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A-SUBMANIFOLDS IN EUCLIDEAN SPACE

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§0. Introduction. In this paper we give a geometrical characterization of A-submanifolds M^n of a Euclidean space E^{n+p} using the $(p-2)$th polar hypersurface $K_m^{(2)}$ of the characteristic hypersurface K of the normal space $T_m^{\perp}M^n$, $m\in M^n$. This leads us to other characterizations involving Lipschitz-Killing curvature and second mean curvature. Several examples of A-submanifolds in E^4 are given. Finally we extend the notion of A-submanifold in a natural way to A_k-submanifold according to the position of the mean curvature vector with respect to $K_m^{(2)}$.

§1. Preliminaries.

Let M^n be an n-dimensional submanifold immersed in an $(n+p)$-dimensional Euclidean space E^{n+p}. At $m\in M^n$ we choose an orthonormal frame $(e_1, e_2, \cdots, e_{n+p})$ such that the vectors $e_1, \cdots, e_n$ span the tangent space T_mM^n and $e_{n+1}, \cdots, e_{n+p}$ span the normal space $T_m^{\perp}M^n$. Then we have

$$dm=\omega^1e_1+\cdots+\omega^ne_n\,,$$

$$de_i=\sum_{j=1}^{n+p}\omega_i^je_j\,,\ \omega_i^j+\omega_j^i=0\,,$$

$$\omega_i^j=\sum_{k=1}^{n}\gamma_{ik}^j\omega^k\,.$$

The second fundamental form is given by

$$h=\sum\gamma_{ij}^k\omega^i\omega^je_k\quad i,\ j=1,\ 2,\ \cdots,\ n,\ k=n+1,\ \cdots,\ n+p\,,$$

and the mean curvature vector by

$$H=\frac{1}{n}\sum_{k=n+1}^{n+p}\Big(\sum_{i=1}^{n}\gamma_{ii}^k\Big)e_k\,. \tag{1}$$

With any normal vector e at $m\in M^n$ we associate the symmetric transformation $A(e)$ of T_mM^n into itself defined by

$$\langle A(e)(X),\ Y\rangle=\langle e,\ h(X,\ Y)\rangle$$

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for all tangent vectors X, Y at m. $A(e)$ is the second fundamental tensor associated with e. The determinant of $A(e)$ is the Lipschitz-Killing curvature $K(m, e)$. M^n is said to be pseudo-umbilical if $A\left(\frac{H}{|H|}\right)=\lambda I$. A-submanifolds of E^{n+p} are defined as follows [2]; we consider a normal vector field u and a local frame field such as

$$u=|u|e_{n+1},\ |u|=\langle u, u\rangle^{1/2}.$$

Then the *allied vector field* of u is given by [2]

$$a(u)=\frac{1}{n}|u|\sum_{r=2}^{p}\operatorname{trace}(A(e_{n+1})A(e_{n+r}))e_{n+r}. \tag{2}$$

The *allied mean curvature vector* is $a(H)$. If the allied mean curvature vector $a(H)=0$, then M^n is called an *A-submanifold* of E^{n+p}. (Minimal submnanifolds, pseudo-umbilical submanifolds and hypersurfaces are A-submanifolds of E^{n+p}).

§ 2. Polar hyperquadric of an A-submanifold.

In the case of a surface M^2 in E^4 the locus of the points N in which $T_m^\perp M^2$ is cut by the neighboring normal plane of M^2 is a conic, which is called the Kommerell conic. In the case of a general M^n is E^{n+p} we obtain an algebraic hypersurface of $T_m^\perp M^n$, denoted by K. This was studied by Perepelkine [8]. The coordinates $X^r(r=1, \cdots, p)$ of a point N belonging to $T_m^\perp M^n$ and to a neighboring normal space are such that

$$-\omega^i+\sum_{r=1}^{p}X^r\omega_i^{n+r}=0,\quad i=1, \cdots, n,$$

and hence the equation of K is

$$\det\left|\sum_{r=1}^{p}X^r\gamma_{ij}^{n+r}-\delta_{ij}\right|=0. \tag{3}$$

(3) defines an algebraic hypersurface of degree n in $T_m^\perp M^n$. We associate in an intrinsic way to the couple (K, m) the successive polar hypersurfaces of K with respect to m. Let $F(X^1, X^2, \cdots, X^{p+1})=0$ be the homogeneous equation of K. Then the equation of the $(p-s)$th polar hypersurface $K_m^{(s)}$ is

$$\left(\sum_{r=1}^{p+1}X^r\frac{\partial}{\partial X^r}\right)^s F(X^1, \cdots, X^{p+1})=0$$

where the values of the partial derivatives are taken at the point $m(0, 0, \cdots, 1)$. For the $(p-2)$th polar hypersurface $K_m^{(2)}$ we obtain

$$\sum_r\left[(X^r)^2\sum_{\lambda\neq\mu}\{\gamma_{\lambda\lambda}^{n+r}\gamma_{\mu\mu}^{n+r}-(\gamma_{\lambda\mu}^{n+r})^2\}\right]+\sum_{r\neq s}X^rX^s\left[\sum_{\lambda\neq\mu}\{\gamma_{\lambda\lambda}^{n+r}\gamma_{\mu\mu}^{n+s}-\gamma_{\lambda\mu}^{n+r}\gamma_{\lambda\mu}^{n+s}\}\right]$$

(4) $$\cdots -2(n-1)\sum_r X^r(\sum_r \gamma_{\lambda\lambda}^{n+r})+n(n-1)=0, \text{ where } \begin{array}{l} r,\ s=1,2,\cdots,p, \\ \lambda,\ \mu=1,2,\cdots,n. \end{array}$$

Further the allied vector field $a(e_{n+1})$ is

$$a(e_{n+1})=\frac{1}{n}\sum_{r=2}^{p} \text{Trace}\,[A(e_{n+1})A(e_{n+r})]e_{n+r},$$

and $\text{Trace}\,[A(e_{n+1})A(e_{n+r})]=\sum_{\lambda}\gamma_{\lambda\lambda}^{n+1}\gamma_{\lambda\lambda}^{n+r}+\sum_{\lambda\neq\mu}\gamma_{\lambda\mu}^{n+1}\gamma_{\lambda\mu}^{n+r}$; $\lambda,\ \mu=1,2,\cdots,n$. By choosing the local frame such that $\frac{H}{|H|}=e_{n+1}$ we get $\sum_{\lambda=1}^{n}\gamma_{\lambda\lambda}^{n+r}=0$ for $r>1$, and $\text{Trace}\,[A(e_{n+1})A(e_{n+r})]=-\sum_{\lambda\neq\mu}\gamma_{\lambda\lambda}^{n+1}\gamma_{\mu\mu}^{n+r}+\sum_{\lambda\neq\mu}\gamma_{\lambda\mu}^{n+1}\lambda_{\lambda\mu}^{n+r}$; $\lambda,\ \mu=1,2,\cdots,n$.

Hence if M^n is an A-submanifold the coefficient of X^1X^r in the equation of $K_m^{(2)}$ vanishes. This proves

THEOREM 1. *A submanifold M^n of a Euclidean space E^{n+p} is an A-submanifold if and only if the mean curvature vector H determines a principal direction of the $(p-2)$th polar hypersurface of K.*

Remark 1. If M^n is a pseudo-umbilical submanifold of E^{n+p} with mean curvature vector $|H|e_{n+1}$ then

$$\gamma_{11}^{n+1}=\gamma_{22}^{n+1}=\cdots=\gamma_{nn}^{n+1}=\gamma;\ \gamma_{\lambda\mu}^{n+1}=0;\ \lambda\neq\mu;\ \lambda,\ \mu=1,2,\cdots,n$$

The terms independent of X^r, $r>1$, in the equation of $K_m^{(2)}$ are,

$$n(n-1)\gamma(X^1)^2-2n(n-1)\gamma X^1+n(n-1)=n(n-1)(\gamma X^1-1)^2,$$

and hence $K_m^{(2)}$ is a quadratic hypercone with vertex at $m+\frac{1}{\gamma}e_{n+1}$ and H as principal axe.

Remark 2. M^n is minimal if and only if $K_m^{(2)}$ has its center at m.

Remark 3. It is possible to give a geometric interpretation of various theorems of B. Y. Chen, L. Verstraelen and K. Yano for submanifolds of codimension 2 with umbilical or quasi-umbilical normal direction. We give some examples.

If $p=2$, K is a curve. For an M^n umbilical w. r. t. a nonparallel normal direction, K degenerates into two straight lines (with multiplicity 1 and $n-2$ respectively) [3].

If $p=2$, $n>4$ and M^n quasi umbilical w. r. t. a non-parallel normal direction, K has a multiple point of order $n-1$ and another of order $>n-3$. Then K degenerates and contains a line with multiplicity $>n-3$, [4].

§ 3. Examples of A-submanifolds in E^4.

Recently [10], G. Vranceanu studied a class of surfaces M^2 in E^4 called *rotation surfaces*. These surfaces are defined by the following equations w. r. t. an orthonormal system of coordinates (x_1, x_2, x_3, x_4): $x_1=r(u)\cos u\cos v$, $x_2=r(u)\cos u\sin v$, $x_3=r(u)\sin u\cos v$, $x_4=r(u)\sin u\sin v$. Now we choose a local moving frame such that e_1, e_2 are in the tangent plane and e_3, e_4 are in the normal plane. For example, we take

$$e_1=\begin{pmatrix} -\cos u\sin v \\ \cos u\cos v \\ -\sin u\sin v \\ \sin u\cos v \end{pmatrix},\quad e_2=\frac{1}{A}\begin{pmatrix} B\cos v \\ B\sin v \\ C\cos v \\ C\sin v \end{pmatrix},\quad e_3=\frac{1}{A}\begin{pmatrix} -C\cos v \\ -C\sin v \\ B\cos v \\ B\sin v \end{pmatrix},$$

$$e_4=\begin{pmatrix} -\sin u\sin v \\ \sin u\cos v \\ \cos u\sin v \\ -\cos u\cos v \end{pmatrix},$$

where $A=\sqrt{r^2+r'^2}$, $B=r'\cos u-r\sin u$, $C=r'\sin u+r\cos u$. Then, with $\omega^1=r\,dv$ and $\omega^2=\sqrt{r^2+r'^2}\,du$, we get

$$\begin{cases} \gamma_{11}^3=\dfrac{1}{\sqrt{r^2+r'^2}}, \quad \gamma_{12}^3=0, \quad \gamma_{22}^3=\dfrac{-rr''+2r'^2+r^2}{(r^2+r'^2)^{3/2}} \\ \gamma_{11}^4=0, \quad \gamma_{12}^4=\dfrac{-1}{\sqrt{r^2+r'^2}}, \quad \gamma_{22}^4=0 \\ \omega_1^2=\omega_3^4=\dfrac{-r'\omega^1}{\sqrt{r^2+r'^2}} \end{cases}$$

Hence we have immediately

Theorem 2. *Rotation surfaces of E^4 are A-submanifolds.*

Remark 1. When $r=e^{au}$ we obtain pseudo-umbilical submanifolds of E^4 and from $\omega_1^2=\omega_3^4$ it follows that these submanifolds are flat.

Remark 2. It is possible to give others examples of A-submanifolds of E^4. For instance R. Calapso [1] studied the M^2 of E^4 for which K is a circle and proved that such M^2 posses a conjugate net of Voss of special type (type c). These submanifolds are A-submanifolds and have constant Lipschitz-Killing curvature at m. L. N. Krivonosov proved that surfaces in E^4 for which K is a circle are in normal correspondance with minimal surfaces of E^4[7].

§4. Lipschitz-killing curvature of A-surfaces.

For a surface M^2 of E^{2+p} the K-variety is an hyperquadric. The Lipschitz-Killing curvature in the direction of the unit normal vector $e(x^1, x^2, \cdots, x^p)$ is given by

$$K(m, e)=\det\left|\sum_{r=1}^{p}\gamma_{ij}^{r}x^{r}\right|.$$

The K-variety has two common points with the line $N=m+\rho e$. These points are determined by the roots ρ_1, ρ_2 of the equation

$$\det\left|\sum_{r=1}^{p}\rho x^{r}\gamma_{ij}^{r}-\delta_{ij}\right|=0.$$

Then $K(m, e)=\dfrac{1}{\rho_1\rho_2}$. It is well known that for a hyperquadric $\dfrac{1}{\rho_1\rho_2}$ has extremal values when e determines a principal direction. This proves

THEOREM 3. *A surface M^2 in E^{2+p} is an A-submanifold if and only if the Lipchitz-killing curvature has an extremal value in the direction of the mean curvature vector.*

This result is analogous to a theorem of C. S. Houh for pseudoumbilical surfaces [6]. In the case of rotation surfaces of E^4 the Lipschitz-Killing curvature for $e=\cos\theta e_3+\sin\theta e_4$ is

$$K(m, e)=\frac{2r^2+3r'^2-rr''}{(r^2+r'^2)^2}\cos^2\theta-\frac{1}{r^2+r'^2}.$$

$K(m, e)$ has maximal value if $2r^2+3r'^2-rr''>0$ and minimal value if $2r^2+3r'^2-rr''<0$. It is possible to construct examples for the two cases. For instance if $r=\cos u$, $K(m, e)$ has maximal value in the direction of the mean curvature vector. If $r=u^{-(1/5)}$, $K(m, e)$ has minimal value in the direction of the mean curvature vector for some values of u. This is a counterexample to a lemma of C. S. Houh in [6].

If $2r^2+3r'^2-rr''=0$, the rotation surfaces are special minimal surfaces (R-surfaces) studied by Eisenhart [5].

It is possible to generalize theorem 3 to M^n in E^{n+p} as follows.

THEOREM 4. *A submanifold M^n in E^{n+p} is an A-submanifold if and if the second mean curvature has an extremal value in the direction of the mean curvature vector.*

Proof. Let $k_1, \cdots, k_p$ be the principal curvatures of M^n w.r.t. a normal unit vector e. The lth mean curvature of M^n w.r.t. e is defined by [2].

$$\binom{p}{l} M_l(e) = \sum k_1 k_2 \cdots k_l\,.$$

By the geometric properties of polar hypersurfaces [9] we see that the point $m+\rho e$ of $K_m^{(2)}$ are determined by the equation

$$\sum_{i \neq j} \left(\frac{1}{\rho} - k_i\right)\left(\frac{1}{\rho} - k_j\right) = 0\,.$$

Hence

$$\frac{1}{\rho_1 \rho_2} = \sum_{i \neq j} k_i k_j = \binom{p}{2} M_2(e)\,.$$

The theorem follows now at once.

§ 5. A_k-submanifolds.

Let A be the symmetric square matrix of order p associated with the $(p-2)$ th polar hypersurface $K_m^{(2)}$. Let E_{λ_i} be the eigenspace associated with each eigenvalue λ_i of A; we get a decomposition of $T_m^{\perp} M^n$ into an orthogonal direct sum $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_s}$. We denote by k the number of nonzero projections of H on the eigenspaces E_{λ_i}. A submanifold M^n of E^{n+p} is said to be an A_k-submanifold of E^{n+p} if k is the integer associated with the mean curvature vector H of M^n.

A_0-submanifolds are minimal submanifolds. A_1-submanifolds are the A-submanifolds. Each submanifold of E^{n+p} is an A_k-submanifold for some k with $0 \leqq k \leqq p$. Now we prove the following theorem on product submanifolds.

THEOREM 5. *If* M^{n_1}(resp. $M^{n_1'}$) *is an* A_{k_1}—(resp. *an* $A_{k_1'}$) *submanifold of* $E^{n_1+p_1}$(resp. $E^{n_1'+p_1'}$) *then the product* $M^{n_1} \times M^{n_1'}$ *is an* A_k-*submanifold of* $E^{n_1+p_1} \times E^{n_1'+p_1'}$ *with* $k \leqq p_1 + p_1'$.

Proof. Let A_1(resp. A_1') the symmetric square matrix of order p_1 associated with the polar hypersurface $K_m^{(2)}$ of K in $T_m^{\perp} M^{n_1}$. There exist k_1 eigenvectors of A_1 denoted by $\xi_1, \cdots, \xi_{k_1}$ such that

$$H_1 = \sum_{i=1}^{k_1} h_i \xi_i\,, \quad h_i \neq 0\,.$$

There exist an orthonormal set of p_1 vectors such that e_{n_1+1} is collinear with the mean curvature vector H_1 of M^{n_1}, $e_{n+r}(r=1, \cdots, k_1)$ belong to the vector space spanned by $\xi_1, \cdots, \xi_{k_1}$, and $e_{n+s}(s=k_1+1, \cdots, p_1)$ are eigenvectors of A_1. We can then write A_1 as

$$A_1 = \left(\begin{array}{c:c} \alpha_1 & 0 \\ \hdashline 0 & a_1 \end{array}\right)$$

where α_1 is a symmetric square matrix of order k_1 and a_1 is a diagonal matrix of order p_1-k_1. The same construction may be done for A'_1 and a short calculation shows that the matrix A associated with the polar hyperquadric of $M^{n_1}\times M^{n'_1}$ is

$$A=\begin{pmatrix} \alpha_1 & & \begin{matrix}\gamma\gamma' \\ 0 \\ \vdots\end{matrix} & \\ & a_1 & & \\ \gamma\gamma'\ 0 \cdots & & \alpha'_1 & \\ & & & a'_1 \end{pmatrix}$$

where $\gamma=\sum_{i=1}^{n}\gamma_{ii}^{n_1+1}$ and $\gamma'=\sum_{i=1}^{n'_1}\gamma_{ii}^{n'_1+1}$.

The mean curvature vector of $M^{n_1}\times M^{n'_1}$ is given by $H=\frac{1}{n_1+n'_1}(\gamma e_{n_1+1}+\gamma' e_{n'_1+1})$. In the general case $A(H)$ belongs to a linear space of dimension $k_1+k'_1$ orthogonal to the eigenvectors $\xi_{k_1+1}, \cdots, \xi_{p_1}, \xi_{k'_1+1}, \cdots, \xi_{p'_1}$ of A. Thus $M^{n_1}\times M^{n'_1}$ is an A_k-submanifold with $k\leqq k_1+k'_1$ and theorem 5 is proved.

Remark. If $k_1=0$ then $\gamma=0$ (M^{n_1} is minimal) and $k=k'_1$.
If $k_1=k'_1=1$ then α_1 and α'_1 are (1, 1) matrices and

$$A(H)=\frac{1}{n_1+n'_1}\{(\alpha_1\gamma+\gamma\gamma'^2)e_{n_1+1}+(\alpha'_1\gamma'+\gamma'\gamma^2)e_{n'_1+1}\}$$

we have $A(H)=\mu H$ if and only if

$$\alpha_1+\gamma'^2=\gamma^2+\alpha'_1\,.$$

But by (4) $\gamma'^2-\alpha'_1=\Big(\sum_{i=1}^{n_1}\gamma_{ii}^{n_1+1}\Big)^2-\sum_{\lambda\neq\mu}\{\gamma_{\lambda\lambda}^{n_1+1}\gamma_{\mu\mu}^{n_1+1}-(\gamma_{\lambda\mu}^{n_1+1})^2\}$

$$=\sum_{\lambda\neq\mu}^{n_1}(\gamma_{ii}^{n_1+1})+\sum_{\lambda\neq\mu}(\gamma_{\lambda\mu}^{n_1+1})^2=\mathrm{Trace}\,(A(e_{n_1+1}))^2\,.$$

So

The product of two A-submanifolds is an A-submanifold if and only if $\mathrm{Trace}\,(A(e_{n_1+1}))^2=\mathrm{Trace}\,(A(e_{n'_1+1}))^2$ [B. Y. Chen [2]]. *In the other cases the product of two A-submanifolds is an* A_2*-submanifold.*

This gives a method to construct examples of A_2-submanifolds. Most of these results can be extended to Riemannian spaces.

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