Indonesian Journal of Electrical Engineering and Computer Science
Vol. 11, No. 3, September 2018, pp.1228∼1235
ISSN: 2502-4752, DOI:10.11591/ijeecs.v11.i3.pp1228-1235

A Study of Some Iterative Methods for Solving Fuzzy Volterra-Fredholm Integral Equations

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Article Info

Article history:
Received May 9, 2018
Revised July 3, 2018
Accepted July 16, 2018

Keyword:
Adomian Decomposition Method
Variational Iteration Method
Homotopy Analysis Method
Fuzzy Volterra-Fredholm Integral Equation.

ABSTRACT

This paper mainly focuses on the recent advances in the some approximated methods for solving fuzzy Volterra-Fredholm integral equations, namely, Adomian decomposition method, variational iteration method and homotopy analysis method. We converted fuzzy Volterra-Fredholm integral equation to a system of Volterra-Fredholm integral equations in crisp case. The approximated methods using to find the approximate solutions of this system and hence obtain an approximation for the fuzzy solution of the fuzzy Volterra-Fredholm integral equation. To assess the accuracy of each method, algorithms with Mathematica 6 according is used. Also, numerical example is included to demonstrate the validity and applicability of the proposed techniques.

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1. INTRODUCTION

Recently, the topics of fuzzy integral equations which attracted increasing interest, in particular in relation to fuzzy control, have been rapidly developed. The concept of fuzzy numbers and arithmetic operations firstly introduced by Zadeh [1], and then by Dubois and Prade [2] . Also, they have introduced the concept of integration of fuzzy functions. The fuzzy mapping function was introduced by Cheng and Zadeh [1]. Moreover, [3] presented an elementary fuzzy calculus based on the extension principle. Later, Goetschel and Voxman [4] preferred a Riemann integral type approach. Kaleva [5] chose to define the integral of fuzzy function, using the Lebesgue-type concept for integration. One of the first applications of the fuzzy integral equation was given by Ma and Wu who investigated the fuzzy Fredholm integral equation of the second kind. Recently, some mathematicians have studied fuzzy integral and integro-differential equation by numerical techniques [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. As we know the fuzzy integral and differential equations are one of the important parts of the fuzzy analysis theory that play a main role in the numerical analysis.

In this work, we will suggests recent advances in the some approximated methods for solving fuzzy Volterra-Fredholm integral equations of the second kind, namely, Adomian decomposition method, variational iteration method and homotopy analysis method.

2. FUZZY VOLTERRA-FREDHOLM INTEGRAL EQUATION

The fuzzy Volterra-Fredholm integral equation of the second kind is as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \mu_1 \int_a^x K_1(x,t) G_1(t,\tilde{u}(t))dt + \mu_2 \int_a^b K_2(x,t) G_2(t,\tilde{u}(t))dt, \quad (1)$$

where $\mu_1, \mu_2 \geq 0, \tilde{f}(x)$ is a fuzzy function of $x; a \leq x \leq b$, and $K_i(x,t), G_i(t, \tilde{u}(t)), i = 1, 2$, are analytic functions on $[a, b]$. For solving in parametric form of Eq.(1), consider $(\underline{f}(x,r), \overline{f}(x,r))$ and $(\underline{u}(x,r), \overline{u}(x,r)), 0 \leq r \leq 1$ and $t \in [a, b]$ are parametric form of $\tilde{f}(x)$ and $\tilde{u}(x)$, respectively. then, parametric form of Eq.(1) is as follows:

$$\underline{u}(x,r) = \underline{f}(x,r) + \mu_1 \int_a^x \underline{K_1(x,t)G_1(t,u(t,r))}dt + \mu_2 \int_a^b \underline{K_2(x,t)G_2(t,u(t,r))}dt, \tag{2}$$

$$\overline{u}(x,r) = \overline{f}(x,r) + \mu_1 \int_a^x \overline{K_1(x,t)G_1(t,u(t,r))}dt + \mu_2 \int_a^b \overline{K_2(x,t)G_2(t,u(t,r))}dt, \tag{3}$$

Let for $a \leq t \leq b$, we have

$$H_1(t,\underline{u},\overline{u}) = \min\{G_1(t,\beta) | \underline{u}(t,r) \leq \beta \leq \overline{u}(t,r)\},$$
$$H_2(t,\underline{u},\overline{u}) = \min\{G_2(t,\beta) | \underline{u}(t,r) \leq \beta \leq \overline{u}(t,r)\},$$
$$F_1(t,\underline{u},\overline{u}) = \max\{G_1(t,\beta) | \underline{u}(t,r) \leq \beta \leq \overline{u}(t,r)\},$$
$$F_2(t,\underline{u},\overline{u}) = \max\{G_2(t,\beta) | \underline{u}(t,r) \leq \beta \leq \overline{u}(t,r)\}.$$

Then,

$$\underline{K_1(x,t)G_1(t,u(t,r))} = \begin{cases} K_1(x,t)H_1(t,\underline{u},\overline{u}), & K_1(x,t) \geq 0, \\ K_1(x,t)F_1(t,\underline{u},\overline{u}), & K_1(x,t) < 0. \end{cases}$$

$$\underline{K_2(x,t)G_2(t,u(t,r))} = \begin{cases} K_2((x,t))H_2(t,\underline{u},\overline{u}), & K_2(x,t) \geq 0, \\ K_2((x,t))F_2(t,\underline{u},\overline{u}), & K_2(x,t) < 0. \end{cases}$$

$$\overline{K_1(x,t)G_1(t,u(t,r))} = \begin{cases} K_1(x,t)F_1(t,\underline{u},\overline{u}), & K_1(x,t) \geq 0, \\ K_1(x,t)H_1(t,\underline{u},\overline{u}), & K_1(x,t) < 0. \end{cases}$$

$$\overline{K_2(x,t)G_2(t,u(t,r))} = \begin{cases} K_2(x,t)F_2(t,\underline{u},\overline{u}), & K_2(x,t) \geq 0, \\ K_2(x,t)H_2(t,\underline{u},\overline{u}), & K_2(x,t) < 0. \end{cases}$$

For each $0 \leq r \leq 1$ and $a \leq x \leq b$. We can see that Eq.(1) convert to a system of Volterra-Fredholm integral equations in crisp case for each $0 \leq r \leq 1$ and $a \leq t \leq b$. Now, we explain Adomian decomposition method, variational iteration method and homotopy analysis method for approximating solution of this system of integral equations in crisp case. Then, we find approximate solutions for $\tilde{u}(x), a \leq x \leq b$.

3. DESCRIPTION OF THE METHODS

Here we will highlight briefly on some reliable methods for solving this type of equations, where details can be found in [16, 17, 21, 22, 23].

3.1. Adomian Decomposition Method (ADM)

The Adomian decomposition method has been applied to a wild class of functional equations [16, 19, 20, 21] by scientists and engineers since the beginning of the 1980s. Adomian gives the solution as a infinite series usually converging to a solution consider the following fuzzy Fredholm-Volterra integral equation of the form

$$\underline{u}(x,r) = \underline{f}(x,r) + \mu_1 \int_a^x \underline{K_1(x,t)G_1(t,u(t,r))}dt + \mu_2 \int_a^b \underline{K_2(x,t)G_2(t,u(t,r))}dt,$$
$$\overline{u}(x,r) = \overline{f}(x,r) + \mu_1 \int_a^x \overline{K_1(x,t)G_1(t,u(t,r))}dt + \mu_2 \int_a^b \overline{K_2(x,t)G_2(t,u(t,r))}dt, \tag{4}$$

The ADM assume an infinite series solution for the unknowns functions $[\underline{u}, \overline{u}]$, given by

$$\underline{u}(x) = \sum_{i=0}^{\infty} \underline{u}_i(x),$$
$$\overline{u}(x) = \sum_{i=0}^{\infty} \overline{u}_i(x). \tag{5}$$

The nonlinear operators $G_1(t,\underline{u}(t)), G_1(t,\overline{u}(t)), G_2(t,\underline{u}(t)), G_2(t,\overline{u}(t))$ into an infinite series of polynomials given by

$$\begin{aligned} G_1(t,\underline{u}(t)) &= \sum_{i=0}^{\infty} \underline{A}_n, \qquad G_1(t,\overline{u}(t)) = \sum_{i=0}^{\infty} \overline{A}_n, \\ G_2(t,\underline{u}(t)) &= \sum_{i=0}^{\infty} \underline{B}_n, \qquad G_2(t,\overline{u}(t)) = \sum_{i=0}^{\infty} \overline{B}_n. \end{aligned} \tag{6}$$

where the $\tilde{A}_n = [\underline{A}_n, \overline{A}_n], \tilde{B}_n = [\underline{B}_n, \overline{B}_n], n \geq 0$, are the so-called Adomian polynomial. Substituting Eqs.(5) and Eqs.(6) into Eq.(4), we get

$$\begin{aligned} \underline{u}_0 &= \underline{f}(x,r), \\ \underline{u}_1 &= \mu_1 \int_a^x \underline{K_1(x,t)A_0} dt + \mu_2 \int_a^b \underline{K_2(x,t)B_0} dt, \\ \underline{u}_{n+1} &= \mu_1 \int_a^x \underline{K_1(x,t)A_n} dt + \mu_2 \int_a^b \underline{K_2(x,t)B_n} dt. \end{aligned}$$

and

$$\begin{aligned} \overline{u}_0 &= \overline{f}(x,r), \\ \overline{u}_1 &= \mu_1 \int_a^x \overline{K_1(x,t)A_0} dt + \mu_2 \int_a^b \overline{K_2(x,t)B_0} dt, \\ \overline{u}_{n+1} &= \mu_1 \int_a^x \overline{K_1(x,t)A_n} dt + \mu_2 \int_a^b \overline{K_2(x,t)B_n} dt. \end{aligned}$$

We approximate $\tilde{u}(x,r) = [\underline{u}(x,r), \overline{u}(x,r)]$ by

$$\begin{aligned} \underline{\varphi}_n &= \sum_{i=0}^{n-1} \underline{u}_i(x,r), \\ \overline{\varphi}_n &= \sum_{i=0}^{n-1} \overline{u}_i(x,r), \end{aligned}$$

where,

$$\lim_{n\to\infty} \underline{\varphi}_n = \underline{u}(x,r), \ \lim_{n\to\infty} \overline{\varphi}_n = \overline{u}(x,r).$$

3.2. Variational Iteration Method (VIM)

The variational iteration method (VIM) is proposed by (He 1997) [18, 23] as a modification of a general Lagrange multiplier method. This method has been shown to solve effectively ,easily and accurately a large class of nonlinear problems with approximations converging rapidly to a accurate solutions. To illustrate its basic idea of the technique, we consider following general nonlinear system:

$$L[u(x)] + N[u(x)] = g(x), \tag{7}$$

Where L is linear operator, N is a nonlinear operator, and $g(x)$ is given continuous function. The basic character of method is to a correction functional for system Eq.(7) which

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\} d\tau, \tag{8}$$

Where $\lambda(\tau)$ is a general Lagrangian multiplier (Kaleva 1987) which can be identified optimally via variational theory, the subscript n denotes the n^{th}-order approximation and $\tilde{u}_n$ is consider a restricted variation, i.e. $\delta\tilde{u}_n = 0$ where

$L = \frac{d}{dt}$. For the integral equation (1), let $w(x)$ be a function such that $w'(x) = \tilde{u}(x)$, noting that $\tilde{u}(x)$ is continuous. Then we have

$$w'(x) = \tilde{f}(x) + \mu_1 \int_a^x K_1(x,t)G_1(t, w'(t))dt + \mu_2 \int_a^b K_2(x,t)G_2(t, w'(t))dt. \quad (9)$$

Consider

$$\mu_1 \int_a^x K_1(x,t)G_1(t, w'(t))dt + \mu_2 \int_a^b K_2(x,t)G_2(t, w'(t))dt, \quad (10)$$

as a restricted variation; we have the iteration sequence

$$w_{n+1} = w_n + \int_0^x \lambda \left[w_n'(s) - \mu_1 \int_a^s K_1(s,t)G_1(t, w'(t))dt - \mu_2 \int_a^b K_2(s,t)G_2(t, w'(t))dt - \tilde{f}(s) \right] ds.$$

Taking the variation with respect to the independent variable w_n and noticing that $\delta w_n(0) = 0$, we get

$$\delta w_{n+1} = \delta w_n + \lambda(s)\delta w_n|_{s=x} - \int_0^x \lambda'(s)\delta w_n ds = 0 \quad (11)$$

Then we apply the following stationary conditions:

$$1 + \lambda(s)|_{s=x} = 0, \quad \lambda'(s)|_{s=x} = 0,$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda = -1$$

and, as a result, we obtain the following iteration formula:

$$w_{n+1} = w_n - \int_0^x \left[w_n'(s) - \tilde{f}(s) - \mu_1 \int_a^s K_1(s,t)G_1(t, w'(t))dt - \mu_2 \int_a^b K_2(s,t)G_2(t, w'(t))dt \right] ds.$$

Therefore, we can write the following iteration formulas

$$\underline{u}_{n+1}(x,r) = \underline{u}_n(x,r) - \int_0^x \left[\underline{u}_n(s,r) - \underline{f}(s,r) - \mu_1 \int_a^s K_1(s,t)G_1(t, \underline{u}(t,r))dt - \mu_2 \int_a^b K_2(s,t)G_2(t, \underline{u}(t,r))dt \right] ds.$$

$$\overline{u}_{n+1}(x,r) = \overline{u}_n(x,r) - \int_0^x \left[\overline{u}_n(s,r) - \overline{f}(s,r) - \mu_1 \int_a^s K_1(s,t)G_1(t, \overline{u}(t,r))dt - \mu_2 \int_a^b K_2(s,t)G_2(t, \overline{u}(t,r))dt \right] ds.$$

3.3. Homotopy Analysis Method (HAM)

Consider,

$$N[\tilde{u}] = 0,$$

where N is a nonlinear operator, $\tilde{u} = [\underline{u}(x,r), \overline{u}(x,r)]$ are unknown functions and x is an independent variable [22]. Let $\underline{u}_0(x,r), \overline{u}_0(x,r)$ denote an initial guess of the exact solution $\underline{u}(x,r), \overline{u}(x,r)$, $h \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x)] = 0$ when $s(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we can construct a homotopy when consider, $N[\underline{u}] = 0$, as follows:

$$(1-q)L[\underline{\phi}(x;q,r) - \underline{u}_0(x,r)] - qhH_1(x)N[\phi(x;q,r)] = \hat{H}[\phi(x;q,r); \underline{u}_0(x,r), H_1(x), h, q]. \quad (12)$$

It should be emphasized that we have great freedom to choose the initial guess $\underline{u}_0(x,r)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x)$. Enforcing the homotopy Eq.(12) to be zero, i.e.,

$$\hat{H}_1[\underline{\phi}(x;q,r); \underline{u}_0(x,r), H_1(x), h, q] = 0, \quad (13)$$

we have the so-called zero-order deformation equation

$$(1-q)L[\underline{\phi}(x;q,r)-\underline{u}_0(x,r)] = qhH_1(x)N[\phi(x;q,r)]. \quad (14)$$

when $q = 0$, the zero-order deformation Eq.(14) becomes

$$\underline{\phi}(x;0,r) = \underline{u}_0(x,r). \quad (15)$$

and when $q = 1$, since $h \neq 0$ and $H_1(x) \neq 0$, the zero-order deformation Eq.(14) is equivalent to

$$\underline{\phi}(x;1,r) = \underline{u}(x,r). \quad (16)$$

Thus, according to Eqs.(15) and (16), as the embedding parameter q increases from 0 to 1, $\phi(x;q,r)$ varies continuously from the initial approximation $\underline{u}_0(x,r)$ to the exact solution $\underline{u}(x,r)$. Such a kind of continuous variation is called deformation in homotopy. Due to Taylor's theorem, $\phi(x;q,r)$ can be expanded in a power series of q as follows

$$\underline{\phi}(x;q,r) = \underline{u}_0(x,r) + \sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m, \quad (17)$$

where,

$$\underline{u}_m(x,r) = \frac{1}{m!}\frac{\partial^m \underline{\phi}(x;q,r)}{\partial q^m}|_{q=0}, \quad (18)$$

Let the initial guess $\underline{u}_0(x,r)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x)$ be properly chosen so that the power series Eq.(17) of $\underline{\phi}(x;q,r)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$\underline{u}(x,r) = \underline{\phi}(x;1,r) = \underline{u}_0(x,r) + \sum_{m=1}^{\infty} \underline{u}_m(x,r). \quad (19)$$

From Eq.(17) , we can write Eq.(14) as follows:

$$\begin{aligned}(1-q)L[\underline{\phi}(x;q,r)-\underline{u}_0(x,r)] &= (1-q)L[\sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m] \\ &= qhH_1(x)N[\underline{\phi}(x;q,r)]\end{aligned} \quad (20)$$

then,

$$L[\sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m] - qL[\sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m] = qhH_1(x)N[\underline{\phi}(x;q,r)]. \quad (21)$$

By differentiating Eq.(20) m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m] - qL[\sum_{m=1}^{\infty} \underline{u}_m(x,r)q^m]\}^{(m)} = hH_1(x)m\frac{\partial^{m-1}N[\underline{\phi}(x;q,r)]}{\partial q^{m-1}}|_{q=0}.$$

Therefore,

$$L[\underline{u}_m(x,r) - \chi_m \underline{u}_{m-1}(x,r)] = hH_1(x)\Re_m(\underline{u}_{m-1}(x,r)), \quad (22)$$

where,

$$\Re_m(u_{m-1}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial q^{m-1}}|_{q=0}, \quad (23)$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

Note that the high-order deformation Eq.(22) is governing the linear operator L, and the term $\Re_m(\underline{u}_{m-1}(x,r))$ can be expressed simply by Eq.(23) for any nonlinear operator N.

To obtain the approximation solution of Eq.(2), according to HAM

$$\begin{aligned}\Re_m(\underline{u}_{m-1}(x,r)) &= \underline{u}_{m-1}(x,r) - \underline{f}(x,r) - \mu_1 \int_a^x \underline{K_1(x,t)G_1(t,u(t,r))}dt \\ &- \mu_2 \int_a^b \underline{K_2(x,t)G_2(t,u(t,r))}dt - (1-\chi_m)\underline{f}(x,r), m \geq 1 \end{aligned} \tag{24}$$

Substituting Eq.(24) into Eq. (22)

$$\begin{aligned}L[\underline{u}_m(x,r) - \chi_m \underline{u}_{m-1}(x,r)] &= hH_1(x)[\underline{u}_{m-1}(x,r) - \mu_1 \int_a^x \underline{K_1(x,t)G_1(t,u(t,r))}dt \\ &- \mu_2 \int_a^b \underline{K_2(x,t)G_2(t,u(t,r))}dt - (1-\chi_m)\underline{f}(x,r)]. \end{aligned} \tag{25}$$

We take an initial guess $\underline{u}_0(x,r) = \underline{f}(x,r)$, an auxiliary linear operator $L\underline{u} = \underline{u}$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x) = 1$. This is substituted into Eq.(25) to give the recurrence relation

$$\begin{aligned}\underline{u}_0(x,r) &= \underline{f}(x,r) \\ \underline{u}_{n+1}(x,r) &= \mu_1 \int_a^x \underline{K_1(x,t)G_1(t,u_n(t,r))}dt + \mu_2 \int_a^b \underline{K_2(x,t)G_2(t,u_n(t,r))}dt, n \geq 0. \end{aligned} \tag{26}$$

Similarly, we can construct a homotopy when consider, $N[\overline{u}] = 0$, to give the recurrence relation

$$\begin{aligned}\overline{u}_0(x,r) &= \overline{f}(x,r) \\ \overline{u}_{n+1}(x,r) &= \mu_1 \int_a^x \overline{K_1(x,t)G_1(t,u_n(t,r))}dt + \mu_2 \int_a^b \overline{K_2(x,t)G_2(t,u_n(t,r))}dt, n \geq 0. \end{aligned} \tag{27}$$

From Eqs.(26), and Eqs.(27) we approximate $\tilde{u}(x,r) = [\underline{u}(x,r), \overline{u}(x,r)]$ by

$$\underline{u}(x,r) = \lim_{n\to\infty} \underline{u}_n, \quad \overline{u}(x,r) = \lim_{n\to\infty} \overline{u}_n.$$

4. NUMERICAL EXAMPLE

In this section, we solve the fuzzy Volterra-Fredholm integral equation of the second kind by the ADM, VIM and HAM.

Example 4.1

Consider the fuzzy Volterra-Fredholm integral equation of the second kind as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x sin(x)sin(\frac{t}{2})\tilde{u}^3(t)dt + \int_0^{0.6} sin(\frac{x}{2})sin(t)(1+\tilde{u}^2(t))dt, \tag{28}$$

whear,

$$\underline{f}(x,r) = \sin(\frac{x}{2})(\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r^3-r),$$
$$\overline{f}(x,r) = \sin(\frac{x}{2})(\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r),$$

and,

$$r = 0.3, \quad \epsilon = 10^{-2}, \quad 0 \leq x,t \leq 0.6.$$

x	$ADM_{(n=11)}$	$VIM_{(n=4)}$	$HAM_{(n=4)}$
0.1	0.2203548375	0.220466127	0.2204663982
0.2	0.3062332542	0.306329751	0.3063488741
0.3	0.4035946723	0.403659665	0.4037996457
0.4	0.5233741235	0.523379658	0.5234862764
0.5	0.5964831157	0.614656263	0.6259432736
0.6	0.6523678927	0.652356871	0.6524855123

Table 1. The Obtained Solutions for Example 4.1

The above table show comparison between the approximate solutions by using ADM, VIM and HAM for results of the example 4.1 .

5. CONCLUSION

We discussed the different methods for solving fuzzy Volterra-Fredholm integral equations, namely, Adomian decomposition method, variational iteration method and homotopy analysis method. To assess the accuracy of each method, the test example with known exact solution is used. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, HAM is the easiest, the most efficient and convenient.

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