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A SUBCLASS OF HARMONIC FUNCTIONS DEFINED BY A CERTAIN FRACTIONAL CALCULUS OPERATORS

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Abstract: In this paper a subclass of p-valent harmonic functions in the open unit disc is introduced by making use of a certain fractional calculus operator and some properties such as coefficient estimates, distortion theorem and extreme points are studied.

Keywords and Phrases: Open Unit disk, Multivalent Functions, Harmonic Functions.

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1. Introduction

A continuous function f = u + iv is a complex valued harmonic function in a complex domain C, if both u and v are real harmonic in C. In any simply connected domain $D \subseteq C$, we can write f=h+$\bar{g}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [3]).

Denote by M(p) the class of functions f=h+$\bar{g}$, that are harmonic multivalent and sense-preserving in the unit disk U=$\{z \in C : |z| < 1\}$. The class M(p) was studied by Ahuja and Jahangiri [1] and class M(p) for p=1 was defined and studied by Jahangiri et. al. in [5]. For f=h+$\bar{g} \in M(p)$, we may express the analytic functions h and g as:

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1}, g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1} \tag{1.1}$$

where $z \in U$,$|b_p| < 1$.
For real number $\mu\,(-\infty < \mu < 1)\,, \gamma\,(-\infty < \gamma < 1)$ and a positive real number η Murugusundramoorthy et al [6] have defined an operator $U_{0,z}^{\mu,\gamma,\eta} : M(p) \longrightarrow M(p)$ as

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = U_{0,z}^{\mu,\gamma,\eta} h(z) + \overline{U_{0,z}^{\mu,\gamma,\eta} g(z)} \tag{1.2}$$

where

$$U_{0,z}^{\mu,\gamma,\eta} h(z) = z^p + \sum_{n=2}^{\infty} \Gamma_{0,z}^{\mu,\gamma,\eta} a_{n+p-1} z^{n+p-1} \tag{1.3}$$

$$U_{0,z}^{\mu,\gamma,\eta} g(z) = \sum_{n=1}^{\infty} \Gamma_{0,z}^{\mu,\gamma,\eta} b_{n+p-1} z^{n+p-1} \tag{1.4}$$

and

$$\Gamma_{0,z}^{\mu,\gamma,\eta} = \frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}} \tag{1.5}$$

For $\alpha \geq 0$ and $0 \leq \beta < 1$, let $\Omega_{0,z}^{\mu,\gamma,\eta}(p, \alpha, \beta)$ denote the family of harmonic functions f of the form (1.1) such that

$$Re\left\{ \frac{(1+\alpha e^{i\Phi}) z (U_{0,z}^{\mu,\gamma,\eta} f(z))'}{U_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha e^{i\Phi} \right\} \geq \beta \tag{1.6}$$

We also let $M\Omega_{0,z}^{\mu,\gamma,\eta}(p, \alpha, \beta) = \Omega_{0,z}^{\mu,\gamma,\eta}(p, \alpha, \beta) \bigcap M(p)$.

Remark. *The above class $U_{0,z}^{\mu,\gamma,\eta} f(z)$ of functions is a subclass of the general case of harmonic univalent functions which was defined by Clunie and Sheil-Small [3] in 1984 and the class $\Omega_{0,z}^{\mu,\gamma,\eta}(1, \alpha, \beta)$ is a subclass of the general case of harmonic starlike functions of order β of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ which was studied by Jahangiri [4] in 1999 for which the condition of starlikeness is given by $\frac{\delta}{\delta\theta}(argf(re^{i\theta})) \geq \beta; 0 \leq \beta < 1$. The arguments used to prove the theorems in the next three sections as well as the coefficients bounds, distortion theorem and extreme points for the general case of harmonic starlike functions of order β can be found in Jahangiri [4].*

2. Coefficient Inequality

We will first derive a sufficient condition for the functions to be in the class $\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

Theorem 2.1. *Let f=h+$\bar{g}$ (h and g being given by 1.1). If*

$$\sum_{n=2}^{\infty}\left[\left\{\frac{(n+p-1-\beta)+\alpha(n+p-2)}{p-\beta}\right\}|a_{n+p-1}|+\right.$$
$$\left.\left\{\frac{(n+p-1+\beta)+\alpha(n+p)}{p-\beta}\right\}|b_{n+p-1}|\right]\Gamma_{0,z}^{\mu,\gamma,\eta} \qquad (2.1)$$

$\leq 1-\left\{\frac{(p+\beta)+\alpha(p+1)}{p-\beta}\right\}|b_p|$, *then* $f\in\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

Proof. Let f=h+$\bar{g}$ in (2.1), we get

$\text{Re}\left\{\frac{(1+\alpha e^{i\Phi})z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-\alpha e^{i\Phi}\right\}=Re\left\{\frac{A(z)}{B(z)}\right\}$

where

A(z)= $(1+\alpha e^{i\Phi})\left[z(U_{0,z}^{\mu,\gamma,\eta}h(z))'-\overline{z(U_{0,z}^{\mu,\gamma,\eta}g(z))'}\right]-\alpha e^{i\Phi}\left[U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}\right]$

and B(z)=$U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}$

Using the fact that Re $\omega(z)\geq\beta$ if and only if $|1-\beta+\omega|\geq|1+\beta-\omega|$ which was first used by Jahangiri[4], in 1999, it is sufficient to show that

$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\geq 0.$

Substituting for A(z) and B(z), we get

$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|=|(1+\alpha e^{i\Phi})\left[z(U_{0,z}^{\mu,\gamma,\eta}h(z))'-\overline{z(U_{0,z}^{\mu,\gamma,\eta}g(z))'}\right]$
$-\alpha e^{i\Phi}\left[U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}\right]+(1-\beta)\left[U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}\right]|$-
$|(1+\alpha e^{i\Phi})\left[z(U_{0,z}^{\mu,\gamma,\eta}h(z))'-\overline{z(U_{0,z}^{\mu,\gamma,\eta}g(z))'}\right]-\alpha e^{i\Phi}\left[U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}\right]$-
$(1+\beta)\left[U_{0,z}^{\mu,\gamma,\eta}h(z)+\overline{U_{0,z}^{\mu,\gamma,\eta}g(z)}\right]|$
$=|(1+\alpha e^{i\Phi})\left[pz^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)a_{n+p-1}z^{n+p-1}\right.$
$\left.-\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)\overline{b_{n+p-1}z^{n+p-1}}\right]-\alpha e^{i\Phi}\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}\right.$
$\left.+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]+(1-\beta)\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}\right.$
$\left.+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]|-|(1+\alpha e^{i\Phi})\left[pz^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)a_{n+p-1}z^{n+p-1}\right.$
$\left.-\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)\overline{b_{n+p-1}z^{n+p-1}}\right]-\alpha e^{i\Phi}\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}\right.$
$\left.+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]+$

$(1+\beta)\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]|$
$=|\left[(p+1-\beta)+\alpha(p-1)e^{i\Phi}\right]z^p+\sum_{n=2}^{\infty}\left[(p+n-\beta)+\alpha(p+n-2)e^{i\Phi}\right]$
$\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}-\sum_{n=1}^{\infty}\left[(p+n-2+\beta)+\alpha(p+n)e^{i\Phi}\right]\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}|$
$-|\left[(p-1-\beta)+\alpha(p-1)e^{i\Phi}\right]z^p+\sum_{n=2}^{\infty}\left[(p+n-2-\beta)+\alpha(p+n-2)e^{i\Phi}\right]$
$\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}-\sum_{n=1}^{\infty}\left[(p+n+\beta)+\alpha(p+n)e^{i\Phi}\right]\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}|$
$\geq[\{(p+1-\beta)-\alpha(p-1)\}+\{(p-1-\beta)+\alpha(p-1)\}]|z|^p$
$-\sum_{n=2}^{\infty}[\{(p+n-\beta)+\alpha(p+n-2)\}+\{(p+n-2-\beta)+\alpha(p+n-2)\}]$
$\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}||z|^{n+p-1}-\sum_{n=1}^{\infty}[\{(p+n-2+\beta)+\alpha(p+n)\}$
$+\{(p+n+\beta)+\alpha(p+n)\}]\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}||z|^{n+p-1}$
$=2(p-\beta)|z|^p-\sum_{n=2}^{\infty}2\left[(p+n-1-\beta)+\alpha(p+n-2)\right]\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}||z|^{n+p-1}$
$-\sum_{n=1}^{\infty}2\left[(p+n-1+\beta)+\alpha(p+n)\right]\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}||z|^{n+p-1}$
$\geq 2(p-\beta)|z|^p[1-\left\{\dfrac{(p+\beta)+\alpha(p+1)}{p-\beta}\right\}]|b_p|$
$-\sum_{n=2}^{\infty}\left\{\dfrac{(p+n-1-\beta)+\alpha(p+n-2)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}||z|^{n-1}$
$-\sum_{n=2}^{\infty}\left\{\dfrac{(p+n-1+\beta)+\alpha(p+n)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}||z|^{n-1}]\geq 0.$

By virtue of (2.1), this implies that $f\in\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

Theorem 2.2: Let f=h+$\bar{g}\in M(p)$. Then $f\in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\left[\left\{\frac{(n+p-1-\beta)+\alpha(n+p-2)}{p-\beta}\right\}|a_{n+p-1}|+\left\{\frac{(n+p-1+\beta)+\alpha(n+p)}{p-\beta}\right\}|b_{n+p-1}|\right]\Gamma_{0,z}^{\mu,\gamma,\eta}\tag{2.2}$$

$\leq 1-\left\{\dfrac{(p+\beta)+\alpha(p+1)}{p-\beta}\right\}|b_p|$, where $0\leq\beta<1$.

Proof: Since $M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)\subset\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$, we only need to prove 'only if' part of the theorem. To this end, for functions f of the form (1.1) with condition (1.6), we notice the condition

Re $\left\{\dfrac{(1+\alpha e^{i\Phi})z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-(\alpha e^{i\Phi}+\beta)\right\}\geq 0$.

The above inequality is equivalent to

$Re[(1+\alpha e^{i\Phi})\left[pz^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)a_{n+p-1}z^{n+p-1}\right.$
$\left.-\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}(n+p-1)\overline{b_{n+p-1}z^{n+p-1}}\right]$
$-(\alpha e^{i\Phi}+\beta)\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]$

$\times\left[z^p+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n+p-1}+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n+p-1}}\right]^{-1}]\geq 0.$

$Re[\{p+\alpha(p-1)-\beta\}+\sum_{n=2}^{\infty}\left\{(p+n-1)+\alpha(p+n-2)e^{i\Phi}-\beta\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n-1}$

$-\frac{\overline{z}^p}{z^p}\sum_{n=1}^{\infty}\left\{(p+n-1)+\alpha(p+n)e^{i\Phi}+\beta\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}}\overline{z}^{n-1}]$

$\times[1+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}a_{n+p-1}z^{n-1}+\frac{\overline{z}^p}{z^p}\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}\overline{b_{n+p-1}z^{n-1}}]^{-1}\geq 0.$

This condition must hold for all values of z, such that $|z|=r<1$. Choosing Φ according to condition and noting that $Re(-\alpha e^{i\Phi})\geq -\alpha|e^{i\Phi}|=-\alpha$, the above inequality reduces to

$$\begin{aligned}
|\{p+\alpha(p-1)-\beta\}-\{p+\alpha(p+1)+\beta\}|b_p|\\
&-\sum_{n=2}^{\infty}[\{(n+p-1)+\alpha(n+p-2)-\beta\}|a_{n+p-1}|\\
&+\{(n+p-1)+\alpha(n+p)+\beta\}|b_{n+p-1}|]\,\Gamma_{0,z}^{\mu,\gamma,\eta}r^{n-1}|\\
&\times[1+\sum_{n=2}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}|r^{n-1}+\sum_{n=1}^{\infty}\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}|r^{n-1}]]^{-1}\geq 0.
\end{aligned}\tag{2.3}$$

Letting $r\longrightarrow 1^-$ and if the condition (2.2) does not hold, then the numerator in (2.3) is negative. This contradicts our assumptions that $f\in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$. Hence $f\in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

For $\sum_{n=2}^{\infty}|x_{n+p-1}|+\sum_{n=1}^{\infty}|y_{n+p-1}|=1$, the harmonic univalent function

$f(z)=z^p+\sum_{n=2}^{\infty}\frac{(p-\beta)}{\{(n+p-1)+\alpha(n+p-2)-\beta\}\Gamma_{0,z}^{\mu,\gamma,\eta}}x_{n+p-1}z^{n+p-1}$

$+\sum_{n=1}^{\infty}\frac{(p-\beta)}{\{(n+p-1)+\alpha(n+p)+\beta\}\Gamma_{0,z}^{\mu,\gamma,\eta}}\overline{y_{n+p-1}z^{n+p-1}}$

shows the equality in the coefficient bound given by (2.2) is sharp.

3. Distortion Theorem

Theorem 3.1. *Let the function f(z) defined by (1.1) be in the class $M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_p|)r^p+\frac{(2-\gamma)(2-\mu+\eta)}{(2-\gamma+\eta)}\left[\frac{(p-\beta)}{2}-\left\{\frac{(p+\beta)+\alpha(p+1)}{2}\right\}|b_p|\right]r^{p+1}\tag{3.1}$$

and

$$|f(z)| \geq (1-|b_p|)r^p - \frac{(2-\gamma)(2-\mu+\eta)}{(2-\gamma+\eta)}\left[\frac{(p-\beta)}{2} - \left\{\frac{(p+\beta)+\alpha(p+1)}{2}\right\}|b_p|\right]r^{p+1}. \tag{3.2}$$

The result is sharp.

Proof. We prove only the left hand inequality. Let $f \in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$. Taking the absolute value of f(z), we have

$|f(z)| \geq (1-|b_p|)r^p - \sum_{n=2}^{\infty}\{|a_{n+p-1}|+|b_{n+p-1}|\}r^{n+p-1}$

$\geq (1-|b_p|)r^p - r^{p+1}\sum_{n=2}^{\infty}\{|a_{n+p-1}|+|b_{n+p-1}|\}$

$\geq (1-|b_p|)r^p - \dfrac{(2-\gamma)(2-\mu+\eta)(p-\beta)}{2(2-\gamma+\eta)}r^{p+1}$

$\sum_{n=2}^{\infty}\dfrac{(2)_{n-1}(2-\gamma+\eta)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}(p-\beta)}\{|a_{n+p-1}|+|b_{n+p-1}|\}$

$\geq (1-|b_p|)r^p - \dfrac{(2-\gamma)(2-\mu+\eta)(p-\beta)}{2(2-\gamma+\eta)}r^{p+1}\sum_{n=2}^{\infty}$

$[\dfrac{(n+p-1-\beta)+\alpha(n+p-2)}{(p-\beta)}|a_{n+p-1}|+\dfrac{(n+p-1+\beta)+\alpha(n+p)}{(p-\beta)}|b_{n+p-1}|]\Gamma_{0,z}^{\mu,\gamma,\eta}$

$\geq (1-|b_p|)r^p - \dfrac{(2-\gamma)(2-\mu+\eta)(p-\beta)}{2(2-\gamma+\eta)}\left[1-\left\{\dfrac{(p+\beta)+\alpha(p+1)}{p-\beta}\right\}|b_p|\right]r^{p+1}$

$= (1-|b_p|)r^p - \dfrac{(2-\gamma)(2-\mu+\eta)}{(2-\gamma+\eta)}\left[\dfrac{(p-\beta)}{2}-\left\{\dfrac{(p+\beta)+\alpha(p+1)}{2}\right\}|b_p|\right]r^{p+1}.$

4. Extreme Points

Theorem 4.1. *A function $f = h + \bar{g} \in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$ if and only if f(z) can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty}(A_n h_n + B_n g_n) \tag{4.1}$$

where

$$h_1(z) = z^p, h_n = z^p + \frac{(p-\beta)}{\{(n+p-1-\beta)+\alpha(n+p-2)\}\Gamma_{0,z}^{\mu,\gamma,\eta}}z^{n+p-1}(n=2,3,...) \tag{4.2}$$

$g_n = z^p + \dfrac{(p-\beta)}{\{(n+p-1+\beta)+\alpha(n+p)\}\Gamma_{0,z}^{\mu,\gamma,\eta}}z^{n+p-1}$ *(n=1,2,3,...),*

$\Gamma_{0,z}^{\mu,\gamma,\eta}$ *is given by (1.5) and* $\sum_{n=1}^{\infty}(A_n+B_n) = 1$ *with* $A_n \geq 0, B_n \geq 0$.

In particular the extreme points of $M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$ are h_n and g_n.

Proof. Let f(z) be of the form (4.1). Then we have

$f(z)=\sum_{n=1}^{\infty}(A_n+B_n)\,z^p+\sum_{n=2}^{\infty}\frac{(p-\beta)}{\{(n+p-1-\beta)+\alpha(n+p-2)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}A_n z^{n+p-1}$

$+\sum_{n=1}^{\infty}\frac{(p-\beta)}{\{(n+p-1+\beta)+\alpha(n+p)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}B_n z^{n+p-1}$

$=z^p+\sum_{n=2}^{\infty}\frac{(p-\beta)}{\{(n+p-1-\beta)+\alpha(n+p-2)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}A_n z^{n+p-1}$

$+\sum_{n=1}^{\infty}\frac{(p-\beta)}{\{(n+p-1+\beta)+\alpha(n+p)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}B_n z^{n+p-1}$

Furthermore, let

$|a_{n+p-1}|=\frac{(p-\beta)}{\{(n+p-1-\beta)+\alpha(n+p-2)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}A_n$

and $|b_{n+p-1}|=\frac{(p-\beta)}{\{(n+p-1+\beta)+\alpha(n+p)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}B_n.$

Now, $\sum_{n=2}^{\infty}\left\{\frac{(n+p-1-\beta)+\alpha(n+p-2)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}|+$

$\sum_{n=1}^{\infty}\left\{\frac{(n+p-1+\beta)+\alpha(n+p)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}|$

$=\sum_{n=2}^{\infty}A_n+\sum_{n=1}^{\infty}B_n=1-A_1\le 1$. So $f\in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

Conversely, suppose that $f\in M\Omega_{0,z}^{\mu,\gamma,\eta}(p,\alpha,\beta)$.

Setting $A_n=\left\{\frac{(n+p-1-\beta)+\alpha(n+p-2)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|a_{n+p-1}|$ (n=2,3,...)

$B_n=\left\{\frac{(n+p-1+\beta)+\alpha(n+p)}{p-\beta}\right\}\Gamma_{0,z}^{\mu,\gamma,\eta}|b_{n+p-1}|$ (n=1,2,...)

and $A_1=1-\sum_{n=2}^{\infty}A_n-\sum_{n=1}^{\infty}B_n$.

Therefore,

$f(z)=z^p+\sum_{n=2}^{\infty}|a_{n+p-1}|z^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}|z^{n+p-1}$

$=z^p+\sum_{n=2}^{\infty}\frac{(p-\beta)}{\{(n+p-1-\beta)+\alpha(n+p-2)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}A_n z^{n+p-1}$

$+\sum_{n=1}^{\infty}\frac{(p-\beta)}{\{(n+p-1+\beta)+\alpha(n+p)\}\,\Gamma_{0,z}^{\mu,\gamma,\eta}}B_n z^{n+p-1}$

$=z^p+\sum_{n=2}^{\infty}\left[A_n\left\{h_n(z)-z^p\right\}\right]+\sum_{n=1}^{\infty}\left[B_n\left\{g_n(z)-z^p\right\}\right]$

So $f(z)=\sum_{n=1}^{\infty}\left(A_n h_n(z)+B_n g_n(z)\right)$.

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