American Journal of Systems and Software, 2014, Vol. 2, No. 1, 9-13
Available online at http://pubs.sciepub.com/ajss/2/1/2

DOI:10.12691/ajss-2-1-2

A Study on the Differential Problem of Trigonometric Functions with Maple

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Received December 08, 2013; Revised December 28, 2013; Accepted January 15, 2014

Abstract This article uses the mathematical software Maple for the auxiliary tool to study the differential problem of two types of trigonometric functions. We can obtain the Fourier series expansions of any order derivatives of these two types of functions by using binomial theorem and differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order derivative values. On the other hand, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple.

Keywords: *derivatives, trigonometric functions, Fourier series expansions, binomial theorem, differentiation term by term theorem, Maple*

Cite This Article: Chii-Huei Yu, "A Study on the Differential Problem of Trigonometric Functions with Maple." *American Journal of Systems and Software* 2, no. 1 (2014): 9-13. doi: 10.12691/ajss-2-1-2.

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, [1-7] can be adopted as references.

In calculus and engineering mathematics curricula, evaluating the n -th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x=c$, in general, needs to go through two procedures: firstly determining the n -th order derivative $f^{(n)}(x)$ of $f(x)$, and then taking $x=c$ into $f^{(n)}(x)$. These two procedures will make us be faced with increasingly complex calculations when calculating higher order derivative values of this function (i.e. n is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we mainly study the differential problem of the following two types of trigonometric functions

$$f(x)=\exp[\lambda\cos(ax+b)] \tag{1}$$

$$g(x)=\exp[\lambda\sin(ax+b)] \tag{2}$$

where λ, a, b are real numbers. We can obtain the Fourier series expansions of any order derivatives of these two types of trigonometric functions by using binomial theorem and differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of calculating their higher order derivative values. For the study of related differential problems can refer to [8-24]. In addition, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and two formulas used in this study.

2.1. Notation

Suppose m,k are non-negative integers, $m \le k$. Define $\binom{k}{m} = \dfrac{k(k-1)\cdots(k-m+1)}{m!}$ and $\binom{k}{0} = 1$.

2.2. Euler's Formula

$e^{iy} = \cos y + i\sin y$, where $i + \sqrt{-1}$, y is any real number.

2.3. Taylor Series Expansion of Exponential Function

$e^{y} = \sum_{k=0}^{\infty} \frac{1}{k!} y^k$, where y is any real number.

The followings are two important theorems used in this paper, we can refer to [25,26] respectively.

2.4. Binomial Theorem

Suppose x, y are complex numbers, and k is any non-negative integer. Then

$$(x+y)^k = \sum_{m=0}^{k} \binom{k}{m} x^{k-m} y^m$$

2.5. Differentiation Term by Term Theorem

For all non-negative integers k , if the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b) , (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) . Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty}\frac{d}{dx} g_k(x)$.

Before deriving the first major result of this study, we need a lemma.

2.6. Lemma 1

Suppose a,b y are real numbers, and k is any non-negative integer. Then the trigonometric function

$$\cos^k(ax+b) = \frac{1}{2^k}\cdot \sum_{m=0}^{k} \binom{k}{m} \cdot \cos[(k-2m)(ax+b)] \quad (3)$$

for all $x \in R$.

Proof

$$\cos^k(ax+b)$$
$$= \left[\frac{1}{2}[e^{i(ax+b)} + e^{-i(ax+b)}]\right]^k$$
(By Euler's formula)
$$= \frac{1}{2^k}\cdot \sum_{m=0}^{k} \binom{k}{m} [e^{i(ax+b)}]^{k-m} [e^{-i(ax+b)}]^m$$
(By binomial theorem)
$$= \frac{1}{2^k}\cdot \sum_{m=0}^{k} \binom{k}{m} e^{i(k-2m)(ax+b)}$$
$$= \frac{1}{2^k}\cdot \sum_{m=0}^{k} \binom{k}{m} \cos[(k-2m)(ax+b)]$$

Next, we determine the Fourier series expansions of any order derivatives of the trigonometric function (1).

2.7. Theorem 1

Suppose λ, a, b are real numbers, and n is any positive integer. Let the domain of the trigonometric function

$$f(x) = \exp[\lambda\cos(ax+b)]$$

be $(-\infty,\infty)$. Then the n -th order derivative of $f(x)$,

$$f^{(n)}(x) = a^n \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{2^k k!} \cdot \sum_{m=0}^{k} \binom{k}{m} \cdot (k-2m)^n \times \cos\left[(k-2m)(ax+b) + \frac{n\pi}{2}\right] \quad (4)$$

for all $x \in R$.

Proof Because

$$f(x) = \exp[\lambda\cos(ax+b)]$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!}[\lambda\cos(ax+b)]^k$$
(By Formula 2.3)
$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\cos^k(ax+b) \quad (5)$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{2^k k!}\cdot \sum_{m=0}^{k} \binom{k}{m} \cdot \cos[(k-2m)(ax+b)]$$
(By Lemma 1)

By differentiation term by term theorem, differentiating n -times with respect to x on both sides of (5), we obtain

$$f^{(n)}(x) = a^n \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{2^k k!} \cdot \sum_{m=0}^{k} \binom{k}{m} \cdot (k-2m)^n \times \cos\left[(k-2m)(ax+b) + \frac{n\pi}{2}\right]$$

for all $x \in R$.

To prove the second major result of this study, we also need a lemma.

2.8. Lemma 2

Let the assumptions be the same as Lemma 1, then

$$\sin^k(ax+b) = \frac{1}{2^k}\cdot \sum_{m=0}^{k} \binom{k}{m} \cdot \cos\left[(k-2m)\left(ax+b-\frac{\pi}{2}\right)\right] \quad (6)$$

for all $x \in R$.

Proof

$$\sin^k(ax+b)$$
$$=\cos^k\left[(ax+b)-\frac{\pi}{2}\right]$$
$$=\frac{1}{2^k}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot\cos\left[(k-2m)\left(ax+b-\frac{\pi}{2}\right)\right]$$

(By Lemma 1)

Finally, we find the Fourier series expansions of any order derivatives of the trigonometric function (2).

2.9. Theorem 2

If the assumptions are the same as Theorem 1. Let the domain of the trigonometric function

$$g(x)=\exp[\lambda\sin(ax+b)]$$

be $(-\infty,\infty)$. Then the n -th order derivative of $g(x)$,

$$g^{(n)}(x)=a^n\cdot\sum_{k=0}^{\infty}\frac{\lambda^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^n\times \cos\left[(k-2m)\left(ax+b-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right] \tag{7}$$

for all $x\in R$.

Proof Because

$$g(x)=\exp[\lambda\sin(ax+b)]$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}[\lambda\sin(ax+b)]^k$$
$$=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sin^k(ax+b) \tag{8}$$
$$=\sum_{k=0}^{\infty}\frac{\lambda^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot\cos\left[(k-2m)\left(ax+b-\frac{\pi}{2}\right)\right]$$

(By Lemma 2)

Also, using differentiation term by term theorem, differentiating n -times with respect to x on both sides of (8), we have

$$g^{(n)}(x)=a^n\cdot\sum_{k=0}^{\infty}\frac{\lambda^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^n\times \cos\left[(k-2m)\left(ax+b-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right]$$

for all $x\in R$.

3. Examples

In the following, for the differential problem of the two types of trigonometric functions in this study, we provide two examples and use Theorems 1, 2 to determine the Fourier series expansions of any order derivatives of these functions and evaluate some of their higher order derivative values. On the other hand, we employ Maple to calculate the approximations of these higher order derivative values and their solutions for verifying our answers.

3.1. Example 1

Suppose the domain of the trigonometric function

$$f(x)=\exp\left[3\cos\left(2x-\frac{\pi}{4}\right)\right] \tag{9}$$

is $(-\infty,\infty)$. By Theorem 1, we obtain any n -th order derivative of $f(x)$,

$$f^{(n)}(x)=2^n\cdot\sum_{k=0}^{\infty}\frac{3^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^n\times \cos\left[(k-2m)\left(2x-\frac{\pi}{4}\right)+\frac{n\pi}{2}\right] \tag{10}$$

for all $x\in R$.

Therefore, the 13 -th order derivative value of $f(x)$ at $x=\frac{5\pi}{4}$,

$$f^{(13)}\left(\frac{5\pi}{4}\right)$$
$$=-2^{13}\cdot\sum_{k=0}^{\infty}\frac{3^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^{13}\sin\frac{(k-2m)\pi}{4} \tag{11}$$

Next, we use Maple to verify the correctness of (11).

```
>f:=x->exp(3*cos(2*x-Pi/4));
>evalf((D@@13)(f)(5*Pi/4),22);
```

$$2.987948637552443836942\cdot 10^{12}$$

```
>evalf(-2^13*sum(3^k/(2^k*k!)*sum(k!/(m!*(k-m)!)*(k-2*m)^13*sin((k-2*m)*Pi/4),m=0..k),k=0..infinity),22);
```

$$2.987948637552443835338\cdot 10^{12}$$
$$-8.003065905698558809678\cdot 10^{-23}I$$

The above answer obtained by Maple appears I ($=\sqrt{-1}$), it is because Maple calculates by using special functions built in. The imaginary part is very small, so can be ignored.

3.2. Remark 1

In Example 1, the first two derivatives of $f(x)$ are

$$f'(x)=-6\sin\left(2x-\frac{\pi}{4}\right)\exp\left[3\cos\left(2x-\frac{\pi}{4}\right)\right] \tag{12}$$

$$f''(x)$$
$$=\left[36\sin^2\left(2x-\frac{\pi}{4}\right)-12\cos\left(2x-\frac{\pi}{4}\right)\right] \exp\left[3\cos\left(2x-\frac{\pi}{4}\right)\right] \tag{13}$$

If we continue to differentiate $f(x)$, we will be faced with a complex calculation. Thus, Theorem 1 gives us a convenient method to evaluate the derivatives of $f(x)$, and hence reduces the difficulty of calculating the higher order derivative values of $f(x)$.

3.3. Example 2

If the domain of the trigonometric function

$$g(x)=\exp\left[5\sin\left(7x+\frac{4\pi}{3}\right)\right] \tag{14}$$

is $(-\infty,\infty)$. Using Theorem 2, we can evaluate any n -th order derivative of $g(x)$,

$$g^{(n)}(x)=7^n\cdot\sum_{k=0}^{\infty}\frac{5^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^n\times \cos\left[(k-2m)\left(7x+\frac{5\pi}{6}\right)+\frac{n\pi}{2}\right] \tag{15}$$

for all $x\in R$.

Thus, the 10 -th order derivative value of $g(x)$ at $x=-\frac{\pi}{6}$,

$$g^{(10)}\left(-\frac{\pi}{6}\right) =-7^{10}\cdot\sum_{k=0}^{\infty}\frac{5^k}{2^k k!}\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot(k-2m)^{10}\cos\frac{(k-2m)\pi}{3} \tag{16}$$

```
>g:=x->exp(5*sin(7*x+4*Pi/3));
>evalf((D@@10)(g)(-Pi/6),28);
```

$$7.018748950306407387413 23\cdot10^{15}$$

```
>evalf(-7^10*sum(5^k/(2^k*k!)*sum(k!/(m!*(k-m)!)*(k-2*m)^10*cos((k-2*m)*Pi/3),m=0..k),k=0..infinity),28);
```

$$7.018748950306407387413 15\cdot10^{15}$$

$$+1.026299999999999990102170936\cdot10^{-8}\,I$$

The imaginary part of the above answer obtained by Maple is very small, so can be ignored.

3.4. Remark 2

In Example 2, the first two derivatives of $g(x)$ are

$$g'(x)=35\cos\left(7x+\frac{4\pi}{3}\right)\exp\left[5\sin\left(7x+\frac{4\pi}{3}\right)\right] \tag{17}$$

$$g''(x) =\left[1225\cos^2\left(7x+\frac{4\pi}{3}\right)-245\sin\left(7x+\frac{4\pi}{3}\right)\right] \exp\left[5\sin\left(7x+\frac{4\pi}{3}\right)\right] \tag{18}$$

Also, if we continue to differentiate $g(x)$, the calculation will be complicated. Therefore, Theorem 2 provide a good method to find any derivatives of $g(x)$, and reduces the difficulty of calculating its higher order derivative values.

4. Conclusion

In this study, we provide a new technique to evaluate any order derivatives of some trigonometric functions. We hope this technique can be applied to solve another differential problems. On the other hand, the binomial theorem and the differentiation term by term theorem play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

References

[1] C. T. J. Dodson and E. A. Gonzalez, *Experiments in mathematics using Maple*, Springer-Verlag, New York, 1995.

[2] D. Richards, *Advanced mathematical methods with Maple*, Cambridge University Press, New York, 2002.

[3] M. L. Abell and J. P. Braselton, *Maple by example*, 3rd ed., Elsevier Academic Press, New York, 2005.

[4] R. J. Stroeker and J. F. Kaashoek, *Discovering mathematics with Maple: an interactive exploration for mathematicians*, engineers and econometricians, Birkhauser Verlag, Basel, 1999.

[5] F. Garvan, *The Maple book*, Chapman & Hall/CRC, London, 2001.

[6] J. S. Robertson, *Engineering mathematics with Maple*, McGraw-Hill, New York, 1996.

[7] C. Tocci and S. G. Adams, *Applied Maple for engineers and scientists*, Artech House, Boston, 1996.

[8] A. Griewank and A. Walther, *Evaluating derivatives: principles and techniques of algorithmic differentiation*, 2nd ed., SIAM, Philadelphia, 2008.

[9] M. Wagner, A. Walther, and B. J. Schaefer, "On the efficient computation of high-order derivatives for implicitly defined functions," *Computer Physics Communications*, vol. 181, issue. 4, pp. 756-764, 2010.

[10] M. A. Patterson, M. Weinstein, and A. V. Rao, "An efficient overloaded method for computing derivatives of mathematical functions in MATLAB," *ACM Transactions on Mathematical Software*, vol. 39, no. 3, 2013.

[11] C. -H. Yu, "A study on the differential problem," *International Journal of Research in Aeronautical and Mechanical Engineering*, vol. 1, issue. 3, pp. 52-57, 2013.

[12] C. -H. Yu, "The differential problem of two types of functions," *International Journal of Computer Science and Mobile Computing*, vol. 2, issue. 7, pp. 137-145, 2013.

[13] C. -H. Yu, "Evaluating the derivatives of two types of functions," International Journal of Computer Science and Mobile Applications, vol. 1, issue. 2, pp. 1-8, 2013.

[14] C. -H. Yu, "Evaluating the derivatives of trigonometric functions with Maple," *International Journal of Research in Computer Applications and Robotics*, vol. 1, issue. 4, pp. 23-28, 2013.

[15] C.-H. Yu, "Application of Maple on solving the differential problem of rational functions," *Applied Mechanics and Materials*, vols. 479-480 (2014), pp 855-860, 2013.

[16] C. -H. Yu, "Application of Maple on the differential problem," *Universal Journal of Applied Science*, vol. 2, no. 1, pp. 11-18, 2014.

[17] C. -H. Yu, "Studying the differential problem with Maple," *Computer Science and Information Technology*, vol. 1, no. 3, pp. 190-195, 2013.

[18] C. -H. Yu, "A study on some differential problems with Maple," *Proceedings of the 6th IEEE/International Conference on Advanced Infocomm Technology*, Taiwan, no. 00291, July 2013.

[19] C. -H. Yu, "Application of Maple on solving some differential problems," *Proceedings of IIE Asian Conference 2013*, Taiwan, vol. 1, pp. 585-592, July 2013.

[20] C. -H. Yu, and B. -H. Chen, "Application of Fourier series on the differential problems," *UHC 2013 The Conference on Ubiquitous Home*, Taiwan, pp. 180-185, December 2013.

[21] C. -H. Yu, and B. -H. Chen, "Evaluating the derivatives of trigonometric functions using binomial theorem," *UHC 2013 The Conference on Ubiquitous Home,* Taiwan, pp. 192-196, December 2013.

[22] C. -H. Yu, "The differential problem of four types of functions," *Academic Journal of Kang-Ning*, vol. 15, pp. 51-63, 2013.

[23] C. -H. Yu, "A study on the differential problem of some trigonometric functions," *Journal of Jen-Teh*, vol. 10, pp. 27-34, 2013.

[24] C. -H. Yu, "The differential problem of some functions," *International Journal of Computer Science and Mobile Applications*, vol. 1, issue. 1, pp. 31-38, 2013.

[25] R. A. Brualdi, *Introductory combinatorics*, Elsevier Science Publishing, New York, p 54, 1977.

[26] T. M. Apostol, *Mathematical analysis*, 2nd ed., Addison-Wesley Publishing, Boston, p 230, 1975.