in,

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho \left(H - H_0 \right) - T_0 \rho \left(s - s_0 \right) - \mathbf{m}_0 \cdot \mathbf{u} - p \right] + \nabla \cdot \left(\mathbf{m} H \right) - \mathbf{m} \cdot \psi \\ - TQ + \mathbf{m} \cdot \nabla \left(H_0 - T_0 s_0 \right) - \nabla \cdot \left[\mathbf{m} \left(H_0 - T_0 s_0 \right) \right] + \mathbf{m} s \cdot \nabla T_0 \\ - \nabla \cdot [\mathbf{m} s T_0] + T_0 Q - \nabla \cdot [\mathbf{m}_0 H] - \mathbf{m}_0 \cdot [\zeta - T \nabla s - \psi] = 0 \end{aligned} \tag{A.90}$$

Pulling out the entropy gradient term from the momentum equation,

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho \left(H - H_0 \right) - T_0 \rho \left(s - s_0 \right) - \mathbf{m}_0 \cdot \mathbf{u} - p \right] + \nabla \cdot \left(\mathbf{m} H \right) - \mathbf{m} \cdot \psi \\ - TQ + \mathbf{m} \cdot \nabla \left(H_0 - T_0 s_0 \right) - \nabla \cdot \left[\mathbf{m} \left(H_0 - T_0 s_0 \right) \right] + \mathbf{m} s \cdot \nabla T_0 \\ - \nabla \cdot [\mathbf{m} s T_0] + T_0 Q - \nabla \cdot [\mathbf{m}_0 H] + \mathbf{m}_0 \cdot T \nabla s - \mathbf{m}_0 \cdot [\zeta - \psi] = 0 \end{aligned} \tag{A.91}$$

further manipulating individual terms,

$$\begin{aligned} \nabla \cdot [\mathbf{m}_0 T s] &= \mathbf{m}_0 T \cdot \nabla s + s \nabla \cdot (T \mathbf{m}_0) \\ &= \mathbf{m}_0 T \cdot \nabla s + s T \nabla \cdot \mathbf{m}_0 + s \mathbf{m}_0 \cdot \nabla T \\ &= \mathbf{m}_0 T \cdot \nabla s + s \mathbf{m}_0 \cdot \nabla T \\ &= \mathbf{m}_0 T \cdot \nabla s + s \mathbf{m}_0 \cdot \nabla T \\ \mathbf{m}_0 T \cdot \nabla s &= \nabla \cdot [\mathbf{m}_0 T s] - s \mathbf{m}_0 \cdot \nabla T \end{aligned} \tag{A.92}$$

and inserting the chain rule expansion,

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\left(H-H_0\right)-T_0\rho\left(s-s_0\right)-\mathbf{m}_0\cdot\mathbf{u}-p\right]+\nabla\cdot\left(\mathbf{m}H\right)\\-\mathbf{m}\cdot\psi-TQ+\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)-\nabla\cdot\left[\mathbf{m}\left(H_0-T_0s_0\right)\right]\\+\mathbf{m}s\cdot\nabla T_0-\nabla\cdot\left[\mathbf{m}sT_0\right]+T_0Q-\nabla\cdot\left[\mathbf{m}_0H\right]+\nabla\cdot\left[\mathbf{m}_0Ts\right]\\-s\mathbf{m}_0\cdot\nabla T-\mathbf{m}_0\cdot\left[\zeta-\psi\right]=0\end{aligned}\tag{A.93}$$

Collecting the terms,

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\left(H-H_0\right)-T_0\rho\left(s-s_0\right)-\mathbf{m}_0\cdot\mathbf{u}-p\right]\\+\nabla\cdot\left[\mathbf{m}H-\mathbf{m}\left(H_0-T_0s_0\right)-\mathbf{m}sT_0-\mathbf{m}_0H+\mathbf{m}_0Ts\right]\\-\left(\mathbf{m}-\mathbf{m}_0\right)\cdot\psi-\left(T-T_0\right)Q+\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)\\+\mathbf{m}s\cdot\nabla T_0-s\mathbf{m}_0\cdot\nabla T-\mathbf{m}_0\zeta=0\end{aligned}\tag{A.94}$$

Then, manipulating the 4th and 3rd to last terms, expand and chain rule,

$$\begin{aligned}\mathbf{m}\cdot\nabla\left(H_0-T_0s_0\right)+\mathbf{m}s\cdot\nabla T_0&=\mathbf{m}\cdot\nabla H_0-\mathbf{m}\cdot\nabla\left(T_0s_0\right)+\mathbf{m}s\cdot\nabla T_0\\&=\mathbf{m}\cdot\nabla H_0-\mathbf{m}s_0\cdot\nabla T_0\\&\quad-\mathbf{m}T_0\cdot\nabla s_0+\mathbf{m}s\cdot\nabla T_0\end{aligned}\tag{A.95}$$

And base momentum,

$$\nabla H_0=\psi_0+T_0\nabla s_0-\zeta_0\tag{A.96}$$

Now into original,

$$
\begin{aligned}
&\mathbf{m}\cdot\nabla H_0 - \mathbf{m}s_0\cdot\nabla T_0 - \mathbf{m}T_0\cdot\nabla s_0 + \mathbf{m}s\cdot\nabla T_0 \\
&\quad = \mathbf{m}\cdot\left(\psi_0 + T_0\nabla s_0 - \zeta_0\right) + \mathbf{m}\left(s - s_0\right)\cdot\nabla T - \mathbf{m}T_0\cdot\nabla s_0 \\
&\quad = \mathbf{m}\cdot\left(\psi_0 - \zeta_0\right) + \mathbf{m}\left(s - s_0\right)\cdot\nabla T - \mathbf{m}T_0\cdot\nabla s_0 + \mathbf{m}T_0\cdot\nabla s_0 \\
&\quad = \mathbf{m}\cdot\left(\psi_0 - \zeta_0\right) + \mathbf{m}\left(s - s_0\right)\cdot\nabla T
\end{aligned} \tag{A.97}
$$

$$\tag{A.98}$$

And back into the energy equation,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\rho\left(H - H_0\right) - T_0\rho\left(s - s_0\right) - \mathbf{m}_0\cdot\mathbf{u} - p\right] \\
&\qquad + \nabla\cdot\left[\mathbf{m}H - \mathbf{m}\left(H_0 - T_0 s_0\right) - \mathbf{m}sT_0 - \mathbf{m}_0 H + \mathbf{m}_0 T s\right] \\
&\qquad\quad - \left(\mathbf{m} - \mathbf{m}_0\right)\cdot\psi - \left(T - T_0\right)Q + \mathbf{m}\cdot\left(\psi_0 - \zeta_0\right) \\
&\qquad\qquad + \left(s - s_0\right)\mathbf{m}\cdot\nabla T_0 - s\mathbf{m}_0\cdot\nabla T - \mathbf{m}_0\zeta = 0
\end{aligned} \tag{A.99}
$$

Looking at the divergence term,

$$
\begin{aligned}
&\nabla\cdot\left[\mathbf{m}H - \mathbf{m}\left(H_0 - T_0 s_0\right) - \mathbf{m}sT_0 - \mathbf{m}_0 H + \mathbf{m}_0 T s\right] \\
&\quad = \nabla\cdot\left[\mathbf{m}\left(H - H_0\right) - \mathbf{m}T_0\left(s - s_0\right) - \mathbf{m}_0 H + \mathbf{m}_0 T s\right] \\
&\quad = \nabla\cdot\begin{bmatrix}\mathbf{m}\left(H - H_0\right) - \mathbf{m}T_0\left(s - s_0\right) - \mathbf{m}_0 H - \mathbf{m}_0 H_0 \\ +\mathbf{m}_0 H_0 + \mathbf{m}_0 T s - \mathbf{m}_0 T s_0 + \mathbf{m}_0 T s_0\end{bmatrix} \\
&\quad = \nabla\cdot\begin{bmatrix}\mathbf{m}\left(H - H_0\right) - \mathbf{m}T_0\left(s - s_0\right) - \mathbf{m}_0\left(H - H_0\right) \\ -\mathbf{m}_0 H_0 + \mathbf{m}_0 T\left(s - s_0\right) + \mathbf{m}_0 T s_0\end{bmatrix} \\
&\quad = \nabla\cdot\begin{bmatrix}\left(\mathbf{m} - \mathbf{m}_0\right)\left(H - H_0\right) - \mathbf{m}T_0\left(s - s_0\right) \\ +\mathbf{m}_0 T\left(s - s_0\right) - \mathbf{m}_0 H_0 + \mathbf{m}_0 T s_0\end{bmatrix}
\end{aligned} \tag{A.100}
$$

Base momentum and entropy,

$$\nabla \cdot (\mathbf{m}_0 H_0) = \mathbf{m}_0 \psi_0 + T_0 Q_0 \tag{A.101a}$$

$$\nabla \cdot (\mathbf{m}_0 s_0) = Q_0 \tag{A.101b}$$

Manipulating,

$$\begin{aligned} \nabla \cdot (\mathbf{m}_0 T s_0) &= T \nabla \cdot (\mathbf{m}_0 s_0) + \mathbf{m}_0 s_0 \cdot \nabla T \\ &= T Q_0 + \mathbf{m}_0 s_0 \cdot \nabla T \end{aligned} \tag{A.102}$$

And we put the previous divergence equation into the form,

$$\begin{aligned} \nabla \cdot \left[(\mathbf{m} - \mathbf{m}_0) (H - H_0) - \mathbf{m} T_0 (s - s_0) + \mathbf{m}_0 T (s - s_0) \right] - \mathbf{m}_0 \psi_0 \\ - T_0 Q_0 + T Q_0 + \mathbf{m}_0 s_0 \cdot \nabla T \\ = \nabla \cdot \left[(\mathbf{m} - \mathbf{m}_0) (H - H_0) - \mathbf{m} T_0 (s - s_0) + \mathbf{m}_0 T (s - s_0) \right] \\ - \mathbf{m}_0 \psi_0 + (T - T_0) Q_0 + \mathbf{m}_0 s_0 \cdot \nabla T \end{aligned} \tag{A.103}$$

We substitute this back into our energy equation.

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \rho (H - H_0) - T_0 \rho (s - s_0) - \mathbf{m}_0 \cdot \mathbf{u} - p \right\} \\ + \nabla \cdot \left[(\mathbf{m} - \mathbf{m}_0) (H - H_0) - \mathbf{m} T_0 (s - s_0) + \mathbf{m}_0 T (s - s_0) \right] \\ - \mathbf{m}_0 \psi_0 + (T - T_0) Q_0 + \mathbf{m}_0 s_0 \cdot \nabla T - (\mathbf{m} - \mathbf{m}_0) \cdot \psi - (T - T_0) Q \\ + \mathbf{m} \cdot (\psi_0 - \zeta_0) + (s - s_0) \mathbf{m} \cdot \nabla T_0 - s \mathbf{m}_0 \cdot \nabla T - \mathbf{m}_0 \zeta = 0 \end{aligned} \tag{A.104}$$

Add in time derivatives of base order since they equal zero, rearrange the source and divergence terms

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\rho\left[H-H_0-T_0\left(s-s_0\right)\right]-\mathbf{m}_0\cdot\left(\mathbf{u}-\mathbf{u}_0\right)-\left(p-p_0\right)\right\}\\+\nabla\cdot\left[\left(\mathbf{m}-\mathbf{m}_0\right)\left[H-H_0+T_0\left(s-s_0\right)\right]+\mathbf{m}_0\left(T-T_0\right)\left(s-s_0\right)\right]\\-\left(\mathbf{m}-\mathbf{m}_0\right)\cdot\left(\psi-\psi_0\right)-\left(T-T_0\right)\left(Q-Q_0\right)\\-\mathbf{m}\cdot\zeta_0-\mathbf{m}_0\zeta+\left(s-s_0\right)\mathbf{m}\cdot\nabla T_0-\left(s-s_0\right)\mathbf{m}_0\cdot\nabla T=0\end{aligned}\quad\text{(A.105)}$$

The heat transfer and viscous terms have to be split into work and source terms. Beginning with the stress terms, we expand them with their tensor form.

$$\begin{aligned}\left(\mathbf{m}-\mathbf{m}_0\right)\cdot\left(\psi-\psi_0\right)&=\left(m_j-m_{0_j}\right)\left(\frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}-\frac{1}{\rho_0}\frac{\partial P_{0ij}}{\partial x_i}\right)\\&=\frac{\partial}{\partial x_i}\left[\left(m_j-m_{0_j}\right)\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\right]-\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\frac{\partial}{\partial x_i}\left(m_j-m_{0_j}\right)\\&\quad+\left(m_j-m_{0_j}\right)\left(\frac{1P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}-\frac{P_{0_{ij}}}{\rho_0^2}\frac{\partial\rho_0}{\partial x_i}\right)\end{aligned}\quad\text{(A.106)}$$

$$\begin{aligned}(T-T_0)(Q-Q_0)&=\left(T-T_0\right)\left(\frac{\phi}{T}-\frac{\phi_0}{T_0}-\frac{\nabla\cdot\mathbf{q}}{T}+\frac{\nabla\cdot\mathbf{q_0}}{T_0}\right)\\&=\left(T-T_0\right)\left(\frac{\phi}{T}-\frac{\phi_0}{T_0}\right)-\nabla\cdot\left[\left(T-T_0\right)\left(\frac{\mathbf{q}}{T}-\frac{\mathbf{q_0}}{T_0}\right)\right]\\&\quad+\left(\frac{\mathbf{q}}{T}-\frac{\mathbf{q_0}}{T_0}\right)\cdot\nabla(T-T_0)-\left(T-T_0\right)\left(\frac{\mathbf{q}\cdot\nabla T}{T^2}-\frac{\mathbf{q_0}\cdot\nabla T_0}{T_0^2}\right)\end{aligned}\quad\text{(A.107)}$$

These relations are separated into work and source terms. Inserting them back into Eqn. (A.105) and separating Energy, Work and Source terms with the form, $\frac{\partial E}{\partial t} + \nabla \cdot W = D$.

$$E = \rho \left[H - H_0 - T_0 \left(s - s_0 \right) \right] - \mathbf{m}_0 \cdot \left(\mathbf{u} - \mathbf{u}_0 \right) - \left(p - p_0 \right) \tag{A.108}$$

$$\begin{aligned} W = {} & \left(\mathbf{m} - \mathbf{m}_0 \right) \left[H - H_0 + T_0 \left(s - s_0 \right) \right] + \mathbf{m}_0 \left(T - T_0 \right) \left(s - s_0 \right) \\ & - \left(m_j - m_{0_j} \right) \left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0} \right) + \left(T - T_0 \right) \left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0} \right) \end{aligned} \tag{A.109}$$

$$\begin{aligned} D = {} & \mathbf{m} \cdot \zeta_0 + \mathbf{m}_0 \zeta - \left(s - s_0 \right) \mathbf{m} \cdot \nabla T_0 + \left(s - s_0 \right) \mathbf{m}_0 \cdot \nabla T \\ & - \left(\frac{P_{ij}}{\rho} - \frac{P_{0_{ij}}}{\rho_0} \right) \frac{\partial}{\partial x_i} (m_j - m_{0_j}) + \left(m_j - m_{0_j} \right) \left(\frac{1 P_{ij}}{\rho^2} \frac{\partial \rho}{\partial x_i} - \frac{P_{0_{ij}}}{\rho_0^2} \frac{\partial \rho_0}{\partial x_i} \right) \\ & + \left(T - T_0 \right) \left(\frac{\phi}{T} - \frac{\phi_0}{T_0} \right) + \left(\frac{\mathbf{q}}{T} - \frac{\mathbf{q_0}}{T_0} \right) \cdot \nabla (T - T_0) \\ & - \left(T - T_0 \right) \left(\frac{\mathbf{q} \cdot \nabla T}{T^2} - \frac{\mathbf{q_0} \cdot \nabla T_0}{T_0^2} \right) \end{aligned} \tag{A.110}$$

Appendix B

B.1 Second Order from the Full Energy Corollary

Recreating the second order expansion from Chapter 3 using the exact energy corollary.

B.1.1 Energy

Starting with the energy term from the general energy corollary in Section 3.4

$$E = \rho \left[H - H_0 - T_0 \left(s - s_0 \right) \right] - \mathbf{m}_0 \cdot \left(\mathbf{u} - \mathbf{u}_0 \right) - \left(p - p_0 \right) \tag{B.1}$$

Expanding each individual term.

$$E = \left(\rho_0 + \rho_1 + \rho_2 \right) \left[H_1 + H_2 - T_0 \left(s_1 + s_2 \right) \right] - \mathbf{m}_0 \cdot \left(\mathbf{u}_1 + \mathbf{u}_2 \right) - \left(p_1 + p_2 \right) \tag{B.2}$$

Collecting the second order terms,

$$E_2 = \rho_0 H_2 + \rho_1 H_1 - T_0 \rho_1 s_1 - T_0 \rho_0 s_2 - \mathbf{m}_0 \cdot \mathbf{u}_2 - p_2 \tag{B.3}$$

Total Enthalpy,

$$H = h + \frac{1}{2}\mathbf{u}^2 \tag{B.4a}$$

$$H_1 = h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 \tag{B.4b}$$

$$H_2 = h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 \tag{B.4c}$$

Applying these relations,

$$E_2 = \rho_0 \left(h_2 + \frac{1}{2}{\mathbf{u}_1}^2 + \mathbf{u}_0 \cdot \mathbf{u}_2 \right) + \rho_1 \left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 \right) - T_0\rho_1 s_1 - T_0\rho_0 s_2 - \mathbf{m}_0 \cdot \mathbf{u}_2 - p_2 \tag{B.5}$$

$$E_2 = \rho_0 h_2 + \frac{1}{2}\rho_0{\mathbf{u}_1}^2 + \rho_0\mathbf{u}_0 \cdot \mathbf{u}_2 + \rho_1 h_1 + \rho_1\mathbf{u}_0 \cdot \mathbf{u}_1 - T_0\rho_1 s_1 - T_0\rho_0 s_2 - \mathbf{m}_0 \cdot \mathbf{u}_2 - p_2 \tag{B.6}$$

Now, $\mathbf{m}_0 \cdot \mathbf{u}_2 = \rho_0\mathbf{u}_0\mathbf{u}_2$, canceling terms,

$$E_2 = \rho_0 h_2 + \frac{1}{2}\rho_0{\mathbf{u}_1}^2 + \rho_1 h_1 + \rho_1\mathbf{u}_0 \cdot \mathbf{u}_1 - T_0\rho_1 s_1 - T_0\rho_0 s_2 - p_2 \tag{B.7}$$

Recalling, $h = e + \frac{p}{\rho}$, $\rho h = \rho e + p$, and collecting the second order terms,

$$\begin{aligned} \rho_0 h_2 + \rho_1 h_1 + \rho_2 h_0 &= (\rho e)_2 + p_2 \\ \rho_0 h_2 + \rho_1 h_1 - p_2 &= (\rho e)_2 - \rho_2 h_0 \end{aligned} \tag{B.8}$$

Recalling,

$$(\rho e)_2 = h_0\rho_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0{a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0{s_1}^2}{2c_p} \tag{B.9}$$

Insert into Eqn B.8,

$$\rho_0 h_2 + \rho_1 h_1 - p_2 = \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0{a_0}^2} + T_0\rho_1 s_1 + \frac{\rho_0 T_0{s_1}^2}{2c_p} \tag{B.10}$$

Then back into Eqn B.6,

$$E_2 = \frac{1}{2}\rho_0 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 - T_0 \rho_1 s_1 - T_0 \rho_0 s_2 + \rho_0 T_0 s_2 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + T_0 \rho_1 s_1 + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \quad \text{(B.11)}$$

Canceling equal terms,

$$E_2 = \frac{1}{2}\rho_0 {\mathbf{u}_1}^2 + \rho_1 \mathbf{u}_0 \cdot \mathbf{u}_1 + \frac{{p_1}^2}{2\rho_0 {a_0}^2} + \frac{\rho_0 T_0 {s_1}^2}{2c_p} \quad \text{(B.12)}$$

And we arrive at the same results as Eqn. 3.24, but with significantly less work than that shown in Chapter 5.

B.1.2 Work

Now the work terms are derived, starting with the general work term,

$$\nabla \cdot W = \nabla \cdot \{(\mathbf{m} - \mathbf{m}_0)\,[H - H_0 - T_0\,(s - s_0)] + \mathbf{m}_0\,(T - T_0)\,(s - s_0)\} \quad \text{(B.13)}$$

Solving for W,

$$W = (\mathbf{m} - \mathbf{m}_0)\,[H - H_0 - T_0\,(s - s_0)] + \mathbf{m}_0\,(T - T_0)\,(s - s_0) \quad \text{(B.14)}$$

Using the expanded terms,

$$W = (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3)\,[H_1 + H_2 + H_3 - T_0\,(s_1 + s_2 + s_3)] + \mathbf{m}_0\,(T_1 + T_2 + T_3)\,(s_1 + s_2 + s_3) \quad \text{(B.15)}$$

Taking only the second order terms,

$$W_2 = \mathbf{m}_1 H_1 - \mathbf{m}_1 T_0 s_1 + \mathbf{m}_0 T_1 s_1 \quad \text{(B.16)}$$

using B.4b, arriving at the second order work term quickly.

$$W_2 = \mathbf{m}_1 \left(h_1 + \mathbf{u}_0 \cdot \mathbf{u}_1 - T_0 s_1\right) + \mathbf{m}_0 T_1 s_1 \tag{B.17}$$

And similarly, the same work term is derived, neglecting viscous and heat transfer effects.

B.1.3 Sources

And to finish, the source terms are recreated as well,

$$\begin{aligned} D = \left(\mathbf{m} - \mathbf{m}_0\right) \cdot \left[\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 + \left(s - s_0\right) \nabla T_0\right] \\ - \left(s - s_0\right) \mathbf{m}_0 \cdot \nabla \left(T - T_0\right) + viscous + h.t. \end{aligned} \tag{B.18}$$

$$\begin{aligned} D = \left(\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3\right) \cdot \left[\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 + \left(s_1 + s_2 + s_3\right) \nabla T_0\right] \\ - \left(s_1 + s_2 + s_3\right) \mathbf{m}_0 \cdot \nabla \left(T_1 + T_2 + T_3\right) \end{aligned} \tag{B.19}$$

The expansion of the terms involving vorticity is given by,

$$\begin{aligned}
\omega \times \mathbf{u} - \omega_0 \times \mathbf{u}_0 &= (\omega_0 + \omega_1 + \omega_2) \times (\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2) - \omega_0 \times \mathbf{u}_0 \\
&= \left\{ \begin{array}{l} \omega_0 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_0 + \omega_2 \times \mathbf{u}_0 \\ +\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_1 \\ +\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_2 + \omega_2 \times \mathbf{u}_2 \end{array} \right\} - \omega_0 \times \mathbf{u}_0 \\
&= \left\{ \begin{array}{l} \omega_1 \times \mathbf{u}_0 + \omega_2 \times \mathbf{u}_0 \\ +\omega_0 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_1 + \omega_2 \times \mathbf{u}_1 \\ +\omega_0 \times \mathbf{u}_2 + \omega_1 \times \mathbf{u}_2 + \omega_2 \times \mathbf{u}_2 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1 \\ +\omega_2 \times \mathbf{u}_0 + \omega_1 \times \mathbf{u}_1 + \omega_0 \times \mathbf{u}_2 \\ +\omega_2 \times \mathbf{u}_1 + \omega_1 \times \mathbf{u}_2 \\ +\omega_2 \times \mathbf{u}_2 \end{array} \right\}
\end{aligned} \tag{B.20}$$

Only the first order terms will continue into the second order equation by multiplying by $\mathbf{m}_1$. All other terms are of a higher order.

$$D_2 = \mathbf{m}_1 \cdot (\omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1) + \mathbf{m}_1 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_1 \tag{B.21}$$

$$D_2 = (\rho_0 \mathbf{u}_1 + \rho_1 \mathbf{u}_0) \cdot (\omega_1 \times \mathbf{u}_0 + \omega_0 \times \mathbf{u}_1) + \mathbf{m}_1 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_1 \tag{B.22}$$

Remembering that $A \cdot (A \times B) = 0$ for all vectors B.

$$\mathbf{u}_0 \cdot \omega_1 \times \mathbf{u}_0 = 0 \tag{B.23a}$$

$$\mathbf{u}_1 \cdot \omega_0 \times \mathbf{u}_1 = 0 \tag{B.23b}$$

Applying to B.22 yields,

$$D_2 = \rho_0 \mathbf{u}_1 \cdot (\omega_1 \times \mathbf{u}_0) + \rho_1 \mathbf{u}_0 \cdot (\omega_0 \times \mathbf{u}_1) + \mathbf{m}_1 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_1 \tag{B.24}$$

Recalling the triple product rules, $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) = -C \cdot (B \times A)$, putting D_2 into a more traditional form,

$$D_2 = \rho_0 \mathbf{u}_0 \cdot (\mathbf{u}_1 \times \omega_1) + \rho_1 \mathbf{u}_1 \cdot (\mathbf{u}_0 \times \omega_0) + \mathbf{m}_1 \cdot s_1 \nabla T_0 - s_1 \mathbf{m}_0 \cdot \nabla T_1 \tag{B.25}$$

Note that Myers has $-D$ in his total energy term. Instead of D as in the formula used, so as with the other terms, the second order energy balance is recreated with significantly less work. This method is reproduced to generate the third order terms as well.

Vita

Eric James Jacob was born in West Allis, WI on July 3rd, 1982. After graduating from Oconomowoc High School in 2000 he attended The Milwaukee School of Engineering of Milwaukee, WI. In the spring of 2004 he received a Bachelors of Science in Mechanical Engineering with a minor in Physics.

In August of 2004 he began graduate studies in the Mechanical, Aerospace and Biomedical Engineering department at the University of Tennessee Space Institute. From 2004 to 2006 Eric designed, built and ran experiments on nonlinear acoustics under the guidance of Dr. Flandro. He graduated with a Master of Science in Aerospace Engineering in the fall of 2006. Since that time he has continued research in Combustion Instability in pursuit of his PhD in Aerospace Engineering.

Throughout his time at UTSI he has acted as the American Institute of Aeronautics and Astronautics (AIAA) Student Chapter President, SGA Senator and Vice President, and President of several clubs. In 2007 he was awarded First Prize in the Masters Division for his work on an "Oxidizer Enhanced Hybrid Rocket Engine" at the AIAA student conference. During the Fall of 2007 he interned at the Deutsches Zentrum für Luft- und Raumfahrt (German Aerospace Center) of Göttingen, Germany. In 2008 he was awarded the AIAA Gordon C. Oates Air Breathing Propulsion Graduate Award.

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