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A SUFFICIENT CONDITION THAT AN OPERATOR ALGEBRA BE SELF-ADJOINT

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1. Introduction. It is well-known, and easily verified, that each of the following assertions implies the preceding ones.

(i) Every operator has a non-trivial invariant subspace.
(ii) Every commutative operator algebra has a non-trivial invariant subspace.
(iii) Every operator other than a multiple of the identity has a non-trivial hyperinvariant subspace.
(iv) The only transitive operator algebra on $\mathscr{H}$ is $\mathfrak{B}(\mathscr{H})$.

Note. Operator means bounded linear operator on a complex Hilbert space $\mathscr{H}$, *operator algebra* means weakly closed algebra of operators containing the identity, *subspace* means closed linear manifold, a *non-trivial* subspace is a subspace other than $\{0\}$ and $\mathscr{H}$, a *hyperinvariant* subspace for A is a subspace invariant under every operator which commutes with A, a *transitive* operator algebra is one without any non-trivial invariant subspaces and $\mathfrak{B}(\mathscr{H})$ denotes the algebra of all operators on $\mathscr{H}$.

All of these statements are true if $\mathscr{H}$ is finite-dimensional; (iv) is Burnside's Theorem [**12**, p. 276] in this case. Statement (i) is trivially true in the case where $\mathscr{H}$ is non-separable. It is not hard to show that the truth of (iv) in the non-separable case would imply its truth in the separable case. (Use the fact that whenever $\mathfrak{A}$ is a transitive operator algebra, so is $\mathfrak{A} \otimes \mathfrak{B}(\mathscr{K})$ for any Hilbert space $\mathscr{K}$; cf. the remarks in the second paragraph of [**16**].) Each statement is known to be true under various additional hypotheses; e.g., for (i) see [**1**; **3**; **10**; **14**; **25**], for (ii) and (iii) see [**7**; **11**; **19**; **21**; **25**]. Arveson [**2**] showed that (iv) is true with the additional hypothesis that the algebra contains a maximal abelian von Neumann algebra (another proof is in [**18**]) or the unilateral shift, and (iv) has subsequently been verified in certain other cases (cf. [**16**; **17**]).[1]

In general it seems likely that (i) is false and hence that the rest are too. In particular, then, a counter-example to (iv) is to be expected. However no counter-example has yet been found in spite of a great deal of interest in this problem during the past five years or so.

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[1] R. G. Douglas and Carl Pearcy have recently shown [8] that (iv) is true with the additional hypothesis that the algebra contains a Hermitian operator which is not of uniform multiplicity $\aleph_0$.

In this paper we introduce another assertion in this chain.

Definition. An operator algebra $\mathfrak{A}$ is *Hermitian* if it (is weakly closed, contains the identity and) has the property that every invariant subspace is reducing (i.e., whenever $\mathscr{M}$ is invariant under $\mathfrak{A}$, then $\mathscr{M}^{\perp}$ is invariant under $\mathfrak{A}$).

Clearly every self-adjoint algebra (i.e., von Neumann algebra) is Hermitian. We consider the converse.

(v) Every Hermitian operator algebra is self-adjoint.

It is easily seen that (v) implies (iv). For if $\mathfrak{A}$ is a transitive operator algebra then $\mathfrak{A}$ is Hermitian; if we assume (v), then $\mathfrak{A}$ is a von Neumann algebra with trivial commutant and hence $\mathfrak{A} = \mathfrak{B}(\mathscr{H})$. Thus, in view of the above remarks, (v) is probably false in general. We feel that a counter-example to (v) would be a good beginning to a counter-example to each of the preceding assertions and thus would be very useful. We have been unable to find such an example, however. In fact, we generalize Arveson's Theorem to show that (v) is true with the additional hypothesis that the algebra contains a maximal abelian von Neumann algebra. We also show that (v) is true in certain other cases, including the finite-dimensional case. Special cases of (v) such as these may prove useful even if (v) turns out to be false in general.

One special case of (v) is already known: Sarason [**24**] has shown that an Hermitian operator algebra consisting of normal operators is self-adjoint; see also [**18**, Remark (iii)].

2. Some basic results. If A is an operator and $\mathscr{M}$ is an invariant subspace of A, then $A|\mathscr{M}$ denotes the restriction of A to $\mathscr{M}$.

LEMMA 1. *If $\mathfrak{A}$ is an Hermitian operator algebra and if P is a projection which commutes with $\mathfrak{A}$, then every invariant subspace of $\mathfrak{A}|P\mathscr{H}$ is reducing (in $P\mathscr{H}$).*

Proof. Let $\mathscr{M}$ be an invariant subspace of $\mathfrak{A}|P\mathscr{H}$. Then $\mathscr{M}$ is also invariant under $\mathfrak{A}$ and thus $\mathscr{M}^{\perp}$ is invariant under $\mathfrak{A}$. Therefore $P\mathscr{M}^{\perp}$ is invariant under $\mathfrak{A}|P\mathscr{H}$ and $P\mathscr{H} \ominus \mathscr{M} = P\mathscr{M}^{\perp}$.

Note that Lemma 1 does not imply that the restriction of an Hermitian operator algebra to a reducing subspace is Hermitian. For this it needs also to be established that $\mathfrak{A}|P\mathscr{H}$ is weakly closed. For von Neumann algebras this is the case [**6**, p. 16], but we have been unable to prove it for Hermitian operator algebras.

THEOREM 1. *If $\mathfrak{A}$ is an Hermitian operator algebra on $\mathscr{H}$ and if there exists a collection $\{\mathscr{M}_i\}_{i=1}^{n}$ of pairwise-orthogonal reducing subspaces for $\mathfrak{A}$ such that $\mathscr{H} = \sum_{i=1}^{n} \oplus \mathscr{M}_i$ and $\mathfrak{A}|\mathscr{M}_i = \mathfrak{B}(\mathscr{M}_i)$ for all i, then $\mathfrak{A}$ is self-adjoint.*

Proof. The proof is by induction on n. The theorem is true for $n = 1$ since $\mathfrak{B}(\mathscr{H})$ is self-adjoint.

Assume that the theorem is known for $n - 1$. We distinguish two cases.

Case (a). There exist positive integers j and k and an algebra isomorphism ϕ of $\mathfrak{B}(\mathcal{M}_j)$ onto $\mathfrak{B}(\mathcal{M}_k)$ such that if $A \in \mathfrak{A}$ then $A|\mathcal{M}_k = \phi(A|\mathcal{M}_j)$.

Since ϕ is an algebra isomorphism, there exists a one-to-one bounded linear operator S mapping $\mathcal{M}_j$ onto $\mathcal{M}_k$ such that $\phi(B) = SBS^{-1}$ (cf. [**20**, Theorem 2.5.19]).

Now let $\mathcal{M}$ be the subspace of $\mathcal{H}$ consisting of all vectors of the form $\sum_{i=1}^{n} \oplus x_i$ such that $x_i \in \mathcal{M}_i$ and $x_k = Sx_j$. A trivial computation shows that $\mathcal{M}$ is invariant under $\mathfrak{A}$ and therefore, since $\mathfrak{A}$ is Hermitian, $\mathcal{M}^\perp$ is also invariant under $\mathfrak{A}$. Another computation shows that $\mathcal{M}^\perp$ consists of all vectors of the form $\sum_{i=1}^{n} \oplus x_i$ such that $x_i = 0$ for $i \neq j, k$ and $-S^*x_k = x_j$. The invariance of $\mathcal{M}^\perp$ under $\mathfrak{A}$ then gives $-S^*SBS^{-1} = -BS^*$ for all $B \in \mathfrak{B}(\mathcal{M}_j)$. Thus $S^*SB = BS^*S$ for all $B \in \mathfrak{B}(\mathcal{M}_j)$. Hence $S^*S = \lambda I$ for some positive number λ. Let $U = S/\sqrt{\lambda}$. Then U is unitary and $\phi(B) = UBU^{-1}$ for all $B \in \mathfrak{B}(\mathcal{M}_j)$.

Now consider $\mathfrak{A}|\mathcal{M}_k^\perp$. This algebra is weakly closed, for if

$$A_{1,\alpha} \oplus \ldots \oplus A_{k-1,\alpha} \oplus A_{k+1,\alpha} \oplus \ldots \oplus A_{n,\alpha}$$

is a net in $\mathfrak{A}|\mathcal{M}_k^\perp$ converging weakly to

$$A_1 \oplus \ldots \oplus A_{k-1} \oplus A_{k+1} \oplus \ldots \oplus A_n,$$

then the net

$$A_{1,\alpha} \oplus \ldots \oplus A_{k-1,\alpha} \oplus UA_{j,\alpha}U^{-1} \oplus A_{k+1,\alpha} \oplus \ldots \oplus A_{n,\alpha}$$

is in $\mathfrak{A}$ and converges weakly to

$$A_1 \oplus \ldots \oplus A_{k-1} \oplus UA_jU^{-1} \oplus A_{k+1} \oplus \ldots \oplus A_n.$$

Hence

$$A_1 \oplus \ldots \oplus A_{k-1} \oplus A_{k+1} \oplus \ldots \oplus A_n$$

is in $\mathfrak{A}|\mathcal{M}_k^\perp$.

Also, Lemma 1 implies that every invariant subspace of $\mathfrak{A}|\mathcal{M}_k^\perp$ is reducing. Hence $\mathfrak{A}|\mathcal{M}_k^\perp$ is Hermitian and by the inductive hypothesis $\mathfrak{A}|\mathcal{M}_k^\perp$ is self-adjoint. Therefore, since $\mathfrak{A}$ is the set of all operators of the form

$$A_1 \oplus \ldots \oplus A_{k-1} \oplus UA_jU^{-1} \oplus A_{k+1} \oplus \ldots \oplus A_n$$

such that

$$A_1 \oplus \ldots \oplus A_{k-1} \oplus A_{k+1} \otimes \ldots \oplus A_n$$

is in $\mathfrak{A}|\mathcal{M}_k^\perp$, $\mathfrak{A}$ is also self-adjoint.

Case (b). There exists no such isomorphism. Then for every pair (i, j) of positive integers less than or equal to n there exists an $A \in \mathfrak{A}$ such that exactly one of the two operators $A|\mathcal{M}_i$ and $A|\mathcal{M}_j$ is zero (since it follows from the fact that $\mathfrak{A}$ is an algebra that every map ϕ from $\mathfrak{B}(\mathcal{M}_i)$ to $\mathfrak{B}(\mathcal{M}_j)$ defined by $\phi(A|\mathcal{M}_i) = A|\mathcal{M}_j$ for $A \in \mathfrak{A}$ is an algebra homomorphism).

We first show that there exists an $A \in \mathfrak{A}$ such that $A|\mathcal{M}_i$ is non-zero for exactly one integer i. To do this choose a non-zero $A \in \mathfrak{A}$ for which the number of integers i such that $A|\mathcal{M}_i = 0$ is maximal. We must show that this maximal number is $n - 1$. By permuting the indices if necessary we can assume that A has the form

$$A_1 \oplus \ldots \oplus A_m \oplus 0 \oplus \ldots \oplus 0,$$

with $A_i \neq 0$ for $i = 1, \ldots, m$.

If $m > 1$, then, by interchanging indices if necessary, we can assume that $\mathfrak{A}$ contains an operator B such that $B|\mathcal{M}_1 \neq 0$ and $B|\mathcal{M}_m = 0$. Then the set $\mathscr{I}$ of all $\hat{C} \in \mathfrak{B}(\mathcal{M}_1)$ such that $\hat{C} = C|\mathcal{M}_1$ for some $C \in \mathfrak{A}$ with $C|\mathcal{M}_m = 0$ is a two-sided ideal in $\mathfrak{B}(\mathcal{M}_1)$ different from $\{0\}$. Hence, as is well-known (cf. [**15**, p. 292]), $\mathscr{I}$ contains all finite-rank operators. Now let $\mathscr{J}$ denote the set of all operators $\hat{C} \in \mathfrak{B}(\mathcal{M}_1)$ such that $\hat{C} = C|\mathcal{M}_1$ for some $C \in \mathfrak{A}$ with $C|\mathcal{M}_i = 0$ if $i > m$. Then $\mathscr{J}$ is also a non-zero two-sided ideal in $\mathfrak{B}(\mathcal{M}_1)$ and, therefore, $\mathscr{J}$ also contains all finite-rank operators. Now let P be any non-zero finite-rank projection in $\mathfrak{B}(\mathcal{M}_1)$. Then $P \in \mathscr{I} \cap \mathscr{J}$ and, therefore, there exist operators in $\mathfrak{A}$ of the form

$$P \oplus A_2 \oplus \ldots \oplus A_m \oplus 0 \oplus \ldots \oplus 0$$

and

$$P \oplus B_2 \oplus \ldots \oplus 0 \oplus B_{m+1} \oplus \ldots \oplus B_n.$$

Thus the product of these two operators

$$P \oplus A_2B_2 \oplus \ldots \oplus A_{m-1}B_{m-1} \oplus 0 \oplus \ldots \oplus 0$$

is in $\mathfrak{A}$, contradicting the maximality of m.

Therefore, $m = 1$ and $\mathscr{J}$ is the ideal consisting of all operators $\hat{C} \in \mathfrak{B}(\mathcal{M}_1)$ such that $C|\mathcal{M}_1 = \hat{C}$ for some $C \in \mathfrak{A}$ with $C|\mathcal{M}_i = 0$ for $i > 1$. Now it is clear that $\mathscr{J}$ is weakly closed and hence $\mathscr{J} = \mathfrak{B}(\mathcal{M}_1)$. Therefore, if $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ is in $\mathfrak{A}$ for some choice of $A_i \in \mathfrak{B}(\mathcal{M}_i)$, then $0 \oplus A_2 \oplus \ldots \oplus A_n$ is also in $\mathfrak{A}$. It follows that $\mathfrak{A}|\mathcal{M}_1^\perp$ is weakly closed and thus, by Lemma 1, is Hermitian.

By the inductive hypothesis, then, $\mathfrak{A}|\mathcal{M}_1^\perp$ is self-adjoint, and the relation between $\mathfrak{A}$ and $\mathfrak{A}|\mathcal{M}_1^\perp$ implies that $\mathfrak{A}$ is self-adjoint.

In particular, Theorem 1 applies in the case where every $\mathcal{M}_i$ is finite-dimensional (the proof can be somewhat simplified in this case); this leads to a proof that (v) is true in the finite-dimensional case.

THEOREM 2. *Every Hermitian algebra of operators on a finite-dimensional space is self-adjoint.*

Proof[2]. Let $\mathfrak{A}$ be an Hermitian operator algebra on a finite-dimensional space $\mathscr{H}$. If $\mathfrak{A}$ has no invariant subspaces, then the result follows by Burnside's Theorem. If $\mathfrak{A}$ has an invariant subspace, then it has a minimal one, say $\mathcal{M}_1$, since $\mathscr{H}$ is finite-dimensional. By Lemma 1, $\mathfrak{A}|\mathcal{M}_1^\perp$ is Hermitian. Then $\mathfrak{A}|\mathcal{M}_1^\perp$

[2]As Dr C. L. Olsen has kindly informed us, Theorem 2 follows immediately from known results (cf. [**4**, p. 127]).

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has a minimal invariant subspace, say $\mathscr{M}_2$. Proceeding in this manner we obtain a decomposition $\sum_{i=1}^n \oplus \mathscr{M}_i$ of $\mathscr{H}$ such that $\mathfrak{A}|\mathscr{M}_i$ is transitive for each i. By Burnside's Theorem $\mathfrak{A}|\mathscr{M}_i = \mathfrak{B}(\mathscr{M}_i)$ for each i; hence Theorem 1 gives the result.

The next theorem shows that statement (v) comes close to reducing to statement (iv) for algebras containing compact operators; the proof is very similar to the proof of [**22**, Theorem 2].

THEOREM 3. *If $\mathfrak{A}$ is an Hermitian operator algebra and if $\mathfrak{A}$ contains a compact operator whose nullspace is finite-dimensional, then there exists a countable collection $\{\mathscr{M}_i\}$ of pairwise-orthogonal reducing subspaces for $\mathfrak{A}$ such that $\mathscr{H} = \sum \oplus \mathscr{M}_i$ and $\mathfrak{A}|\mathscr{M}_i$ is transitive for each i.*

Proof. Note that here $\mathscr{H}$ must be separable; on a non-separable space every compact operator has infinite-dimensional nullspace. Let A denote the compact operator in $\mathfrak{A}$.

We first prove a preliminary result: given any such $\mathfrak{A}$ on an infinite-dimensional space there exists a reducing subspace $\mathscr{M} \neq \{0\}$ such that $\mathfrak{A}|\mathscr{M}$ is transitive. For this we follow the proof of [**22**, Theorem 2]. Let $\mathscr{F}$ denote the family of subspaces $\mathscr{N}$ that reduce $\mathfrak{A}$ and have the property that the norm of $A|\mathscr{N}$ is equal to the norm of A. Choose a maximal chain $\{\mathscr{N}_\alpha\}$ in $\mathscr{F}$ and let $\mathscr{M} = \bigcap \mathscr{N}_\alpha$. We will be finished if we show that $\mathfrak{A}|\mathscr{M}$ is transitive. By [**22**, Lemma, p. 827] there exists a countable subfamily $\{\mathscr{N}_{\alpha_i}\}$ of $\{\mathscr{N}_\alpha\}$ such that $\mathscr{N}_{\alpha_{i+1}} \subseteq \mathscr{N}_{\alpha_i}$ for each i and $\mathscr{M} = \bigcap_{i=1}^\infty \mathscr{N}_{\alpha_i}$. Since a compact operator attains its norm, for each i there is a vector $x_i \in \mathscr{N}_{\alpha_i}$ such that $||x_i|| = 1$ and $||Ax_i|| = ||A||$. Choose a subsequence of $\{x_i\}$ that converges weakly to some x. Then $||Ax|| = ||A||$ and also $x \in \mathscr{M}$ since $\{x_i\}$ is eventually in each $\mathscr{N}_{\alpha_i}$. Therefore, $\mathscr{M} \in \mathscr{F}$. If $\mathfrak{A}|\mathscr{M}$ were not transitive, then, by Lemma 1, $\mathfrak{A}|\mathscr{M}$ would have a non-trivial reducing subspace, say $\mathscr{L}$. But then at least one of the two subspaces $\mathscr{L}$ and $\mathscr{M} \cap \mathscr{L}^\perp$ would be in $\mathscr{F}$, which is impossible since $\mathscr{M}$ is a minimal subspace in $\mathscr{F}$. This establishes the preliminary result.

Zorn's Lemma implies that there exists a maximal family $\{\mathscr{M}_i\}$ of pairwise-orthogonal reducing subspaces for $\mathfrak{A}$ such that no $\mathscr{M}_i$ is $\{0\}$ and such that $\mathfrak{A}|\mathscr{M}_i$ is transitive for each i. Since $\mathscr{H}$ is separable, the family $\{\mathscr{M}_i\}$ is countable. Let $\mathscr{K}$ denote the orthogonal complement of the span of $\{\mathscr{M}_i\}$. We must show that $\mathscr{K} = \{0\}$. If $\mathscr{K} \neq \{0\}$ then $\mathscr{K}$ reduces $\mathfrak{A}$ and, by Lemma 1, every invariant subspace of $\mathfrak{A}|\mathscr{K}$ is reducing. If $\mathscr{K}$ were finite-dimensional, then it would follow as in the proof of Theorem 2 that $\mathscr{K}$ contains a reducing subspace $\mathscr{L}$ such that $\mathfrak{A}|\mathscr{L}$ is transitive; this contradicts the maximality of $\{\mathscr{M}_i\}$. If $\mathscr{K}$ were infinite-dimensional, then by the preliminary result established above, $\mathscr{K}$ would contain a reducing subspace $\mathscr{L}$ such that $\mathfrak{A}|\mathscr{L}$ was transitive (note that the preliminary result does not require the algebra to be closed). This would also contradict the maximality of $\{\mathscr{M}_i\}$. Hence $\mathscr{K} = \{0\}$ and the theorem is proven.

3. The main result. We shall now prove that an Hermitian operator algebra which contains a maximal abelian self-adjoint algebra is self-adjoint. For this we need some of the notation from [**18**]. If $\mathcal{S}$ is a collection of operators, let Lat $\mathcal{S}$ denote the set of all subspaces invariant under every operator in $\mathcal{S}$. The usual direct sum of n copies of the Hilbert space $\mathcal{H}$ is denoted by $\mathcal{H}^{(n)}$; vectors in $\mathcal{H}^{(n)}$ are written $(x_1, \ldots, x_n)$. If A is an operator on $\mathcal{H}$, the operator $\sum_{i=1}^{n} \oplus A_i$ on $\mathcal{H}^{(n)}$ with $A_i = A$ for all i will be denoted by $A^{(n)}$ and if $\mathfrak{A}$ is an algebra of operators, $\mathfrak{A}^{(n)}$ will denote the algebra $\{A^{(n)}: A \in \mathfrak{A}\}$.

The (closed) span of a family $\{\mathcal{M}_\alpha\}$ of subspaces will be denoted by $\vee_\alpha \mathcal{M}_\alpha$. We shall use the abbreviation "m.a.s.a" for "maximal abelian self-adjoint algebra".

The following lemma is a special case of a well-known lemma (cf. [**18**, Lemma 1]).

LEMMA 2. *An operator algebra $\mathfrak{A}$ is self-adjoint if and only if $\mathfrak{A}^{(n)}$ is Hermitian for every positive integer n.*

Proof. Suppose that $\mathfrak{A}^{(n)}$ is Hermitian for all n. Let $A \in \mathfrak{A}$. If

$$U = \{B: \|Bx_i - A^*x_i\| < \epsilon, i = 1, \ldots, n\}$$

is any basic strong neighbourhood of A^*, then the invariant subspace of $\mathfrak{A}^{(n)}$ generated by $(x_1, \ldots, x_n)$ is invariant under $A^{*(n)}$ by hypothesis. Hence there exists some $B \in \mathfrak{A}$ such that $\|B^{(n)}(x_1, \ldots, x_n) - A^{*(n)}(x_1, \ldots, x_n)\| < \epsilon$. Thus $B \in U$. Since $\mathfrak{A}$ is weakly closed it follows that $A^* \in \mathfrak{A}$.

The converse, of course, is trivial.

LEMMA 3. *If $\mathcal{N}$ is a subspace and B an operator and if there exists a net $\{P_\alpha\}$ of projections converging to the identity such that $P_\alpha \mathcal{N} \subseteq \mathcal{N}$ and $P_\alpha \mathcal{N} \in$ Lat $P_\alpha B P_\alpha$ for all α, then $\mathcal{N} \in$ Lat B.*

Proof. Fix $x \in \mathcal{N}$ and $y \in \mathcal{N}^\perp$. We must show that $(Bx, y) = 0$. Note that $P_\alpha y \perp P_\alpha \mathcal{N}$ and thus

$$0 = (P_\alpha B P_\alpha x, P_\alpha y) = (BP_\alpha x, P_\alpha y).$$

Now

$$\begin{aligned} |(Bx, y) - (BP_\alpha x, P_\alpha y)| &= |(Bx, y) - (Bx, P_\alpha y) + (Bx, P_\alpha y) - (BP_\alpha x, P_\alpha y)| \\ &\leqq |(Bx, y - P_\alpha y)| + |(x - P_\alpha x, B^* P_\alpha y)| \\ &\leqq \|Bx\| \, \|y - P_\alpha y\| + \|x - P_\alpha x\| \, \|B^*\| \, \|y\|. \end{aligned}$$

Therefore, $(Bx, y) = 0$.

LEMMA 4. *Let $\mathfrak{R}$ be a* m.a.s.a. *on $\mathcal{H}$ and let $\mathcal{M} \in$ Lat $\mathfrak{R}^{(n)}$ such that $(x_1, \ldots, x_n) \in \mathcal{M}$ and $x_1 = 0$ imply that $x_2 = \ldots = x_n = 0$. Let $\mathcal{D}$ be the linear manifold consisting of all the first co-ordinates of vectors in $\mathcal{M}$. Then there exist (possibly unbounded) linear transformations T_i, $i = 1, \ldots, n - 1$, all defined on $\mathcal{D}$ and each commuting with $\mathfrak{R}$, such that*

$$\mathcal{M} = \{(x, T_1 x, \ldots, T_{n-1} x): x \in \mathcal{D}\}.$$

(Note: "commuting" here means that $\mathscr{D}$ is an invariant linear manifold of $\Re$ and $T_iRx = RT_ix$ for all $R \in \Re$ and $x \in \mathscr{D}$.)

Proof. The hypotheses imply that each x_i is uniquely and linearly determined by x_1 so $\mathscr{M}$ has the form exhibited above for some linear transformations T_i defined on $\mathscr{D}$. It follows from the invariance of $\mathscr{M}$ under $\Re^{(n)}$ that each T_i commutes with $\Re$.

LEMMA 5. *If* $\mathscr{M} = \{(x, T_1x, \ldots, T_{n-1}x): x \in \mathscr{D}\}$ *is in* Lat $\Re^{(n)}$ *as in Lemma* 4 *and if* $\mathscr{U}$ *is any strong neighbourhood of the identity operator on* $\mathscr{H}$, *then there exists a projection* $P \in \mathscr{U} \cap \Re$ *such that* $P\mathscr{H} \subseteq \mathscr{D}$, $P\mathscr{H}$ *is invariant under* T_i and $T_i|P\mathscr{H}$ *is a bounded normal operator for each* i.

Proof. Let $\mathscr{U} = \{A: \|(A - I)y_j\| < \epsilon, j = 1, \ldots, k\}$ be any basic strong neighbourhood of I in $\mathfrak{B}(\mathscr{H})$ and let Q be the projection of $\mathscr{H}$ onto the subspace $\vee_{j=1}^{k} [\Re y_j]$. Then $Q \in \Re$ and the m.a.s.a. $Q\Re Q$ is countably decomposable. This allows us to identify $Q\mathscr{H}$ with $\mathscr{L}^2(X, \mu)$ and $Q\Re Q$ with multiplications by $\mathscr{L}^\infty(X, \mu)$ functions, where μ is a totally finite measure [**6**]. Thus each T_i is multiplication by an everywhere defined measurable function ϕ_i [**2**, Lemma 3.2]. For each positive integer k let

$$\mathscr{S}_k = \{s \in X: |\phi_i(s)| \leqq k, i = 1, \ldots, n - 1\}$$

and let P_k be the projection on $Q\mathscr{H}$ defined by multiplication by the characteristic function of $\mathscr{S}_k$. Since $\bigcup_{k=1}^{\infty} \mathscr{S}_k = X$, we have $\{\mu(X - \mathscr{S}_k)\} \to 0$. Hence $P_k \in \mathscr{U}$ for sufficiently large k. Let $P = P_k$ for such a k.

We must show that $P\mathscr{H} = PQ\mathscr{H} \subseteq \mathscr{D}$. The invariance of $\mathscr{M}$ under $P^{(n)}$ implies that the linear manifold

$$\{(y, T_1y, \ldots, T_{n-1}y): y \in P\mathscr{D}\}$$

is contained in $\mathscr{M}$ and thus so is its closure. But its closure is exactly

$$\{(y, T_1y, \ldots, T_{n-1}y): y \in P\mathscr{H}\},$$

because $\overline{P\mathscr{D}} = P\mathscr{H}$ and the T_i are bounded when restricted to $P\mathscr{D}$. Thus $\mathscr{D}$ contains $P\mathscr{H}$.

Now, obviously, $T_i|P\mathscr{H}$ is a bounded normal operator for each i.

THEOREM 4. *An Hermitian operator algebra containing a* m.a.s.a. *is self-adjoint.*

Proof. Let $\mathfrak{A}$ be an Hermitian operator algebra containing a m.a.s.a. $\Re$. To show that $\mathfrak{A}$ is self-adjoint, it suffices, by Lemma 2, to show that

$$\text{Lat } \mathfrak{A}^{(n)} \subseteq \text{Lat } A^{*(n)}$$

for all $A \in \mathfrak{A}$ and all n.

Suppose that the desired inclusion holds for all $k < n$ and consider a member $\mathscr{M}$ of Lat $\mathfrak{A}^{(n)}$. Let $\mathscr{M}_1$ be the set of all those elements $(x_1, \ldots, x_n)$ of $\mathscr{M}$ for which $x_1 = 0$. Then $\mathscr{M}_1 \in$ Lat $\mathfrak{A}^{(n)}$. The special form of $\mathscr{M}_1$ makes it evident that $\mathscr{M}_1$ reduces $\mathfrak{A}^{(n)}$ by the induction hypothesis. Let $A \in \mathfrak{A}$; we need only show that $\mathscr{M} \ominus \mathscr{M}_1 \in \text{Lat}\, A^{*(n)}$. Let $\mathscr{N} = \mathscr{M} \ominus \mathscr{M}_1$. Then $\mathscr{N} \in \text{Lat}\, \mathfrak{A}^{(n)}$ and $\mathscr{N}$ satisfies the hypothesis of Lemma 4. Given any strong neighbourhood $\mathscr{U}$ of I in $\mathfrak{B}(\mathscr{H})$ choose a $P \in \mathscr{U} \cap \mathfrak{R}$ as in Lemma 5. Then $P^{(n)}\mathscr{H}^{(n)} \in \text{Lat}\, P^{(n)}\mathfrak{A}^{(n)}P^{(n)}$, for $P^{(n)}$ commutes with the projection onto $\mathscr{N}$. Therefore, PT_iP commutes with PAP for each i. Fuglede's Theorem [**9**, p. 68] implies that

$$PT_iPPA^*P = PA^*PPT_iP.$$

Hence $P^{(n)}\mathscr{N} \in \text{Lat}\, P^{(n)}A^{*(n)}P^{(n)}$ for each such P.

Now we can easily show that $\mathscr{N} \in \text{Lat}\, A^{*(n)}$. For it follows from the fact that each P as above is in $\mathfrak{A}$ that $P^{(n)}\mathscr{N} \subseteq \mathscr{N}$. Thus, by the above, there exists a net $P_\alpha^{(n)}$ converging to the identity on $\mathscr{H}^{(n)}$ and satisfying the hypotheses of Lemma 3. Hence $\mathscr{N} \in \text{Lat}\, A^*$.

COROLLARY 1. (Arveson's Theorem) *A transitive operator algebra which contains a* m.a.s.a. *is* $\mathfrak{B}(\mathscr{H})$.

Proof. By Theorem 4 such an algebra is self-adjoint and the only transitive von Neumann algebra is $\mathfrak{B}(\mathscr{H})$.

The next result is about triangular operator algebras (see [**13**] for definitions and basic theory).

COROLLARY 2. *If* $\mathfrak{A}$ *is a triangular operator algebra and if* $\mathfrak{A}$ *is also Hermitian, then* $\mathfrak{A}$ *is a* m.a.s.a.

Proof. Since $\mathfrak{A}$ is triangular, $\mathfrak{A} \cap \mathfrak{A}^*$ is a m.a.s.a. But $\mathfrak{A}$ is self-adjoint, by Theorem 4, and thus $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{A}^*$.

4. Some conjectures. Other generalizations of Arveson's Theorem in addition to Theorem 4 are known (cf. [**5**; **18**]). All of these results are special cases of the following conjecture.

Conjecture 1. If $\mathfrak{A}$ is an operator algebra containing a m.a.s.a., then $\mathfrak{A}$ is reflexive. (See [**18**] for definitions and basic results about reflexive algebras.)

We have been unable to prove Conjecture 1, although it seems very unlikely that it could be false given the fact that the special cases mentioned above have been proven.

Conjecture 2. A weakly closed triangular operator algebra is hyperreducible.

Conjecture 2 is trivially true if the algebra is Hermitian, by Corollary 2. It is shown in [**23**] that Conjecture 2 is true if the algebra is a maximal triangular algebra. It is also observed that Conjecture 1 implies Conjecture 2; on the other hand it is very conceivable that Conjecture 2 could be proven without Conjecture 1 being decided.

Conjecture 3. If $\mathfrak{A}$ and $\mathfrak{D}$ are m.a.s.a.'s such that no projection in $\mathfrak{A}$ other than

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0 is a subprojection of a projection in $\mathfrak{D}$ other than I, then the weak closure of the linear space of all finite sums of the form $\sum A_iB_i$, $A_i \in \mathfrak{A}$, $B_i \in \mathfrak{D}$, is the set of all operators.

Conjecture 3 is also implied by Conjecture 1; one needs only to consider the weak closure of the set of all operators of the form

$$\begin{pmatrix} A & \sum A_iB_i \\ 0 & B \end{pmatrix}$$

with the A's in $\mathfrak{A}$ and the B's in $\mathfrak{D}$.

Conjecture 4. If $\mathfrak{A}$ is Hermitian and if P is the projection onto a reducing subspace of $\mathfrak{A}$, then $\mathfrak{A}|P\mathscr{H}$ is Hermitian.

The problem in Conjecture 4 is proving that $\mathfrak{A}|P\mathscr{H}$ is weakly closed; see the comments after Lemma 1 above.

Conjecture 5. If $\mathfrak{A}$ is an Hermitian operator algebra containing a finite-multiplicity unilateral shift, then $\mathfrak{A}$ is self-adjoint.

Conjecture 5 would follow from Conjecture 4, Theorem 1, and Nordgren's result [**16**] in the transitive case. It seems likely, however, that Conjecture 5 can be established without Conjecture 4, by strengthening Nordgren's techniques.

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