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A study of the generalized Faraday effect in several media

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Abstract. The Faraday effect has been investigated in various fields of study since 1845. In this paper, we provide a unified formulation of the effect for several media. In the classical case, the direction of propagation is parallel to the magnetic field. Here we generalize to any direction of propagation. When describing the state of wave polarization, we find it convenient to use the complex polarization ratio R plane. On the complex R plane, the change of polarization as the wave propagates follows a circle. The resulting polarization loci are illustrated for three different lossless media: a biaxial medium, a chiral medium, and a magnetoionic medium. Through the analysis of the patterns of these loci, we can visualize the generalized Faraday effect on the polarization state diagram and also gain some physical insights into these media in terms of their wave polarization characteristics.

1. Introduction

In 1845, M. Faraday discovered experimentally that when a linearly polarized light wave is incident on a medium (such as a block of glass) with an externally applied longitudinal magnetic field, the plane of polarization of this light wave undergoes a rotation [*Faraday*, 1846]. This phenomenon has been called the Faraday effect. The rotational angle Ω experienced by the wave is proportional to the propagation path length z in the medium and to the component of the magnetic field in the propagation direction B_0, namely,

$$\Omega = VB_0z \tag{1}$$

where V is called the Verdet constant [*Verdet*, 1856]. The value of V for many media has been measured and is tabulated [e.g., *Condon and Odishaw*, 1958; *Yariv and Yeh*, 1984]. The theoretical explanation of the Faraday effect in a dielectric medium was first given by *Bacquerel* [1897] using the classical theory of the Zeeman effect. To our knowledge, the theory of Faraday effect in a magnetoionic medium such as the ionosphere was first given by *Pedersen* [1927]. This theory was extended to the quasi-longitudinal case for propagation in a slowly varying medium by *Browne et al.* [1956] in connection with the Moon bounce experiment. Their theory currently stands as the basis for measuring the Faraday total electron content (TEC) in the ionosphere using radio beacon satellites [e.g., *Davis*, 1989].

Strictly speaking, the Faraday effect occurs only in a medium where the two characteristic waves are both circularly polarized but rotating in the opposite sense. This is certainly the case in many media when $\vec{B}_0$ is in the direction of propagation. If, however, $\vec{B}_0$ is perpendicular to the direction of propagation, the changing polarization is described as the Cotton-Mouton effect [*Cotton and Moutton*, 1907], which is not simply a continuous rotation of the plane of polarization. In general, for an arbitrary direction of propagation, the two characteristic waves are elliptically polarized. The change of polarization as the wave propagates becomes rather complicated and is the topic of investigation in this paper.

The state of an elliptically polarized wave can be represented by a point on the Poincaré sphere [e.g.,

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Booker et al., 1951; *Kong*, 1990]. For application to optical interference problems, see *Pancharatnam* [1956] and *Klein et al.* [1994]. Unfortunately, the Poincaré sphere is three dimensional and is difficult to visualize for our purpose. To demonstrate the changing state of polarization as the wave propagates, it is easier to use the locus on the complex polarization ratio R plane, first used by *Booker* [1934]. Each point on the complex R plane represents one state of elliptical polarization. With $R = E_y/E_x$, the origin represents a linear polarization in the x axis and the infinity represents a linear polarization in the y axis. For a time dependence of the form $e^{j\omega t}$, points in the upper half plane represent elliptical polarization rotating in the left-handed sense, while points in the lower half plane represent right-handed elliptical polarization. This is demonstrated in Figure 1.

The problem of this investigation is formulated in section 2, where the changing state of the wave polarization is shown describable as a circle in the R plane. Since a circle is completely determined by its radius and the coordinates of its center, formulas for the radius and center location are found in section 2. As examples, these general formulas are applied, in section 3, to three different media: biaxial, chiral, and magnetoionic. How the wave polarization transforms as it propagates in each medium is investigated in section 3 with the help of the complex R plane. The paper is then concluded in section 4.

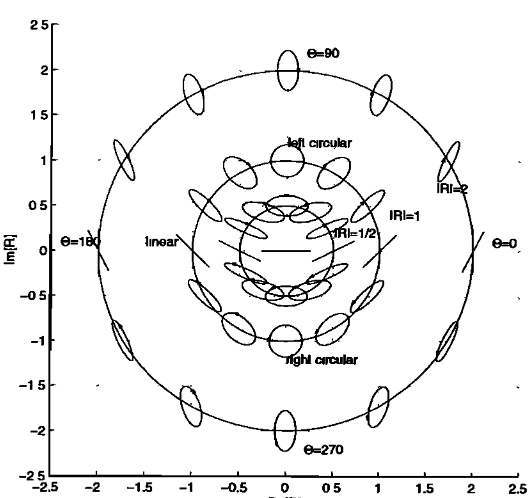


Figure 1. Polarization state representation in the complex R plane where, by definition, $R = E_y/E_x$.

2. Formulation of the Problem

In the study of propagation of a uniform plane wave in the form $e^{j(\omega t - \vec{k}\cdot\vec{r})}$ in a lossless and uniform medium, a characteristic equation of the form

$$\overline{\overline{\mathcal{D}}}(\vec{k},\omega)\cdot\vec{E} = 0 \tag{2}$$

generally results [*Yeh and Liu*, 1972b; *Kong*, 1990; *Fedorov*, 1994]. For electromagnetic waves, $\vec{E}$ in (2) is the electric field and $\overline{\overline{\mathcal{D}}}$ is a 3×3 dyad. For nontrivial solutions to (2), we require

$$\det\overline{\overline{\mathcal{D}}}(\vec{k},\omega) = 0 \tag{3}$$

This relation, when solved, relates $\vec{k}$ to ω and is known as the dispersion relation. Setting the determinant of $\overline{\overline{\mathcal{D}}}$ to zero will result in a high-order algebraic equation in $\vec{k}$. In many cases, equation (3) is a quadratic equation in k^2, indicating the existence of two characteristic waves. The solution k^2 for a given ω and a given direction of propagation are real, showing propagation when $k^2 > 0$ and evanescence when $k^2 < 0$. In the following the existence of only two characteristic waves will be assumed, both propagating. The characteristic vector $\vec{a}_\alpha$, corresponding to the αth characteristic wave ($\alpha = 1, 2$), satisfies the characteristic equation

$$\overline{\overline{\mathcal{D}}}(\vec{k}_\alpha,\omega)\cdot\vec{a}_\alpha = 0 \tag{4}$$

Depending on the properties of $\overline{\overline{\mathcal{D}}}$, the characteristic vectors may be orthonormalized in a certain manner. This will be introduced later.

Consider a field propagating in some direction, say, the z direction. At some point, say, $z = 0$, the field is given by $\vec{E}(0)$. This field is decomposed into a sum of two characteristic components,

$$\vec{E}(0) = \sum_{\alpha=1}^{2} E_\alpha \vec{a}_\alpha \tag{5}$$

The components transverse to $\vec{k}$, that is, transverse to the z direction, are

$$\begin{aligned}\vec{E}_\perp(0) &= \sum_{\alpha=1}^{2} E_\alpha \vec{a}_{\alpha\perp}\\ &= E_1(\hat{x}a_{1x} + \hat{y}a_{1y}) + E_2(\hat{x}a_{2x} + \hat{y}a_{2y})\end{aligned} \tag{6}$$

where $\vec{a}_{\alpha\perp}$ is the part of $\vec{a}_\alpha$ that is perpendicular to $\vec{k}$ (i.e., $\hat{z}$); $\hat{x}$ and $\hat{y}$ are the two unit vectors along the

x axis and the y axis, respectively; and $a_{\alpha x}$ and $a_{\alpha y}$ are the x component and y component of $\vec{a}_\alpha$, $\alpha = 1$ and 2, respectively. The initial wave polarization in the transverse plane is given by the wave polarization ratio, which is defined as

$$R(0) = \frac{E_y(0)}{E_x(0)} = \frac{a_{1y}E_1 + a_{2y}E_2}{a_{1x}E_1 + a_{2x}E_2} \tag{7}$$

The ratio in (7) is generally complex; as shown in Figure 1, each value on the complex R plane corresponds to one state of elliptical polarization with linear and circular polarizations as two special limiting cases. In our problem, the initial field $\vec{E}(0)$ is supposedly given. This is equivalent to knowing the initial polarization ratio (7). As this initial field $\vec{E}(0)$ propagates a distance z, the field at z becomes

$$\vec{E}(z) = \sum_{\alpha=1}^{2} E_\alpha \vec{a}_\alpha e^{-jk_\alpha z} \tag{8}$$

The transverse polarization ratio of the electric field after propagating a distance z becomes

$$\begin{aligned} R(z) &= E_y(z)/E_x(z) \\ &= \frac{a_{1y}E_1 + a_{2y}E_2 e^{-j(k_2-k_1)z}}{a_{1x}E_1 + a_{2x}E_2 e^{-j(k_2-k_1)z}} \end{aligned} \tag{9}$$

For a given medium, the characteristic vector $\vec{a}_\alpha$ and its corresponding wavenumber k_α, $\alpha = 1$ or 2, can be computed. The characteristic amplitude ratio E_1/E_2 depends on the initial field $\vec{E}(0)$ and can also be computed from (7). Thus the polarization ratio (9) at any z can be computed easily. What is of interest is the behavior of $R(z)$ as a function of z. To this end, it is interesting to note that (9) is of the form

$$\mathbf{w} = \frac{\alpha\mathbf{z} + \beta}{\gamma\mathbf{z} + \delta} \tag{10}$$

which in complex variables is known as the bilinear or Möbius transformation [e.g., *Churchill*, 1948; *Smirnov*, 1964]. In (10), α, β, γ, and δ are four complex constants, and $\mathbf{w}$ and $\mathbf{z}$ are two complex variables. The readers should be cautioned that the symbol z is used to denote the z coordinate, while the symbol $\mathbf{z}$ is used to represent a complex variable, as in (10). This should not cause any great confusion as they can be easily differentiated by the context. As a mater of fact, comparing (9) and (10) shows that

$$\mathbf{z} = e^{-j(k_2-k_1)z} \tag{11}$$

When z increases, it traces out a circle of unit radius with its center at the origin in the complex $\mathbf{z}$ plane.

The properties of the bilinear transformation have been extensively studied in mathematics because of the prominent role it plays in some parts of geometry. The most important property that is needed in this application is the fact that the transformation (10) maps circles in the complex $\mathbf{z}$ plane into circles in the complex $\mathbf{w}$ plane in a one-to-one correspondence. This property becomes especially transparent when transformation (10) is recast in the form, for $\gamma \neq 0$,

$$\frac{\gamma^2}{\beta\gamma - \alpha\delta}(\mathbf{w} - \alpha/\gamma) = \frac{1}{\mathbf{z} + \delta/\gamma} \tag{12a}$$

and, for $\gamma = 0$,

$$\mathbf{w} - \beta = \alpha\mathbf{z} \tag{12b}$$

In (12b), without loss of generality, the assumption $\delta = 1$ is made. In this form, it can be easily demonstrated that a centered circle of unit radius in the complex $\mathbf{z}$ plane is mapped into a circle in the complex $\mathbf{w}$ plane. The radius of the circle ρ and the coordinates of the center W are given by

$$\rho = \begin{cases} |\beta\gamma - \alpha\delta|/(|1 - |\delta/\gamma|^2||\gamma^2|) & \gamma \neq 0 \\ |\alpha| & \gamma = 0 \text{ and } \delta = 1 \end{cases} \tag{13}$$

and

$$W = \begin{cases} \frac{\alpha}{\gamma} - \frac{\delta^*(\beta\gamma - \alpha\delta)}{\gamma^*\gamma^2(1 - |\delta/\gamma|^2|)} & \gamma \neq 0 \\ \beta & \gamma = 0 \text{ and } \delta = 1 \end{cases} \tag{14}$$

In (14), the asterisks are used to denote complex conjugation. Comparing (9) and (10), these results can be translated to mean that as the propagation distance z increases, the polarization ratio $R(z)$ traces out a circle in the complex R plane with

$$\rho = \begin{cases} \dfrac{\left|\left(\frac{a_{1y}}{a_{2x}} - \frac{a_{2y}a_{1x}}{a_{2x}^2}\right)\frac{E_1}{E_2}\right|}{\left|1 - \left|\frac{a_{1x}E_1}{a_{2x}E_2}\right|^2\right|} & a_{2x}E_2 \neq 0 \\ |a_{2y}E_2/(a_{1x}E_1)| & a_{2x}E_2 = 0 \end{cases} \tag{15}$$

and

$$W = \begin{cases} \frac{a_{2y}}{a_{2x}} - \frac{a_{1x}^*(a_{1y}a_{2x} - a_{2y}a_{1x})}{a_{2x}^* a_{2x}^2\left(1 - \left|\frac{a_{1x}E_1}{a_{2x}E_2}\right|^2\right)} \left|\frac{E_1}{E_2}\right|^2 & a_{2x}E_2 \neq 0 \\ a_{1y}/a_{1x} & a_{2x}E_2 = 0 \end{cases} \tag{16}$$

To proceed further, we can use the properties of characteristic vectors $\vec{a}_\alpha$.

For propagation of uniform plane waves in an anisotropic dielectric medium, the coefficient dyad $\overline{\overline{\mathcal{D}}}$ can be derived from Maxwell's equations and is found to be [e.g., *Yeh and Liu*, 1972b]

$$\overline{\overline{\mathcal{D}}} = -k_0^2\overline{\overline{K}}(\omega) - \vec{k}\vec{k} + k^2\overline{\overline{I}} \tag{17}$$

where $k_0 = \omega/c$, $\overline{\overline{K}}(\omega)$ is a 3×3 relative permittivity dyad and is assumed to have only temporal dispersion but no spatial dispersion, and $\overline{\overline{I}}$ is a 3×3 unit dyad. For a dissipation-less medium, $\overline{\overline{K}}$ must be Hermitian, which makes $\overline{\overline{\mathcal{D}}}$ also Hermitian. In this case it can be proven that the transverse characteristic vectors $\vec{a}_{\alpha\perp}$ can be made to satisfy the orthonormal condition [*Shafranov*, 1967; *Deschamps and Kesler*, 1967]

$$\vec{a}_{\alpha\perp}\cdot\vec{a}^*_{\beta\perp} = \delta_{\alpha\beta}, \quad \alpha,\beta = 1,2 \tag{18}$$

The symbol $\delta_{\alpha\beta}$ denotes a Kronecker delta. If one writes the complex polarization ratio for the characteristic wave 1 in the polar form as

$$R_1 = \frac{a_{1y}}{a_{1x}} = re^{j\theta} \tag{19}$$

the orthogonal part of condition (18) requires that the complex polarization ratio for the characteristic wave 2 to be

$$R_2 = \frac{a_{2y}}{a_{2x}} = -\frac{1}{r}e^{j\theta} \tag{20}$$

The two characteristic waves are generally elliptically polarized. The fact that the polarization ratios are related by $R_1R_2^* = -1$, as indicated by (19) and (20), implies that these two characteristic ellipses have the same major-to-minor axial ratio, perpendicular major axes, and opposite sense of rotation. This is a consequence of the orthogonal part of condition (18). Furthermore, the normalization part of condition (18) can be satisfied if

$$|a_{1x}|^2 = |a_{2y}|^2 = 1/(1+r^2) \tag{21}$$

and

$$|a_{1y}|^2 = |a_{2x}|^2 = r^2/(1+r^2) \tag{22}$$

These relations still leave a_{1x}, a_{2x}, a_{1y}, and a_{2y} with some arbitrariness in phase. One possible set of choices is

$$\vec{a}_{1\perp} = (1+r^2)^{-1/2}(\hat{x}+\hat{y}re^{j\theta})e^{j\phi_2} \tag{23a}$$

$$\vec{a}_{2\perp} = (1+r^2)^{-1/2}(-\hat{x}re^{-j\theta}+\hat{y})e^{j(\phi_2-\phi_1)} \tag{23b}$$

where θ is the phase angle of the polarization ratio of the first characteristic wave defined by (19). In (23a) and (23b), ϕ_1 is the phase of a_{2y} relative to a_{1x}, and ϕ_2 is the absolute phase of a_{1x}. Both ϕ_1 and ϕ_2 play no role in determining the locus in the complex R plane. When (23a) and (23b) are substituted into (15) and (16), respectively, one finds, after some calculation, that

$$\rho = \begin{cases} \left|\frac{(1+r^{-2})|E_1/E_2|}{1-r^{-2}|E_1/E_2|^2}\right| & a_{2x}E_2 \neq 0 \\ |E_2/E_1| & a_{2x}E_2 = 0 \end{cases} \tag{24}$$

and

$$W = \begin{cases} -r^{-1}e^{j\theta}\left(1+\frac{1+r^2}{r^2-|E_1/E_2|^2}\left|\frac{E_1}{E_2}\right|^2\right) & a_{2x}E_2 \neq 0 \\ re^{j\theta} & a_{2x}E_2 = 0 \end{cases} \tag{25}$$

in the complex R plane. It is interesting to note that the special condition $a_{2x}E_2 = 0$ can occur under two conditions.

The first condition is $E_2 = 0$. This corresponds to the case when only one characteristic wave (namely, characteristic wave 1) is excited initially. In this case, from (24) and (25), the radius is zero and the circle is centered at the point specified by the characteristic wave 1. Thus the excited wave for this special case propagates without change in polarization as expected.

The second condition is $a_{2x} = 0$. This corresponds to the case when the characteristic wave 2 is polarized linearly in the y direction, and, via the orthonormal condition (18), the characteristic wave 1 must be necessarily polarized also linearly but in the x direction, i.e., $a_{1y} = 0$ and thus $r = 0$. In this case, from (24) and (25), the locus in the complex $R(z)$ plane as the wave propagates along z is a centered circle with a radius equal to the amplitude ratio of the characteristic wave 2 to that of the characteristic wave 1 at $z = 0$.

With the initial polarization ratio $R(0)$ expressed in the polar form

$$R(0) = R_0e^{j\Theta} \tag{26}$$

the complex amplitude ratio of the two characteristic waves E_1/E_2 that appears in (24) and (25) can be found from (7) as

$$\frac{E_1}{E_2} = -\frac{1 + rR_0 e^{j(\Theta-\theta)}}{r - R_0 e^{j(\Theta-\theta)}} e^{-j\theta} \tag{27}$$

In the next section the results (24) and (25) will be applied to several examples.

3. Several Examples

Many crystals found in nature are anisotropic. Some naturally isotropic media can be turned into anisotropic media with the application of an external static electric field (electro-optical effect), an external static magnetic field (magneto-optical effect), or a stress (photo-elastic effect). Discussions of these effects and the resulting anisotropic properties are given by many authors [e.g., *Nye*, 1985; *Dmitrieve et al.*, 1991; *Guenther*, 1990; *Theocaris and Gdoutos*, 1979]. More recently there has been increased interest in chiral media [e.g., *Lindell et al.*, 1993] (see also the *Journal of Electromagnetic Waves and Applications* special issue on wave interaction with chiral and complex media (*6*(5/6) 537-798, 1992)). In the following, several such examples are selected for our investigation and discussion.

3.1. Biaxial Media

The dielectric tensor of biaxial media is a real symmetric matrix which can be diagonalized when the principal axes are aligned with the symmetric axes of the crystal. Thus, along the principal axes, the relative dielectric tensor can be written in the form

$$\overline{\overline{K}} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} \tag{28}$$

For waves propagating in the z direction in such a medium, the characteristic equation (2) yields two characteristic waves: Wave 1 is polarized in the x direction with $\vec{a}_1 = \hat{x}$, $k_1^2 = k_0^2 K_1$, and $\vec{E}_1 = \hat{x}E_1 e^{-jk_1 z}$; and wave 2 is polarized in the y direction with $\vec{a}_2 = \hat{y}$, $k_2^2 = k_0^2 K_2$, and $\vec{E}_2 = \hat{y}E_2 e^{-jk_2 z}$. Since $a_{2x} = 0$, this corresponds to condition 2 discussed in the previous section. That is, a field of arbitrary initial polarization ratio $R(0)$ in the transverse plane will undergo a polarization transformation as it propagates along z. This polarization transformation is described by a locus on the complex R plane as a centered circle of radius $R_0 = |R(0)|$. The radius of the circle depends only on the initial polarization ratio. As $R(0) = E_2/E_1 = R_0 e^{j\Theta}$, the starting point has a polar angle Θ. The circular locus rotates in a counterclockwise sense if $k_1 > k_2$ or $K_1 > K_2$ and in a clockwise sense if $k_1 < k_2$ or $K_1 < K_2$ as z increases. The locus makes a complete circle for every distance increment $\Delta z = 2\pi/|k_1 - k_2| = \lambda_0/|K_1 - K_2|$, where λ_0 is the free-space wavelength.

To be specific, we choose a biaxial medium from *Dmitrieve et al.* [1991]. The relative dielectric tensor is

$$\begin{bmatrix} 2.51064 & 0 & 0 \\ 0 & 2.32746 & 0 \\ 0 & 0 & 2.29371 \end{bmatrix} \tag{29}$$

where $k_0 = \pi \times 10^7$. When the wave propagates in the z direction, it has $k_1/k_0 = 1.5845$ and $k_2/k_0 = 1.5256$. Hence the polarization locus rotates in a counterclockwise sense. Besides, the polarization state will change periodically for each $\Delta z = 3.3956$ μm. Depicted in Figure 2 are several examples showing how the wave polarization transforms along z for four cases of initial polarizations.

3.2. Chiral Media

There has been strong interest in chiral media because of their interesting properties. A recent list of

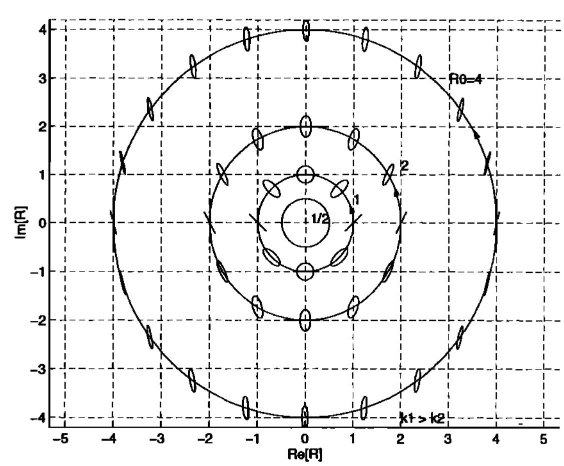


Figure 2. The transformation of wave polarization in a biaxial medium is illustrated for several examples, with magnitude of the initial polarization ratio R_0 equal to 1/2, 1, 2, and 4 in the complex $R(z)$ plane. The relative dielectric tensor is given by equation (29).

related references contains 1605 entries, which can overwhelm any reader easily [*Unrau*, 1997]. For the purpose of this paper, the constitutive relations of an isotropic chiral medium are given by [e.g., *Koivisto et al.*, 1993]

$$\vec{D} = \epsilon \vec{E} - j\xi \vec{B} \tag{30a}$$

$$\vec{H} = -j\xi \vec{E} + \vec{B}/\mu \tag{30b}$$

where ξ is called the chirality factor, ϵ is the electric permittivity, and μ is the magnetic permeability. The parameters ξ, ϵ, and μ are assumed to be real and positive. Substituting the constitutive relations (30a) and (30b) into Maxwell's equations, for uniform plane waves, a characteristic equation of the form (2) is obtained, with the coefficient dyad given by

$$\overline{\overline{\mathcal{D}}}(\vec{k}, \omega) = (-k^2 + k_0^2)\overline{\overline{I}} - j2\omega\mu\xi\vec{k}\times \tag{31}$$

Here $\vec{k}\times$ represents an antisymmetric dyad which makes $\overline{\overline{\mathcal{D}}}$ Hermitian symmetric. As in (3), setting the determinant of $\overline{\overline{\mathcal{D}}}$ to zero yields the dispersion relation. For transverse waves, one obtains for characteristic wave 1 propagating in the positive z direction

$$k_1 = \omega(-\mu\xi + \sqrt{\mu^2\xi^2 + \mu\epsilon}) \tag{32a}$$

$$\vec{a}_1 = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}j)e^{j\phi_2} \tag{32b}$$

and one obtains for the characteristic wave 2 also propagating in the positive z direction

$$k_2 = \omega(\mu\xi + \sqrt{\mu^2\xi^2 + \mu\epsilon}) \tag{33a}$$

$$\vec{a}_2 = \frac{1}{\sqrt{2}}(\hat{x}j + \hat{y})e^{j(\phi_2 - \phi_1)} \tag{33b}$$

Thus characteristic wave 1 is circularly polarized in the left-handed sense, and characteristic wave 2 is circularly polarized in the right-handed sense. Note that the characteristic vectors (32b) and (33b) satisfy the orthonormal condition (18). In the following, two special cases are considered first before looking at the general case.

3.2.1. Case 1: $R(0) = 0$. This is the case where, initially at $z = 0$, the wave is polarized linearly in the x direction, i.e., $E_y(0) = 0$. For this case, one obtains $R_0 = 0$, $r = 1$, $\theta = \pi/2$, and $E_1/E_2 = j$. When these values are inserted into (24) and (25), the degenerate case of $\rho = \infty$ and $W = \infty$ is obtained. Actually, for this special case, the polarization ratio $R(z)$ can be computed directly from (9), which yields

$$R(z) = -\tan[(k_2 - k_1)z/2] = -\tan(\omega\mu\xi z) \tag{34}$$

In general, the argument of tangent in (34) contains the phase ϕ_1. In order to satisfy the initial condition $R(0) = 0$, it is required that $\phi_1 = 0$.

In the complex R plane, (34) occupies the real axis, each point on which represents a linear polarization with a different tilt angle. At $z = 0$, the plane of polarization is horizontal. As the wave propagates in the positive z direction the plane of polarization rotates in the left-hand sense, with a tilt angle Ω at a point z given by

$$\Omega = -\omega\mu\xi z \tag{35}$$

That is, the plane of polarization rotates through an angle equal to one half of the relative phase shift between two characteristic waves. This is sometimes referred to as the Faraday rotation. The wave becomes vertically polarized when $z = \pi/2\omega\mu\xi$ and horizontally polarized again when $z = \pi/\omega\mu\xi$. As a matter of fact, the plane of polarization undergoes a complete rotation for every distance increment $\Delta z = \pi/\omega\mu\xi$.

If the wave meets a perfect conductor normally, the reflected wave will also undergo a Faraday rotation. Since the rotation is left-handed relative to the direction of propagation and the reflected wave has the direction of propagation reversed, the plane of polarization of the reflected wave will retrace that of the incident wave. That is, at any point, the plane of polarization of the reflected wave is the same as that of the incident wave. This is a property of the isotropic medium. If the medium is anisotropic, this is no longer the case, as we will find later.

3.2.2. Case 2: $R(0) = j$. This is the case where, initially at $z = 0$, the wave is polarized circularly in the left-handed sense. For this case, one obtains $E_2 = 0$, $R_0 = r = 1$, and $\Theta = \theta = \pi/2$. With these values, the locus of the polarization ratio on the complex R plane has, from (24) and (25), $\rho = 0$ and $W = j$. Thus, as the wave propagates, its polarization is unchanged. It is expected because the wave has the characteristic polarization of wave 1.

The dual case $R(0) = -j$ yields $\rho = 0$ and $W = -j$. This implies that if the wave has the characteristic polarization of wave 2, it propagates without change in its polarization, as expected.

3.2.3. Case 3: $R(0) = R_0 e^{j\Theta}$. In this case, the initial phase at $z = 0$ is arbitrarily polarized. In this simple chiral medium, $r = 1$ and $\theta = \pi/2$, which gives, from (27),

$$\frac{E_1}{E_2} = j\frac{1 - R_0^2 - j2R_0 \cos\Theta}{1 + R_0^2 - 2R_0 \sin\Theta} \tag{36}$$

As the wave propagates, its polarization ratio in the complex R plane traces out a circle with

$$\rho = \frac{\sqrt{(1 + R_0^2)^2 - 4R_0^2 \sin^2\Theta}}{2R_0 |\sin\Theta|} \tag{37}$$

from (24) and with

$$W = j\frac{1 + R_0^2}{2R_0 \sin\Theta} \tag{38}$$

from (25). The center is therefore on the imaginary axis. It is on the positive portion of the imaginary axis above the point j if $0 < \Theta < \pi$ and on the negative portion of the imaginary axis below the point $-j$ if $\pi < \Theta < 2\pi$. Since the numerator of (37) is smaller than that of (38) and their denominators are the same, these circles never cross the real axis. This implies that as the wave propagates, the elliptical polarization will change with z but its sense of rotation stays the same. The circles actually form mirror images relative to the real R axis. The locus on the R plane depends only on the initial polarization and not on the parameters of the medium.

However, as the wave propagates, the locus traces out a circular arc in the clockwise direction for an initial polarization represented by a point in the upper half R plane and in the counterclockwise direction for that in the lower half R plane. Then, the polarization repeats itself over every distance increment $\Delta z = \pi/\omega\mu\xi$, which depends on the frequency and the chirality factor. Several examples are illustrated in Figure 3.

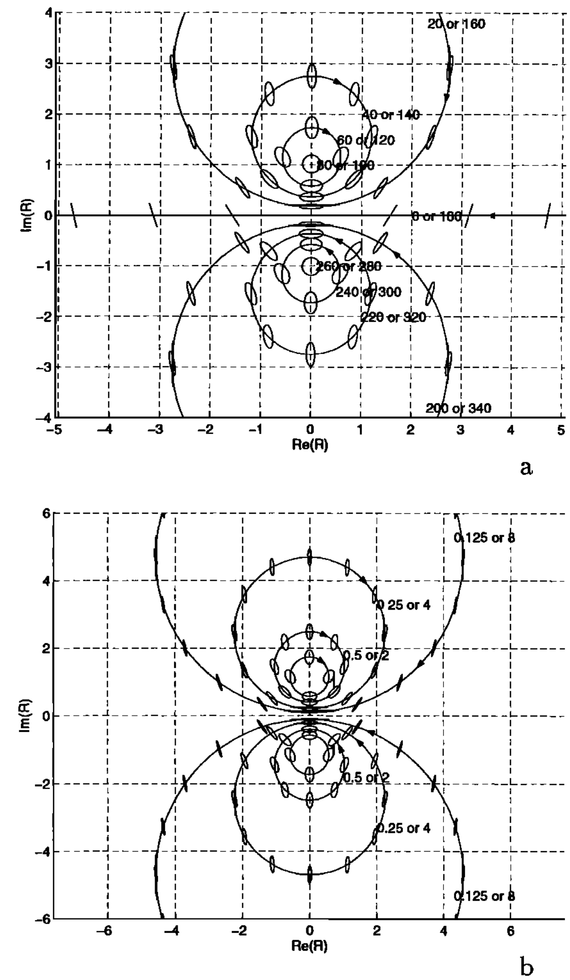


Figure 3. The transformation of wave polarization in a chiral medium is illustrated for several examples in the complex R plane. In Figure 3a, R_0 is equal to one for various values of Θ. In Figure 3b, the upper half plane corresponds to $\Theta = 60^o$ and various values of R_0. The lower half plane corresponds to $\Theta = 120^o$ and various values of R_0.

3.3. Magnetoionic Media

A magnetoionic medium is a cold plasma permeated by a steady magnetic field $\vec{B}_0$. In the collisionless case, its relative dielectric dyad $\overline{\overline{K}}$ with $\vec{B}_0$ parallel to the z axis has the elements given by [*Ratcliff*, 1959; *Budden*, 1961; *Yeh and Liu*, 1972a, b]

$$\begin{aligned} K_{xx} &= K_{yy} = 1 - X/(1 - Y^2) \\ K_{zz} &= 1 - X \\ K_{xy} &= -K_{yx} = -jXY/(1 - Y^2) \\ K_{xz} &= K_{yz} = K_{zx} = K_{zy} = 0 \end{aligned} \tag{39}$$

where in magnetoionic notations, X and Y are frequency ratios defined by $X = \omega_p^2/\omega^2$, the squared ratio of plasma frequency to radio frequency, and $Y = \omega_B/\omega$, the ratio of gyrofrequency to radio frequency.

Substituting (39) into (17), one obtains the coefficient dyad $\overline{\overline{\mathcal{D}}}$. Setting the determinant of $\overline{\overline{\mathcal{D}}}$ to zero yields the dispersion relation. For a wave propagating at an angle θ_B relative to $\vec{B}_0$, the wave number for the αth mode is $k_\alpha = k_0 n_\alpha$, $\alpha = 1, 2$, where the refractive indices n_α are given by the well-known Appleton-Lassen formulas [*Ratcliff*, 1959; *Rawer and Suchy*, 1976]

$$n_\alpha^2 = 1 - X[1 - \frac{Y^2 \sin^2\theta_B}{2(1-X)} \pm \sqrt{\frac{Y^4 \sin^4\theta_B}{4(1-X)^2} + Y^2\cos^2\theta_B}]^{-1} \qquad (40)$$

For definiteness regarding the signs appearing in (40), the following convention is chosen: $\alpha = 1$ corresponds to the upper sign, and $\alpha = 2$ corresponds to the lower sign, both for $0 \le \theta_B \le \pi$. The characteristic vectors $\vec{a}_\alpha$ can be obtained by inserting (40) into the characteristic equation (2). Rotating the coordinate system so that $\vec{k}$ is parallel to the z axis and $\vec{B}_0$ is in the yz plane making an angle θ_B relative to $\vec{k}$, the two transverse characteristic vectors in the new coordinate system become (23a) and (23b), respectively, where r and θ are given by

$$r = Y|\cos\theta_B|[\sqrt{\frac{Y^4 \sin^4\theta_B}{4(1-X)^2} + Y^2\cos^2\theta_B} - \frac{Y^2\sin^2\theta_B}{2(1-X)}]^{-1} \qquad (41a)$$

and

$$\theta = \begin{cases} \pi/2 & 0 \le \theta_B < \pi/2 \\ -\pi/2 & \pi/2 < \theta_B \le \pi \end{cases} \qquad (41b)$$

In the following, two special cases are considered first, before the consideration of the general case. Some early considerations of the Faraday effect are given by *Brandstatter* [1963].

3.3.1. Case 1: $\theta_B = 0$. This is known as the longitudinal propagation case where $\vec{k}$ and $\vec{B}_0$ are both parallel to the z axis. For this case, (41a) yields $r = 1$; the refractive indices (40) and the characteristic vectors (23a) and (23b) for each mode become

$$n_1^2 = 1 - X/(1+Y) \qquad (42a)$$

$$\vec{a}_1 = 2^{-1/2}(\hat{x} + \hat{y}j)e^{j\phi_2} \qquad (42b)$$

respectively, and

$$n_2^2 = 1 - X/(1-Y) \qquad (43a)$$

$$\vec{a}_2 = 2^{-1/2}(\hat{x}j + \hat{y})e^{j(\phi_2 - \phi_1)} \qquad (43b)$$

Thus, as with the propagation in a chiral medium, wave 1 is circularly polarized in the left-handed sense (compare (42b) with (32b)), and wave 2 is circularly polarized in the right-handed sense (compare (43b) with (33b)). For a wave with an arbitrary initial polarization, the transformation of polarization as it propagates along z can be discussed in a manner similar to that in section 3.2. In particular, if the wave is initially linearly polarized along the x axis (i.e., $R(0) = 0$), its polarization for any z continues to be linear and has a polarization ratio

$$R(z) = -\tan[k_0(n_2 - n_1)z/2] \qquad (44)$$

That is, the plane of polarization of this linearly polarized wave is making an angle Ω relative to the x axis given by

$$\Omega = k_0(n_1 - n_2)z/2 \qquad (45)$$

The angle Ω is known as the Faraday rotation angle, which as given by (45) is equal to one half of the differential phase shift between the two characteristic waves. In the high-frequency limit (i.e., $\omega \gg \omega_p, \omega_B$), (45) reduces to

$$\Omega \cong \omega_p^2 \omega_B z/2c\omega^2 \qquad (46)$$

This formula is used in Faraday rotation experiments as a basis for measuring the total electron content when generalized to a slowly varying ionosphere [*Yeh and Liu*, 1972b; *Davies*, 1989]. Note that this rotation is right-handed for this case i.e., the case when the direction of propagation and the steady magnetic field are both in the positive z direction.

Suppose the linearly polarized wave is reflected normally from a perfect conductor: The reflected wave would undergo a Faraday rotation as well. A similar analysis can be carried out to show that the Faraday rotation for the reflected wave becomes left-handed. As the reflected wave has the direction of propagation reversed from that of the incident wave, a left-handed Faraday rotation will rotate in the same direction as that of the incident Faraday rotation, which is right-handed. Thus, for a given z, the plane of polarization of the reflected wave will continue its rotation (the same direction as the incident wave), and when it reaches its starting point, its plane of polarization may be different from that of the incident wave. This is contrary to propagation in the isotropic chiral medium discussed in section 3.2, case 1.

3.3.2. Case 2: $\theta_B = \pi/2$. This is known as the transverse propagation case, where $\vec{k}$ is along $\vec{z}$ and $\vec{B}_0$ is along $\vec{y}$. A straightforward calculation of r using (41a) yields the zero-over-zero indeterminate situation. However, one can use any form of limiting calculations to obtain $r = \infty$. The two characteristic waves have their refractive indices and characteristic vectors given by

$$n_1^2 = 1 - X \tag{47a}$$

$$\vec{a}_{1\perp} = \hat{y} j e^{j\phi_2} \tag{47b}$$

and

$$n_2^2 = 1 - X(1-X)/(1 - X - Y^2) \tag{48a}$$

$$\vec{a}_{2\perp} = \hat{x} j e^{j(\phi_2 - \phi_1)} \tag{48b}$$

Here wave 1 is the ordinary wave, as its refractive index is not affected by the steady magnetic field, and wave 2 is the extraordinary wave, which has a refractive index dependent on the strength of the steady magnetic field. In the transverse plane, both characteristic waves are linearly polarized, wave 1 along the y axis (i.e., parallel to $\vec{B}_0$) and wave 2 along the x axis. Thus the wave polarizations of the characteristic vectors are similar to those in the biaxial medium discussed in section 3.1. For a wave of initial polarization ratio given by $R(0) = R_0 e^{j\Theta}$ as it propagates to a point z, its polarization ratio becomes

$$R(z) = R_0 e^{j\Theta} e^{-j(k_1 - k_2)z} \tag{49}$$

In the complex R plane as z increases, (49) will trace out a centered concentric circle with a radius R_0 as shown in Figure 2.

3.3.3. Case 3: Arbitrary θ_B. In this case, r and θ can be found from (41a) and (41b), respectively. The normalized transverse characteristic vectors $\vec{a}_{1\perp}$ and $\vec{a}_{2\perp}$ can then be calculated using (23a) and (23b). The refractive indices for the two characteristic waves are given in (40). With these quantities computed, the polarization ratio $R(z)$ for a given $R(0)$ can be computed from (9) for each value of z. As z varies, $R(z)$ will trace out a circle in the complex R plane with its radius and center given by, from (24) and (25),

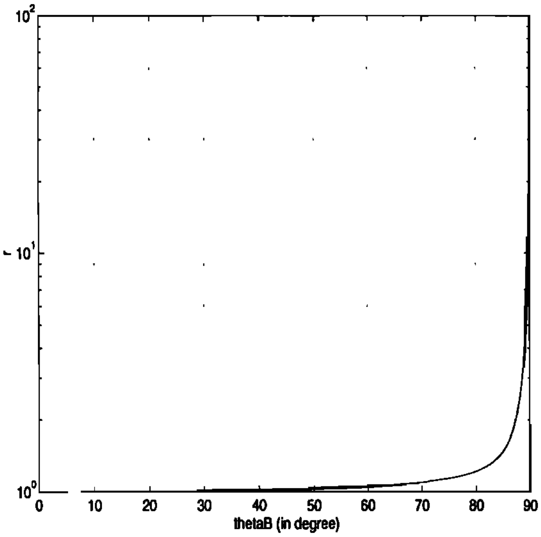
$$\rho = \left| (r^2 - 2rR_0 \sin\Theta + R_0^2)^{1/2} (1 + 2rR_0 \sin\Theta + r^2 R_0^2)^{1/2} (r^2 - 2rR_0 \sin\Theta - 1)^{-1} \right| \tag{50a}$$

$$W = \mp j \frac{r(1 + R_0^2)}{r^2 - 2rR_0 \sin\Theta - 1} \tag{50b}$$

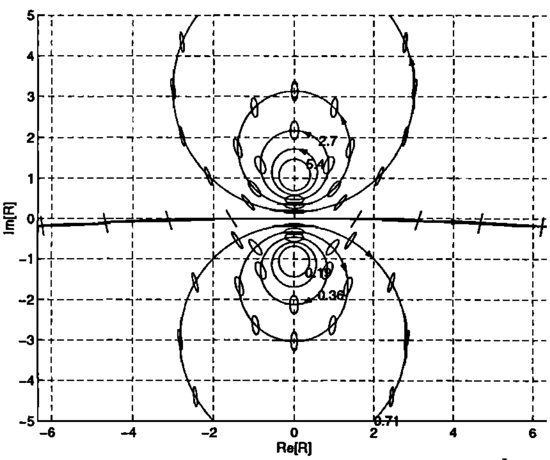
where in (50b) the upper sign applies for $0 \le \theta_B < \pi/2$ and the lower sign applies for $\pi/2 < \theta_B \le \pi$. Thus, in all cases, the center falls on the imaginary R axis.

To quantitatively illustrate the transformation of polarization as a wave of any initial polarization propagates, some numerical values must be chosen. In the parameter space, there exist two regions where both characteristic waves propagate in all directions [*Yeh and Liu*, 1972a]. In the first region, we have $X < 1$, $Y^2 < 1$, and $X < 1 - Y$. In this region, the refractive indices of both characteristic waves are very close to unity. To be specific, we pick $f = 25$ MHz, $f_p = 8$ MHz and $f_B = 1.6$ MHz, which correspond to $X = 0.1024$ and $Y^2 = 0.004096$. These values are applicable to the ionospheric conditions. The results are plotted in Figure 4. As r is very close to 1 for almost all values of θ_B (see Figure 4a), the quasi-longitudinal condition can be applied for all directions of propagation except within a few degrees from the exact perpendicular condition. This is well known in ionospheric studies. As depicted in Figure 4, the transition of Faraday rotation from the longitudinal behavior to the transverse behavior takes place rather suddenly. The transformation of polarization for any initial polarization is illustrated by ellipses attached to the circles shown in Figures 4b-4f. Here the loci are computed for several values of the amplitude ratio of the two characteristic waves $|E_1/E_2|$.

The second region in the parameter space where both characteristic waves propagate in all directions occurs when $X < 1$ and $Y^2 > 1$. For this case, shown in Figure 5, we pick $f = 10$ MHz, $f_p = 7.07$ MHz and $f_B = 11.9$ MHz. This corresponds to $X = 0.500$ and $Y^2 = 1.416$. In this second region, the refractive indices of the two characteristic waves can differ substantially, resulting in a much smaller distance of propagation over which the wave polarization repeats itself (0.021 km for this example versus 25.563 km for the previous example, a thousandfold

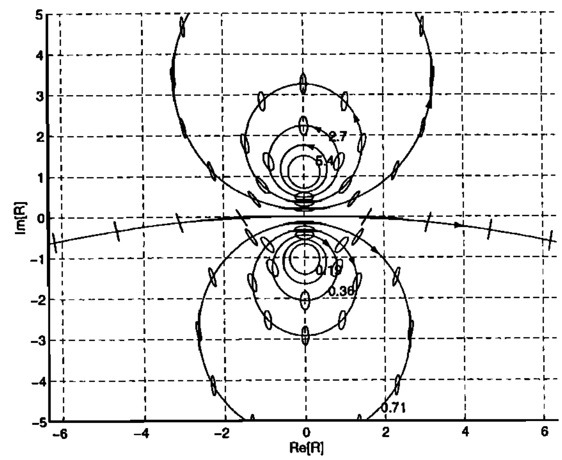


a



b

Figure 4. The transformation of the polarization for several θ of the first example in magnetoplasma. In this example, $k_1 > k_2$. The parameters chosen correspond to $f = 25$ MHz, $f_p = 8$ MHz, and $f_B = 1.6$ MHz. (a) The magnitude of the polarization ratio for characteristic wave 1 in a magnetoionic medium as a function of θ_B; (b) $\theta_B = 30.000^o$, $\Delta z = 1.053$ km (in this and the following four cases, circles are computed for several values of the amplitude ratio $|E_1/E_2|$); (c) $\theta_B = 50.000^o$, $\Delta z = 1.417$ km; (d) $\theta_B = 70.000^o$, $\Delta z = 2.654$ km; (e) $\theta_B = 89.000^o$, $\Delta z = 22.965$ km; and (f) $\theta_B = 89.999^o$, $\Delta z = 25.563$ km.



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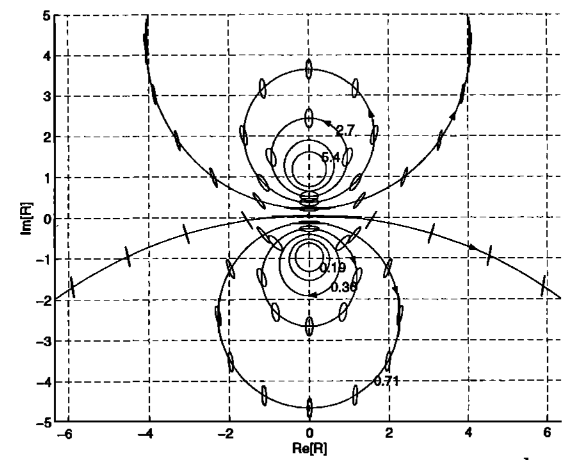

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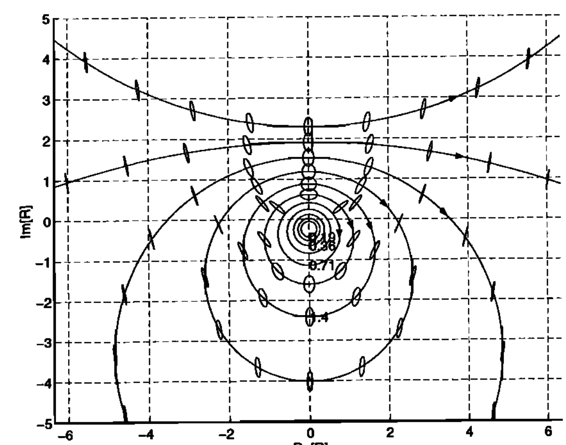


e

Figure 4. (continued)

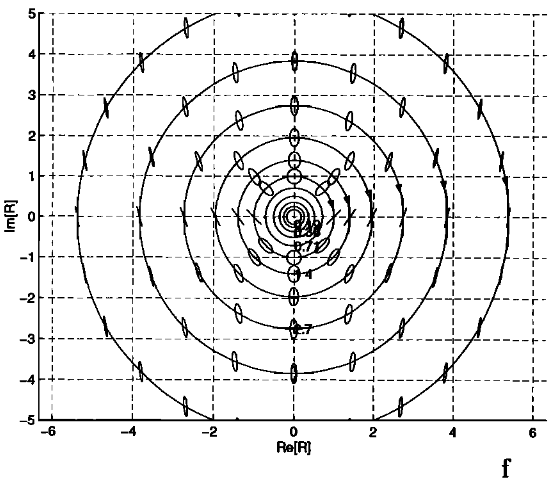


Figure 4. (continued)

change). The transition from the longitudinal behavior to the transverse behavior takes place more gradually. Again, small ellipses are attached to the circles to illustrate the polarization transformations as depicted in Figures 5b-5f. As in Figure 4, the parameter is the amplitude ratio of the two characteristic waves $|E_1/E_2|$.

4. Summary and Conclusions

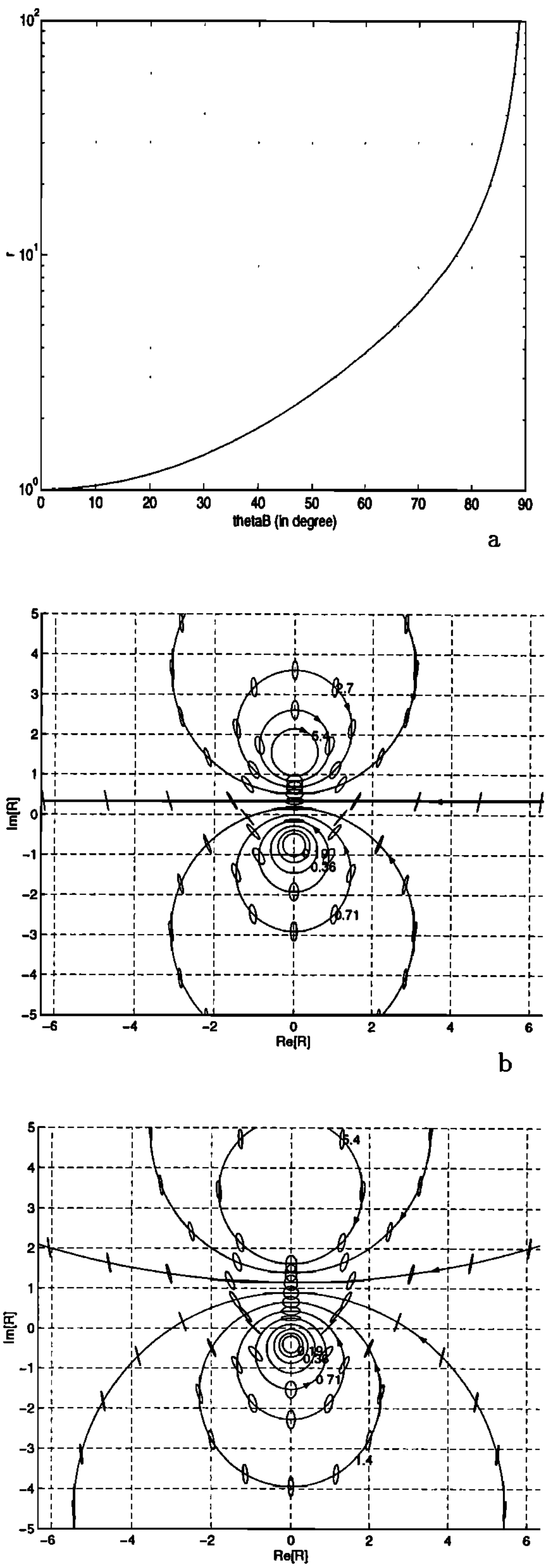
In this paper we have revisited an old problem, a problem first discovered experimentally by *Faraday* [1846]. This discovery was thought to be of great historical importance because it connected the areas between optics and magnetism, up to that time two entirely distinct areas of study, and also because of its strong support for the electromagnetic nature of light [*Sommerfeld*, 1954; *Condon and Odishaw*, 1958]. In modern times, the Faraday effect continues to play an important role because of its applications. By measuring the Faraday rotation, it is possible to learn the electromagnetic characteristics of the propagating medium. The measurement of the Verdet constant is one example; the measurement of TEC in the ionosphere or the space plasma is another example. On the other hand, in engineering applications, we may desire one wave polarization while given with a different initial wave polarization. An anisotropic medium is usually used to achieve that purpose. What occurs in the anisotropic medium is, of course, the generalized Faraday effect. In recent years, there is great interest in making artificial electromagnetic media. This interest is reflected in the large collection of references given by *Unrau* [1997]. The Faraday effect in such artificial media is another topic of current interest.

The formulation of the generalized Faraday effect is made in section 2. In order to show the changing state of the wave polarization, the polarization ratio R plane is used. A wave of arbitrary initial polarization will propagate with a changing wave polarization. The changing state of polarization can be described by a circular locus in the complex R plane. A circle is completely determined if the coordinates of its center and the radius of the circle are both known. Formulas for radius and center location have been derived. The use of these formulas will reveal how the wave polarization transforms as it propagates. The formulas are then applied to three different examples in section 3.

In a biaxial medium, the two characteristic waves are linearly polarized but orthogonal to each other. A wave of arbitrary polarization initially will transform according to the locus of a centered circle going through the point in the R plane. This implies that as the wave propagates, the elliptically polarized wave will have its wave polarization changing in such a manner that its polarization ellipses can all be circumscribed in the same rectangle. Whatever its initial sense of rotation is, it will become opposite after passing through the state of linear polarization. The tilt angle of the ellipse is zero (if $|R_0| > 1$) or 90^o (if $|R_0| < 1$) when the axial ratio minimizes; the magnitude of the tilt angle maximizes when the polarization is linear. The pattern repeats itself after propagating a certain distance so that the differential phase between the two characteristic waves shifts by 2π.

The second medium considered is an isotropic chiral medium. The two characteristic polarizations in such a medium are both circular but of opposite sense. In this case, an initially linearly polarized wave will remain linearly polarized, except its plane of polarization will rotate, the classical Faraday effect. If the initial polarization is elliptical with a right-hand (or left-hand) sense of rotation, the sense of rotation will remain right-handed (or left-handed) as it propagates, even though the axial ratio changes continuously . Thus the sense of rotation never changes, contrary to what occurs in a biaxial medium.

The third medium considered is a magnetoplasma. The characteristic polarization depends on the propagation angle θ_B relative to the direction of the steady magnetic field. When the propagation is lon-



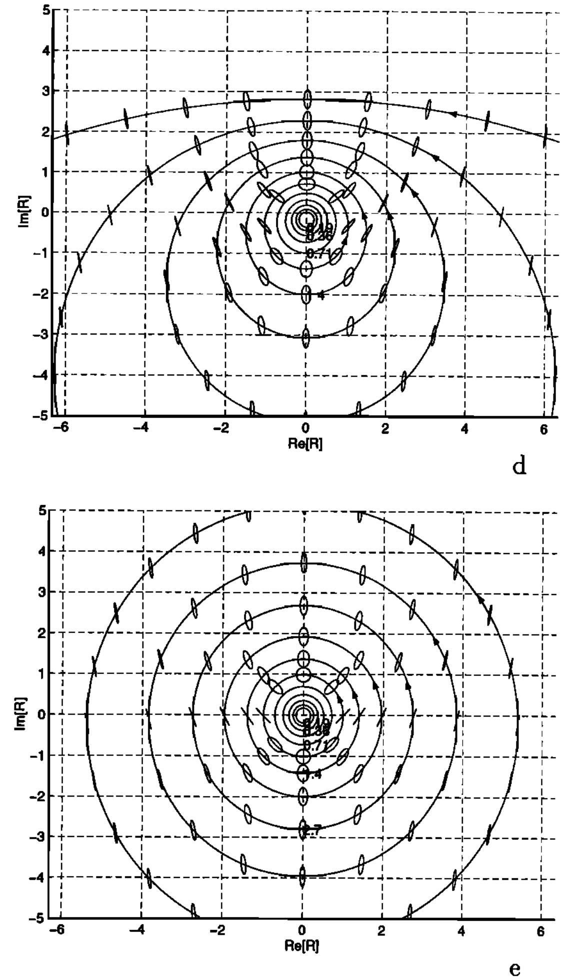


Figure 5. The transformation of the polarization for several θ_B of the second example in a magnetoplasma. In this example, $k_1 < k_2$. The parameters chosen correspond to $f = 10$ MHz, $f_p = 7.07$ MHz, and $f_B = 11.9$ MHz. (a) The magnitude of the polarization ratio for characteristic wave 1 in a magnetoionic medium as a function of θ_B; (b) $\theta_B = 30.000^o$, $\Delta z = 0.021$ km (in this and the following three cases, circles are computed for several values of the amplitude ratio $|E_1/E_2|$); (c) $\theta_B = 50.000^o$, $\Delta z = 0.033$ km; (d) $\theta_B = 70.000^o$, $\Delta z = 0.038$ km; and (e) $\theta_B = 89.000^o$, $\Delta z = 0.039$ km.

gitudinal (i.e., $\theta_B = 0$), the two characteristic polarizations are circular of opposite sense. The associated Faraday effect is quite similar to what occurs in a chiral medium. When the propagation is transverse (i.e., $\theta_B = \pi/2$), the two characteristic polarizations are linear but orthogonal. The resulting Faraday effect is much like what occurs in a biaxial medium.

As the angle θ_B increases from 0 to $\pi/2$, the properties of the generalized Faraday effect must change between these two limiting cases. Two numerical examples have been considered. Parameters are chosen so that the wave propagates in all directions for the two characteristic waves in both examples.

In the first example, the chosen parameters are suitable for applications in ionospheric propagation (f = 25 MHz, f_p = 8 MHz, and f_B = 1.6 MHz). Since the radio frequency is high, refractive indices for both characteristic waves are close to unity. Under such a condition the quasi-longitudinal approximation is known to be accurate for all angles except within a few degrees from the exact perpendicular condition [*Ratcliff*, 1959; *Yeh and Liu*, 1972a]. In this case, the transition of the Faraday effect from longitudinal behavior to transverse behavior takes place rapidly in a few degrees from $\theta_B = \pi/2$, as shown in Figure 4. On the other hand, in the second example, the parameters are chosen such that the transition of longitudinal behavior to transverse behavior takes place slowly as θ_B increases from 0 to $\pi/2$ (f = 10 MHz, f_p = 7.07 MHz, and f_B = 11.9 MHz). This is reflected in Figure 5. The periodic distances for these loci are shown in Figures 4 and 5 as Δz.

A comparison of these two examples reveals differences in three aspects: The transition from longitudinal behavior to transverse behavior takes place over different ranges of θ_B angles; the propagation distances Δz over which the wave polarization will repeat itself can be a thousandfold different; and the direction of polarization transformation indicated by loci on the R plane as a function of propagation distance is opposite. In plots shown in Figures 4 and 5, the amplitude ratio $|E_1/E_2|$ is used as a parameter. In some applications, it may be desirable to use initial polarization as a parameter. For this purpose and as a demonstration, the initial polarization is assumed to be linear, with a tilt angle equal to 0°, 30°, 60°, or 90°. The plasma parameters are chosen to be identical to those of Figure 5, i.e., f = 10 MHz, f_p = 7.07 MHz, and f_B = 11.9 MHz. Depicted in Figure 6a is the case when $\theta_B = 0°$. This is the classical Faraday effect situation. All four curves merge into one, as a linearly polarized wave of any initial tilt angle will have its tilt angle rotated as the wave propagates. When θ_B is increased to 30°, the situation becomes very different. As depicted in Figure 6b, the four curves corresponding to initial polarizations with four different tilt angles are now separated. The four circles start at $\tan\theta_B$, θ_B = 0°, 30°, 60°, or

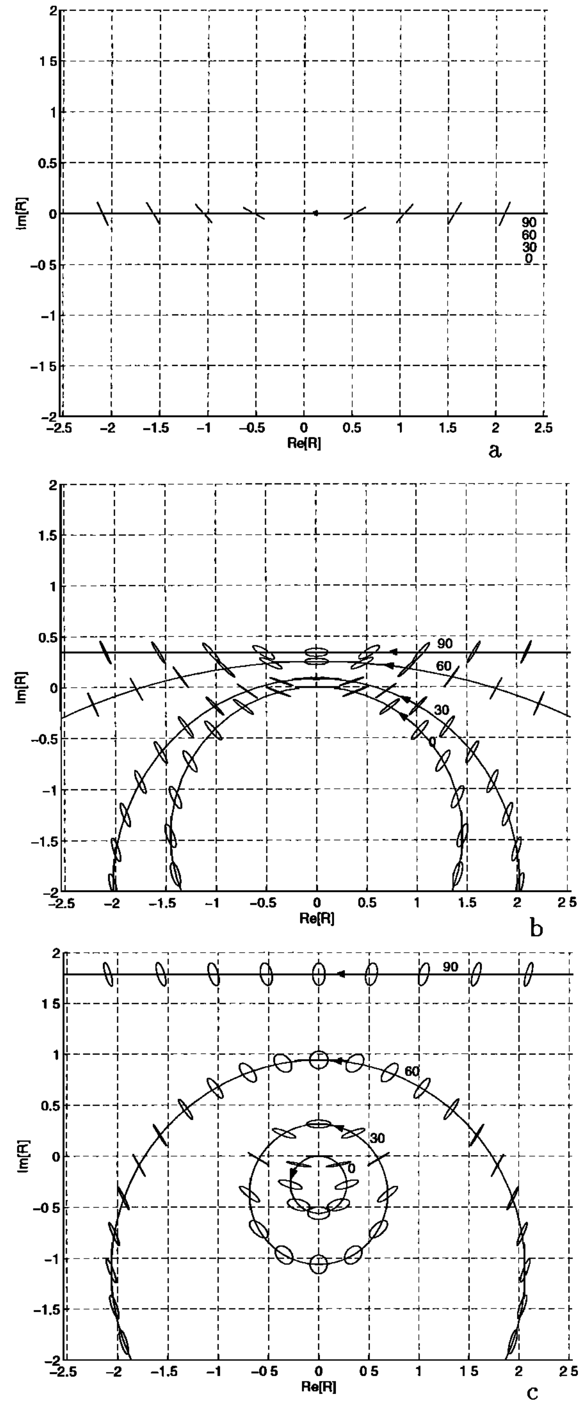


Figure 6. The transformation of an initially linearly polarized wave with a tilt angle equal to 0°, 30°, 60°, or 90° in a magnetoplasma. The plasma parameters chosen are identical to those for Figure 5, i.e., f = 10 MHz, f_p = 7.07 MHz, and f_B = 11.9 MHz. Here θ_B equals (a) 0.000°, (b) 30.000°, (c) 60.000°, and (d) 90.000°.

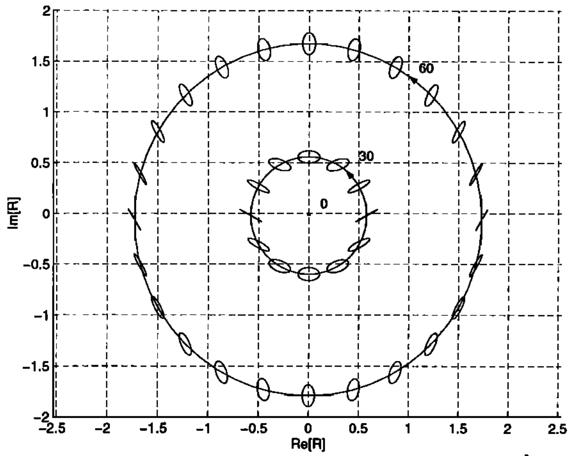


Figure 6. (continued)

90^o, and each undergoes its polarization transformation when the wave propagates, as indicated by the attached ellipses. As θ_B increases farther to 60^o, the radii of all circles decrease (see Figure 6c). When $\theta_B = 90^o$, the 0^o tilt angle (corresponding to an initially horizontally polarized wave) circle shrinks to a point, as depicted in Figure 6d. The 30^o and 60^o circles are centered, passing through $\tan 30^o$ and $\tan 60^o$ points, respectively, on the real R axis. The 90^o circle has an infinite radius and cannot be shown in the finite R plane. Thus, Figure 6d has reached the transverse propagation condition.

The theory presented in this paper is applicable in a general anisotropic lossless and homogeneous dielectric medium. If the medium is lossy, the Faraday locus as a function of propagation distance z in the complex R plane will not repeat itself, but will follow some spiral path toward the point representing the less attenuated characteristic wave eventually.

If the medium is inhomogeneous but slowly varying, the situation can become highly complex, especially when the wave packets are considered. After propagating a certain distance, each characteristic wave packet, the sum of which makes up the original wave packet initially, follows its own group path and may become far enough separated as to render negligible interactions between them. When that happens, the continued separate propagation of the characteristic wave packet has at least three features worth noting [*Dong and Yeh*, 1992]: (1) Pulse dispersion can be described as direction-dependent stretching of the pulse shape, (2) the wave polarization can become position-dependent relative to the pulse center, and (3) the wave packet generally involves three-dimensional energy flow. On the other hand, before the separation occurs, the interference of characteristic waves and the polychromatic nature of a wave packet can make the wave polarization highly complex, in general.

The Faraday effect also occurs in ferrites in the presence of a magnetic field. A parallel analysis can be carried out for ferrites by allowing the permeability to take the dyadic form. In engineering applications, the polarizers usually have a finite dimension. The reflection and transmission effects at interfaces must then be taken into account. This problem has been investigated before. Currently, we are also looking into such problems with some artificial media in mind.

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