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A SUM-DIVISION ESTIMATE OF REALS

LIANGPAN LI AND JIAN SHEN

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ABSTRACT. Let A be a finite set of positive real numbers. We present a sum-division estimate:

$$|A+A|^2|A/A| \geq \frac{|A|^4}{4}.$$

1. Introduction

Let A be a finite set of positive real numbers throughout. The sum-set, product-set and ratio-set of A are defined respectively to be

$$A+A = \{a+b : a, b \in A\},$$
$$AA = \{ab : a, b \in A\},$$
$$A/A = \{a/b : a, b \in A\}.$$

A famous conjecture of Erdös and Szemerédi [6] asserts that for any $\alpha < 2$, there exists a constant $C_\alpha > 0$ such that

$$\max\{|A+A|, |AA|\} \geq C_\alpha |A|^\alpha.$$

In a series of papers [1, 2, 7, 11, 12, 13], upper bounds on α were found by many authors. One highlight in this direction was a proof by Elekes [2], that α can be taken to be $\frac{5}{4}$. His argument utilized a clever application of the Szemerédi-Trotter theorem on point-line incidences. Recently, using the concept of multiplicative energy and an ingenious geometric observation, Solymosi [14] obtained that if A is not a singleton, then

$$|A+A|^2|AA| \geq \frac{|A|^4}{4\lceil \log_2 |A| \rceil}, \tag{1}$$

which yields

$$\max\{|A+A|, |AA|\} \geq \frac{|A|^{4/3}}{2\lceil \log_2 |A| \rceil^{1/3}}. \tag{2}$$

One cannot completely drop the logarithmic term in (2), since if we choose $\widetilde{A} = \{1, 2, \ldots, n\}$, then [4, 5, 8, 15]

$$|\widetilde{A}\widetilde{A}| = \frac{n^2}{(\ln n)^{\beta+o(1)}}, \quad \beta = 1 - \frac{1+\ln\ln 2}{\ln 2} = 0.0860713.... \tag{3}$$

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There is a subtle difference between $|\widetilde{A}\widetilde{A}|$ and $|\widetilde{A}/\widetilde{A}|$. In fact, Elekes and Ruzsa [3] showed that there exists a universal constant $\gamma > 0$ such that

$$|A+A|^6|A/A| \geq \gamma|A|^8, \tag{4}$$

which yields

$$|\widetilde{A}/\widetilde{A}| \geq \frac{\gamma}{64}|\widetilde{A}|^2$$

by choosing $A = \widetilde{A}$. This leads to a natural question: how to give a joint estimate on $|A+A|$ and $|A/A|$? It is not difficult to use the Szemerédi-Trotter theorem on point-line incidences to show that

$$|A+A||A/A| \geq C|A|^{5/2} \tag{5}$$

holds for some universal constant $C > 0$. Besides, if we carefully analyze Solymosi's proof of (1), then

$$|A+A|^2|A/A| \geq \frac{|A|^4}{4\lceil \log_2 |A| \rceil}. \tag{6}$$

The main purpose of this paper is to drop the term $\lceil \log_2 |A| \rceil$ in (6).

Theorem 1. *Let A be a finite set of positive real numbers. Then*

$$|A+A|^2|A/A| \geq \frac{|A|^4}{4}.$$

This implies a sum-division estimate

$$\max\left\{|A+A|, |A/A|\right\} \geq \frac{|A|^{4/3}}{2}.$$

There is an explanation of Theorem 1 in plane geometry. View $\mathbb{R}^2$ naturally as the complex plane $\mathbb{C}$. Given a finite set A of positive real numbers, denote by $\mathrm{Rad}(A\times A)$ and $\mathrm{Ang}(A\times A)$ respectively the radius-set and the angle-set of $A \times A$. Applying Theorem 1 with $\widehat{A} = \{a^2 : a \in A\}$ yields

$$\max\left\{|\mathrm{Rad}(A\times A)|, |\mathrm{Ang}(A\times A)|\right\} \geq \frac{|A|^{4/3}}{2}.$$

This shows the angle-set and the radius-set of $A \times A$ cannot be small simultaneously.

2. Proof of the main result

Suppose $|A/A| = y$ and $A/A = \left\{z_i\right\}_{i=1}^{y}$. Suppose z_i has m_i representations in $A \times A$, that is,

$$m_i = \left|\left\{(a,b) \in A\times A : \frac{a}{b} = z_i\right\}\right| \quad (i = 1, 2, \ldots, y).$$

Without loss of generality we may order all m_i's as follows:

$$m_1 \leq m_2 \leq \cdots \leq m_y. \tag{7}$$

Since $|A|^2 = \sum_{i=1}^{y} m_i$, there exists a unique integer k, $1 \leq k \leq y$, such that

$$\sum_{i=1}^{k-1} m_i < \frac{|A|^2}{2} \leq \sum_{i=1}^{k} m_i \leq k m_k.$$

Hence

$$|A/A| \geq k \geq \frac{|A|^2}{2m_k} \tag{8}$$

and

$$\sum_{i=k}^{y} m_i = \Big(|A|^2 - \sum_{i=1}^{k-1} m_i\Big) \geq \frac{|A|^2}{2}. \tag{9}$$

By (7) and Solymosi's geometric observation [14],

$$|A+A|^2 = |(A\times A)+(A\times A)| \geq m_k \sum_{i=k}^{y} m_i. \tag{10}$$

Multiplying (8), (9) and (10) yields

$$|A+A|^2|A/A| \geq \frac{|A|^4}{4}.$$

This proves Theorem 1.

Remark. Let $F_n = \{a/q : 1 \leq a \leq q \leq n, (a,q)=1\}$ be the set of Farey fractions of order n. It is well-known ([10]) that $|F_n| \sim \frac{3}{\pi^2}n^2$ as $n \to \infty$. Besides, it is not difficult to deduce from (3) (see also [8, 9]) that

$$\max\{|F_n+F_n|, |F_n-F_n|, |F_nF_n|, |F_n/F_n|\} \leq \frac{n^4}{(\ln n)^{\beta+o(1)}} \qquad (n\to\infty).$$

This shows generally that one cannot expect the estimate

$$\max\{|A+A|, |A/A|\} \asymp |A|^2 \qquad (|A|\to\infty).$$

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Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China – and – Department of Mathematics, Texas State University, San Marcos, Texas 78666
E-mail address: liliangpan@yahoo.com.cn

Department of Mathematics, Texas State University, San Marcos, Texas 78666
E-mail address: js48@txstate.edu