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A Study of the Galitskii–Feynman *T* Matrix for Liquid ^{3}He

H. B. Ghassib

*Department of Theoretical Physics, University of Manchester, Manchester, England, and Department of Physics, University of Jordan, Amman, Jordan**

R. F. Bishop

Department of Theoretical Physics, University of Manchester, Manchester, England, and Science Research Council, Daresbury Laboratory, Daresbury, Warrington, England

and M. R. Strayer

Science Research Council, Daresbury Laboratory, Daresbury, England

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The Galitskii–Feynman T *matrix, which sums the infinite ladder series in a many-fermion system for both particle–particle and hole–hole scattering, is studied in detail for a family of realistic He–He interactions. The structure of the* S*-wave bound-state singularity, reported previously, and its dependence on the bare interaction are documented at length. Special attention is devoted to the* T *matrix in the scattering region, where the c.m. energy of the interacting pair is positive. In particular, the on-energy-shell* T *matrix in this region is parametrized in terms of real "effective" phase shifts incorporating many-body effects. The critical behavior discussed previously in the bound-state region manifests itself clearly in the zero-energy limit of these phase shifts for the* S *wave. Below (above) a certain critical density, which is a function of both temperature and c.m. momentum, this limit approaches the value* $0(-\pi)$ *radians. A generalized Levinson's theorem relates this behavior to the existence of fermion–fermion pairing. An especially striking feature of these many-body phase shifts is the cusp behavior exhibited at the Fermi surface in the low-temperature limit, which turns out to arise essentially from the structure of the particle and hole occupation probabilities. Throughout this study the temperature dependence of the* T *matrix is particularly emphasized.*

*Permanent address.

1. INTRODUCTION

A dense Fermi system such as liquid ^{3}He may be studied theoretically within the framework of either a phenomenological or a microscopic approach. Phenomenological theory, typified by the Landau theory of Fermi fluids,[1] has long provided a most lucid, self-consistent description of normal Fermi fluids. A relatively few parameters are determined experimentally and then used to predict other phenomena. The validity and applications of this theory, together with its manifold extensions and finite-temperature corrections,[2] form the substance of an impressive corpus.

The microscopic approach, on the other hand, starts with the properties of the constituent ^{3}He atoms and usually, but not necessarily, with the assumption that these atoms interact pairwise via a static binary potential. Since the electronic excitation energies of the ^{3}He atom[3] are typically 10^5 times as large as the binding energy of the liquid per atom,[4] the excited states of the atoms need not appear explicitly in our theoretical description of the liquid state. The atoms, then, may be treated as the elementary particles of the many-body system from which the bulk and transport properties can be derived using one of three major approaches. In the first place there is the generalized variational treatment in the manner of Feenberg and his school.[5] Secondly, there are the powerful methods of quantum statistical mechanics which Mohling and his collaborators[6] have developed, following the pioneering work of Lee and Yang,[7] and which are highly reminiscent of the more familiar techniques used by Ursell and Mayer in their theory of the equation of state of a classical real gas.[8] Finally, we consider such generalized perturbation approaches as the Brueckner–Bethe–Goldstone[9] and Galitskii–Feynman[10] formalisms, whose fundamental starting point is the evaluation of some density and temperature-dependent effective interaction for the quasiparticles in the many-body medium. The self-energy, and hence all physical observables, of the system can then be expressed in terms of this effective interaction.[11]

In this study we focus our main attention on the Galitskii–Feynman (GF) formalism, which has been the subject of two recent communications[12] (henceforth referred to as I and II, respectively). In these two papers the zero and finite-temperature GF formalisms have been outlined, and preliminary results presented for the GF T matrix in the bound-state region. Since this matrix plays the role of the effective interaction in the theory, a thorough investigation of its properties is essential prior to any more comprehensive calculations of physical observables. Unlike the analogous quantity in the Brueckner–Goldstone (BG) theory which sums repeated particle–particle scatterings outside the Fermi sea, the GF T matrix sums these as well as repeated hole–hole scatterings inside the sea. The holes and particles are, therefore, treated symmetrically in the GF formalism. Another aesthetically very appealing advantage of this formalism is the simple manner in which it incorporates the temperature dependence, which enters solely through the one-body Green's functions representing particle and hole propagation.

In this paper we undertake a detailed investigation of the GF T matrix. We both document at length the structure of the S-wave bound-state singularity appearing as a first-order pole in this matrix, as established in I and II, and also extend our treatment to the scattering region where the c.m. energy of the interacting pair is positive. In particular, we use a generalized unitarity relation to parametrize our on-energy-shell T matrix in this region in terms of real "effective" phase shifts incorporating many-body effects. These phase shifts turn out to be very useful in analyzing the critical behavior reported previously.[12] Below (above) a certain critical density, which is a function of both temperature and c.m. momentum, the zero-energy limit of these phase shifts approaches the value $0(-\pi)$ radians. A generalized Levinson's theorem[13] then relates this limit unambiguously to the existence of pairing in the GF formalism corresponding to pairing between quasiparticles which are linear combinations of particle and hole states. This behavior is found to exist in the S wave only, for the particular potentials studied.

An especially striking feature of these many-body S-wave phase shifts is the cusp behavior exhibited at the Fermi surface in the low-temperature limit ($\lesssim 0.2$ K). This effect arises essentially from our definition of these phase shifts which involves the particle and hole occupation probabilities: it is not related in any direct manner to the cusps associated with two-channel scattering processes,[14] nor to the resonance phenomena familiar in, for example, nuclear and atomic physics.[15]

In passing we introduce a new integration mapping which improves, in certain cases, the accuracy of matrix-inversion techniques[16] in solving the GF integral equation, which is singular in the scattering region. We believe that persistent attempts of this kind are essential in the continuing attempt to find more efficient methods for solving singular integral equations of the kind encountered here.

The remainder of this paper is divided as follows. In Sec. 2 we outline the main ingredients of the GF formalism for an arbitrary temperature. We then present details of our numerical methods (Sec. 3) followed by our results (Sec. 4). Finally we discuss these results in Sec. 5.

2. THE GF FORMALISM

We consider an extended system of N ^{3}He atoms, mass M each, occupying a volume Ω at zero pressure and an arbitrary temperature. This system has already been examined, in its zero-temperature limit, within the framework of both the Brueckner–Bethe–Goldstone[17] and Mohling[18] formalisms. As usual when studying such a many-body system, we shall be interested in the thermodynamic limit $N \to \infty, \Omega \to \infty$. The ratio N/Ω, however, remains constant as the limit is taken, and is simply the normal density of the system defined by

$$\rho = N/\Omega = k_F^3/3\pi^2 \qquad (1)$$

for spin-$\frac{1}{2}$ particles with no other internal quantum numbers, and where k_F is the Fermi momentum.*

It is now well established experimentally that this system has a "λ point"[19]: at zero pressure, it undergoes a phase transition at a temperature of 0.93 mK. However, since our formalism is not sufficiently well developed at present to handle this and other related phenomena in this field,[20] we shall ignore them. In fact, our intention in this section is merely to outline briefly the principal ingredients of the T-matrix formalism, since this has already been described in I and II.

In the c.m. frame of the interacting pair the GF T matrix is defined operationally by the equation

$$T(s, \mathbf{P}, \beta) = u - u[Q(\mathbf{P}; \beta)g_0(s) - \bar{Q}(\mathbf{P}; \beta)g_0^\dagger(s)]T(s, \mathbf{P}, \beta) \qquad (2)$$

where the parametric dependence on s, $\mathbf{P}$, and β is shown explicitly, and where these parameters denote, respectively, the total energy of the interacting pair in the c.m. frame, their average momentum, and the inverse temperature. The operator $u = M\hbar^{-2}V = \frac{1}{2}V$, where V is the bare He–He interaction. The free two-body Green's function $g_0(s)$ is specified by $g_0(s) = (\frac{1}{2}H_0 - s - i\eta)^{-1}$, H_0 being the kinetic energy operator and η a positive infinitesimal in the scattering region ($s > 0$) and zero otherwise, and where the factor $\frac{1}{2}$ is equal to twice the reduced mass in our system of units. Finally, $Q(\bar{Q})$ is the product of two single-particle (hole) occupation probabilities. In momentum space the single-particle occupation probability is just the Fermi–Dirac distribution function, which reduces to the unit-step function at zero temperature. When subtracted from unity, this yields the single-hole occupation probability.

The integral counterpart of Eq. (2) is trivially obtained for our translationally-invariant system in the relative momentum representation, using closure

$$T(\mathbf{p}, \mathbf{p}'; s, \mathbf{P}, \beta) = u(\mathbf{p} - \mathbf{p}') - \int \frac{d\mathbf{k}}{(2\pi)^3} u(\mathbf{p} - \mathbf{k})[Q(\mathbf{k}; \mathbf{P}, \beta)g_0(k, s) - \bar{Q}(\mathbf{k}; \mathbf{P}, \beta)g_0^\dagger(k, s)]T(\mathbf{k}, \mathbf{p}'; s, \mathbf{P}, \beta) \qquad (3)$$

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*Throughout this work we use units such that $\hbar = 2M = k_B = 1$, where k_B is the Boltzmann constant. The conversion unit is $\hbar^2/2M = 8.0425$ K Å^2.

where the relative momenta $\mathbf{p}$, $\mathbf{p}'$ and the average momentum $\mathbf{P}$ are related to the individual momenta of the interacting pair by the equations

$$\begin{aligned} \mathbf{p} &= \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2); \qquad \mathbf{p}' = \tfrac{1}{2}(\mathbf{p}_1' - \mathbf{p}_2') \\ \mathbf{P} &= \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) = \tfrac{1}{2}(\mathbf{p}_1' + \mathbf{p}_2') \end{aligned} \tag{4}$$

To proceed further we then make the customary approximation of angle-averaging the Q and $\bar{Q}$ operators. The resulting expressions are listed in Table I for convenience. This approximation renders it possible to decompose T into the uncoupled partial-wave expansion

$$T(\mathbf{p}, \mathbf{p}'; s, \mathbf{P}, \beta) = \sum_l (2l + 1) T_l(p, p'; s, P, \beta) P_l(\hat{p} \cdot \hat{p}') \tag{5}$$

With this and a similar equation for u, Eq. (3) can finally be reduced to

$$T_l(p, p'; s, P, \beta) = u_l(p, p') - \int_0^\infty \frac{k^2}{2\pi^2} dk\, u_l(p, k) \left[\frac{Q(k, P, \beta)}{k^2 - s - i\eta} - \frac{\bar{Q}(k, P, \beta)}{k^2 - s + i\eta} \right] \times T_l(k, p'; s, P, \beta) \tag{6}$$

where

$$u_l(p, p') = 4\pi \int_0^\infty r^2\, dr\, j_l(pr) u(r) j_l(p'r) \tag{7}$$

and $j_l(x)$ is the spherical Bessel function of order l.

TABLE I

Résumé of the Angle-Averaged Q and $\bar{Q}$ Operators for $\beta^{-1} \geq 0$

β^{-1}	$Q(k, P, \beta)$		$\bar{Q}(k, P, \beta)$	
$\beta^{-1} = 0$	$P < k_F$			
	0.	$0 \leq k < (k_F^2 - P^2)^{1/2}$	1;	$0 \leq k < k_F - P$
	$(P^2 + k^2 - k_F^2)/2Pk$;	$(k_F^2 - P^2)^{1/2} \leq k \leq k_F + P$	$(k_F^2 - P^2 - k^2)/2Pk$;	$k_F - P \leq k \leq (k_F^2 - P^2)^{1/2}$
	1;	$k > k_F + P$	0;	$k > (k_F^2 - P^2)^{1/2}$
	$P > k_F$			
	1.	$0 \leq k \leq P - k_F$	0;	everywhere
	$(P^2 + k^2 - k_F^2)/2Pk$;	$P - k_F \leq k \leq P + k_F$		
	1;	$k > P + k_F$		
$\beta^{-1} > 0$	$\{2\beta Pk[1 - \exp(-2\beta X)]\}^{-1} L(k, P)$		$\{2\beta Pk[\exp(2\beta X) - 1]\}^{-1} L(k, P)$	
	$L(k, P) \equiv \ln \left(\frac{\cosh \{\frac{1}{2}\beta[(P + k)^2 - k_F^2]\}}{\cosh \{\frac{1}{2}\beta[(P - k)^2 - k_F^2]\}} \right)$			
	$X \equiv P^2 + k^2 - k_F^2$			

The solution of Eq. (6) represents our fully-off-shell GF T matrix pertaining to a relative partial-wave l. The on-energy-shell counterpart $T_l(p; P, \beta)$ is obtained directly from this by setting $p = p'$ and $s = p^2$. The T matrix is an implicit function of k_F, this dependence arising via the Q and $\bar{Q}$ operators. On setting $\bar{Q} = 0$, hole–hole scattering is switched off and the BG T matrix is obtained. Since liquid ^{3}He is a relatively dense system, in the sense that the range of the two-body interaction is commensurate with the ^{3}He–^{3}He interatomic spacing, hole–hole scattering is not expected to be insignificant. Accordingly, the GF formalism should describe this system better than BG. The accurate solution of the integral Eq. (6) in both the bound-state ($s < 0$) and scattering ($s > 0$) regions provides our central theme in this work. We note that in the free-scattering limit $Q(\bar{Q}) \to 1(0)$, so that T reduces to the familiar Lippmann–Schwinger t matrix.

In the formal theory of scattering it is well known[13] that the probability amplitude for a transition of a system from some initial to another final state may be described by a scattering matrix S. The unitarity of S for Hermitian hamiltonians, representing conservation of the flux of particles participating in the scattering process, leads to a parametrization of this matrix in terms of real phase shifts $\delta_l(p)$. This, in turn, leads to the following parametrization of the on-energy-shell Lippmann–Schwinger t matrix[21]:

$$t_l(p) = -(4\pi/p) \exp [i\delta_l(p)] \sin \delta_l(p) \tag{8}$$

Likewise, we may use a generalized unitarity relation to parametrize $T_l(p; P, \beta)$ in terms of real "effective" many-body shifts $\delta_l^E(p; P, \beta)$ according to[22]

$$\begin{aligned} T_l(p; P, \beta) = &-(4\pi/p)[Q(p; P, \beta) + \bar{Q}(p; P, \beta)]^{-1} \\ &\times \exp [i\delta_l^E(p; P, \beta)] \sin \delta_l^E(p; P, \beta) \end{aligned} \tag{9}$$

whence

$$\delta_l^E(p; P, \beta) = \arctan [\mathrm{Im}\, T_l(p; P, \beta)/\mathrm{Re}\, T_l(p; P, \beta)] \tag{10}$$

Re T_l(Im T_l) denoting the real (imaginary) part of T_l.

These latter two quantities can most easily be extracted from Eq. (6) through a real matrix K defined by an integral equation identical to (6), save for taking the principal value (PV) of the integral. Using the standard relations

$$\mathrm{PV}[g_0(s)] = \lim_{\eta \to 0} \tfrac{1}{2}[g_0(s) + g_0^\dagger(s)] \tag{11}$$

$$1/(k^2 - s \mp i\eta) = \mathrm{PV}/(k^2 - s) \pm i\pi\delta(k^2 - s) \tag{12}$$

and

$$\delta(q^2 - k^2) = (1/2q)\delta(q - k) \tag{13}$$

where PV indicates that the principal value of the implied integral should be taken, we finally obtain, after some simple algebra,

$$\operatorname{Re}[T_l(p, p'; s, P, \beta)] = K_l(p, p'; s, P, \beta) - \frac{f_l^2(s, P, \beta)}{1 + f_l^2(s, P, \beta)} \times \left[\frac{K_l(p, \kappa; s, P, \beta) K_l(\kappa, p'; s, P, \beta)}{K_l(\kappa, \kappa; s, P, \beta)}\right] \tag{14}$$

and

$$\operatorname{Im}[T_l(p, p' : s, P, \beta)] = -\frac{f_l(s, P, \beta)}{1 + f_l^2(s, P, \beta)} \times \left[\frac{K_l(p, \kappa : s, P, \beta) K_l(\kappa, p'; s, P, \beta)}{K_l(\kappa, \kappa; s, P, \beta)}\right] \tag{15}$$

for $s > 0$, where

$$f_l(s, P, \beta) \equiv (\kappa/4\pi)[Q(\kappa : P, \beta) + \bar{Q}(\kappa; P, \beta)] K_l(\kappa, \kappa; s, P, \beta) \tag{16}$$

and

$$\kappa \equiv +s^{1/2} \tag{17}$$

For $s < 0$ we define

$$\operatorname{Re}[T_l(p, p' : s, P, \beta)] = K_l(p, p' : s, P, \beta) \tag{18}$$

and

$$\operatorname{Im}[T_l(p, p'; s, P, \beta)] = 0 \tag{19}$$

The on-energy-shell quantities Re $T_l(p; P, \beta)$ and Im $T_l(p; P, \beta)$ can then be immediately obtained from (14) and (15) by putting $p = p' = \kappa$.

We have now all the essential tools for a numerical study of the GF *T* matrix.

3. THE ^{3}He–^{3}He POTENTIAL AND NUMERICAL METHODS

The dynamic interaction between two ^{3}He atoms is represented by a binary potential, as usual in nonrelativistic physics. This is purely central, unlike the nucleon–nucleon potential which is generally spin and isospin dependent and which has tensor components and other state dependence. The importance of the He–He potential can hardly be exaggerated here, in view of the sensitivity of many-body helium calculations to its exact shape,[17] and since it is the basic input to any microscopic formalism for liquid helium. The standard philosophy in this respect is based on the following dichotomy: if we could trust our two-body potential, we might affirm something about our many-body formalism; alternatively, if we could trust this formalism, we might end up with some conclusions about the He–He potential itself. However, since the theory we are attempting to construct within the GF formalism is still in its infancy, and since the helium literature abounds with potentials all professing to be suitable candidates for liquid ^{3}He, we are forced to employ more than one potential in this study. In fact, we have investigated five different potentials, including the most highly acclaimed. These are listed in Table II. We have presented a critical survey of their prime characteristics elsewhere.[27] Note that these are all Fourier-transformable; in particular, the MFM potential has an analytic Fourier transform given by

$$u_l(p, p') = 4\pi[A_1 I_l^{(1)}(p, p') + A_2 I_l^{(2)}(p, p')] \tag{20}$$

where

$$\begin{aligned} I_l^{(1)}(p, p') &= [\mu/2(pp')^2]Q_l'(z_\mu) \\ I_l^{(2)}(p, p') &= (1/2pp')Q_l(z_\mu) \\ z_\mu &= (p^2 + p'^2 + \mu^2)/2pp' \end{aligned} \tag{21}$$

and where $(A_1; A_2; \mu) = (-2.11495 \times 10^4\ \text{Å}^{-2};\ 5.60304 \times 10^4\ \text{Å}^{-2};\ 2.68792)$. The Q_l's are Legendre functions of the second kind, and the prime denotes their first derivative with respect to z_μ. We have evaluated Q_l in terms of which Q_l' may be expressed, using standard[28] recursion relations for z_μ less than some $z_{\min}$ (~ 2.0), and hypergeometric functions for $z_\mu > z_{\min}$. As for the other potentials quoted in Table II, we have computed their Fourier transforms numerically using Gaussian quadrature, together with an appropriate mapping.[12,17]

Once again we have found the matrix inversion method of Haftel and Tabakin[16] most helpful in solving the GF integral equation (6) in both the bound-state and scattering regions. As a check this procedure was first tested in the free-scattering limit.[27] To achieve fairly stable and accurate

TABLE II

List of the He–He Potentials Studied in This Work

Potential	Abbreviation	Ref.
Frost–Musulin of Bruch and McGee (1970)	FDD-1	23
Morse of Bruch and McGee (1970)	MDD-2	23
Beck (1968)	B	24
Sposito (1970)	S	25
Modified Frost–Musulin (1954, 1974)	MFM	26, 12

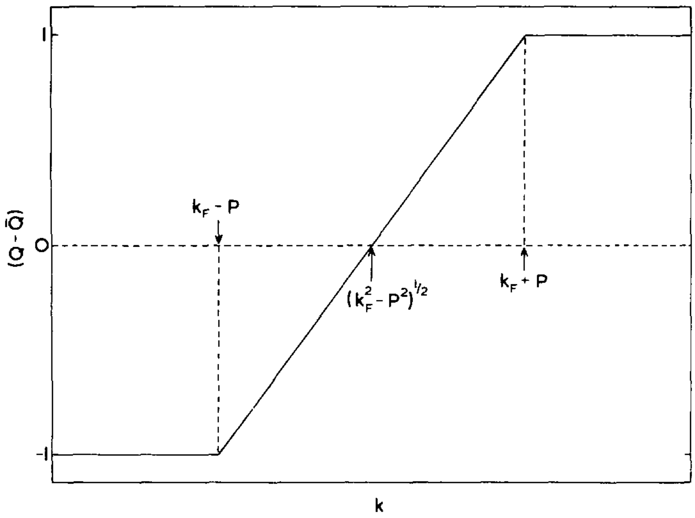

Fig 1. The structure of the angle-averaged operator $(Q - \bar{Q})$ as a function of k, illustrating its symmetry about the point $(k_F^2 - P^2)^{1/2}$ (cf. Table I).

results we have mainly used Gaussian meshes of size ≥ 96 points. The sole additional feature to the method (as expounded in I) is the scaling factor $\lambda(k_F, P)$ we have associated here with the mapping

$$k = \lambda(k_F, P) \tan \tfrac{1}{4}\pi(1 + x) \tag{22}$$

which maps a set of Gaussian quadrature points x_i given on the interval $(-1, 1)$ into a set of points k_i suitable for integrals over the interval $(0, \infty)$, and where

$$\lambda(k_F, P) = (k_F^2 - P^2)^{1/2} \tag{23}$$

Our motivation for introducing this mapping is its property of distributing the given Gaussian points symmetrically about the point $(k_F^2 - P^2)^{1/2}$. This is highly desirable in view of the structure of the operator $(Q - \bar{Q})$ needed in evaluating the principal value integral in Eq. (6). This is sketched in Fig. 1.

4. RESULTS

We are concerned here mainly with the properties of the S-wave GF T matrix in both the bound-state and scattering regions, since the interesting physics we shall be discussing below is confined to this relative partial wave only. However, for completeness we shall occasionally allude to the P and D waves.

In the bound-state region ($s < 0$) the GF integral equation (6) is nonsingular and the T matrix is real [Eqs. (18) and (19)]. As established in I and II, this matrix has a singularity in the s plane appearing as a first-order pole at s_0 with factorizable residue at the pole. The quantity s_0, denoting the binding energy of the bound-state pair embedded in the many-fermion medium, is a function of k_F (or, equivalently, ρ), P, and β^{-1}. The basic point is that for fixed P and β^{-1} there exists a critical Fermi momentum $k_{F_c}(P, \beta^{-1})$ at which s_0 is zero and below which no binding occurs. Alternatively, for fixed k_F and β^{-1} (or P) a critical c.m. momentum $P_c(k_F, \beta^{-1})$ [or a critical temperature $\beta_c^{-1}(k_F, P)$] exists above which no binding occurs.

This behavior has been amply discussed in I and II. Motivated by our new choice of mapping, Eqs. (22) and (23), we have repeated our former calculations for the MFM potential and extended them for the other four potentials listed in Table II. Our new results for $s < 0$ are displayed in Tables III and IV and in Figs. 2 through 4. It is gratifying to note that the previous numerical instabilities (conspicuous, for example, in Fig. 2 of I) have vanished, thereby vindicating our expectations of the new mapping. This is also evident from Table III, where we compare the values $k_{F_c}(P = 0, \beta^{-1} = 0)$ for the five potentials obtained with both the new and old mappings. We note that with the latter mapping these quantities are difficult to determine accurately: they seem to lie exactly at the mapped mesh points.[16] Table III also shows how close the values k_{F_c} are for all five potentials.

Similar conclusions apply to P_c and β_c^{-1}. However, as is clear from Figs. 2 and 3, the values β_c^{-1} correspond to temperatures sufficiently high for quantum effects to be negligible by the correspondence principle; the system then becomes purely classical in behavior. For orientation purposes in this respect we list in Table V some temperatures appropriate to normal liquid ^{3}He.

TABLE III

The Critical Fermi Momentum k_{F_c} for $P = 0$ Å^{-1} and $\beta^{-1} = 0$ for the Five Potentials Listed in Table II[a]

	$k_{F_c}(P = 0, \beta^{-1} = 0)$, Å^{-1}	
Potential	New mapping	Old mapping
FDD-1	0.742	0.753
MDD-2	0.718	0.715
B	0.722	0.715
S	0.689	0.678
MFM	0.714	0.715

[a] This should be compared with the experimental value $k_F = 0.788$ Å^{-1} (Ref. 29)

TABLE IV

The Critical c.m. Momentum P_c as a Function of k_F for $\beta^{-1} = 0$[a]

k_F, Å^{-1}	$P_c(k_F; \beta^{-1} = 0)$, Å^{-1}
0.75	0.362
0.78	0.467
0.82	0.566
0.86	0.645
1.00	0.857
1.50	1.44

[a] The potential is the MFM. Note that P_c increases with increasing k_F (cf. Figs. 3 and 4 for β_c^{-1}).

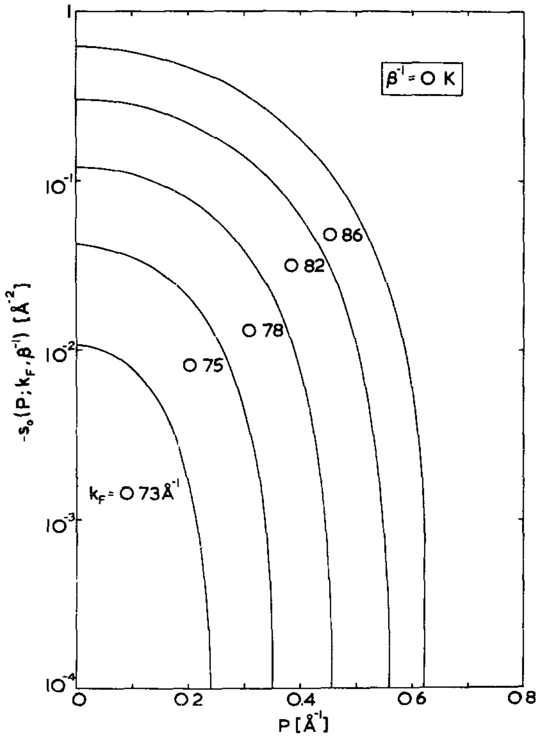

Fig. 2. The position of the bound-state singularity $s_0(P; k_F, \beta^{-1})$ in the S-wave GF T matrix as a function of P for $\beta^{-1} = 0$ and for a family of k_F. The potential here and in the remaining figures is the MFM.

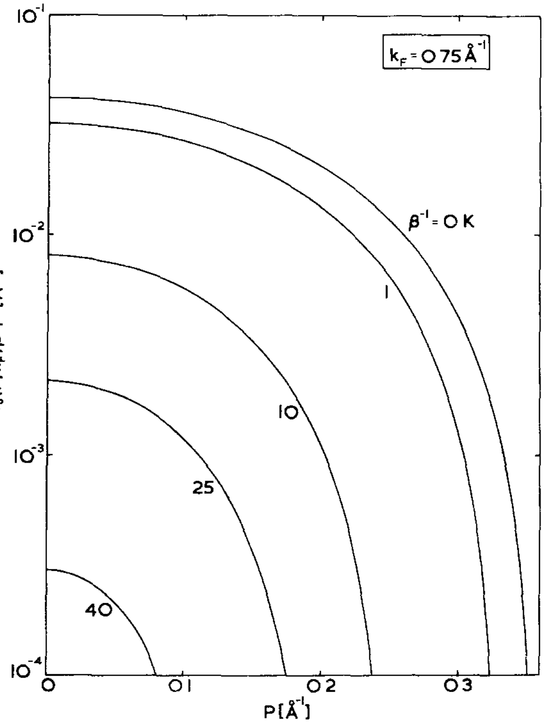

Fig. 3 $s_0(P, k_F, \beta^{-1})$ as a function of P for $k_F = 0.75$ Å^{-1} and for a family of β^{-1}.

On the other hand, in the scattering region ($s > 0$) the GF equation is singular and T is complex [Eqs. (14) and (15)]. Both Re T_0 and Im T_0 are well behaved in the s plane. However, as soon as we examine the effective phase shifts δ_l^E defined by Eq. (10) the picture abruptly becomes more complex. This is illustrated in Figs. 5 through 8. There are two pertinent features worthy of note here. The first is the peak in δ_0^E occurring at the Fermi surface. In the low-temperature limit this manifests itself as a cusp (Figs. 5 and 6) which becomes gradually depleted as β^{-1} increases. This peak seems to persist for higher relative partial waves as well (cf. Fig. 8 for the P and D waves), although it no longer occurs at the Fermi surface.

The second, physically more interesting, feature is the behavior of δ_l^E at zero relative momentum, p. Below k_{F_c} (or above P_c or β_c^{-1}), $\delta_0^E(p = 0;$

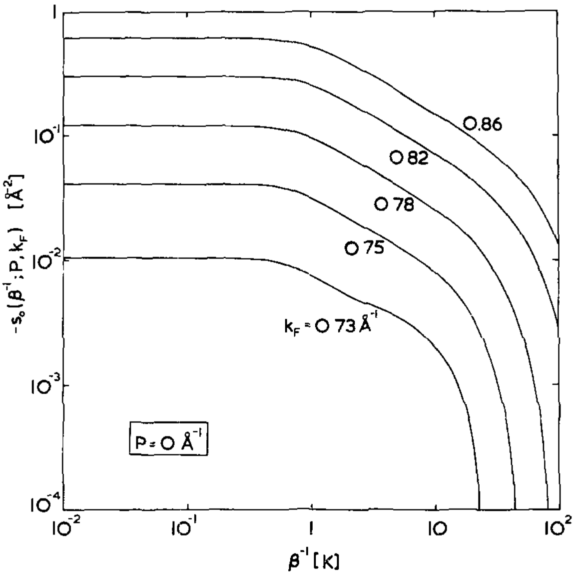

Fig. 4. $s_0(\beta^{-1}, P, k_F)$ as a function of β^{-1} for $P = 0$ Å^{-1} and for a family of k_F

$P, \beta^{-1}) = 0$; whereas above k_{F_c} (or below P_c or β_c^{-1}), $\delta_0^E(p = 0; P, \beta^{-1}) = -180°$. This critical behavior occurs only for the S wave. We shall presently proceed to analyze these results. Prior to this, however, we emphasize again the identical qualitative behavior of all five He–He potentials. In particular, for the parameters we have selected in plotting Figs. 5–8, it is not possible to distinguish between the potentials on the scale used.

TABLE V

Some Temperatures Appropriate to Liquid ^{3}He

Description	Temperature, K
Liquid–gas critical temperature[a]	3.32
Normal boiling point[b]	3.19
The Fermi temperature for a dilute ^{3}He gas[b]	~5.0
The experimental degeneracy temperature (at, or below, which quantum effects manifest themselves)[b]	~0.1

[a] Reference 30.
[b] Reference 2.

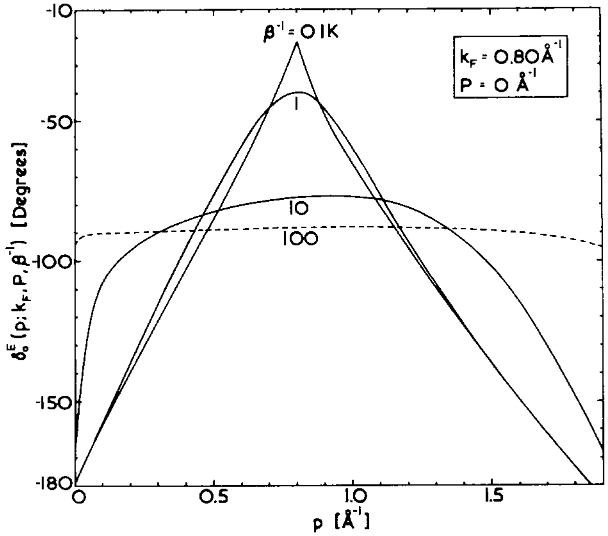

Fig. 5. The S-wave effective phase shifts δ_0^E, given in degrees, as a function of the relative momentum p, showing the temperature dependence. $P = 0$ and $k_F = 0.80$ Å^{-1}

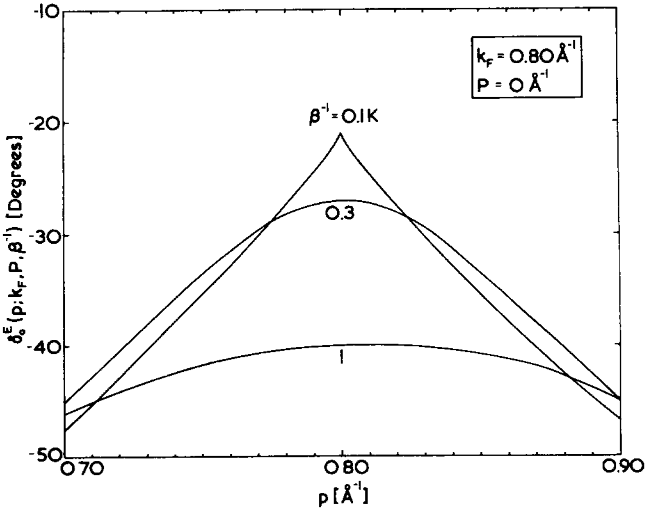

Fig 6. The cusps in Fig. 5 shown here magnified.

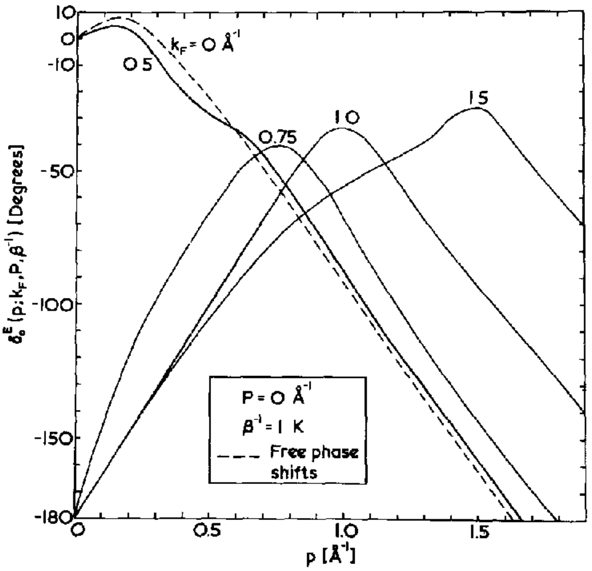

Fig. 7 The dependence of δ_0^E on k_F. $P = 0$ Å^{-1} and $\beta^{-1} = 1$ K.

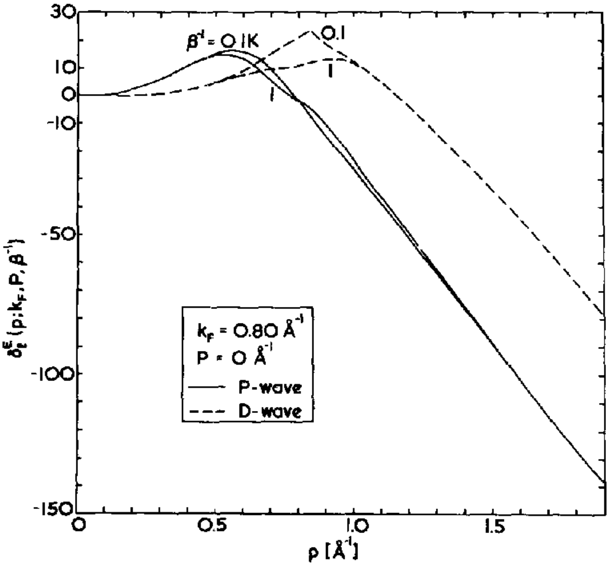

Fig. 8. δ_l^E for the P and D relative partial waves for the two temperatures 0.1 and 1.0 K. $P = 0$ and $k_F = 0.80$ Å^{-1}.

5. DISCUSSION AND CONCLUSIONS

We first dismiss the above cusp behavior as an effect arising from our definition of δ_l^E [Eqs. (9) and (10)]. It is intimately related to the structure of the Q and $\bar{Q}$ operators. This we have rigorously shown elsewhere,[22] and we shall not discuss this matter any further here.

The principal conclusion of our results in the bound-state region is that above some onset density, which is fairly close to the experimental density, our system is unstable against the formation of fermion–fermion pairs in the first-order GF theory.

That this pairing occurs is also confirmed on inspecting closely our results for δ_0^E. To demonstrate this we first recall that, in scattering theory, the number N_l of bound states for a given l and the corresponding phase shift $\delta_l(0)$ at zero energy (defined as the limit from positive p) are related via Levinson's theorem,[13] which is valid for the type of two-body interaction we are interested in here (see, however, Ref. 31). The basic content of this theorem can be summarized by the statement

$$\delta_l(0) - \delta_l(\infty) = \pi N_l \tag{24}$$

where $\delta_l(\infty)$ is normally set to zero by convention. We have generalized Levinson's theorem to our many-body system[22] in a form suitable for interpreting the above results. More specifically we have shown that for the case under consideration here,

$$\delta_0^E(0; P, \beta^{-1}) - \delta_0^E(\infty; P, \beta^{-1}) = -\pi N_0 \tag{25}$$

where $\delta_0^E(\infty; P, \beta^{-1}) = 0$, by definition. It is clear, then, that the above critical behavior exhibited by our S-wave effective phase shifts is simply a reflection of the existence of fermion–fermion pairs in our system.

We are now studying this pairing phenomenon on a deeper level from two different, but complementary, angles. In the first place, regarding our bound ^{3}He–^{3}He pairs as boson impurities, we are examining the possible consequences of their existence in our many-fermion system. In particular, we are attempting to establish whether these boson impurities may condense to form a macroscopically occupied state. In this respect, it is interesting to note that Kohn and Sherrington[32] have categorized bosons into two basic types. Type I bosons are bound complexes of an even number of fermions (or fermion-holes), such as ^{4}He atoms. Type II bosons are elementary excitations which are bound complexes of fermions and their holes, such as excitons and spin waves (see, e.g., Ref. 33). The important point is that when a system of the first type condenses, a superfluid state results, unlike the case with the second type, where no superfluidity results, but merely a change in spatial order (affecting, however, other properties of the system). The superfluid state in the former case is characterized by the so-called off-diagonal long-range order—a concept introduced by Yang.[34] (Perhaps Bose–Einstein condensation is the simplest form of this order.) Our investigations along these lines should also be enlightening in the context of two other most interesting systems—namely, ^{4}He atoms in liquid ^{3}He and superdense nucleonic matter containing pion impurities. The latter system is presently a subject of intense interest—especially in connection with neutron-star theory (see, e.g., Ref. 35).

Secondly, we have outlined a program to evaluate the proper self-energy, Σ^* (and, hence, the thermodynamic properties), of the system in the following manner. We have expressed Σ^* in terms of its real and imaginary components. Our plan is to compute as accurately as possible the real component, and to then use it to determine the imaginary component (and, hence, time-dependent properties) via an appropriate dispersion relation. We have separated the real component, in its turn, into a "background" part and another pole-dominated part, recalling in the latter case that[12]

$$T_0(p, p'; s, P, \beta^{-1}) \xrightarrow[s \to s_0]{} -\frac{R(p; P, \beta^{-1})R(p'; P, \beta^{-1})}{s - s_0(k_F, P, \beta^{-1})} \tag{26}$$

with the product of the functions R being the (factorizable) residue at the pole, s_0.

Our preliminary results are encouraging and we shall report them elsewhere.

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