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A STUDY OF THE HILBERT SPACE PROPERTIES OF THE VENEZIANO MODEL OPERATOR FORMALISM

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ABSTRACT

The domains of definition of the operators used to factorize the generalized Veneziano model are studied within the Hilbert space defined by the harmonic oscillator creation and annihilation operators $a_\mu^{(r)\dagger}, a_\mu^{(r)}$. These individual operators may not be well behaved although, of course, the matrix elements used in the conventional operational factorization are well defined.

Concerning the individual operators it is shown that the ground-state vertex written as

$$V(p) = \exp\left(-\sum_{r=1}^{\infty}(p.a^{(r)\dagger})/\sqrt{r}\right)\exp\left(\sum_{r=1}^{\infty}(p.a^{(r)})/\sqrt{r}\right)$$

is nowhere defined within the Hilbert space; the product with a twisting operator $\Omega(q)V(p)$ is, however, densely defined, as is the symmetrical three-reggeon vertex. The propagator $D(p)$ is bounded everywhere, away from its poles. The twisting operator $\Omega(p)$ is undefined on finite occupation states, but is densely defined on a subset of coherent states; its Hermitian conjugate $\Omega^+(p)$ is densely defined on both finite occupation and coherent states. It is found that a suitable re-written form of the product $D(q)V(p)$ is densely defined for certain values of momenta; this relates to the fact that off-mass shell states satisfying $(L_0 - L_{-r} - 1)|\phi\rangle = 0$, where L_n are the conventional gauge operators, are better defined than those satisfying $(L_0 - L_{-r} + r - 1)|\phi'\rangle = 0$.

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INTRODUCTION

The study of the properties of the N particle generalization of Veneziano's beta function dual model for two-body scattering has been greatly facilitated by the harmonic oscillator operator formalism [1]. This operator formalism makes manifest the factorization properties and the spectrum of states which are not obvious in the original integral representation. It can be written in a form where both the factorization and Möbius invariance properties are displayed simultaneously [2]. For a review of the formalism we refer the reader to the article of Alessandrini et al., [3] and the references cited therein.

In the present paper we shall be concerned with the operator formalism developed in Ref. 1), together with the twisting operator and symmetric three-reggeon vertex of Ref. 4). With the three operators: the propagator $D(p)$, the symmetric vertex $V(p_1p_2p_3)$ and the twisting operator $\Omega(p)$ one can, aside from the gauge identities (which we shall consider towards the end of the article in discussing the physical states), build up the whole theory including loops.

We shall discuss the mathematical basis of the operator formalism, in particular the properties within the Hilbert space defined by the Fock space of harmonic oscillator states.

Concerning the three principal operators, regarded as operators acting on Hilbert space states, the results may be summarized: the propagator $D(p)$ is bounded over the whole space; the ground state vertex $V(p)$ is nowhere defined, while the symmetric vertex $\Omega(q)V(p)$ and its generalization to the symmetrical three-reggeon vertex $V(p_1p_2p_3)$ are densely defined; the twisting operator $\Omega(p)$ is not defined except on states with null four-momentum, while its adjoint $\Omega^+(p)$ is densely defined.

Of course, the Hilbert space is rather a restricted concept and similar difficulties of staying within a Hilbert space occur already in non-relativistic quantum mechanics [5]. The conventional usage of the operator formalism involves always matrix elements of strings of operators (VDVD...V) and these are well defined in terms of generalized beta functions and their analytic continuation; thus the results of the present paper do not, of course, cast any doubt on the validity of the normal use of the operator formalism (i.e., on the matrix elements). The mathematical properties of the specific operators are important to know, if one wants to extract as much as possible out of the operator formalism. It is also of importance in understanding dual models and might provide a means for further developments and for construction of other more realistic dual models.

The organization of the paper is as follows: in Section 2 we give some mathematical definitions of what we mean by certain classes of vectors within the Hilbert space and introduce some terminology useful for discussing the domains of definition of operators. We study the propagator and twisting operator in Section 3, while in Section 4 the vertex is investigated, firstly the ground-state vertex and then the fully symmetric three-reggeon vertex. Section 5 is concerned with the redefinition of the product $D(q)V(p)$, and with the alternative definitions of physical states. The final Section 6 is devoted to some discussion.

2. MATHEMATICAL DEFINITIONS

When we, in the following sections, are going to claim that certain operators are defined what we shall mean is really only that they are defined as operators mapping a Hilbert space into itself (or possibly into another Hilbert space).

The Hilbert space of interest for us is the Fock space in the operator formalism [1] of the Veneziano model. Let us first consider a set of occupation number states of the type

$$\left| \{ \ell_1^1, \ell_1^2, \dots, \ell_1^0; \ \ell_2^1, \dots \ \} \right\rangle = \prod_{n=1}^{\infty} \prod_{\mu=0}^{3} \frac{\left(a_\mu^{(n)+}\right)^{\ell_n^\mu}}{\sqrt{\ell_n^\mu !}} \left| 0 \right\rangle \tag{2.1}$$

where

$$\left[a_\mu^{(n)}, a_\nu^{(m)+} \right] = - \delta_{nm} \, g_{\mu\nu} \tag{2.2}$$

with

$$g_{\mu\nu} = - \delta_{\mu\nu} (-1)^{\delta_{\mu 0}}$$

Here the ℓ_n^{μ}'s are zero for n sufficiently large. The state with all occupation numbers identically equal to zero is called the vacuum state $|0\rangle$.

We define the space $\mathfrak{F}$ as the vector space consisting of all (finite) linear combinations of the vectors (2.1). We call $\mathfrak{F}$ the space of finite occupation states. A typical state $|f\rangle \in \mathfrak{F}$ may be written

$$|f\rangle = \sum_{\{\ell\}} C_{\{\ell\}} \, |\{\ell\}\rangle$$

where only finitely many of the coefficients $C_{\{\ell\}}$ are non-zero.

The space is made a pre-Hilbert space by defining the inner product

$$\langle f' | \Gamma | f\rangle = \sum_{\{\ell\}} C'^{*}_{\{\ell\}} \, C_{\{\ell\}} \tag{2.3}$$

which follows from (2.1) when we put

$$\Gamma = (-1)^{\sum_{n=1}^{\infty} a_0^{(n)+} a_0^{(n)}} \tag{2.4}$$

A norm is defined by $\| \, |f\rangle \| = \sqrt{\langle f | \Gamma | f\rangle}$. Note that this definition of the metric is not Lorentz invariant.

Completing the space $\mathfrak{F}$ we obtain the full Hilbert Fock-space $\mathfrak{H}$. The points $|h\rangle$ of the Hilbert space $\mathfrak{H}$, which we will consider in this article, can be written as formally infinite linear combinations of states of type (2.1), i.e.,

$$|h\rangle = \sum_{\{\ell\}} C_{\{\ell\}} \, |\{\ell\}\rangle \tag{2.5}$$

but now an infinite number of coefficients $C_{\{\ell\}}$ may be different from zero. However, the norms in $\mathfrak{H}$ are bounded and it is thus required that

$$\sum_{\{e\}} |c_{\{e\}}|^2 < \infty \tag{2.6}$$

As a consequence of the non-covariance of the metric Γ, Eq. (2.4), the Hilbert space $\mathcal{H}$ is not covariant.

An example of a state in $\mathcal{H}$ is the coherent state defined by

$$|\alpha^{(n)}\rangle = \exp\left(\sum_n \alpha^{(n)} \cdot a^{(n)\dagger} \right) |0\rangle \tag{2.7}$$

where we have required

$$\sum_{n=1}^{\infty} \sum_{\mu=0}^{3} |\alpha_\mu^{(n)}|^2 < \infty \tag{2.8}$$

We shall denote the set of all finite linear combinations of finite-norm coherent states as $\mathcal{C}$.

A function $\mathfrak{J}$ that maps every vector $|h\rangle \in \mathfrak{D}$ into a vector $\mathfrak{J}|h\rangle \in \Delta \subseteq \mathcal{H}$ is called an operator in the space $\mathcal{H}$ defined on the domain $\mathfrak{D}$ and Δ is called the image when it is required that each element in Δ has the form $\mathfrak{J}|h\rangle$.

In the following sections we shall be interested in whether the domain $\mathfrak{D}$ is a dense set in $\mathcal{H}$, i.e., whether

$$\overline{\mathfrak{D}} = \mathcal{H} \tag{2.9}$$

where $\overline{\mathfrak{D}}$ is the closure of $\mathfrak{D}$, according to the topology defined by the norm.

We shall also be interested in whether the operators are bounded. A linear operator $\mathfrak{J}$ is bounded when

$$\sup_{\substack{\| |h\rangle \| \leq 1 \\ |h\rangle \in \mathfrak{D}}} \| \mathfrak{J} |h\rangle \| < \infty \tag{2.10}$$

3. MÖBIUS GROUP OPERATORS

In the operator formalism of the dual resonance model certain representations of the group of Möbius transformations leaving a circle invariant $SL(2,R) \approx SO(2,1)$ homomorphic to $SU(1,1)$ play an important role [3]. In the conventional model the generators are

$$\begin{aligned} L_0(p) &= -p^2 - \sum_{n=1}^{\infty} n\, a^{(n)+} \cdot a^{(n)} \\ L_+(p) &= -\sqrt{2}\, a^{(1)+} \cdot p - \sum_{n=1}^{\infty} \sqrt{n(n+1)}\; a^{(n+1)+} \cdot a^{(n)} \\ L_-(p) &= \left(L_+(p)\right)^{+} \end{aligned} \tag{3.1}$$

Of particular interest are the following functions of these generators: the propagator

$$D = (L_0 - 1)^{-1} \tag{3.2}$$

and the twisting operators

$$\begin{aligned} \Omega &= (-1)^{L_0} e^{-L_+} = e^{L_+} (-1)^{L_0} \\ \Omega^{+} &= e^{-L_-} (-1)^{L_0} = (-1)^{L_0} e^{L_-} \end{aligned} \tag{3.3}$$

In the upper half of the Table the boundedness and domain properties of such operators are summarized.

We now indicate how these entries in the Table were obtained.

i) $D(p)$ and $D^{-1}(p)$

On the space $\mathfrak{F}$, $D(p)$ is bounded, since the eigenvalue of $(L_0 - 1)^{-1}$ of an $\mathfrak{F}$ state is bounded. This is because the eigenvalue of $(L_0 - 1)^{-1}$ on a state $|f\rangle$ is given by

$$(L_o-1)^{-1}\,|f,p\rangle = \sum_{\{\ell\}} c_{\{\ell\}}\,(L_o-1)^{-1}\,|\{\ell\},p\rangle$$

$$= \sum_{\{\ell\}} c_{\{\ell\}}\,\Big(-p^2-\sum_{n,\mu} n\ell_n^{\mu}-1\Big)^{-1}\,|\{\ell\},p\rangle \tag{3.4}$$

and the norm of this state $\|(L_o-1)^{-1}|f>\|$ is always bounded off-mass shell. Now we can apply a theorem about bounded operators to be found, for example, in Naimark's book [6].

Theorem

In a Hilbert space $\mathfrak{H}$, a bounded linear operator A is extendible by continuity from its domain $\mathfrak{D}_A$ to a bounded linear operator with $\overline{\mathfrak{D}}_A$, i.e., the closure of $\mathfrak{D}_A$, as its domain of definition.

In the present case, since $D(p)$ is bounded on $\mathfrak{F}$ it can be extended, therefore, by continuity to be bounded on $\overline{\mathfrak{F}} = \mathfrak{H}$, the full Hilbert space.

$D^{-1}(p)$ is unbounded on $\mathfrak{F}$, but it is defined there, i.e., $\mathfrak{D}_{D^{-1}(p)} \supseteq \mathfrak{F}$, which is easily seen from Eq. (3.4) written for $(L_o - 1)$. Now we consider $D^{-1}(p)$ acting on a coherent state

$$D^{-1}(p)\,|\alpha^{(n)}\rangle = (L_o-1)\,|\alpha^{(n)}\rangle$$

$$= \Big(-p^2-\sum_{n=1}^{\infty} n\,a_n^{+}\cdot\alpha_n-1\Big)\,|\alpha^{(n)}\rangle \tag{3.5}$$

The norm of this state is given by

$$\big\|D^{-1}(p)\,|\alpha^{(n)}\rangle\big\|^2 =$$

$$(p^2+1)^2\langle\alpha^{(n)}|\alpha^{(n)}\rangle + 2(p^2+1)\sum_{n=1}^{\infty} n|\alpha^{(n)}|^2\,\langle\alpha^{(n)}|\alpha^{(n)}\rangle$$

$$+\Big(\Big[\sum_{n=1}^{\infty} n|\alpha^{(n)}|^2\Big]^2-\sum_{n=1}^{\infty} n^2|\alpha^{(n)}|^2\Big)\langle\alpha^{(n)}|\alpha^{(n)}\rangle \tag{3.6}$$

and the summations can diverge, while still $< \alpha'_n | \alpha_n >$ is finite. Therefore, $D^{-1}(p)$ is not defined on $\mathcal{C}$. It is, however, defined on a dense subset, for example, the dense subset $\{ | \alpha_n = \hat{n} z^n > \}$ where $|z| < 1$ and $\hat{n}$ is a constant four-vector.

ii) $\underline{\Omega^+(p)}$

On a state of the $\mathcal{F}$ space we have

$$\Omega^+(p)\,|f> = (-1)^{L_0(p)}\, e^{L_-(p)}\, |f> \tag{3.7}$$

Now the exponential in $|L_-(p)|$ becomes a polynomial; also the mode summation in $L_-(p)$ is finite. Therefore, the state $e^{L_-(p)}|f>$ has finite but unbounded norm. The operator $(-1)^{L_0(p)}$ is unitary and, therefore, norm preserving. It follows that $\Omega^+(p)$ is defined on $\mathcal{F}$, but unbounded on both $\mathcal{F}$ and $\bar{\mathcal{F}} = \mathcal{H}$. Since $|0> \subset \mathcal{F}$, $\Omega^+(p)$ is defined on the vacuum.

For a coherent state $|\alpha_n, p>$

$$\Omega^+(p)\,|\alpha_n, p> = (-1)^{L_0(p)}\, e^{L_-(0)} \exp\Big(-\sum_{n=1}^{\infty} (a_n \cdot p)/\sqrt{n}\Big)\, |\alpha_n, p> \tag{3.8}$$

Using the canonical formalism of Alessandrini et al. [3] we now have

$$e^{L_-(0)}\, |\alpha_n\, p> = \Big| \sum_m c_{nm}\, \alpha_m,\, p> \tag{3.9}$$

where

$$\sum_{m=1}^{\infty} c_{nm} \frac{z^m}{\sqrt{m}} = \frac{1}{\sqrt{n}} \left(\frac{z}{1-z}\right)^n \tag{3.10}$$

Thus,

$$\| e^{L_-(0)}\, |\alpha_n, p> \|^2 = \exp\Big(\sum_{n=1}^{\infty} \Big| \sum_{m=n}^{\infty} \sqrt{\frac{n}{m}} \binom{m}{n} \alpha_m \Big|^2 \Big) =$$

$$= \exp\Big(\sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} \sqrt{n(n+1)}\, |\alpha_{n+1}|^2 + \dots \Big) \tag{3.11}$$

which for all $\alpha_n > 0$ can be divergent even for $\sum |\alpha_n|^2$ finite. To show that it can be defined on a dense set of $\mathcal{C}$ we use $\alpha_n = \hat{n}\, z^n/\sqrt{n}$ with $|z| < 1$; then we have

$$\| e^{L_-(p)} |\alpha_n\, p\rangle \|^2 = \exp\left(|\hat{n}|^2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left|\frac{z}{1-z}\right|^{2n} \right) \times$$

$$\times \exp\left(-2\,\mathrm{Re}\left[\sum_{n=1}^{\infty} \frac{z^n}{n}\ \hat{n}\cdot p \right] \right) \tag{3.12}$$

where $|\hat{n}|^2 = \hat{n}_0^2 + \underline{\hat{n}}^2$, which is convergent for $\mathrm{Re}\, z < \frac{1}{2}$. Therefore, $\Omega^+(p)$ is defined on a dense set in $\mathcal{C}$.

iii) $\underline{\Omega(p)}$

For general p^2 we have

$$\| \Omega(p) |0\rangle \|^2 = \exp\left(|p|^2 \sum_{n=1}^{\infty} \frac{1}{n} \right) \tag{3.13}$$

Note that for $p^\mu = 0$ we have $\Omega(0)|0\rangle = |0\rangle$. Thus, on vacuum, $\Omega(p)$ is not defined within the Hilbert space for $p_\mu \neq 0$.

For an occupation state $|f\rangle$ we investigate first a singly occupied level, where

$$\| \Omega(0)\, a_q^+ |0,0\rangle \|^2 = \| e^{-L_+(0)}\, a_q^+ |00\rangle \|^2 =$$

$$= \sum_{n=0}^{\infty} \left| \frac{(q+n)!}{q!\, n!} \sqrt{\frac{q}{q+n}} \right|^2 = \infty, \quad \text{for any } q \geq 1 \tag{3.14}$$

where we have used the explicit form of $L_+(0)$, Eq. (2.1), and the fact that $(-1)^{L_0(0)}$ is unitary. It is not difficult to convince oneself that this argument generalizes to any $|f\rangle$ state. Therefore, $\Omega(0)$ is unbounded on $\mathcal{F}$, and on its closure $\overline{\mathcal{F}} = \mathcal{H}$.

For a coherent state we can write

$$\left\| \Omega(p)\,|\alpha_n\rangle \right\|^2 = \left\| (-1)^{L_0(p)}\, e^{-L_+(0)} \exp\left(\sum_{n=1}^{\infty} (a_n^{+}\cdot p)/\sqrt{n}\right) |\alpha_n\rangle \right\|^2$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{m=1}^{n} \binom{n}{m} \sqrt{m}\,(-1)^m \alpha_m^{\mu} - p^{\mu} \right|^2 \right) \tag{3.15}$$

If we choose $\alpha_m^{\mu} = -z^m p^{\mu}/\sqrt{m}$ such that $|z| < 1 \wedge |1-z| < 1$ then this leads to a finite norm. By adding to α_m^{μ} lower powers of z (obtained by differentiating α_m as is discussed in more detail for the vertex in Section 4 below) we can show that $\Omega(p)$ is defined on a dense subset of $\mathcal{C}$, although not on $\mathcal{C}$ itself since $\mathcal{C}$ includes states with α_m^{μ} such that, for example, $0 > z > -1$.

4. THE VERTEX

In this Section we consider the conventional untwisted vertex, $V(p)$ for emission of a scalar ground state meson; the result will be that $V(p)$ is nowhere defined within the Hilbert space. We shall, however, find that the twisted vertex $\Omega(p+q)V(p)$ is a densely defined unbounded operator. More generally the cyclically symmetric Caneschi-Schwimmer-Veneziano vertex will be found to be defined for a certain dense set of coherent states in the sense that putting in one type of coherent state on one leg, together with another type on a second leg, one will obtain at the third leg a normalizable state.

We shall first give some rather simple and convincing arguments that the vertex $V(p)$ is not defined as long as the momentum p has no time component (which is possible for special spacelike momentum), and further is not defined, for general momentum, on any $\mathcal{C}$ or $\mathcal{F}$ state. Only then shall we introduce a more abstract approach to demonstrate that $V(p)$ is quite generally undefined.

The conventional ground state vertex in the operator formalism is written formally [1)]

$$V(p) = \exp\left(-\sum_{n=1}^{\infty} \frac{(a^{(n)+}\cdot p)}{\sqrt{n}}\right) \exp\left(\sum_{n=1}^{\infty} \frac{(a^{(n)}\cdot p)}{\sqrt{n}}\right) \tag{4.1}$$

so the N point function can then be written formally (in a multiperipheral configuration)

$$A_N = \langle 0|\, V(k_2)\, D(s_{12})\, V(k_3)\, D(s_{13}) \ldots\ldots \quad V(k_{N-2})\, D(s_{N-1,N})\, V(k_{N-1})\, |0\rangle \tag{4.2}$$

where $S_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2$, and the bra and ket vacuum states have momenta k_1, k_N respectively.

We now will show the unhappy result that the operator $V(p)$ is not defined anywhere within the Hilbert space. Define an operator

$$V_x(p) = \exp\left(-\sum_{n=1}^{\infty} \frac{a^{(n)+}\cdot p}{\sqrt{n}}\, x^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{a^{(n)}\cdot p}{\sqrt{n}}\, x^n\right) \tag{4.3}$$

so that $\lim_{x\to 1} V_x(p) = V(p)$; also define, for momentum with no time component, the unitary operator

$$\mathcal{U}(p,x) = \exp\left(p\cdot \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\, x^n \left(a^{(n)} - a^{(n)+}\right)\right) \tag{4.4}$$

$U(p,x)$ is unitary (and bounded) in $\mathcal{F}$. Now, we use the fact

$$\forall\ |\psi\rangle \in \mathcal{H} \quad \exists \quad \text{a sequence } \{|\phi_n\rangle\}_{n=1}^{\infty}$$

such that

$$\lim_{n\to\infty} |\phi_n\rangle = |\psi\rangle \quad \text{where all } |\phi_i\rangle \in \mathcal{F} \tag{4.5}$$

Now we observe that

$$V_x(p) = c(p,x)\, \mathcal{U}(p,x) \tag{4.6}$$

with the c number

$$c(p,x) = \exp\left(-\frac{p^2}{2}\sum_{n=1}^{\infty}\frac{x^n}{n}\right) \tag{4.7}$$

Now since $U(p,x)$ is unitary for all x we can consider

$$\lim_{n\to\infty} \left\| \mathcal{U}(p,x)\,|\phi_n\rangle \right\|^2 = \lim_{n\to\infty} \left\| |\phi_n\rangle \right\|^2 = \left\| |\psi\rangle \right\|^2 \tag{4.8}$$

and then we see that

$$\begin{aligned} \left\| V_x(p)\,|\phi_n\rangle \right\|^2 &= |c(p,x)|^2 \left\| U(p,x)\,|\phi_n\rangle \right\|^2 \\ &= |c(p,x)|^2 \left\| |\phi_n\rangle \right\|^2 \end{aligned} \tag{4.9}$$

Therefore $V_x(p)$ is bounded by $|c(p,x)|$ and, for $|x|<1$, $V_x(p)$ is defined on $\mathfrak{F}$. For $x\to 1$, however, on a general state $|\psi\rangle$

$$\begin{aligned} \lim_{x\to 1} \left\| V_x(p)\,|\psi\rangle \right\|^2 &= \lim_{x\to 1} |c(p,x)|^2 \left\| |\psi\rangle \right\|^2 \\ &= \infty \end{aligned} \tag{4.10}$$

Thus we deduce, for momentum with no time component, that $V(p)$ is undefined everywhere in the Hilbert space.

For general p_μ, it is easy to show that $V(p)$ is not defined on any coherent state or any $\mathfrak{F}$ state, as a Hilbert space operator. In the case of a coherent state $|\{\alpha_n\}, q\rangle$ there occurs, in the norm $\| V(p)|\{\alpha_n\}, q\rangle \|$, an exponential of a term

$$\left[(p^2 + p_0^2)\sum_{n=1}^{\infty}\frac{1}{n}\right]$$

which cannot be cancelled by any choice of the α_n such that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$$

That $V(p)$ is not defined on $\mathcal{F}$ states is seen by noticing that the harmonic oscillators above a certain mode number are excited by $V(p)$ in just the same way as mentioned already for the coherent state.

The more abstract general proof for the non-existence of $V(p)$ proceeds in three steps (i) the derivation of an operator identity for $V_x(p)^{\dagger}\Gamma V_x(p)$; (ii) the proof that the expectation value of an exponential of a Hermitian operator is strictly greater than zero; and (iii) the deduction from (i) and (ii) that any image of $V(p)$ has infinite norm.

(i) Using the commutation rules, Eq. (2.2), for the harmonic oscillator operators, it is straightforward to find that (formally), for $p_\mu = (p_{0\perp}, \underline{p})$,

$$V_x(p)^{\dagger}\ \Gamma\ V_x(p) = \exp\left(p_\mu \sum_{n=1}^{\infty} \frac{a_\mu^{n\dagger} x^n}{\sqrt{n}}\right) \exp\left(-p_\mu \sum_{n=1}^{\infty} \frac{a_{n\mu} x^n}{\sqrt{n}}\right) \Gamma \cdot$$

$$\cdot \exp\left(-p_\mu \sum_{n=1}^{\infty} \frac{a_\mu^{n\dagger} x^n}{\sqrt{n}}\right) \exp\left(p_\mu \sum_{n=1}^{\infty} \frac{a_\mu^{n} x^n}{\sqrt{n}}\right)$$

$$= (1-x^2)^{-3p_0^2 - \underline{p}^2}\ \tilde{\mathcal{U}}_x \exp(-H_x)\, \Gamma \exp(H_x)\ \tilde{\mathcal{U}}_x \tag{4.11}$$

as an operator identity, where

$$\tilde{\mathcal{U}}_x = \exp\left(p_0 \sum_{n=1}^{\infty} \frac{a_{n0}^{\dagger} + a_{n0}}{\sqrt{n}}\ x^n\right) \tag{4.12}$$

is Hermitian with respect to the indefinite metric

$$\tilde{\mathcal{U}}_x = \tilde{\mathcal{U}}_x^{\dagger} \tag{4.13}$$

and unitary with respect to the Γ metric

$$\tilde{\mathcal{U}}_x^{-1} = \Gamma \, \tilde{\mathcal{U}}_x^{+} \, \Gamma \tag{4.14}$$

and where

$$H_x = p_0 \sum_{n=1}^{\infty} \frac{a_{n0} - a_{n0}^{+}}{\sqrt{n}} \, x^n \tag{4.15}$$

is Hermitian in the Γ metric

$$H_x = \Gamma \, H_x^{+} \, \Gamma \tag{4.16}$$

In deriving Eq. (4.11) we used

$$\exp\left(2p_0 \sum_{n=1}^{\infty} \frac{a_{n0} x^n}{\sqrt{n}}\right) = (1-x^2)^{-p_0^2} \exp(H_x) \, \tilde{\mathcal{U}}_x$$

The Γ norm squared of $V(p)|h>$ where $|h> \in \mathcal{H}$ is the expectation value of this operator (4.11) and from the ultimate step in Eq. (4.11) we see that this expectation value is equal to a diverging c number multiplied by the expectation value of an exponential $\exp(2H_x)$ for the state $\tilde{U}_x|h>$. Since $\tilde{U}_x$ is unitary and so bounded this state $\tilde{U}_x|h>$ exists for all $|h> \in \mathcal{H}$ and is in fact different from zero, since $\| \tilde{U}_x |h> \| = \| h> \|$.

(ii) Now we wish to show that the expectation value $\langle h|\exp(H)|h\rangle$, for H a Hermitian operator, is greater than zero.

According to the spectral theorem of von Neumann [7] for Hermitian operators we may write the Stieltjes integral form of H

$$H = \int_{-\infty}^{\infty} t \; dI_t \tag{4.17}$$

where I_t is the family of projectors for the operator H. More generally we may make such an integral representation for any operator function of H, using the same spectral function I_t; in particular,

$$\exp(H) = \int_{-\infty}^{\infty} \exp(t) \; dI_t \tag{4.18}$$

The resolution of the identity, I_t, is defined such that $\lim_{t\to-\infty} I_t = 0$ and $\lim_{t\to+\infty} I_t = 1$. Therefore, there exists a t' such that

$$\langle h| \, (1 - I_{t'}) \, |h\rangle > 0$$

whereupon writing

$$\langle h| \exp(H) |h\rangle = \int_{-\infty}^{t'} \exp(t) \; \langle h| \, dI_t \, |h\rangle + \int_{t'}^{\infty} \exp(t) \; \langle h| \, dI_t \, |h\rangle$$

we see that the first term on the right-hand side is greater than or equal to zero, while the second term satisfies

$$\int_{t'}^{\infty} \exp(t) \langle h | dI_t | h \rangle \geqslant \exp(t') \int_{t'}^{\infty} \langle h | dI_t | h \rangle$$

$$= \exp(t') \langle h | (1 - I_{t'}) | h \rangle \quad > 0$$

(4.20)

It follows that

$$\langle h | \exp(H) | h \rangle \quad > 0 \qquad \text{Q.E.D.}$$

(4.21)

(iii) Now we combine the results of Eqs. (4.11) and (4.21) to deduce that

$$\| V(p) | h \rangle \|^2 = \lim_{x \to 1} \Big[(1-x^2)^{-3p_0^2 - \not{p}^2} \cdot$$

$$\langle h | \tilde{u}_x^{+} \Gamma \exp(2H_x) \tilde{u}_x | h \rangle \Big] =$$

$$= \Big[\lim_{x \to 1} (1-x^2)^{-3p_0^2 - \not{p}^2} \Big] \cdot \langle h | \tilde{u}_1^{+} \Gamma \exp(2H_1) \tilde{u}_1 | h \rangle$$

$$= \infty$$

(4.22)

By this we have shown that the vertex $V(p)$ is non-existent within the Hilbert space.

Despite the bad properties of $V(p)$ the operator ΩV corresponding to the cyclically symmetric vertex can in fact be defined somewhere even for spacelike momentum of the ground state particle. Note that we must regard the operator ΩV as a single entity rather than as a product of two operators, if we wish to remain in the Hilbert space. In fact let

$$|\alpha, q\rangle = \prod_{n=1}^{\infty} e^{\alpha_n \cdot a^{(n)\dagger}} |0, q\rangle \tag{4.23}$$

be a coherent state with four-momentum q, the twisted ground state vertex is formally written $\Omega(q+p)V(p)$ when acting on $|\alpha, q\rangle$ and

$$\Omega(p+q)V(p)|\alpha, q\rangle = \Omega(0)\, e^{(p+q)\cdot\sum \frac{a^{(n)\dagger}}{\sqrt{n}}} \cdot$$

$$e^{-p\cdot\sum \frac{a^{(n)\dagger}}{\sqrt{n}}}\; e^{p\cdot\sum \frac{a^{(n)}}{\sqrt{n}}}\; |\alpha, q\rangle =$$

$$= e^{-q\sum_{n=1}^{\infty}\frac{a^{(n)\dagger}}{\sqrt{n}}} : \exp\left(-\sum_{n,m} a^{(n)\dagger}(C_{nm}-\delta_{nm})\, a^{(m)}\right): \cdot$$

$$e^{p\cdot\sum\frac{a^{(n)}}{\sqrt{n}}}\; |\alpha, q\rangle$$

$$= e^{p\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{\sqrt{n}}}\; \left|\left\{-\sum_j C_{ij}\alpha_j - q\frac{1}{\sqrt{i}}\right\}_{i=1,2,\ldots}, q\right\rangle \tag{4.24}$$

Here

$$C_{nm} = \sqrt{\frac{m}{n}}\,(-1)^m \binom{n}{m}$$

It is rather easy to see that by taking for instance

$$\alpha_n = \frac{z^n}{\sqrt{n}}\cdot q \qquad \text{with} \qquad |z|<1 \wedge |1-z|<1 \tag{4.25}$$

and using [3]

$$\sum_{m=1}^{\infty} C_{nm}\frac{z^m}{\sqrt{m}} = \frac{(1-z)^n - 1}{\sqrt{n}} \tag{4.26}$$

that the expression for the formal symbol $\Omega(p+q)V(p)|\alpha, q\rangle$ becomes

$$e^{p\cdot q \sum_{n=1}^{\infty} \frac{z^n}{n}} \mid \{ - \frac{(1-z)^i}{\sqrt{i}} q \}_{i=1,2,\ldots}, q > \tag{4.27}$$

which is a finite norm coherent state. To be specific the norm is

$$\| e^{p\cdot q \sum_{n=1}^{\infty} \frac{z^n}{n}} \mid \{ - \frac{(1-z)^i}{\sqrt{i}} q \}_{i=1,2,\ldots}, q > \| =$$

$$= \exp \left(\mathrm{Re}\, p\cdot q \sum_{n=1}^{\infty} \frac{z^n}{n} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{(1-z)^{2i}}{i} |q|^2 \right) \tag{4.28}$$

and this exponential is finite when $|z| < 1$ and $|1 - z| < 1$.

The possibility (4.25) is only one out of infinitely many since we can add to the series α any series for which $C.\beta$ has finite norm, i.e., for which

$$\sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} C_{nm} \beta_m \right|^2 < \infty \tag{4.29}$$

Using (4.26) it is easy by differentiation for example to find an infinite number of such series β, namely

$$\beta_n^{(r)} = n(n-1)\ldots(n-r+1) \frac{z^{n-r}}{\sqrt{n}} \tag{4.30}$$

where $r = 1,2,\ldots$ and $|z| < 1 \wedge |1-z| < 1$. In fact one finds

$$\sum_{m=1}^{\infty} C_{nm} \beta_m^{(r)} = (-1)^r \frac{(1-z)^{n-r} n(n-1)\ldots(n-r+1)}{\sqrt{n}} \tag{4.31}$$

which is obviously of finite norm because $|1-z| < 1$. We remark for its own interest that for $z = \frac{1}{2}$ the series β_m is an eigenvector of the matrix C with eigenvalue $(-1)^r$.

We now want to show that by means of the coherent states

$$| \{ \frac{z^i}{\sqrt{i}} q \}_{i=1,2..} + \sum_{r=1}^{\text{finite}} c_r \beta^{(r)}, q > \tag{4.32}$$

where c_r are four-vectors, it is possible to argue that the operator ΩV is defined on a dense domain in the Hilbert space.

Since any coherent state $|\alpha>$ (with $\sum |\alpha_n|^2 < \infty$) can be approximated by another one $|\alpha'>$ provided we can approximate the series α by α' in the norm of the Hilbert space ℓ_2 of series with convergent square sum, we can show that the states of type (4.32) can approximate any coherent state. By choosing z in (4.30) small but different from zero one easily shows that linear combinations of series β can approximate any series which has only zeros after a certain step and thus any series at all. By choosing z sufficiently small we can make the norm of

$$\{ \frac{z^i}{\sqrt{i}} q \}_{i=1,2,\dots}$$

arbitrarily small too and so states of the form (4.32) can approximate any coherent state and so by taking finite linear combinations we can approximate all states in the Hilbert space by states for which ΩV is defined. That is to say ΩV is densely defined.

It is rather easy to see that also the Caneschi-Schwimmer-Veneziano three-reggeon vertex [4] is defined on a dense set in the following sense: there exists a dense set of vectors $|1>$ for which the "vertex operator"

$$_2<0|\, _1<0|\, V(P_1, P_2, P_3)\, |1> |2> |0>_3 \tag{4.33}$$

mapping the space 2 into space 3 is densely defined, i.e., has a dense domain in Hilbert space 2.

We shall see that the domain in space 2 can be chosen the same whatever the state $|1>$ is as long as $|1>$ belongs to the dense set mentioned in space 1.

In fact we can, for the cyclically symmetric vertex

$$V(p_1, p_2, p_3) = {}_a\langle 0| \, {}_b\langle 0| \exp[-p_1 \cdot c^+ + p_2 \cdot a + p_3 \cdot b$$
$$+ [ab]_- - [bc^+]_- - [c^+a]_-] \, |a>|b>|0>_c \tag{4.34}$$

in which

$$[\alpha\beta]_- = \sum_{m,n=1}^{\infty} \frac{\alpha_n \beta_m}{\sqrt{nm}\, B(-n,m)}$$

show that it is defined for the following coherent state:

$$|a> = |\{ \frac{z^n}{\sqrt{n}} p_1 + \sum_{r=1}^{\text{finite}} c_r \beta_n^{(r)} \}_{n=1,2,\ldots}, p_1> \tag{4.35}$$

[a state of type (4.21)], where

$$|z| < 1 \;\wedge\; |1-z| < 1 \tag{4.36}$$

and a state $|b>$ that is either a finite occupation number state or a coherent state with only finitely many modes excited. We shall prove it explicitly for the latter case.

It is in fact rather easy to check that with such states $|a>$ and $|b>$ the vector (4.34) becomes of finite norm. First it is noticed that both $|a>$ and $|b>$ are eigenstates of respectively $e^{p_2 \cdot a}$ and $e^{p_3 \cdot b}$ with finite eigenvalues because of respectively the exponential convergence z^n and the cut-off. Secondly

$$\exp([ab]_-)\, |a>|b> = \exp(\text{convergent c number})\, |a>|b> \tag{4.37}$$

because of the cut-off in the excitations in $|b\rangle$ and the exponential decrease from $|a\rangle$. Thirdly, also in the factors involving $c^\dagger$ can the a's and b's be replaced by c numbers and the overlap with the vacuum states ${}_a\langle 0|\,{}_b\langle 0|$ just results in a finite c number too. So the whole expression becomes a formal coherent state in space 3, i.e., of the form

$$\exp\left(\sum_{n=1}^{\infty} \gamma_n c^{(n)\dagger}\right)|0\rangle_c \tag{4.38}$$

and the only thing to be checked is that the norm is finite, and that will be the case provided

$$\sum_{n=1}^{\infty} |\gamma_n|^2 < \infty \tag{4.39}$$

The contribution to $\sum_{n=1}^{\infty} \gamma_n c^{(n)\dagger}$ from $[bc^\dagger]_-$ is only different from zero for a finite number of γ_n, i.e., $n \leq N$. It is thus not able to spoil (4.39). The main trick is that we have arranged it so that although both $-[c^\dagger a]$ and $-p_1 \cdot c^\dagger$ give contributions that violate (4.39) the sum

$$\{-p_1 \cdot c^\dagger - [c^\dagger a]_-\}$$

gives a contribution that obeys restriction (4.39). In fact the term $(z^n/\sqrt{n})p_1$ is accurately constructed to provide this cancellation; the terms $\sum_{r=1}^{\text{finite}} c_r \beta^{(r)}$ give rise through $-[c^\dagger a]_-$ only to a contribution obeying (4.39). This completes the proof that the cyclically symmetric vertex is densely defined.

5. THE PRODUCT D(q)V(p); PHYSICAL STATES

We have shown that the ground state vertex $V(p)$ is in general non-existent in the Hilbert space. However, consider the combination

$$\begin{aligned} D(q)V(p) &= \int_0^1 dx\; x^{L_0-2}\, V(p) \\ &= \int_0^1 dx\; V(p,x)\, x^{L_0-2} \end{aligned} \tag{5.1}$$

where

$$V(p,x) = \exp\left(-\sum_{r=1}^{\infty} \frac{a^{+}_{r\mu} p^{\mu}}{\sqrt{r}} x^{r}\right) \exp\left(\sum_{r=1}^{\infty} \frac{a_{r\mu} p^{\mu}}{\sqrt{r}} x^{-r}\right) \tag{5.2}$$

We can use the combination of operators in Eq. (5.1), defined by the integral expression containing $V(p,x)$. The rule should be that the integration is done after the integrand has operated. Consider this operator acting now on a coherent state

$$\left\| \int_0^1 dx\, e^{-\sum_r p\cdot a^{+}_r x^{r}/\sqrt{r}}\, e^{\sum_r p\cdot a_r\cdot x^{-r}/\sqrt{r}}\, x^{L_0(q)-2}\, |\alpha_n\rangle \right\|^2$$

$$= \int_0^1 dx_1 dx_2\, (x_1x_2)^{-2-q^2}\, (1-x_1x_2)^{-|p|^2} \times$$

$$\exp\left(\sum_{n=1}^{\infty} (x_1x_2)^n \left(|\alpha_n|^2 - 2\frac{\mathrm{Re}\,|\alpha_n\cdot p|}{\sqrt{n}}\right)\right) \times$$

$$\exp\left(2p_\mu \sum_{n=1}^{\infty} \frac{\alpha_n^{\mu}}{\sqrt{n}}\right) \tag{5.3}$$

where $|p|^2 = p_o^2 + \underline{p}^2$ and $|\alpha_n\cdot p| = (\alpha_{no}p_o + \underline{\alpha}_n\cdot\underline{p})$. The last two exponentials are finite on a dense subset of $\mathcal{C}$ (namely provided $\sum \alpha_n/\sqrt{n} < \infty$).

Hence, on this dense set the norm is finite if $q^2 < -1$ and $|p|^2 < +2$ to avoid singularities at the lower and upper end-points respectively. This redefinition of the product $D(q)V(p)$ is thus defined on a dense subset of $\mathcal{C}$ for these momentum values.

The fact that the product of propagator times ground state vertex is better defined than the vertex alone has some interesting consequences on the definition of an off-mass shell physical state, if we require that the off-shell state remains normalizable within the Hilbert space. In general a physical state defined by its coupling to N ground state particles

$$|\phi\rangle = D\,V(k_1)\,D\,V(k_2)\,D \ldots D\,V(k_{N-1})\,|0\rangle \tag{5.4}$$

satisfies the gauge condition [8)]

$$W_{-r}\,|\phi\rangle = (L_0 - L_{-r} - 1)\,|\phi\rangle = 0 \tag{5.5}$$

If we redefine a physical state without the final propagator then it satisfies instead the conditions

$$|\phi'\rangle = V(k_1)\,D\,V(k_2)\,D \ldots D\,V(k_{N-1})\,|0\rangle$$

$$W_{-r}\,|\phi'\rangle = (L_0 - L_{-r} + r - 1)\,|\phi'\rangle = 0 \tag{5.6}$$

In view of the non-existence of $V(p)$ as a Hilbert space operator we expect that the $|\emptyset'\rangle$ states defined by Eq. (5.6) be not normalizable, and it is amusing to confirm this by constructing such states within the irreducible representations of the gauge algebra.

In Ref. 9), it is described how to analyze the spectrum of states in terms of irreducible representations of the Virasoro algebra, with generators L_r $r = 0, \pm1, \pm2, \pm3, \ldots$. For the present purpose we note that an exactly similar analysis for spacelike momentum can be made using irreducible representations of the Gliozzi algebra [10)] with generators L_0, $L_{\pm1}$. Each irreducible representation of the Gliozzi algebra then contains one and only one state (that having lowest L_0 eigenvalue) which satisfies $L_{-1}|\emptyset''\rangle = 0$. All other states, obtained by raising with L_{+1} are the σ states.

Within each representation of the Gliozzi algebra it is straightforward to determine the unique state which is a physical state according to the definitions (5.5), (5.6) respectively (for $r = 1$). We may write for the former case

$$|\phi\rangle = \sum_{n=0}^{\infty} \alpha_n (L_1)^n \; |\phi'', c\rangle \tag{5.7}$$

where

$$L_0 \, |\phi'', c\rangle = c\,|\phi'', c\rangle \quad ; \quad L_{-1}\,|\phi'', c\rangle = 0$$

Using the commutation relation

$$[\, L_{-1} , L_1^n] = 2 \sum_{q=0}^{n-1} L_1^q \, L_0 \, L_1^{n-q-1} \tag{5.8}$$

one finds that the condition (5.5) for $r = 1$, gives, putting $\alpha_0 = 1$,

$$\alpha_n = \frac{1}{n!} \; \frac{\Gamma(c+n-1)\,\Gamma(2c)}{\Gamma(c-1)\,\Gamma(2c+n)} \tag{5.9}$$

If

$$\| \; |\phi'', c\rangle \| = 1$$

one then finds, always for spacelike momentum with zero energy, that

$$\langle\phi|\phi\rangle = \sum_{n=0}^{\infty} \left[\prod_{q=1}^{n} \frac{(c+q-2)^2}{q(2c+q-1)} \right] \tag{5.10}$$

For large n the square bracket behaves as

$$\prod_{q=1}^{n} \frac{(c+q-2)^2}{q(2c+q-1)} \quad \overset{n\to\infty}{\sim} \quad \frac{1}{n^3} \tag{5.11}$$

so that the state $|\phi\rangle$ has finite norm.

If we write, however,

$$|\phi'\rangle = \sum_{n=0}^{\infty} \alpha_n' \, (L_1)^n \; |\phi'', c\rangle \tag{5.12}$$

satisfying (5.6) then one finds

$$|\phi'\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(c+n)\,\Gamma(2c)}{\Gamma(2c+n)\,\Gamma(c)} \,(L_1)^n\, |\phi\,;c\rangle \tag{5.13}$$

and hence

$$\langle\phi'|\phi'\rangle = \sum_{n=0}^{\infty} \left[\prod_{q=1}^{n} \frac{(c+q-1)^2}{q\,(2c+q-1)} \right] \underset{n \text{ large}}{\sim} \sum^{\infty} \frac{1}{n} \tag{5.14}$$

giving a logarithmically divergent norm.

We deduce that unless c is a negative integer, whereupon the summations in Eqs. (5.7) and (5.12) may cut off, all states satisfying Eq. (5.6) are not normalizable. The states satisfying Eq. (5.5), on the other hand, can be normalizable. Thus, the $\mathcal{W}_{-r}$ condition for an off-shell physical state is more satisfactory in this respect than the W_{-r} definition. This is as expected from the better definition of $D(q)V(p)$ in Eq. (5.1) than that of $V(p)$ alone of Eq. (4.1).

6. SUMMARY AND DISCUSSION

We have studied the properties of the three fundamental operators in the operator formalism: $D(p)$, $V(p_1 p_2 p_3)$ and $\Omega(p)$. We have found that the vertex for ground state emission, $V(p)$, is strictly speaking non-existent within the Hilbert space. The twisting operator $\Omega(p)$ is also non-existent when acting on any state with non-null four-momentum; its Hermitian conjugate is, however, densely defined. The propagator $D(p)$ is everywhere defined off-mass shell.

We have noted that the product $\Omega(q)V(p)$ and, more generally, the symmetric three-reggeon vertex $V(123)$ are densely defined. Also, the product $D(q)V(p)$ can be well defined and this was related to the recognition that states annihilated by $\mathcal{W}_{-r} = (L_0 - L_{-r} - 1)$ were more suitable candidates for off-shell physical states than states annihilated by $W_{-r} = (L_0 - L_{-r} + r - 1)$.

The fact that not all operators and their image states can be represented within the Hilbert space is not very surprising because similar difficulties already occur in non-relativistic quantum mechanics, where, for example, the position operator acting on a square-integrable wave function can give a new function outside of the Hilbert space spanned by the set of all square-integrable functions [5]. In that case, extension to a larger space has proved useful [11].

To conclude, we re-emphasize that the usual operator factorization (with matrix elements taken) of the generalized Veneziano model is well defined; it is only when we study the operators D, V and Ω in isolation, as Hilbert space operators that the question of good definition arises. The matrix elements usually considered are scattering amplitudes, and for these we know the analytic structure and can continue analytically to any kinematical region. If we isolate operators or operator products then there are no similar analyticity assumptions for these and, therefore, we have to understand their mathematical properties in order to use them correctly.

BOUNDEDNESS AND DOMAIN PROPERTIES OF OPERATORS

Operator	Bounded on $\mathcal{H}$, the full Hilbert space	Defined on finite occupation states, i.e., $\mathcal{D} \supseteq \mathcal{F}$	Defined on coherent states, i.e., $\mathcal{D} \supseteq \mathcal{C}$	Defined on a dense subset of $\mathcal{C}$	Defined (i.e., bounded) on the vacuum, i.e., $\mathcal{D} \supseteq \vert 0\rangle$
$D(p)$	Yes *)	Yes *)	Yes *)	Yes *)	Yes *)
$D^{-1}(p)$	No	Yes	No	Yes	Yes
$\Omega^{\dagger}(p)$	No	Yes	No	Yes	Yes
$\Omega(p)$	No	No	No	Yes	(a)
$V(p)$	No	No	No	No	No
$\Omega(q)V(p)$	No	No	No	Yes	(a)
$D(q)V(p)\begin{pmatrix}\vert p\vert^2 < +2 \\ q^2 < -1\end{pmatrix}^{+)}$	No	Yes	no	Yes	Yes

*) We are always working off-mass shell for $D(p)$.

(a) Yes, when $p_\mu = 0$; No, otherwise.

+) $|p|^2 = p_0^2 + \underline{p}^2$.

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