CAUCHY – JURNAL MATEMATIKA MURNI DAN APLIKASI
Volume 4 (4) (2017), Pages 146-154
p-ISSN: 2086-0382; e-ISSN: 2477-3344

A Super (A,D)-Bm-Antimagic Total Covering Of Ageneralized Amalgamation Of Fan Graphs

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ABSTRACT

We assume finite, simple and undirected graphs in this study. Let G, H be two graphs. By an (a,d)-H-antimagic total graph, we mean any obtained bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ such that for each subgraph H' which is isomorphic to H, their total H-weights $w(H) = \sum_{v \in E(H')} f(v) + \sum_{v \in E(H')} f(e)$ show an arithmetic sequence $\{a, a+d, a+2d, \ldots., a+(m-1)d\}$ where $a, d > 0$ are integers and m is the cardinality of all subgraphs H' isomorphic to H. An (a, d)-H-antimagic total labeling f is called super if the smallest labels are assigned in the vertices. In this paper, we will study a super (a, d)-B_m-antimagicness of a connected and disconnected generalized amalgamation of fan graphs in which a path is a terminal.

Keywords: Super (a, d)-B_m-antimagic total covering, generalized amalgamation of fan graphs, connected and disconnected

INTRODUCTION

In [1], Dafik *et.al.* defined an amalgamation of graphs as follows: Let G_i be a finite collection of graphs and suppose each G_i has a fixed vertex v_j called a terminal. The amalgamation G_i where v_j as a terminal is formed by taking all the G_i's and identifying their terminal. When G_i are all isomorphic connected graphs, for any positive integer m, we denote such amalgamation by $Amal(G, m)$, where m denotes the number of copies of G. If we replace the terminal vertex v_j by a subgraph $P \subset G$ then such amalgamation is said to be a generalized amalgamation of G and denoted by $amal(G, P, m)$.

Furthermore, Baca *et. al.* in [2] and Dafik *et, el.* [3] defined an (a, d)-edge-antimagic total labeling of G as a mapping $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$, such that the set of edge-weights $\{f(u) + f(uv) + f(v) | \, uv \in E(G)\}$ is equal to the set $\{a, a + d, a + 2d, \ldots, a + (|E(G)| - 1)d\}$ for some positive integers a and d. Combining the two previous labelings, [1], [4], [5], [6], [7] introduced the (a,d)-H- antimagic total labeling. A graph G is said to be an (a, d)-H-antimagic total graph if there exist a bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the total H-weights $w(H) = \sum_{v \in E(H')} f(v) + \sum_{v \in E(H')} f(e) = \gamma$ form an arithmetic

Submitted: 14 Nopember 2016 Reviewed: 29 Nopember 2017 Accepted: 19 May 2017
DOI: http://dx.doi.org/10.18860/ca.v4i4.3758

progression $\{a, a+d, a+2d, \dots, a+(m-1)d\}$, where *a, d* > 0 are integers and *m* is the number of all subgraphs *H'* isomorphic to *H*. An (*a, d*)-*H*-antimagic total labeling *f* is called super if the smallest labels are assigned in the vertices.

There are many results show the existence of the (*a, d*)-*H*-antimagic total labeling, see [1], [4], [7], [8], [9], and [10]. In this paper, we will study a super (*a, d*)-B_m-antimagicness of an amalgamation of fans of order *m* when a path of order *n* is a terminal, denoted by *Amal*(F_n, P_n, *m*) as well as the disjoint union of multiple *s* copies of *Amal*(F_n, P_n, *m*). The cover *H'* is a book of order two, thus $H = B_m$. In other word, we will show the existence of super (*a, d*)-B_m-antimagic total labeling of *Amal*(F_n, P_n, *m*) and disjoint union of multiple *s* copies of *Amal*(F_n, P_n, *m*) denoted by *sAmal*(F_n, P_n, *m*).

LITERATURE REVIEW

Prior to showing the research result on the existence of super (*a,d*)-B_m-antimagic total labeling *sAmal*(F_n, P_n, *m*), we will rewrite a known lemma excluding the proof that will be useful for determining the necessary condition for a graph to be super (a,d)-B_m-antimagic total labeling. This lemma proved by [2] provides an upper bound for feasible value of *d*, and it is a sharp.

Lemma1. [2] *Let G be a simple graph of order p_G and size q_G. If G is super* (*a, d*)-*H- antimagic total labeling then* $d \le \frac{(p_G - p_{H'})p_{H'} + (q_G - q_{H'})q_{H'}}{t-1}$, *for H' are subgraphs isomorphic to H.* $|V(G)| = p_G, |E(G)| = q_G, |V(H')| = p_{H'}, |E(H')| = q_{H'}$, *and* $t = |H'_j|$.

RESULTS AND DISCUSSIONS

The Connected Graph. An amalgamation of fan graphs, denoted by *Amal*(F_n, P_n, *m*), is a connected graph with vertex set *V* (*Amal*(F_n, P_n,*m*)) $=\{A_j, x_i; 1 \le j \le m, 1 \le i \le n\}$ and *E*(*Amal*(F_n, P_n, *m*)) $= \{A_j, x_i; 1 \le j \le m, 1 \le i \le n\} \cup \{x_i x_{i+1}; 1 \le i \le n-1\}$. Since we study a super (*a, d*)-*H*- antimagic total labeling for $H' = B_m$ isomorphic to *H*, thus p_G = |*V* (*Amal*(F_n, P_n, *m*)) | = *m* + *n*, q_G = |*E*(*Amal*(F_n, P_n, *m*))| =*mn* + *n* - 1, $p_{H'}$ = |*V* (B_m)| = *m* + 2, $q_{H'}$ = |*E*(B_m)| = 2*m* + 1, *t* = |H'_j | =|B_m| = *n* - 1.

If amalgamation of fan graphs *Amal*(F_n, P_n, *m*) has a super (*a, d*)-B_m- antimagic total labeling then for p_G = |*V*(*Amal*(F_n, P_n, *m*))|= *m*+*n*, q_G = |*E*(*Amal*(F_n, P_n,*m*))|= *mn*+*n*-1, $p_{H'}$ = |*V*(F_n, P_n, *m*)|= *m*+2, $q_{H'}$ = |*E* (F_n, P_n, *m*)|= 2*m* + 1, *t* =|H'_j | = *n* - 1, it follows from Lemma 1.1 the upper bound of $d \le 2m^2 + 4m + 3$.

Now we start to describe the result of the super (*a,d*)-*H*-antimagic total labeling of amalgamation of fan graph with the following theorems. Figure. 1 shows an illustrasion of graph *Amal*(F_n, P_n,*m*).

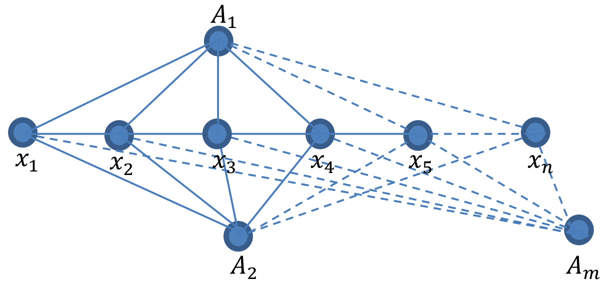

Figure. 1 illustrasion of graph *Amal* (F_n, P_n,*m*)

Theorem 2.1. *For m, n ≥2, the graph Amal(F_n, P_n, m) admits a super* $\left(\left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{9}{2}\right)m+n+2m+3+1, 2m+3\right)$*-$B_m$-antimagic total labeling.*

Proof. For G = *Amal*(F_n, P_n, *m*), define the vertex labeling f_1, as follow: $f_1\ (A_j)=j$ and $f_1(x_i)\ =m+i;\ 1\le j\le m, 1\le i\le n$, and the edge labeling as follows:

$$f_1(A_jx_i\ )=m+n+(j-1)n+i;\ 1\le j\le m, 1\le i\le n$$
$$f_1(x_i\ \ x_{i+1})=m+n+nm+i+i; 1\le i\le n-1$$

The vertex and edge labelings f_1 are a bijective function $f_1\colon V\ (G)\cup E(G)\to\{1,2,3,\dots,3mn-m+1\}$. The H-weights of *Amal*(F_n, P_n,*m*), for $1\le j\le m, 1\le i\le n$ under the labeling f_1, constitute the following sets $w_{f_1}=\bigcup_{i=1}^{n-1}\{f_1(A_j)+f_1(x_i)\}=\{\bigcup_{i=1}^{n-1}\{(2m+2i+1+\left(\frac{m^2+m}{2}\right)\}$, and the total H-weights of *Amal*(F_n, P_n,*m*) constitute the following sets $W_{f_1}=\bigcup_{i=1}^{n-1}\{w_{f_1}+\sum_{j=1}^{m}f_1(A_j\,x_i)+f_1(x_ix_{i+1})\}=\bigcup_{i=1}^{n-1}\{\left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{9}{2}\right)m+n+(2m+3)i+1\}$. It is easy to observe that the set $Wf_1=\{\left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{9}{2}\right)m+n+(2m+4), \left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{9}{2}\right)m+n+4m+7, \left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{9}{2}\right)m+n+6m+10,\dots,\left(n+\frac{5}{2}\right)m^2+\left(4n+\frac{5}{2}\right)m+4n-2\}$. It gives the desired proof.

■

Theorem 2.2. *For m, n ≥2, the graph Amal(F_n, P_n,m) admits a super* $\left(\left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{5}{2}\right)m+2n+2, 2m+1\right)$*-$B_m$-antimagic total labeling.*

Proof. For G = *Amal*(F_n, P_n, *m*), define the vertex labeling f_2, as follow: $f_2\ (A_j)=\{n+j;\ 1\le j\le m\}\ and\ f_2(x_i)\ =\ i;\ 1\le i\le n$, and the edge labeling as follows:

$$f_2(A_jx_i\ )=2n+m-1+(j-1)n+i;\ 1\le j\le m, 1\le i\le n$$
$$f_2(x_i\ \ x_{i+1})=2n+m-i; 1\le i\le n-1$$

The vertex and edge labelings f_2 are a bijective function $f_2\colon V\ (G)\cup E(G)\to\{1,2,3,\dots,3mn-m+1\}$. The H-weights of *Amal*(F_n, P_n,*m*), for $1\le j\le m, 1\le i\le n$ under the labeling f_2, constitute the following sets $w_{f_2}=\bigcup_{i=1}^{n-1}\{f_2(x_i)+f_2(x_{i+1})+\sum_{j=1}^{m}f_2(A_j)\}=\{\bigcup_{i=1}^{n-1}\{\frac{1}{2}m^2+j+1\}$, and the total H-weights of *Amal*(F_n, P_n,*m*) constitute the following sets $W_{f_2}=\bigcup_{i=1}^{n-1}\{w_{f_2}+\sum_{j=1}^{m}f_2(A_j)+f_2(x_ix_{i+1})\}=\bigcup_{i=1}^{n-1}\{\left(n+\frac{5}{2}\right)m^2+4nm+\frac{1}{2}m+2n+1+i(2m+1)\}$. It is easy to observe that the set $W_{f_2}\ =\{\left(n+\frac{5}{2}\right)m^2+\left(4n+\frac{5}{2}\right)m+2n+2, \left(n+\frac{5}{2}\right)m^2+\left(4n+\frac{9}{2}\right)m+2n+3, \left(n+\frac{5}{2}\right)m^2+\left(4n+\frac{13}{2}\right)m+2n+4,\dots,\left(n+\frac{5}{2}\right)m^2+\left(6n-\frac{3}{2}\right)m+3n\}$. Therefore, the graph *Amal*(F_n, P_n,*m*) admits a super $\left(\left(n+\frac{5}{2}\right)m^2+\left(2n+\frac{5}{2}\right)m+2n+2, 2m+1\right)$-$B_m$- antimagic total labeling, *For* $m,n\ \ge 2$ ■

Theorem 2.3. *For m, n ≥2, the graph Amal(F_n, P_n,m) admits a super* $\left(\frac{5}{2}(m^2+m)+4nm+6+2m^2, 2m^2+3\right)$*-$B_m$-antimagic total labeling.*

Proof. For $G = Amal(F_n, P_n, m)$, define the vertex labeling f_3, as follow: $f_3\ (A_1) = 1,\ f_3(x_i) = i + 1;\ 1 \le i \le n$ and $f_3(xA_j)\ = n + j;\ 2 \le j \le m$ and the edge labeling as follows:

$$f_3(A_j x_i\ ) = n + mi + j;\ 1 \le j \le m, 1 \le i \le n$$
$$f_3\big(x_i\ \ x_{i+1}\big) = m + n + nm + i; 1 \le i \le n - 1$$

The vertex and edge labelings f_3 are a bijective function $f_3: V\ (G) \cup E(G) \to \{1, 2, 3, \ldots, 3mn - m + 1\}$. The H-weights of $Amal(F_n, P_n, m)$, for $1 \le j \le m, 1 \le i \le n$ under the labeling f_3, constitute the following sets $w_{f_3} = \cup_{i=1}^{n-1}\{\sum_{j=2}^{m} f_3\big(A_j\big) + f_3(x_i) + f_3(x_{i+1}) + f_3(A_1)\} = \cup_{i=1}^{n-1}\{\frac{1}{2}m^2 + \frac{1}{2}m + (m-1)n + 2i + 3\}$, and the total H-weights of $Amal(F_n, P_n, m)$ constitute the following sets $W_{f_3} = \cup_{i=1}^{n-1}\{w_{f_3} + f_3(x_i x_{i+1}) + \sum_{j=1}^{m} f_3(A_j\ x_i) + f_3(A_j x_{i+1})\} = \cup_{i=1}^{n-1}\{\frac{5}{2}m^2 + \frac{5}{2}m + 4nm + 3 + (2m^2 + 3)i\}$. It is easy to observe that the set $Wf_3 = \left\{\frac{5}{2}(m^2 + m\right) + 4nm + 2m^2 + 6, \frac{5}{2}(m^2 + m) + 4nm + 4m^2 + 9, \ldots, 3n^2\left(2m - \frac{1}{2}\right) + n\left(\frac{15}{2} - 6m\right) + 5m - 5\}$. It gives the desired proof ■

Theorem 2.4. *For n ≥2, the graph Amal(Fn, Pn, 2) admits a super* $\left(\frac{29n+32}{2}, 0\right)$*-B2-antimagic total labeling for n even and super* $\left(\frac{29n+32}{2}, 0\right) - B_2$*-antimagic total labeling for n odd.*

Proof. Define the vertex and edge labeling f_4 as follows:

$$f_4(a) = 1; f_4(b) = 2$$

$$f_4(x_i) = \begin{cases} \dfrac{i+5}{2}, for\ 1 \le i \le n, i\ odd \\ \dfrac{n+i+4}{2}, for\ 1 \le i \le n, i\ even, n\ even \\ \dfrac{n+i+5}{2}, for\ 1 \le i \le n, i\ even, n\ odd \end{cases}$$

$$f_4(x_i x_{i+1}) = 2n - i + 2, for\ 1 \le i \le n - 1$$

$$f_4(bx_i) = 2n - i + 1, for\ 1 \le i \le n$$
$$f_4(ax_i) = 4n - i + 2, for\ 1 \le i \le n$$

The vertex and edge labelings f_4 are a bijective function $f_4: V\ (Amal(F_n, P_n, 2)) \cup E\big(Amal(F_n, P_n, 2)\big) \to \{1, 2, 3, \ldots, 4n + 1\}$. The H-weights of $Amal(F_n, P_n, 2)$, for $1 \le i \le n$ under the labeling f_4, constitute the following sets $w_{f_4} = f_4(a) + f_4(b) + f_4(x_i) + f_4(x_{i+1}) = \frac{n+2i+16}{2}$, for n even and $w_{f_4} = f_4(a) + f_4(b) + f_4(x_i) + f_4(x_{i+1}) = \frac{n+2i+17}{2}$ for n odd and the total H-weights of $Amal(F_n, P_n, 2)$ constitute the following sets $W_{f_4} = wf_4 + f_4(x_i x_{i+1}) + f_4(bx_i) + f_4(bx_{i+1}) + f_4(ax_i) + f_4(ax_{i+1}) = \frac{29n+32}{2}$, for n even and $W_{f_4} = wf_4 + f_4(x_i x_{i+1}) + f_4(bx_i) + f_4(bx_{i+1}) + f_4(ax_i) + f_4(ax_{i+1}) = \frac{29n+25}{2}$ for n odd. It is easy to observe that the set $Wf_4 = \{\frac{29n+32}{2}, \frac{29n+32}{2}, \ldots, \frac{29n+32}{2}\}$ for n even and $Wf_4 = \left\{\frac{29n+25}{2}, \frac{29n+25}{2}, \ldots, \frac{29n+25}{2}\right\}$for n odd. Therefore, the graph $Amal(F_n, P_n, 2)$ admits a super $\left(\frac{29n+32}{2}, 0\right) - B_2$- antimagic total labeling for $n \ge 2$ for n even, and the graph $Amal(F_n, P_n, 2)$ admits a super $\left(\frac{29n+25}{2}, 0\right) - B_2$ -antimagic total labeling for $n \ge 2$ for n odd It gives the desired proof . ■

Theorem 2.5. *For $n \geq 2$, the graph Amal(F_n, P_n, 2) admits a super $(13n+19,1)$-B_2 -antimagic total labeling.*

Proof. Define the vertex and edge labeling f_5 as follows:

$$f_5(a) = 1; f_5(b) = n+2$$

$$f_5(x_i) = i+2, for\ 1 \leq i \leq n$$

$$f_5(bx_i) = 2n - i + 3, for\ 1 \leq i \leq n$$
$$f_5(ax_i) = 2n + i + 2, for\ 1 \leq i \leq n$$
$$f_5(x_i x_{i+1}) = 4n - i + 2, for\ 1 \leq i \leq n-1$$

The vertex and edge labelings f_5 are a bijective function $f_5: V\ (Amal(F_n, P_n, 2)) \cup E\big(Amal(F_n, P_n, 2)\big) \rightarrow \{1, 2, 3, \ldots, 4n+1\}$. The H-weights of Amal(F_n, P_n, 2), for $1 \leq i \leq n$ under the labeling f_5, constitute the following sets $w_{f_5} = f_5(a) + f_5(b) + f_5(x_i) + f_5(x_{i+1}) = n + 2i + 6$, and the total H-weights of Amal(F_n, P_n, 2) constitute the following sets $W_{f_5} = wf_5 + f_5(x_i x_{i+1}) + f_5(bx_i) + f_5(bx_{i+1}) + f_5(ax_i) + f_5(ax_{i+1}) = 13n + i + 18$. It is easy to observe that the set Wf_5 = $\{\frac{29n+32}{2}, \frac{29n+32}{2}, \ldots, \frac{29n+32}{2}\}$ for n even and Wf_5 = $\{13n+19, 13n+20, \ldots, 14n+18\}$. Therefore, the graph $Amal(F_n, P_n, 2)$ admits a super $(13n + i + 18, 1) - B_2 -$ antimagic total labeling for $n \geq 2$ It gives the desired proof ∎

The Disconnected Graph. A disjoint union of amalgamation of fan graphs, denoted by *sAmal*(F_n, P_n,m), is a disconnected graph with vertex set V (*sAmal*(F_n, P_n, m)) $= A_j^k, x_i^k; 1 \leq j \leq m, 1 \leq i \leq n; 1 \leq k \leq s\}$ and E(*sAmal*(F_n, P_n, m)) $= A_j^k, x_i^k; 1 \leq j \leq m, 1 \leq i \leq n; 1 \leq k \leq s\}$ Since we study a super (a, d)-H- antimagic total labeling for $H' = B_m$ isomorphic to H, thus $p_G = |V\ (sAmal(F_n, P_n,m))| = s(m+n)$, $q_G = |E(sAmal(F_n, P_n,m))| = s(mn+n-1)$, $p_{H'} = |V(B_m)| = m+2$, $q_{H'} = |E(B_m)| = 2m+1$, $t = |H'_j| = |B_m| = s(n-1)$.

If amalgamation of fan graphs *sAmal*(F_n, P_n,m) has a super (a, d)-B_m- antimagic total labeling then for $p_G = s(m+n)$, $q_G = s(mn+n-1)$, $p_{H'} = m+2$, $q_{H'} = 2m+1$, $t = s(n-1)$, it follows from Lemma 1.1 the upper bound of

$$d \leq \frac{[m^2(2sn+s-5) + 4snm + 3sn - 8m - s - 5}{s(n-1)-1}$$

Theorem 2.6. *For m, n ≥2, s ≥ 2 and m is even integer, the sAmal(F_n, P_n,m) admits a super* $\left((3+n)m^2 s + (2m+1)ns - 2s + \frac{m}{2} + (2m+3)(s+1), 2m+3\right)$-$B_m$- *antimagic total labeling.*

Proof. For G = *sAmal*(F_n, P_n, m), define the vertex labeling f_6, for $1 \leq j \leq m, 1 \leq i \leq n$ (m is even integer), $1 \leq k \leq s$ as follow:

$$f_6(x_i^k) = s(m+i-1) + k$$
$$f_6(A_j^k) = \begin{cases} k + (j-1)s; & for\ \ 1 \leq k \leq s, 1 \leq j \leq m, j\ odd \\ (m-4)s + 1 + js - k & for\ \ 1 \leq k \leq s, 1 \leq j \leq m, j\ even \end{cases}$$

and edge labeling as follow:

for $1 \leq j \leq m, 1 \leq i \leq n$ (m is even integer), $1 \leq k \leq s$

$$f_6\big(A_j^k x_i^k\big) = s(m + nj + i - 1) + k$$

for $1 \leq i \leq n-1, 1 \leq k \leq s$

$$f_6\big(x_i^k x_{i+1}^k\big) = s(m + n + nm + i - 1) + k$$

The vertex and edge labelings f_6 are a bijective function $f_6: V(G) \cup E(G) \to \{1, 2, 3, \dots, 3mns - ms + s\}$. The H-weights of $sAmal(F_n, P_n, m)$, for $1 \le j \le m, 1 \le i \le n$ (m is even integer), $1 \le k \le s$ under the labeling f_6, constitute the following sets $w_{f_6} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{f_6(x_i^k) + f_6(x_{i+1}^k)\} + \sum_{j=1}^{m}(A_j^k) = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{s(2m + 2i - 1) + 2k + \frac{m}{2}(2ms - 4s + 1)\}$, and the total H-weights of $sAmal(F_n, P_n, m)$ constitute the following sets:
$W_{f_6} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{w_{f_6} + f_6(x_i^k x_{i+1}^k) + \sum_{j=1}^{m}[f_6(A_j^k) + f_6(A_j^k x_{i+1}^k)] = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{s(3m + n + nm + 3i - 2) + 3k + \frac{m}{2}(2ms - 4s + 1) + \sum_{j=1}^{m}[s(m + jn + i - 1) + k + s(m + jn + i) + k]\} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{(3 + n)m^2 s + (2m + 1)ns - 2s + \frac{m}{2} + (2m + 3)(si + k)\}\}$. It is easy to observe that the set $W_{f_6} = \{(3 + n)m^2 s + (2m + 1)ns - 2s + \frac{m}{2} + (2m + 3)(s + 1), (3 + n)m^2 s + (2m + 1)ns - 2s + \frac{m}{2} + (2m + 3)(s + 2), (3 + n)m^2 s + (2m + 1)ns - 2s + \frac{m}{2} + (2m + 3)(s + 3), \dots, 2ms(2n^2 - 2n + 1) - s\left(n^2 - n - \frac{5}{2}\right) - \frac{1}{2}(n^2 - n - 3) + (n^2 + 2n - 3)(ms + s)\}$. It gives the desired proof. ■

Theorem 2.7. *For m, n ≥2, s ≥ 2 and m is even integer, the sAmal(Fn, Pn, m) admits a super* $\left(\frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 1), 2m + 1\right)$*-$B_m$-antimagic total labeling.*
Proof. For $G = sAmal(F_n, P_n, m)$, define the vertex labeling f_5, for $1 \le j \le m, 1 \le i \le n, 1 \le k \le s$ as follow:

$$f_7(x_i^k) = si + k - s$$

$$f_7(A_j^k) = \begin{cases} s(j - 1) + sn + k; & for\ 1 \le k \le s, 1 \le j \le m, j\ odd \\ sn + 1 + js - k & for\ 1 \le k \le s, 1 \le j \le m, j\ even \end{cases}$$

and edge labeling as follow:
for $1 \le j \le m, 1 \le i \le n, 1 \le k \le s$

$$f_7(A_j^k x_i^k) = s(2n + m) + 1 - si - k$$

for $1 \le i \le n - 1, 1 \le k \le s$

$$f_7(x_i^k x_{i+1}^k) = s(2n + m - 2 + (j - 1)n + i) + k$$

The vertex and edge labelings f_7 are a bijective function $f_7: V(G) \cup E(G) \to \{1, 2, 3, \dots, 3mns - ms + s\}$. The H-weights of $sAmal(F_n, P_n, m)$, for $1 \le j \le m, 1 \le i \le n$ (m is even integer), $1 \le k \le s$ under the labeling f_5, constitute the following sets $w_{f_7} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{f_7(x_i^k) + f_7(x_{i+1}^k)\} + \sum_{j=1}^{m}(A_j^k) = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{\frac{1}{2}(sm^2 + m) + s(mn - 1) + 2(si + k)\} + 2k + \frac{m}{2}(2ms - 4s + 1)\}$, and the total H-weights of $sAmal(F_n, P_n, m)$ constitute the following sets $W_{f_7} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{w_{f_7} + f_7(x_i^k x_{i+1}^k) + \sum_{j=1}^{m}[f_7(A_j^k) + f_5(A_j^k x_{i+1}^k)] = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{\frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(si + k)\}$. It is easy to observe that the set $W_{f_7} = \{\frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 1), \frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 2), \frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 3), \dots, \frac{m^2 s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)sn\}$. It gives the desired proof. ■

Theorem 2.8. *For m, n ≥2, s ≥ 2 and m is even integer, the sAmal(F_n, P_n, m) admits a super* $\left(\frac{5}{4}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)s + 2m - 1{,}2m-1\right)$-$B_m$- *antimagic total labeling.*

Proof. For G = $sAmal(F_n, P_n, m)$, define the vertex labeling f_8, for $1 \le j \le m, 1 \le i \le n$ (m is even integer), $1 \le k \le s$ as follow:

$$f_8(A_1^k) = k$$

$$f_8(x_i^k) = s(n+2) + 1 - si - k$$

$$f_8(A_j^k) = \begin{cases} sn + 1 + js - k; & for\ \ 1 \le k \le s, 1 \le j \le m, j\ odd \\ s(n+j-1) + k; & for\ \ 1 \le k \le s, 1 \le j \le m, j\ even \end{cases}$$

and edge labeling as follow:
for $1 \le j \le m, 1 \le i \le n$ (m is even integer), $1 \le k \le s$

$$f_8(A_j^k x_i^k) = s(n + mi + j - 1) + k$$

for $1 \le i \le n-1, 1 \le k \le s$

$$f_8(x_i^k x_{i+1}^k) = s(n + m + nm + i - 1) + k$$

The vertex and edge labelings f_8 are a bijective function $f_8: V(G) \cup E(G) \to \{1, 2, 3, \ldots, 3mns - ms + s\}$. The H-weights of $sAmal(F_n, P_n, m)$, for $1 \le j \le m, 1 \le i \le n$ (m is even integer), $1 \le k \le s$ under the labeling f_8, constitute the following sets $w_{f_8} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{f_8(A_1^k) + f_8(x_i^k) + f_8(x_{i+1}^k) + \sum_{j=2}^{m} f_8(A_j^k) + \sum_{j=3}^{m} f_8(A_j^k)\} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \left\{s\left(\frac{m^2}{2} + n - 2i + mn + 3\right) + \frac{m}{2} + 2 - 2k\right\}$, and the total H-weights of $sAmal(F_n, P_n, m)$ constitute the following sets $W_{f_8} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{w_{f_8} + f_8(x_i^k x_{i+1}^k) + \sum_{j=1}^{m} [f_8(A_j^k x_i^k) + f_8(A_j^k x_{i+1}^k)]\} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{s} \{\frac{5}{2}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)si + (2m-1)k\}$. It is easy to observe that the set $W_{f_8} = \{\frac{5}{2}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)s + (2m-1), \frac{5}{2}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)s + 4m - 2, \frac{5}{2}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)s + 6m - 3, \ldots, \frac{5}{2}m^2s + sn(4m+2) + 2s + \frac{m}{2} + 2 + (2m^2-1)s(n-1) + (2m-1)s\}$. It gives the desired proof. ■

Theorem 2.9. *For n ≥2, the graph sAmal(F_n, P_n, 2) admits a super* $(12sn + 16s + 5, 1)$-B_2-*antimagic total labeling.*

Proof. Define the vertex and edge labeling f_9 as follows:

$$f_9(a^j) = s - j + 1, for\ 1 \le j \le s$$

$$f_9(b^j) = s + j, for\ 1 \le j \le s$$

$$f_9(x_i{}^j) = si + s + j, for\ 1 \le i \le n, 1 \le j \le s$$

$$f_9(a^j x_i{}^j) = 2sn + 3s - si - j + 1, for\ 1 \le i \le n, 1 \le j \le s$$

$$f_9(b^j x_i{}^j) = si + 2sn + s + j, for\ 1 \le i \le n, 1 \le j \le s$$

$$f_9(x_i{}^j x_{i+1}{}^j) = 4sn - si + 2s - j + 1, for\ 1 \le i \le n-1, 1 \le j \le s$$

The vertex and edge labelings f_9 are a bijective function $f_9: V(sAmal(F_n, P_n, 2)) \cup E(sAmal(F_n, P_n, 2)) \to \{1, 2, 3, \ldots, 4sn + s\}$. The H-weights of s$Amal$(F_n, P_n, 2), for $1 \le i \le n$ and $1 \le j \le s$ under the labeling f_9, constitute the following sets $w_{f_9} = f_9(a^j) + f_9(b^j) + f_9(x_i{}^j) + f_9(x_{i+1}{}^j) = 5s + 2j + 2si + 1$, and the total H-weights of s$Amal$(F_n, P_n,

2) constitute the following sets $W_{f_9} = w_{f_9} + f_9(a^j x_i{}^j) + f_9(a^j x_{i+1}{}^j) + f_9(b^j x_i{}^j) + f_9(b^j x_{i+1}{}^j) + f_9(x_i{}^j x_{i+1}{}^j) = 15s + j + si + 4 + 12sn$. It is easy to observe that the set $Wf_9 = \{12sn + 16s + 5, 12sn + 16s + 6, \dots, 13sn + 16s + 4\}$. Therefore, the graph $sAmal(F_n, P_n, 2)$ admits a super $(12sn + 16s + 5, 1) - B_2$- antimagic total labeling for $m, n \geq 2$ It gives the desired proof. ■

Theorem 2.10. *For n ≥2, the graph sAmal(F_n, P_n, 2) admits a super* $(11sn + 17s + 6, 3)$-*B_2-antimagic total labeling.*

Proof. Define the vertex and edge labeling f_{10} as follows:

$$f_{10}(a^j) = s - j + 1, for\ 1 \leq j \leq s$$
$$f_{10}(b^j) = s + j, for\ 1 \leq j \leq s$$
$$f_{10}(x_i{}^j) = si + s + j, for\ 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_{10}(a^j x_i{}^j) = 2sn + 3s - si - j + 1, for\ 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_{10}(b^j x_i{}^j) = si + 2sn + s + j, for\ 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_{10}(x_i{}^j x_{i+1}{}^j) = si + s + 3sn + j, for\ 1 \leq i \leq n - 1, 1 \leq j \leq s$$

The vertex and edge labelings f_9 are a bijective function $f_{10}: V\ (sAmal(F_n, P_n, 2)) \cup E(sAmal(F_n, P_n, 2)) \to \{1, 2, 3, \dots, 4sn + s\}$. The H-weights of $sAmal(F_n, P_n, 2)$, for $1 \leq i \leq n$ and $1 \leq j \leq s$ under the labeling f_{10}, constitute the following sets $w_{f_{10}} = f_{10}(a^j) + f_{10}(b^j) + f_{10}(x_i{}^j) + f_{10}(x_{i+1}{}^j) = 5s + 2j + 2si + 1$, and the total H-weights of $sAmal(F_n, P_n, 2)$ constitute the following sets $W_{f_{10}} = w_{f_{10}} + f_{10}(a^j x_i{}^j) + f_{10}(a^j x_{i+1}{}^j) + f_{10}(b^j x_i{}^j) + f_{10}(b^j x_{i+1}{}^j) + f_{10}(x_i{}^j x_{i+1}{}^j) = 3mi + 14m + 3j + 3 + 11sn$. It is easy to observe that the set $Wf_{10} = \{11sn + 17s + 6, 11sn + 17s + 9, \dots, 14sn + 17s + 3\}$. Therefore, the graph $sAmal(F_n, P_n, 2)$ admits a super $(11sn + 17s + 6, 3) - B_2$ -antimagic total labeling for $m, n \geq 2$ It gives the desired proof.

CONCLUSIONS

In this paper, the result show that super (*a, d*)-B_m-antimagic total labeling of $Amal(F_n, P_n, m)$ and $sAmal(F_n, P_n, m)$ for some feasible d are respectively $d \in \{2m + 1, 2m + 3, 2m^3 + 3\}$ and $d \in \{2m + 3, 2m + 1, 2m - 1\}$. Apart from obtained d above, we haven't found any result yet, so we propose the following open problem:

Let sG = sAmal(F_n, P_n, m) , for m, n ≥2, s ≥ 2, and *s odd, does sG admit a super (a, d)-B_m-antimagic total labeling for feasible d?*

Acknowledgement. We gratefully acknowledge the support from DP2M HIKOM Grant and research group CGANT 2017, University of Jember, Indonesia.

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