A Study on Anti Fuzzy Sub-Bigroup

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ABSTRACT
In this paper, we made an attempt to study the algebraic nature of anti fuzzy sub-bigroup of a bigroup and anti fuzzy sub-bigroup of a group and discussed some of its properties with counter example.
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Keywords
Fuzzy set , anti fuzzy subgroup , anti fuzzy sub-bigroup of a bigroup, anti fuzzy sub-bigroup of a group, bi-lower level subset, homomorphism, anti-homomorphism.

INTRODUCTION
The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld gave the idea of fuzzy subgroups and Ranjith Biswas gave the idea of anti fuzzy subgroups.

The notion of bigroup was first introduced by P.L.Maggu in 1994. W.B. Vasantha Kandasamy and D.Meiyappan introduced concept of fuzzy sub-bigroup of a bigroup and fuzzy sub-bigroup of a group.

In this paper, we introduce the concept of anti fuzzy sub-bigroup of a bigroup and bi-lower level subset of an anti fuzzy sub-bigroup and anti fuzzy sub-bigroup of a group and prove some results with counter examples. Also we establish the relationship between anti fuzzy sub-bigroup of a group and anti fuzzy subgroup of a group.

1. Preliminaries

This section contains some definitions and results to be used in the sequel.

1.1 Definition

Let S be a set. A fuzzy subset A of S is a function A: $S \rightarrow [0,1]$.

1.2 Definition

Let G be a group. A fuzzy subset A of G is called an anti fuzzy subgroup if for x, y ϵG,

(i) $A(xy) \leq \max \{ A(x) , A(y)\}$,

(ii) $A(x^{-1}) = A(x)$.

1.3 Definition

Let A be a fuzzy subset of S. For t ϵ [0,1], the lower level subsets of A is the set,

$$\bar{A}t = \{ x \in S : A(x) \leq t \}.$$

1.4 Definition

Let G be a finite group of order n. and A be an anti fuzzy subgroup of G .

Let $Im(A) = \{ ti : A(x) = ti \text{ for some } x \in G \}$. Then $\{ \bar{A}ti \}$ are the only lower level subgroups of A.

1.5 Definition

A set (G , + , •) with two binary operation ‘ + ‘ and ‘• ‘ is called a bigroup if there exist two proper subsets G1 and G2 of G such that

i. $G = G_1 \cup G_2$

ii. $(G_1 , +)$ is a group.

iii. $(G_2 , \bullet)$ is a group.

A non-empty subset H of a bigroup (G , + , •) is called a sub-bigroup, if H itself is a bigroup under ‘ + ‘ and ‘• ‘ operations defined on G.

1.6 Definition

Let G = (G , + , •) be a bigroup. Then $A : G \rightarrow [0 ,1]$ is said to be an anti fuzzy sub-bigroup of the bigroup G if there exists two fuzzy subsets A1 of G1 and A2 of G2 such that

i. $A = A_1 \cup A_2$

ii. $(A_1 , +)$ is an anti fuzzy subgroup of $(G_1 , +)$.

iii. $(A_2 , \bullet)$ is a an anti fuzzy subgroup $(G_2 , \bullet)$.

1.1 Example

Let G1 = { 0 , a , b , a + b } be a group under the operation ‘+ ‘ with a + a = b + b = 0 and a + b = b + a

Let G2 = { 1 , -1 , i , - i } be a group under the operation • .

Define $A : G \rightarrow [0 ,1]$ by

$$A(x) = \begin{cases} 0.2 & \text{for } x = 0 \\ 0.3 & \text{for } x = a,1 \\ 0.4 & \text{for } x = b, a+b, -1 \\ 0.5 & \text{for } x = i, -i \end{cases}$$

It is easy to verify that A is an anti fuzzy sub-bigroup of the bigroup G, for we can find A1 : G1 → [0, 1] by

$$A_1(x) = \begin{cases} 0.2 & \text{for } x = 0 \\ 0.3 & \text{for } x = a \\ 0.4 & \text{for } x = b, a+b \end{cases}$$

and A2 : G2 → [0 ,1] is given by

$$A_2(x) = \begin{cases} 0.3 & \text{for } x = 1 \\ 0.4 & \text{for } x = -1 \\ 0.5 & \text{for } x = i, -i \end{cases}$$

Clearly A is an anti fuzzy sub-bigroup of the bigroup G.

1.1 Theorem

Every t – lower level subsets of an anti fuzzy sub-bigroup A of a bigroup G need not in general be a sub-bigroup of the bigroup G.

Proof: We prove this by an example.

Take G= {-1,0,1} to be a bigroup under the operation ‘ + ‘ and ‘• ‘ where G1 = {0} and G2 = {-1,1} are groups respectively with respect to usual addition and usual multiplication.

Define A : G → [0 ,1]

$$A(x) = \begin{cases} 0.5 & \text{for } x = -1, 1 \\ 0.75 & \text{for } x = 0 \end{cases}$$

Clearly (A,+, •) is an anti fuzzy sub-bigroup of the bigroup (G, ,+, •). Now consider the lower level subset Āt for t = 0.5 of the anti fuzzy sub-bigroup A

For t = 0.5,

$$\bar{A}_t = \{ x \in G : A(x) \leq 0.5 \}.$$

$$\bar{A}_t = \{ -1, 1 \}.$$

It is easy to verify that { -1,1 } is not a sub-bigroup of the bigroup (G, ,+, •). Hence the lower level subset Āt for t = 0.5 of an anti fuzzy sub-bigroup A is not a sub-bigroup of the bigroup (G, ,+, •).

We introduce the notion of bi-lower level subset of an anti fuzzy sub-bigroup.

1.7 Definition

Let G = (G1 ∪ G2 ,+, •) be a bigroup and A = (A1 ∪ A2) be an anti fuzzy sub-bigroup of the bigroup G. The bi-lower level subset of the anti fuzzy sub-bigroup A of the bigroup G is defined as

$$\bar{A}t = \bar{A}1t \cup \bar{A}2t \text{ for every } t \in [\max\{A1(e1), A2(e2)\}, 1].$$

Where e1 denotes the identity element of the group (G1, +) and e2 denotes the identity element of the group (G2, •).

Remark

The condition t ϵ [max {A1(e1), A2(e2) },1] is essential for the bi-lower level subset to be a sub-bigroup, for if t ∉ [max {A1(e1), A2(e2) },1], the bi-lower level subset need not in general be a sub-bigroup of the bigroup G which is evident from the following example.

1.2 Example

Consider Example 1.1, the bi-lower level subset Āt for t = 0.2 of the anti fuzzy sub-bigroup A is given by Āt = {0} which is not a sub-bigroup of the bigroup G. Therefore the bi-lower level subset Āt for t = 0.2 is not a sub-bigroup of the bigroup G.

1.2 Theorem

Every bi-lower level subset of an anti fuzzy sub-bigroup A of a bigroup G is a sub-bigroup of the bigroup G.

Proof

Let A = (A1 ∪ A2) be an anti fuzzy sub-bigroup of a bigroup G = (G1 ∪ G2 ,+, •). Consider the bi-lower level subset Āt of an anti fuzzy sub-bigroup A for every t ϵ [max {A1(e1), A2(e2) },1], where e1 denotes the identity element of the group (G1, +) and e2 denotes the identity element of the group (G2, •). Then Āt = Ā1t ∪ Ā2t where Ā1t and Ā2t are subgroups of G1 and G2 respectively. Hence by the definition of sub-bigroup Āt is a sub-bigroup of the bigroup (G, ,+, •).

We illustrate the following by an example.

1.3 Example

Let G = {0,±1, ±i} is a bigroup with respect to addition modulo 2 and multiplication. Clearly G1 = {0,1} and G2 = {±1, ±i} are groups with respect to addition modulo 2 and multiplication respectively. Define A : G → [0 ,1] by

$$A(x) = \begin{cases} 0.1 & \text{for } x = 0 \\ 0.5 & \text{for } x = \pm 1 \\ 0.7 & \text{for } x = \pm i \end{cases}$$

It is easy to verify that A is an anti fuzzy sub-bigroup of the bigroup G s there exist two anti fuzzy subgroups A1 : G1 → [0 ,1] , A2 : G2 → [0 ,1] such that A = (A1 ∪ A2) where

$$A_1(x) = \begin{cases} 0.1 & \text{for } x = 0 \\ 0.4 & \text{for } x = 1 \end{cases}$$

and

$$A_2(x) = \begin{cases} 0.5 & \text{for } x = \pm 1 \\ 0.7 & \text{for } x = \pm i \end{cases}$$

We now calculate the bi-lower level subset Āt for t = 0.5,

$$\bar{A}_t = \bar{A}_{1t} \cup \bar{A}_{2t} = \{ x \in G_1 : A_1(x) \le 0.5 \} \cup \{ x \in G_2 : A_2(x) \le 0.5 \}$$

$$= \{ 0,1\} \cup \{-1,1 \}$$

$$= \{ 0,\pm 1\}$$

$$\bar{A}_t = \{ 0 , \pm 1 \}$$

It is easily verified that $\bar{A}_t$ is a sub-bigroup of the bigroup G.

1.3 Theorem

Let G be a bigroup and A1, A2 be fuzzy subsets of A such that A = (A1 ∪ A2). The bi-lower level subset Āt of A is a sub-bigroup of G ,t ∈ [max {A1(e1), A2(e2) },1], where e1 denotes the identity element of G1 and e2 denotes the identity element of G2 respectively. Then A is an anti fuzzy sub-bigroup of a bigroup G.

Proof

Let G = $(G_1 \cup G_2)$ be a bigroup.

Given that the bi-lower level subset Āt = Ā1t ∪ Ā2t is a sub-bigroup of G.

Clearly Ā1t is a subgroup of G1 , A1 is an anti fuzzy subgroup of G1.

Clearly Ā2t is a subgroup of G2 , A2 is an anti fuzzy subgroup of G2.

Hence A = (A1 ∪ A2) and hence A is an anti fuzzy sub-bigroup of G.

1.8 Definition

Let A = (A1 ∪ A2) be an anti fuzzy sub-bigroup of a bigroup G = (G1 ∪ G2). The sub-bigroups Āt, for t ∈[max {A1(e1), A2(e2) },1] ,where e1 denotes the identity element of G1 and e2 denotes the identity element of G2 respectively, are called bi-lower level sub-bigroups of A.

1.4 Theorem

Let A = (A1 ∪ A2) be an anti fuzzy sub-bigroup of a bigroup G = (G1 ∪ G2). Two bi-lower level subgroups Āα, Āβ, α, β ∈[max {A1(e1), A2(e2) },1] , where e1 denotes the identity element of G1 and e2 denotes the identity element of G2 respectively with α < β are equal iff there is no x in G such that α < A(x) ≤ β.

Proof

Let A = (A1 ∪ A2) be an anti fuzzy sub-bigroup of a bigroup G = (G1 ∪ G2).

Consider the two bi-lower level subgroups Āα, Āβ, α, β ∈[max {A1(e1), A2(e2) },1] where e1 denotes the identity element of G1 and e2 denotes the identity element of G2 respectively with α < β.

Let $\bar{A}_\alpha = \bar{A}_\beta$.

We have to prove that there is no x in G such that

α < A(x) ≤ β.

Suppose that there is an x in G such that α < A(x) < β then x ∈ Āβ and x ∉ Āα .

This implies Āα ⊂ Āβ, which contradicts the assumption that Āα = Āβ.

Hence there is no x in G such that α < A(x) ≤ β.

Conversely, suppose that there is no x in G such that

α < A(x) ≤ β.

Then , by definition, Āα ⊂ Āβ.

Let x ∈ Āβ and there is no x in G such that α < A(x) ≤ β.

Hence x ∈ Āα.

(i.e) Āβ ⊂ Āα.

Hence Āα = Āβ.

1.5 Theorem

A fuzzy subset A of G is an anti fuzzy sub-bigroup of G iff the bi-lower level subsets Āt, $t \in$ Image A, are sub-bigroups of G.

Proof

It is clear.

1.6 Theorem

Any sub-bigroup H of a bigroup G can be realized as a bi-lower level sub-bigroup of some anti fuzzy sub-bigroup of a bigroup G.

Proof: Let $G = (G1 \cup G2, +, \bullet)$ be a bigroup.

Let $H = (H1 \cup H2, +, \bullet)$ be a sub-bigroup of G.

Let A1 and A2 be a fuzzy subsets of A defined by

$$A_1(x) = \begin{cases} 0 \text{ if } x \in H_1 \\ t \text{ if } x \notin H_1 \end{cases} ; \qquad A_2(x) = \begin{cases} 0 \text{ if } x \in H_2 \\ t \text{ if } x \notin H_2 \end{cases}$$

where $t \in [\max\{A1(e1), A2(e2)\}, 1]$, and e1 denotes the identity element of G1 and e2 denotes the identity element of G2 respectively.

We shall prove that $A = (A1 \cup A2)$ is an anti fuzzy sub-bigroup of a bigroup G.

Suppose $x, y \in H$, then

i. $x, y \in H_1 \Rightarrow x + y \in H_1$ and $x + (-y) \in H_1$.

$A_1(x) = 0$, $A_1(y) = 0$ and $A_1(x + (-y)) = 0$, then

$A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}$.

ii. $x, y \in H_2 \Rightarrow xy \in H_2$ and $xy^{-1} \in H_2$.

$A_2(x) = 0$, $A_2(y) = 0$ and $A_2(xy^{-1}) = 0$,

Then $A_2(xy^{-1}) \leq \max\{A_2(x), A_2(y)\}$.

iii. $x \in H_1$ and $y \notin H_1 \Rightarrow x + y \notin H_1$

and $x + (-y) \notin H_1$.

$A_1(x) = 0$, $A_1(y) = t$ and $A_1(x + (-y)) = t$, then

$A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}$.

iv. $x \in H_2$ and $y \notin H_2 \Rightarrow xy \notin H_2$ and $xy^{-1} \notin H_2$.

$A_2(x) = 0$, $A_2(y) = t$ and $A_2(xy^{-1}) = t$,

Then $A_2(xy^{-1}) \leq \max\{A_2(x), A_2(y)\}$.

Suppose $x, y \notin H$, then

i. $x, y \notin H_1$ then $x + y \in H_1$ or $x + y \notin H_1$.

$x, y \notin H_1$ then $x + (-y) \in H_1$ or $x + (-y) \notin H_1$.

$A_1(x) = t$, $A_1(y) = t$ and $A_1(x + (-y)) = 0$ or t,

then $A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}$.

ii. $x, y \notin H_2 \Rightarrow xy \in H_2$ or $xy \notin H_2$.

$x, y \notin H_2 \Rightarrow xy^{-1} \in H_2$ or $xy^{-1} \notin H_2$.

$A_2(x) = t$, $A_2(y) = t$ and $A_2(xy^{-1}) = 0$ or t,

then $A_2(xy^{-1}) \leq \max\{A2(x), A2(y)\}$.

Thus in all cases,

$$\begin{cases} (A1, +) \text{ is an anti fuzzy subgroup of } (G1, +) \text{ and} \\ (A2, \bullet) \text{ is an anti fuzzy subgroup of } (G2, \bullet). \end{cases}$$

Hence $A = (A1 \cup A2)$ is an anti fuzzy sub-bigroup of a bigroup G. For this anti fuzzy sub-bigroup, $Āt = Ā1t \cup Ā2t = H$.

Remark

As a consequence of the above Theorems, the bi-lower level sub-bigroups of an anti fuzzy sub-bigroup A form a chain. Since $A(e1) \leq A(x)$ for all x in G1 or $A(e2) \leq A(x)$ for all x in G2, therefore Āt0, where $t0 = \max\{A1(e1), A2(e2)\}$, is the smallest and we have the chain :

$\{e1, e2\} = Āt0 \subset Āt1 \subset Āt2 \subset \ldots \subset Ātn = G$, where $t0 < t1 < t2 < \ldots < tn$.

2. Now we proceed on to define the anti fuzzy bigroup of a group.

2.1 Definition

A fuzzy subset A of a group G is said to be an anti fuzzy sub-bigroup of the group G if there exist two anti fuzzy subgroups A1 and A2 of A ($A1 \neq A$ and $A2 \neq A$) such that $A = A1 \cup A2$.

2.1 Theorem

Every anti fuzzy sub-bigroup of a group G is an anti fuzzy subgroup of a group G but not conversely.

Proof

It is clear. Since the union of any two anti fuzzy subgroup is an anti fuzzy subgroup.

Converse part is not true in general and is proved by the following example.

2.1 Example

Consider the group G = { 1,-1, i, -i} under usual multiplication. Define $A : G \to [0,1]$ by

$$A(x) = \begin{cases} 0 \text{ for } x = \{1, -1\} \\ 1 \text{ for } x = \{i, -i\} \end{cases}$$

Clearly A is an anti fuzzy subgroup as all of its lower level subsets are subgroups of G. Further o(Im(A)) = 2.

If Ak is an anti fuzzy subgroup of A such that Ak is a subset of A.(Ak ≠ A). Then Ak takes following form.

$$A_k(x) = \begin{cases} 0 & \text{for } x = \{1,-1\} \\ \alpha_k & \text{for } x = \{i, -i\} \end{cases}$$

with $0 \leq \alpha_k < 1$ for every i ϵ I.

It is easy to verify that Aj ∪ Ak ≠ A for any j,k ϵ I. Thus there does not exist two anti fuzzy subgroups A1 and A2 of A (A1 ≠ A and A2 ≠ A) such that A = A1 ∪ A2

2.2 Theorem

Let A be an anti fuzzy subgroup of a group G with 3 ≤ o(Im(A)) < ∞ then there exist two anti fuzzy subgroups A1 and A2 of A (A1 ≠ A and A2 ≠ A) such that A = A1 ∪ A2

Proof

Let A be an anti fuzzy subgroup of a group G. Suppose Im(A) = {a1,a2 ,…. an} with a1 < a2 < ….< an and 3 ≤ n < ∞.

Choose b1 , b2 ϵ [0 1] be such that b1 ≤ a1 < b2 < a2 < …. < an and define A1 , A2 : G → [0 ,1] by ,

$$A_1(x) = \begin{cases} a_1 & \text{for } x \in \bar{A}_{a1} \\ b_2 & \text{for } x \in \bar{A}_{a2} - \bar{A}_{a1} \\ A(x) & \text{otherwise} \end{cases}$$

and

$$A_2(x) = \begin{cases} b_1 & \text{for } x \in \bar{A}_{a1} \\ a_2 & \text{for } x \in \bar{A}_{a2} - \bar{A}_{a1} \\ A(x) & \text{otherwise} \end{cases}$$

Clearly A_1 and A_2 are anti fuzzy subgroups.

2.3 Theorem

Let A be a fuzzy subset of a group G with 3 ≤ o(Im(A)) < ∞. Then A is an anti fuzzy subgroup of the group G iff A is an anti fuzzy sub-bigroup of G.

Proof

It is clear from **Theorem 2.1** and **Theorem 2.2**

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