A STUDY ON MATHEMATICAL AND STATISTICAL ASPECTS OF LINEAR MODELS

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Abstract: The primary objective of this research article is to present the mathematical and statistical aspects of linear models and their characteristic properties. Linear model is the most common modeling used in science. Actually linear models have many different meanings depend on the context. Linear model is often preferred than other model such as quadratic model because of its ability to interpret easily. In the other hand most of the real life cases have linear relationship .Modeling the cases using linear model will able us to determine the relative influence of one or more independent variables to the dependent variable. In the present talk an attempt has been made to propose the specific forms of simple and multiple linear regression models. In this conversation mathematical aspects of linear models have been extensively depicted. Different types of mathematical models are discussed here and the methods of fitting transformed models are proposed.Furthermore specific form of linear statistical model is presented and the crucial assumptions of general linear model are extensively discussed.At the last stage of this article the method of ordinary least squares estimation of parameters of a linear model has been proposed

Keywords: Linear Regression Model, Response variable,Predictor,Discrete and continuous models, Biasedness,Non-Stochastic data matrix

1.Introduction

Model refers to a set of functional or structural relationships between two or more characteristics. These characteristics may be either measuremental or non measuremental in nature. The measuremental characteristics which assume different values in a specified range are known as variables. Generally, a set of functional relationships between two or more variables may be expressed in terms of mathematical equations, which is called a mathematical model. This model may be either in the form of a set of linear equations (linear model) or in the form of a set of nonlinear equations (nonlinear model). By introducing a random error variable or a random disturbance term, the mathematical model becomes a statistical model or a regression model. Hence one may have either linear regression model or nonlinear regression model.

Regression analysis is a statistical method to establish the relationship between variables. Regression analysis has a wide number of applications in almost all fields of science, including Engineering, Physical and Chemical Sciences; Economics, Management, Social, Life and Biological Sciences. In fact, regression analysis may be the most frequently used statistical technique in practice.

Suppose that there exists a linear relationship between a dependent variable Y and an independent variable X. In the scatter diagram, if the points cluster around a straight line then the mathematical form of the linear model may be specified as

$$Y_i = \beta_0 + \beta_1 X_i, \quad i = 1, 2, ... n. \qquad (1)$$

where β_0 is the intercept and β_1 is the slope.
Generally the data points in the scatter diagram do not fall exactly on a straight line, so equation (2.1.1) should be modified to account for this. Let the difference between the observed value of Y and the straight line $\left(\beta_0 + \beta_1 X\right)$ be an error ε. It is convenient to think of ε as a statistical error; that is, it is a random variable that accounts for the failure of the model to fit the data exactly. The error may be made up of the effects of other variables, measurement errors and so forth. Thus, a more plausible model may be specified as

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, i = 1, 2, ..., n. \qquad (2)$$

Equation (2) is called a Linear Regression Model or Linear Statistical Model. Customarily X is called the independent variable and Y is called the dependent variable. However, this often causes confusion with the concept of statistical independence, so we refer to X as the Predictor or Regressor variable and Y as the Response variable. Since the equation (1) involves only one Regressor variable, it is called a 'Simple Linear Regression Model' or a 'Two-Variable Linear Regression Model'.

A Three – variable Linear Regression Model may be written as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i, i = 1, 2, ..., n \quad \text{............(3)}$$

This linear regression model contains two regressor variables. The term linear is used because eq. (2.1.3) is a linear function of the unknown parameters β_0, β_1 and β_2.

In general, the response variable Y may be related to k regressor or predictor variables. The model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_k X_{ki} + \varepsilon_i, i = 1, 2, ...n \quad \text{......(4)}$$

is called a 'Multiple Linear Regression Model' with k independent variables. The parameters β_j, j=0,1,2..,k are known as regression coefficients. This model describes a hyperplane in the k – dimensional space of the independent variables X_j's. The parameter β_j represents the expected change in the dependent variable Y per unit change in X_j, when all of the remaining predicted variables X_q's (q≠j) are held constant. Thus, the parameters β_j, j = 1,2,…,k are often known as 'Partial Regression Coefficients.

Multiple linear regression models are often used as empirical models or approximating functions. That is, the exact relationship between Y and $X_1, X_2, …, X_k$ is unknown but over certain ranges of the independent variables, the linear regression model is an adequate approximation to the true unknown function.

In practice, certain nonlinear regression models such as cubic polynomial models and response surface models may often still be analyzed by multiple linear regression techniques. For instance, consider the cubic polynomial model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \varepsilon_i, i = 1, 2, ..., n \quad \text{............(5)}$$

Let $X_1 = X$, $X_2 = X^2$ and $X_3 = X^3$ then eq. (5) can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i, i = 1, 2, ..., n \quad \text{......(6)}$$

which is a multiple linear regression model with three independent variables

Consider a model containing interaction effects as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_{12} X_{1i} X_{2i} + \varepsilon_i, i = 1, 2, ..., n \quad \text{......(7)}$$

Let $X_3 = X_1 X_2$ and $\beta_3 = \beta_{12}$ then eq (7) can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i, i = 1, 2, ..., n \quad \text{.....................(8)}$$

which is a multiple linear regression model with three regressors. Consider the second – order response surface model with interaction,

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_{11} X_{1i}^2 + \beta_{22} X_{2i}^2 + \beta_{12} X_{1i} X_{2i} + \varepsilon_i, i = 1, 2, ..., n \quad \text{......(9)}$$

Let $X_3 = X_1^2, X_4 = X_2^2, X_5 = X_1 X_2, \beta_3 = \beta_{11}, \beta_4 = \beta_{22}$ and $\beta_5 = \beta_{12}$ then eq. (2.1.9) can be rewritten as a multiple linear regression model as follows:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \beta_5 X_{5i} + \varepsilon_i, i = 1, 2, ..., n$$

2. MATHEMATICAL ASPECTS OF LINEAR MODEL

The main purpose of mathematical modelling is to solve real practical problems. The success of mathematical modelling depends on getting things right from the start, and as in most other scientific endeavours, one is more likely to succeed if one adopts a methodical approach. In practice, it is found to complete the following steps.

(i) Clarify the problem;
(ii) List the factors;
(iii)　　List the assumptions; and
(iv)　　Formulate a precise problem statement

An essential part of the mathematical modelling technique is to translate verbal statements about variables along with assumptions into precise mathematical relationships between the variables represented by symbols. Thus, the mathematical statements become amenable to manipulation by mathematical techniques.

For instance, the simplest model is obtained by assuming that Y is proportional to X. The corresponding mathematical statement is then Y $\propto$ X or as a mathematical equation $Y = \beta_0 X$, where β_0 is the constant of proportionality. Now, the graph of Y against X shows a straight line through the origin.

Another simplest model is the linear form $Y = \beta_0 + \beta_1 X$ in which Y increases by β_1 units for every unit increase in X and that $Y = \beta_0$ when X = 0. This also includes the situation where Y decreases as X increases. In that situation the parameter β_1 is negative.

Consider the situation where 'Y decreases as X increases' by inverse proportion

$$Y \propto \frac{1}{X} \text{ or } Y = \frac{\beta_0}{X} \quad \dots\dots\dots\dots\dots(10)$$

It reveals that Y decreases more steeply with X that is the situation in the linear model. One may test the validity of this assumption by examining whether XY remains nearly constant. Another way is that if the plot of Ln Y against Ln X is a straight line of slope "-1".

Thus, under mathematical modelling technique, first represent the variables by the mathematical symbols and then make the assumptions about the relationships among the variables. Further, translate the assumptions into mathematical equations or inequalities.

One of the main uses of mathematical modelling is to predict the future development of the system. Such model relies on assuming that the rate of change of a variable Y is linked to or caused by some or all of the present value of Y, previous values of Y, values of other variables, the rate of change of other variables and time 't'. Here, the mathematical model describes how Y itself varies with time 't'. There are mainly two types of such mathematical models namely,

    (i) Discrete Models and (ii) Continuous Models

DISCRETE MODELS:

For discrete models, one may write the form

(ii)

$$Y_{n+1} = f(Y_n, Y_{n-1}..., t)$$

It is usually known as a difference equation. The simplest type of difference equation is the first order linear constant coefficient equation which is given by

$$Y_{n+1} = \beta_0 + \beta_1 Y_n$$

A difference equation containing relationship between Y_{n+1} and Y_n and no other Y values is known as a 'First order difference equation. If the difference equation also involves Y_{n-1} or Y_{n+2}, then it is said to be second order difference equation.

Sometimes, a linear difference equation may be in the form

$$Y_{n+1} = \beta_1 Y_n + \beta_2 Y_{n-1} + n^2 + \beta_0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(11)$$

Linear difference equations involving more than one variable can be compactly expressed by using vectors and matrices.

Simultaneous linear difference equations can be expressed in the form

$$\overline{Y}_{n+1} = \overline{M}\,\overline{Y}_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(12)$$

where $\overline{M}$ is the co-efficient matrix; $\overline{Y}_{n+1}$ and $\overline{Y}_n$ are the vectors.

The solution can be written as $\overline{Y}_n = \overline{M}^n \overline{X}_o \quad \dots\dots\dots\dots\dots\dots\dots...(13)$

CONTINUOUS MODELS

A variable, which is allowed to take any value within a range, is known as a continuous variable. One advantage of using continuous variables is that one may use powerful mathematical tools such as Calculus.

The linear models are the simplest continuous models. The simplest linear model relating two variables is characterized by mathematical equation of the form
Y = β_0 + β_1X, by having a straight line graph.

Under linear interpolation, if x_1 and x_2 are consecutive values of X and   x is some value between them, then the graph of f(x) may be approximating from X = x_1 to X = x_2 by a linear model.

3. TYPES OF LINEAR MATHEMATICAL MODEL
Linear Models with Several Independent Variables:

If the value of a dependent variable Y depends on the values of other variables $X_1, X_2, ..., X_k$ then a way of expressing the dependence through a linear model is of the form

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_k X_{ki}$$ …………………..........................(14)

Here Y changes by equal amounts for equal changes in any one of the independent variables. This model can be considered the generalization of the simple two- variable linear model

Simultaneous Linear Models:

Sometimes there may be two or more dependent variables, all of which are modelled as linear functions of independent variables. Here, some dependent variables can be considered independent variables in some linear functions (or equations) of the simultaneous linear equations system. This system of simultaneous linear models can be solved by using matrix methods such as Cramer's Rule, Inverse Matrix method etc.

Piecewise Linear Models:

It is a model that does not have to be represented by the same single formula for all values of the independent variable X. Here, two different linear expressions agree at some value of X, so there is no sudden jump (discontinuity) at the changeover point, usually X may be a discrete variable in this model, sometimes, one may model a non-linear function approximately by a piecewise linear function.

Transformed Linear Models:

When a dependent variable Y does not change by equal amounts for equal changes in the independent variable then a linear model may not be suitable for this situation. For instance, the quadratic function or a second degree parabola is a simple nonlinear model.

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2, \ i = 1, 2, ..., n$$

Three separate pieces of information are needed to determine the three parameters β_0, β_1 and β_2. The value of β_2 determines whether the curve is concave upwards ($\beta_2 > 0$) or concave downwards, There is a vertical axis of symmetry at
$$X = \frac{-\beta_1}{2\beta_2}$$
which is also the X value at which the graph has global maximum or minimum value. The value of the parameter β_0 affects the vertical position of the curve relative to the coordinate axes.

A more general higher degree polynomial model can be written as

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + ... + \beta_k X_i^k, \ i = 1, 2, ..., n$$ ………………..(15)

This model can be transformed into a general linear model as

$$Y_i = \beta_0 + \beta_1 Z_{1i} + \beta_2 Z_{2i} + \beta_3 Z_{3i} + ... + \beta_k Z_{ki}$$ ………………………………….(16)

where, $Z_1 = X, Z_2 = X^2, Z_3 = X^3, ... Z_k = X^k$

Likewise, some nonlinear models such as power function model ($Y = \beta_0 X^{\beta_1}$), Exponential Model ($Y = \beta_0 e^{\beta_1 X}$ or $Y = \beta_0 \beta_1^X$) etc, can be transformed into logarithmic linear models $Ln \ Y = Ln \ \beta_0 + \beta_1 Ln \ X;$ and

$Ln \ Y = Ln \ \beta_0 + \beta_1 X$ or $Ln \ Y = Ln \ \beta_0 + (Ln\beta_1)X$ etc., respectively.

When the rate of change of Y is assumed to be proportional to the difference between the present value of Y and some fixed value c, one may write the linear first order differential equation model as

$Y' = k(c - Y)$, which has the solution $Y(t) = c + (Y_0 - c)e^{-kt}$ ....(17)

with $Y \to c$ as $t \to \infty$.

4. FITTING OF TRANSFORMED LINEAR MODELS

In fitting of time series models or Growth curves to the time series data, the following points may be useful to specify the type of the model:

(i) When the time series Y_t is formed to be increasing or decreasing by equal absolute amounts, the straight line times series model is used.

$Y_t = \beta_0 + \beta_1 t$ ……………………………………………...…(18)

(ii) When the time series Y_t is increasing or decreasing by a constant percentage rather than a constant absolute amount, the logarithmic straight line time series model is used.

$Y_t = \beta_0 \beta_1^t$ ……………………………………………….(19)

or $\text{Ln } Y_t = \text{Ln}\beta_0 + (\text{Ln}\beta_1)t$ ………………………………………..(20)

In this case, the data plotted on a semi-logarithmic scale graph gives a straight line graph.

The approximations about the type of the curve to be fitted can be made by using the following theorem based on finite differences

$$\Delta^n Y_t, \Delta^n (\text{Ln } Y_t), \Delta^n \left(\frac{1}{Y_t}\right)$$

"The nth differences of any general polynomial Y_t of nth degree in t is constant and $(n+1)^{th}$ differences are equal to zero".

i.e., If Δ is the difference operator given by

$\Delta Y_t = [Y_{t+h} - Y_t]$, h being the interval of differencing and $\Delta^k Y_t$ is the kth difference of Y_t then for a polynomial Y_t of nth degree in t, the theorem states that

$\Delta^k Y_t =$ constant, $k = n$
$ = 0, k > n$

The following tests based on the calculus of finite differences may be applied in choosing approximations about the type of curve to be fitted:

(i) If $\Delta Y_t =$ constant, then linear model $Y_t = \beta_0 + \beta_1 t$ may be used;

(ii) If $\Delta^2 Y_t =$ constant, then a second degree polynomial function model $Y_t = \beta_0 + \beta_1 t + \beta_2 t^2$ may be used;

(iii) If $\Delta[\text{Log } Y_t] =$ constant, then an exponential or logarithmic linear model $Y_t = \beta_0 e^{\beta_1 t}$ or $Y_t = \beta_0 \beta_1^t$ or $\text{Ln } Y_t = \text{Ln } \beta_0 + \beta_1 t$ or $\text{Ln } Y_t = \text{Ln } \beta_0 + (\text{Ln}\beta_1)t$ may be used;

(iv) If $\Delta^2[\text{Log } Y_t] =$ constant, then second degree curve fitted to logarithms model

$Y_t = \beta_0 \beta_1^t \beta_2^{t^2}$ or

$\text{Ln } Y_t = \text{Ln } \beta_0 + [\text{Ln } \beta_1]t + [\text{Ln } \beta_1]t^2$ may be used;

(v) If $\dfrac{\Delta Y_t}{\Delta Y_{t-1}} =$ constant, then a modified exponential function model $Y_t = \beta_0 + \beta_1 \beta_2^t$ may be used;

(vi) If $\left[\dfrac{\Delta \text{Log } Y_t}{\Delta \text{Log } Y_{t-1}}\right] =$ constant, then Gompertz model $Y_t = \beta_0 \beta_1^{\beta_2^t}$ or a logarithmic modified exponential function model

$\text{Ln } Y_t = \text{Ln}\beta_0 + [\text{Ln}\beta_1]\beta_2^t$ may be used;

(vii) If $\left[\dfrac{\Delta\left(1/Y_t\right)}{\Delta\left(1/Y_{t-1}\right)}\right]=$ constant, then logistic function model

$$Y_t=\left[\frac{k}{1+e^{\beta_0+\beta_1 t}}\right],\ \beta_1<0$$

or

or $\dfrac{1}{Y_t}=\left[\dfrac{1}{k}\right]\left[1+e^{\beta_0+\beta_1 t}\right]$ or a modified exponential function model $\left[1/Y_t\right]=\gamma_0+\gamma_1\gamma_2^t$ may be used.

where $\gamma_0=\dfrac{1}{k},\gamma_1=\dfrac{1}{k}e^{\beta_0},\gamma_2=e^{\beta_1}$ are constants

(viii) If ΔY_t tends to decrease by a constant percentage then a modified exponential function model may be used;

(ix) If ΔY_t shows a skewed frequency curve, then a Gompertz model or a logarithmic modified exponential function model may be used.

5. LINEAR STATISTICAL MODEL:

Suppose there exists a linear relationship between a dependent variable Y and (k-1) independent variables X_2, X_3,......., X_k and a random error term or disturbance term ε. For a sample of n observations on Y and X's, one may specify linear regression model as

$$Y_i=\beta_1+\beta_2 X_{2i}+\beta_3 X_{3i}+...+\beta_k X_{ki}+\varepsilon_i, i=1,2,...,n.$$ ....(21)

where β's are known as regression co-efficients or unknown parameters of the linear regression model

The set of above 'n' linear equations can be expressed in the matrix notation as

$$Y_{nx1}=X_{nxk}\ \beta_{kx1}+\varepsilon_{nx1}$$ ....(22)

$$Y=\begin{bmatrix}Y_1\\Y_2\\.\\.\\.\\Y_n\end{bmatrix}_{nx1},X=\begin{bmatrix}1&X_{21}&X_{31}&...&X_{k1}\\1&X_{22}&X_{32}&...&X_{k2}\\.&.&.&...&.\\.&.&.&...&.\\.&.&.&...&.\\1&X_{2n}&X_{3n}&...&X_{kn}\end{bmatrix}_{nxk}\beta=\begin{bmatrix}\beta_1\\\beta_2\\.\\.\\.\\\beta_k\end{bmatrix}_{kx1}\text{and }\varepsilon=\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\.\\.\\.\\\varepsilon_n\end{bmatrix}_{nx1}$$

where

Here, **Y**: (nx1) vector of observations on a random dependent random variable (observation vector)

X: (nxk) matrix of known observations on a set of independent variables (Data Matrix)

β: (kx1) vector of unknown parameters including intercept parameter (parametric vector)

and **ε:** (nx1) vector of observations on an error random variable. (Error vector or Disturbance vector)

This model is known as a k- variable linear model or a general linear model (or linear statistical model)

6 . CRUCIAL ASSUMPTIONS OF GENERAL LINEAR MODEL

(i) Linearity Assumption:

The basic assumption of the linear model is that the dependent variable vector or observation vector Y may be expressed as a linear function of the sample of observations on independent variables X's and error vector ε i.e.,

$Y=X\beta+\varepsilon$ ....(23)

(ii) Unbiasedness Assumption

$$\mathrm{E}(\varepsilon) = \mathrm{O}_{nx1} \text{ or } \mathrm{E}(\mathrm{Y}) = \mathrm{X}\beta$$

…………………………………...……(24)

where O_{nx1} is (nx1) vector of zeros

$$\Rightarrow \mathrm{E}(\varepsilon_i) = 0 \quad \forall i = 1, 2, ..., n$$

ε_i are having zero means

(iii) Assumption of Homoscedastic and Uncorrelated Errors

$$\mathrm{E}(\varepsilon\varepsilon') = \sigma^2 \mathrm{I}_n$$

…………………………………………………(25)

$$\Rightarrow \mathrm{cov}(\varepsilon_i \varepsilon_j) = \sigma^2, \forall i = j = 1, 2, ...n$$
$$= 0, \forall i \neq j$$

Thus, each ε_i distribution has the same unknown variance; the errors ε_i and ε_j, $\forall i \neq j$ are pairwise uncorrelated.

Here, σ^2 is unknown error variance,

I_n is an Identity matrix of order n ,i.e., ε has covariance matrix $\sigma^2 \mathrm{I}_n$

(iv) Assumption of Linear Independence of Explanatory Variables X's:
Rank of X is k, where k<n
Thus, there is no linear dependence among the Explanatory variables.

(v) Assumption of Non- Stochastic Data Matrix:
The Data matrix X is a non-stochastic matrix.
In other words, X is a fixed known coefficients matrix.

(vi) Assumption of Non- Measuremental Errors:
There are no errors involved in the explanatory variables. In other words, all the independent variables X's are measured without error. Further, X is uncorrelated with ε

(vii) Normality Assumption:
The error vector ε follows a multivariate normal distribution with null vector O_{nx1} as mean vector and $\sigma^2 \mathrm{I}_n$ as the variance covariance matrix. Here, I_n is a unit matrix of order n.

The Linear Statistical Model along with the above crucial assumptions is known as
 'Standard Linear Statistical Model'
or 'Classical Linear Statistical Model'
or 'Gauss-Markoff Linear Statistical Model'
or 'Standard General Linear Model'.
7.ORDINARY LEAST SQUARES ESTIMATION OF PARAMETERS OF LINEAR MODEL

Consider the Classical Linear Regression model

$$Y_{nx1} = X_{nxk}\beta_{kx1} + \varepsilon_{nx1}$$

……………………………………………………….(26)

with usual assumptions such as

$$E(\varepsilon) = 0, \ E(\varepsilon\varepsilon') = \sigma^2 \mathrm{I}_n$$

……………………………………………………..(27)

Write the residual sum of squares as

$$e'e = (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

…………………………………………………...(28)

$$= Y'Y - \hat{\beta}'X'Y - Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$

$$\Rightarrow e'e = Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta} \qquad \left[\because Y'X\hat{\beta} = \hat{\beta}'X'Y\right]$$

where $\hat{\beta}$ is the least squares estimator of β

By the least squares estimation method, $\hat{\beta}$ minimizes the residual sum of squares $e'e$.

First order condition : $\dfrac{\partial}{\partial \hat{\beta}}(e'e) = O \Rightarrow -2X'Y + 2X'X\hat{\beta} = O$

$$\Rightarrow X'X\hat{\beta} = X'Y \qquad\qquad\qquad\qquad\qquad\qquad\qquad (29)$$

The system (2.7.4) contains 'n' simultaneous linear equations, which is called the 'System of Normal Equations'.

Since, the system of normal equations is always consistent, these exists atleast a non zero solution of $\hat{\beta}$,which gives the ordinary least squares (OLS) estimator of β .

i.e., $\hat{\beta} = \left(X'X\right)^{-1} X'Y \qquad\qquad\qquad\qquad\qquad\qquad (30)$

Further, consider the OLS residual vector

$$e = Y - X\hat{\beta} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (31)$$

$$= X\beta + \varepsilon - X\left(X'X\right)^{-1} X'(X\beta + \varepsilon)$$

$$= \varepsilon - X\left(X'X\right)^{-1} X'\varepsilon$$

$$= \left(I_n - X\left(X'X\right)^{-1} X'\right)\varepsilon$$

[I_n is a unit matrix of order n]

$$\Rightarrow e = M\varepsilon \qquad\qquad\qquad\qquad\qquad\qquad\qquad (32)$$

where $M = \left(I_n - X\left(X'X\right)^{-1} X'\right)$ is a symmetric idempotent matrix such that $M'M = M$, $M' = M$ and MX=O.

Now, consider the OLS residual sum of squares

$$e'e = \left(M\varepsilon\right)'\left(M\varepsilon\right) = \varepsilon'M\varepsilon$$

$$\Rightarrow E\left(e'e\right) = E\left(\varepsilon'M\varepsilon\right)$$

$$= E\left(trace\ \varepsilon'M\varepsilon\right)\ \left[\because \varepsilon'M\varepsilon\ is\ a\ scalar\right]$$

$$= E\left(trace\ M\varepsilon\varepsilon'\right)$$

$$= \left(trace\ M\right)E\left(\varepsilon\varepsilon'\right)$$

$$= \sigma^2 trace\ M \qquad \left[\because E\left(\varepsilon\varepsilon'\right) = \sigma^2 I_n\right]$$

$$= \sigma^2 trace\left(I_n - X\left(X'X\right)^{-1} X'\right)$$

$$= \sigma^2 \left[trace\ I_n - trace\left(X'X\right)^{-1}\left(X'X\right)\right]$$

$$= \sigma^2 \left[n - trace\ I_k\right]$$

$$\Rightarrow E\left(e'e\right) = \sigma^2\left(n-k\right)$$

$$or\ E\left(\frac{e'e}{n-k}\right) = \sigma^2$$

$$E\left(S^2\right) = \sigma^2$$

$$S^2 = \frac{e'e}{n-k}$$ is an unbiased estimator of σ^2.

8. Conclusion and Future Research

In the above talk mathematical aspects of linear models have been extensively depicted. Different types of mathematical models are discussed here and the methods of fitting transformed models are proposed. Furthermore specific form of linear statistical model is presented and the crucial assumptions of general linear model are extensively discussed. At the last stage of this article the method of ordinary least squares estimation of parameters of a linear model has been proposed. In the context of future research one may discuss Gauss-Markoff theorem for linear estimation andmean vector and covariance matrix of blue.

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