Commun. Math. Phys. 111, 373–392 (1987)

Communications in Mathematical Physics

A Superanalog of the Selberg Trace Formula and Multiloop Contributions for Fermionic Strings

A. M. Baranov[1], Yu, I. Manin[2], I. V. Frolov[3], and A. S. Schwarz[1]

[1] Institute of Physical Engineering, Moscow
[2] Steklov Mathematical Institute, Moscow
[3] Institute of Minerals and Raw Materials, Moscow

Abstract. An analog of the classical Selberg trace formula is given for discrete groups, acting on the upper complex half-superplane. Applications to the fermionic string measure on the moduli superspace are discussed.

Introduction

Quantum string theory is being discussed now with growing interest both by physicists and mathematicians. Common expectations are that one of the superstring models will correctly describe Planck scale physics and, after multiple symmetry breaking, will lead to the well known effective lagrangians in the low-energy limit.

Mathematically, quantum string theory exists in two versions. Much work was done with canonical quantization, which opened a very interesting new chapter of representation theory. However, it is notably difficult to construct a consistent picture of interacting strings in the operator approach. The Polyakov path integral formalism [1] is devoid of this shortcoming, since to account for interactions in this approach it suffices to sum over all world sheets with topologies compatible with boundary conditions. In particular, the partition function and the amplitudes in the critical dimension $d=26$ are expressed as a sum of a series, where the g-loop contribution is given by an integral over the moduli space M_g of conformal classes of Riemannian surfaces of genus g. In this way a measure on M_g arises, which is expressed as a certain combination of determinants of the Laplace operators (cf. [2]). For $g>1$ the conformal moduli space M_g may be considered as a space of surfaces of constant negative curvature. By means of this identification, one can express the Polyakov measure on the moduli space through the geometric invariants of the constant curvature metrics, namely lengths of geodesics (cf. [3]). To this end one uses the Selberg trace formula [4] and certain recent results on the Weil-Petersson metric of the moduli space. These formulas, announced in [3], are briefly reviewed in Sect. 2 of this paper. A detailed presentation with slightly more precise formulas has since appeared in [5]. An objective of this work is to develop similar tools for the quantum fermionic string theory in the critical dimension

$d=10$. On the way, we give an exposition of the basics of superconformal geometry. It plays an auxiliary part here but is of interest by itself. Our presentation follows [6].

This paper is structured as follows. Section 1 is devoted to the notions of superriemannian and superconformal manifold of real dimension $2|2$. Various physical amplitudes for fermionic strings are represented by path integrals taken over the superspace of the string field and superriemannian metrics. These path integrals can be reduced to the finite-dimensional integrals over the superconformal moduli space, i.e. superspace of riemannian supermetrics modulo superconformal equivalence. The main content of Sect. 1 is an exposition of necessary information on the superconformal moduli space and on the measure on it, arising in fermionic string theory. We utilize here an analog of the realization of the classical moduli space by surfaces of constant negative curvature, following [3] and [6]. We consider only the case $g>1$, our surfaces being closed and oriented, thus restricting ourselves to the multiloop contribution in the theory of closed string. However, the principal notions of Sect. 1 can be readily generalized.

Section 2 contains a discussion of the fermionic string measure on the conformal moduli superspace. We first propose a formula for it in terms of values of the superanalog of Selberg's zeta function. However, this formula involves points outside of the convergence domain. Unfortunately, we were unable to establish an analytic continuation of our zeta. Therefore our formula remains conjectural.

Section 3 presents in some detail our main results on the Selberg supertrace formula.

We must stress that all our calculations refer to the fermionic string in the Polyakov formalism. They do not apply directly to the Green-Schwarz superstring. In the end of Sect. 1 we discuss the relation of the fermionic string measure to the superstring measure.

We adopted a physicist's writing style in the main body of the paper. In particular, our treatment of superspaces can be made more mathematically acceptable with the help of definitions given in [17, 18], which are close in spirit to the physical language. In the Appendix we explain a different approach to the supergeometry, based on the generalization of the usual notions of algebraic and analytic geometry.

1. Superconformal Moduli Space

The bosonic Polyakov string path integrals can be reduced to finite-dimensional ones thanks to two basic facts of two-dimensional geometry of world sheets:

a) To give an orientation and a conformal class of metrics on a surface is the same as to define complex local coordinates on it with holomorphic transition functions.

b) There are only three connected and simply-connected riemannian surfaces: riemannian sphere, riemannian plane, and upper half-plane.

Using a) and b) together in uniformization theory, one can realize the space of conformal compact surfaces essentially as a space of discrete subgroups of $SL(2,\mathbb{R})$.

To extend this picture to fermionic strings, we adopt the following definition. A superriemannian metric on a 2|2-dimensional superdomain with coordinates $Z^M=(x^1,x^2,\theta^1,\theta^2)$ is an odd complex vector field $\hat{e}=e^M(Z)\partial/\partial Z^M$ with the following properties:

i) anticommutator of $\hat{e}$ with its complex conjugate vector field $\hat{\bar{e}}$ is a linear combination of $\hat{e}$ and $\hat{\bar{e}}$:

$$\{\hat{e},\hat{\bar{e}}\}=\bar{\tau}\hat{e}+\tau\hat{\bar{e}}, \tag{1}$$

ii) vector fields $\hat{E}=\hat{e}^2,\hat{\bar{E}},\hat{e},\hat{\bar{e}}$ form a basis of the space of all vector fields over the ring of superfunctions.

Two metrics are called equivalent, if they differ by a phase factor: $\hat{e}'=\exp(i\lambda(Z))\hat{e}$, where λ is a real superfield. A metric $\hat{e}$ defines a Berezin volume element corresponding to the vierbein $\hat{e},\hat{E},\hat{\bar{e}},\hat{\bar{E}}$, the same for equivalent metrics.

A superriemannian metric on a general 2|2-dimensional manifold is defined by a family of metrics on coordinate charts, such that induced metrics on pairwise intersections of charts are equivalent. In the following we shall not distinguish between two equivalent metrics.

It is conceivable that a metric depends also a certain odd constants. An adequate mathematical language is that of "families," for which we defer to the Appendix.

In the absence of odd constants, the notion of superriemannian metric is essentially equivalent to that of spinor structure. Namely, given a riemannian surface with a spinor structure, we can construct a 2|2-dimensional superriemannian manifold as follows. Let the riemannian metric be written as $ds^2=|\varrho dz|^2$, where z is a complex coordinate. Then the corresponding superriemannian metric is given in complex coordinates $\bar{z}=x^1+ix^2$, $\theta=\theta^1+i\theta^2$ by the vector field

$$\hat{e}=(\sqrt{\varrho})^{-1}(\partial/\partial\theta+\theta\partial/\partial z). \tag{2}$$

In general, let a riemannian surface M be defined by an atlas U_α with complex coordinates $z^{(\alpha)}$ and holomorphic patching functions $z^{(\alpha)}=f^{\alpha\beta}(z^{(\beta)})$. Let the riemannian metric be $ds^2=|\varrho_\alpha dz^{(\alpha)}|^2$ in U_α. Then the associated 2|2-dimensional supermanifold $\mathscr{M}$ is covered by superdomains $\tilde{U}_\alpha$ with complex coordinates $(z^{(\alpha)},\theta^{(\alpha)})$, and patchings

$$z^{(\alpha)}=f^{\alpha\beta}(z^{(\beta)}),$$
$$\theta^{(\alpha)}=(\partial z^{(\alpha)}/\partial z^{(\beta)})^{1/2}\theta^{(\beta)}.$$

The spinor structure on M defines the choice of square roots.

An important example is a superanalog $\mathscr{H}$ of the Lobachevsky plane H. It is a superdomain in $\mathbb{R}^{2|2}=\mathbb{C}^{1|1}$, defined by $\operatorname{Im}z>0$, with superriemannian metric

$$\hat{e}=Y^{1/2}(\partial/\partial\theta+\theta\partial/\partial z), \tag{3}$$

where $Y=\operatorname{Im}(z-\frac{1}{2}\theta\bar{\theta})$, (z,θ) complex coordinates. The associated volume element can be conventionally written as

$$dV=Y^{-1}dzd\bar{z}d\theta d\bar{\theta}. \tag{4}$$

The supergroup $\mathscr{A}$, consisting of transformations

$$\tilde{z}=\frac{az+b-a(\varepsilon_1+\varepsilon_2 z)\theta}{cz+d-c(\varepsilon_1+\varepsilon_2 z)\theta},\qquad \theta=\frac{1}{cz+d}(\theta+\varepsilon_1+\varepsilon_2 z+\tfrac{1}{2}\varepsilon_1\varepsilon_2\theta) \tag{5}$$

conserves the metric (3) up to equivalence, if a, b, c, d are even real parameters with $ad-bc=1$ and $\varepsilon_1, \varepsilon_2$ are odd real parameters. We shall call transformations (5) superprojective. Introducing homogeneous coordinates (z_1, z_2, ζ), for which $z=z_1 z_2^{-1}$, $\theta=\zeta z_2^{-1}$, we can rewrite (5) as a linear transformation,

$$\begin{pmatrix}\tilde{\zeta}\\ \tilde{z}_1\\ \tilde{z}_2\end{pmatrix}=T\begin{pmatrix}\zeta\\ z_1\\ z_2\end{pmatrix};\quad T=D^{-1}\begin{pmatrix}D(1+\frac{3}{2}\varepsilon_1\varepsilon_2) & D\varepsilon_2 & D\varepsilon_1\\ b\varepsilon_2-a\varepsilon_1 & a & b\\ d\varepsilon_2-c\varepsilon_1 & c & d\end{pmatrix} \tag{6}$$

for which $\operatorname{Ber} T=1$ and

$$D^2=(ad-bc)(1+\varepsilon_1\varepsilon_2).$$

We have clearly $\mathscr{A}_{\text{red}}=SL(2,\mathbb{R})$.

Consider now a discrete subgroup Γ of $\mathscr{A}$ which may depend on some odd parameters. Using the language of the Appendix, we can make this more precise as follows.

Consider a ring A of superfunctions on a superspace S. Let Γ be a subgroup of A-points of the supergroup $\mathscr{A}$ with the following property: at each point of S the reduced subgroup $\Gamma_{\text{red}}\subset SL(2,\mathbb{R})$ consists of hyperbolic elements, acts discretely upon $H=\mathscr{H}_{\text{red}}$ and has a compact quotient space $\Gamma_{\text{red}}\backslash H$. Then $\Gamma\backslash H\times S$ is a family of $1|1$-dimensional complex compact supermanifolds, parametrized by S.

As is well known, the fundamental group of a compact Riemann surface of genus g can be defined by $2g$ generators satisfying one relation

$$A_1 B_1 A_1^{-1} B_1^{-1}\ldots A_g B_g A_g^{-1} B_g^{-1}=1. \tag{7}$$

Since the dimension of $\mathscr{A}$ equals $3|2$, one can assume that (7) defines a superspace in the product of $2g$ copies of $\mathscr{A}$ of dimension $6g-3|4g-2$. This naive count can be justified by a rigorous argument, at least at points, where Γ_{red} is a classical fuchsian group with compact quotient space. Conjugating Γ by an element of $\mathscr{A}$ we get an isomorphic quotient space. Again, one can prove the existence of a quotient space of fuchsian groups of genus g, which is a supermanifold of dimension $6g-6|4g-4$. This space $\mathscr{T}$ is a base of a family of $1|1$-dimensional complex manifolds, endowed with superriemannian metrics, induced by the canonical metric on $\mathscr{H}$.

On the other hand, in order to calculate fermionic string path integrals, we must consider a superspace T, which can naively be described as a quotient space of the space of all superriemannian metrics on a compact $2|2$-dimensional supermanifold of genus g by a supergroup, generated by superdiffeomorphisms (alias, reparametrizations) and super-Weyl transformations $\hat{e}'=\Lambda(Z)e$, where $\Lambda(Z)$ is an arbitrary even invertible superfield. In the Appendix we explain how to envisage such objects as functors. But the existence of a good finite-dimensional quotient space is a subtle question even in the classical "bosonic" geometry. Therefore for the time being we shall simply work with $\mathcal{T}$ instead of T and use notation T indiscriminately for both spaces. We shall also forget about the discrete quotient of the diffeomorphism group since we shall be working locally on T.

In the following, we usually omit in notation a base space, like T. Thus, we write $\Gamma\backslash\mathcal{H}$ instead of $\Gamma\backslash\mathcal{H}\times T$ etc.

We now recall the heuristic arguments leading to an evaluation of the fermionic string path integrals.

The action of the fermionic string equals

$$S=\int_{\mathcal{M}}\hat{e}X^{\mu}\hat{\hat{e}}X^{\mu}dV\,, \tag{8}$$

where $X^1(Z),\ldots,X^d(Z)$ are even superfields on the 2|2-dimensional world sheet, $\hat{e}$ a superriemannian metric on it, dV the associated volume element. We fix a genus $g>1$ (corresponding to the loop number in operator formalism) and do not include the topological part of the action, whose contribution reduces to multiplication by a constant, depending on g only. We consider closed oriented strings, i.e. compact oriented $\mathcal{M}$.

The g-loop contribution to the partition function is a path integral of $\exp(-S)$, taken over the superspace of X^{μ} and $\hat{e}$. The standard arguments show that in the critical dimension $d=10$, where the superconformal anomaly vanishes, this integral reduces to an integral over the conformal moduli superspace T. Realizing it by means of superuniformization, as above, we can represent the integration measure in the following form:

$$d\pi=|\det{}'\Box_0(\Gamma)|^{-5}|\det{}'\Box_2(\Gamma)|dv\,. \tag{9}$$

Here dv is a superanalog of the Petersson-Weil measure and $\Box_m$ are super-Laplacians on $\mathcal{H}$, defined as follows.

For $\gamma\in\Gamma$, put $(z',\theta')=\gamma(z,\theta)$ and define the superanalog of the classical automorphy factor by

$$F_{\gamma}=(\partial/\partial\theta+\theta\partial/\partial z)\theta'\,.$$

A field of type k on $\Gamma\backslash\mathcal{H}$ is, by definition, a superfunction $\Psi(z)$ with the property $\Psi(\gamma Z)=F_{\gamma}^{k}\Psi(Z)$ for all $\gamma\in\Gamma$. A scalar product in the space of fields of type k is defined by the formula

$$\begin{aligned}(\Psi_1,\Psi_2)&=\int dV\,Y^{-k}\Psi_1\bar{\Psi}_2\,,\\ Y&=\mathrm{Im}(z-1/2\theta\bar{\theta})\,,\end{aligned} \tag{10}$$

where the integral is taken over a fundamental domain U of Γ. The invariant Laplacian on superfunctions is

$$\Box_0=2iYD\bar{D}=\hat{\hat{e}}^{+}\hat{e}\,,$$

where $\hat{\hat{e}}^{+}$ is adjoint to $\hat{e}$. For the general definition of $\Box_k$ on fields of type k see the Appendix.

The operators $\Box_k$ may have zero modes which are not taken into account in the calculation of the determinants. In particular, constants are zero modes for $\Box_0$. For simplicity we assume that there are no more.

In [6] it is shown that cotangent bundle of T can be identified with the bundle of odd superanalytic fields of type -3 on $\Gamma\backslash\mathscr{H}$. The restriction of the scalar product (10) on them is the superanalog of the Petersson-Weyl metric. The measure dv in (9) corresponds to it.

In a forthcoming work of Baranov, Frolov, and Schwarz a different approach to the construction of these objects is developed, which generalizes to the supergeometry the results of W. M. Goldman [Adv. Math. **54**:2, 200–225 (1984)]. In this approach the tangent sheaf to T is identified with $H^1(\Gamma, \mathrm{Lie}\mathscr{A})$ and the symplectic form corresponding to the Kählerian Petersson-Weyl metric corresponds to the cup product pairing

$$H^1(\Gamma, \mathrm{Lie}\mathscr{A}) \otimes H^1(\Gamma, \mathrm{Lie}\mathscr{A}) \to H^2(\Gamma, \mathbb{R}) = \mathbb{R}$$

induced by an invariant form on $\mathrm{Lie}\mathscr{A}$.

As we already mentioned, the measure (9) corresponds to the fermionic string, i.e. to the theory with action (8). The Green-Schwarz superstring is closely connected to the fermionic one but is not equivalent to it. The difference stems from the fact that in the superstring theory one should perform an independent summation over left and right spinor structures, which is not reflected in the prescription (8). In the euclidean approach "left" and "right" means "analytic" and "antianalytic" respectively. Therefore one must investiagte the analytic properties of the measure (9) in order to connect it with the superstring measure.

The analytic properties of the bosonic string measure were established by Belavin and Knizhnik. Their theorem expresses the measure on the moduli space via the so called Mumford form. The Mumford form establishes a $1-1$ correspondence between complex volume forms on the complex linear spaces L_2 and $13L_1$, where L_m is the space of holomorphic m differentials on a compact complex one-dimensional manifold X. This form analytically depends on X. On L_1 there is a scalar product depending only on the complex structure of X. This hermitian metric generates a measure on L_1 considered as a real space. The Mumford form, or rather its modulus squared, allows one to transform this measure into that on L_2. Since L_2 is the cotangent space to the bosonic moduli space at X, the Mumford form generates a measure on this space. Belavin and Knizhnik proved that it coincides with the bosonic string measure. This made it possible to calculate this measure for small genuses via theta-functions [11] and for arbitrary genuses via theta-functions [12] or via holomorphic differentials and their zeroes [Beilinson-Manin: Commun. Math. Phys. **107**, 359–376 (1986)].

The fermionic analog of the Mumford form was recently constructed by A. A. Voronov (to be published), following the suggestion stated in Yu. I. Manin's Berkeley ICM talk. When $\Box_0$ has only constant zero modes, this form establishes a $1-1$ correspondence between complex volume superforms on the superspaces L_{-3} and $5L_{-1}$, where L_m is the space of odd holomorphic fields of type m on a superconformal space X. This form is the holomorphic section of the corresponding bundle on the moduli superspace. Since L_{-1} has an intrinsic scalar product and L_{-3} is the cotangent space to the moduli superspace, the Mumford superform generates a measure on T. In [15] it is shown, that this measure equals (9). This is an analog of the Belavin-Knizhnik theorem. Voronov has also given an expression for the Mumford superform, similar to the Beilinson-Manin formula.

It is convenient to rephrase the Belavin-Knizhnik theorem and its superanalog in terms of an extended moduli space $\tilde{T}$. In the bosonic case it is defined as the total space of the fibration over T, whose fibre is the product of 13 copies of the Jacobian. The tangent space to a point of $\tilde{T}$ contains $13L_1$ and the factor space is L_2^*. Hence the Mumford form gives a complex analyticc volume form on $\tilde{T}$, which we denote π. From the Belavin-Knizhnik theorem it follows that the partition function of the bosonic string coincides with the volume of the extended moduli space $\tilde{T}$ with respect to the modulus squared of π. I.e. the partition function can be represented as the measure of the diagonal in $\tilde{T}\times\bar{\tilde{T}}$ (bar denotes the complex conjugate complex structure).

In a forthcoming paper by Baranov and Schwarz [15] this construction is generalized to the fermionic case. The partition function of the fermionic string equals to the integral of unity over the diagonal of the appropriate superspace $\tilde{T}'\times\bar{\tilde{T}}'$ with respect to a Berezin measure.

It is likely that the partition function of the superstring should be defined as a similar integral of unity taken over a different subspace $A\times B\subset\tilde{T}'\times\bar{\tilde{T}}'$, where $A\subset\tilde{T}'$, $B\subset\bar{\tilde{T}}'$.

In this way one can make sense of the independent summation over left and right spinor structures.

2. Measure and Selberg's Zeta

For the bosonic Polyakov string in the critical dimension $d=26$ the measure on the conformal moduli space at a point, corresponding to the surface $\Gamma\backslash H$, where $\Gamma\subset PSL(2,\mathbb{R})$, can be presented in the form

$$d\pi=\mathrm{const}(Z'_\Gamma(1))^{-13}Z_\Gamma(2)dv\,, \tag{1}$$

where $Z_\Gamma(s)$ is the Selberg zeta function and dv is the Petersson-Weil measure. This follows from the Ray-Singer calculation of analytic torsion, as was explained, e.g., in [3]. The zeta function is defined by the product

$$Z_\Gamma(s)=\prod_{\{\gamma\}}\prod_{k=0}^{\infty}(1-N_\gamma^{-s-k})\,, \tag{2}$$

taken over the set of primitive conjugacy classes $\{\gamma\}$ in the fuchsian group Γ of genus $g>1$, consisting of hyperbolic elements. The element $\gamma\neq 1$ is called primitive, if it cannot be presented in the form $\gamma=\beta^k$, $\beta\in\Gamma$, $k>1$. Its norm is $N_\gamma>1$, if it is conjugate to the transformation $z'=N_\gamma z$; it can be calculated from the equation $N_\gamma^{1/2}+N_\gamma^{-1/2}=|\mathrm{tr}\,\gamma|^\gamma$.

In order to state the superanalog of the formula (1), we start with the generic fuchsian subgroup Γ of the superprojective group $\mathscr{A}$, defined in Sect. 1. On the open subsuperspace of $\mathscr{A}$, consisting of hyperbolic transformations (i.e. hyperbolic at each point), an even superfunction N and a function χ, taking values ± 1, are defined by the relations

$$\chi(\gamma)(N_\gamma^{1/2}+N_\gamma^{-1/2})=(a+d)(1-1/2\varepsilon_1\varepsilon_2)-\varepsilon_1\varepsilon_2\,,\qquad N_{\gamma,\mathrm{red}}>1$$

if γ is taken as in (1.5), or by the relations

$$\chi(\gamma)(N_\gamma^{1/2}+N_\gamma^{-1/2})=1+\mathrm{str}\,T\,,\qquad N_{\gamma,\mathrm{red}}>1\,,$$

if γ is taken as in (1.6). This definition is motivated by the fact that a hyperbolic γ is conjugate in $\mathscr{A}$ to the transformation

$$z'=N_\gamma z\,,\qquad \theta'=\chi(\gamma)N_\gamma^{1/2}\theta\,,\qquad N_{\gamma,\mathrm{red}}^{1/2}>0\,.$$

Note thatNote that $N_{\gamma,\mathrm{red}}$ equals the norm of γ_{red}.

Now we introduce the Selberg zeta functions of the generic fuchsian group Γ as even real analytic superfunctions on the product of $\mathbb{C}$ and the supermanifold T:

$$\begin{aligned} Z_\Gamma(s)&=\prod_{\{\gamma\}}\prod_{k=0}^{\infty}(1-N_\gamma^{-s-k})\,,\\ \tilde{Z}_\Gamma(s)&=\prod_{\{\gamma\}}\prod_{k=0}^{\infty}(1-\chi(\gamma)N_\gamma^{-s-k})\,, \end{aligned} \tag{3}$$

where $\{\gamma\}$ again runs over primitive conjugacy classes in Γ.

In the next section we shall show that for $a\gg 0$ we have

$$\begin{aligned} &\frac{\det(-\Box_m^2(\Gamma)+a^2)}{\det(-\Box_m^2(\Gamma')+a^2)}\\ &=\frac{Z_\Gamma(1+m/2+a)Z_\Gamma(-m/2+a)\tilde{Z}_\Gamma^{-1}((1+m)/2+a)\tilde{Z}_\Gamma^{-1}((1-m)/2+a)}{Z_{\Gamma'}(1+m/2+a)Z_{\Gamma'}(-m/2+a)Z_{\Gamma'}^{-1}((1+m)/2+a)Z_{\Gamma'}^{-1}((1-m)/2+a)}\,, \end{aligned} \tag{4}$$

where Γ' is a copy of Γ, both sides being considered as superfunctions on $T\times T\times\mathbb{R}$.

However, for applications to the fermionic string measure we shall have to apply (4) (or its limiting cases) outside the convergence domain of (3). Therefore we must resort here to a fundamental conjecture which we were unfortunately unable to prove.

Conjecture. a) $Z_\Gamma(s)$, $\tilde{Z}_\Gamma(s)$ admit an analytic continuation at least to the domain $\mathrm{Re}\,s>-\varepsilon$ [for $Z_\Gamma(s)$] or $\mathrm{Re}\,s>-1/2-\varepsilon$ [for $\tilde{Z}_\Gamma(s)$].

b) On an open dense subset of T the orders of zero of Z_Γ (respectively $\tilde{Z}_\Gamma$) at $s=0,1$ (respectively at $s=-1/2,1/2$) are constant.

Taking this for granted, we can deduce from (4) the following formula for the measure:

$$\begin{aligned} d\pi=dv\cdot Z_\Gamma(2)\tilde{Z}_\Gamma(3/2)^{-1}Z_\Gamma^{(1)}(1)^{-5}\tilde{Z}_\Gamma(1/2)^{10}\\ \times Z_\Gamma^{(2g-1)}(0)\tilde{Z}_\Gamma(-1/2)^{-1}Z_\Gamma^{(6g-6)}(-1)/(\Gamma\to\Gamma')\,. \end{aligned} \tag{5}$$

3. The Selberg Trace Formula

In this section we set to investigate trace of a function of $\Box_m(\Gamma)$. As before, we work over a base space, omitted in notation. Instead of $\Box_m(\Gamma)$, we shall deal with a unitarily equivalent operator $\hat{\Box}_m+m/2$, where $\hat{\Box}_m$ acts on superfunctions on $\mathscr{H}$:

$$\hat{\Box}_m=2iYD\bar{D}-\tfrac{1}{2}m(\theta-\bar{\theta})(D+\bar{D})\,. \tag{1}$$

For its relation to $\Box_m$, see (4) below.

Component decomposition allows us to reduce an analysis of its spectral properties to that of the classical Laplacian $\hat{\Delta}_m$. Let

$$\varphi=A+y^{-1/2}(\theta\chi+\bar{\theta}\tilde{\chi})+iy^{-1}\theta\bar{\theta}B$$

be an even differentiable superfunction on $\mathscr{H}$ (or rather $\mathscr{H}\times S$), where A, B are even and $\chi, \tilde{\chi}$ odd. Set

$$\hat{\Delta}_m = -4y^2\partial^2/\partial z\partial\bar{z} - imy\partial/\partial x\,.$$

A calculation shows the following equivalence:

$$\hat{\Box}_m\varphi = s\varphi \Leftrightarrow \begin{cases} \hat{\Delta}_m A = s(1-s)A\,, \quad B = s/2A\,, \\ \hat{\Delta}_{m-1}\chi = (1/4-s^2)\chi\,, \quad \hat{\Delta}_{m+1}\tilde{\chi} = (1/4-s^2)\tilde{\chi}\,, \\ \tilde{\chi}(s-m) = -2iy\partial\chi/\partial\bar{z} + (m-1/2)\chi\,. \end{cases} \tag{2}$$

Here s is an even superfunction on S.

Operators $\hat{\Delta}_m$ are studied in Chap. IY of Hejhal's book [4] (our $\hat{\Delta}_m$ is $-\Delta_{-m}$ in Hejhal's notation). As is easily checked, $(y+\theta_1\theta_2)^s$ and $\theta_2 y^{-s}$ verify (2).

Automorphic superfields of weight m with respect to a fuchsian group Γ are, by definition, differentiable superfields on $\mathscr{H}$, satisfying a functional equation

$$\varphi(\gamma Z) = j_\gamma^m(Z)\varphi(Z)\,,$$

where

$$j_\gamma^m = (F_\gamma)^m|F_\gamma|^{-m} = F_\gamma^{m/2}\bar{F}_\gamma^{-m/2}\,, \tag{3}$$

where the reduced square root in right-hand side of (3) should be positive. Let A^m be the space of such superfields. It is easily seen that $\hat{\Box}_m$ acts on it. There is a linear isomorphism between A^m and the space of type m superfields, defined in Sect. 1, given by $\varphi = Y^{-m/2}\Phi$ for $\varphi\in A^m$. We have

$$\Box_m = Y^{m/2}(\hat{\Box}_m + m/2)Y^{-m/2}\,. \tag{4}$$

A corresponding scalar product in A^m is

$$(\varphi_1, \varphi_2) = \int dV\varphi_1\bar{\varphi}_2\,, \tag{5}$$

where the integral is taken over a fundamental domain of Γ. With respect to this product the linear isomorphism, described above, is an isometry. Hence (4) establishes an unitary equivalence of $\Box_m$ and $\hat{\Box}_m + m/2$ and allows us to study traces of functions of $\hat{\Box}_m$ instead of those of $\Box_m$.

Following Selberg, we start with constructing two-point invariants, i.e. superfunctions on $\mathscr{H}\times\mathscr{H}$, which are invariant with respect to simultaneous action of $\mathscr{A}$ on both points. They can be considered as superfunctions on the quotient space $\mathscr{H}\times\mathscr{H}/\mathscr{A}$. Since heuristically its dimension should be $1|2 = 4|4 - 3|2$, we can expect existence of one basic even invariant R and two basic odd ones ϱ_1 and ϱ_2. They can be constructed, using some results of [16]. We shall need only R and $r = \varrho_1\varrho_2\frac{iR}{2}$:

$$\begin{aligned} R(z,\theta; w, v) &= \frac{|z-w-\theta v|^2}{\operatorname{Im}(z-1/2\theta\bar{\theta})\operatorname{Im}(w-1/2v\bar{v})}, \\ r &= \frac{(\theta_1-v_1)\theta_2}{\operatorname{Im} z} + \frac{(v_1-\theta_2)v_2}{\operatorname{Im} w} + \frac{\theta_2 v_2 \operatorname{Re}(z-w-\theta v)}{\operatorname{Im} z \operatorname{Im} w}. \end{aligned} \tag{6}$$

We have also

$$\varrho_1=(\mathrm{Im}(z-\bar{z}-\theta\bar{\theta}))^{1/2}\left(\frac{\theta-v}{z-w}-\frac{\theta-\bar{\theta}}{z-\bar{z}}\right),$$
$$\varrho_2=\bar{\varrho}_1, \tag{7}$$
$$r=(z-z-\theta\bar{\theta})(4R)^{-1}DR\bar{D}R.$$

For $\theta=v=0$ we have $R(z,0;w,0)=2chd(z,w)-2$, where $d(z,w)$ is the Lobachevsky distance. The invariant r reduces to zero.

We now choose real functions of one variable Φ, Ψ, sufficiently decreasing at infinity, and define the Selberg integral operator $\hat{K}$ on superfunctions φ on $\mathscr{H}$ by the formula

$$\hat{K}\varphi(Z)=\int_{\mathscr{H}} dV(W)k(Z,W)\varphi(W), \tag{8}$$

where

$$k(Z,W)=J^m(Z,W)(\Phi(R)+r\Psi(R)),$$
$$J^m(Z,W)=\left(\frac{z-\bar{w}-\theta\bar{v}}{\bar{z}-w-\bar{\theta}v}\right)^{m/2}. \tag{9}$$

One can check that $\hat{K}$ acts upon A^m, using the relation

$$J^m(\gamma Z,\gamma W)=j_\gamma^m(Z)J^m(Z,W)j_\gamma^{-m}(W).$$

Therefore, we can consider the operator $\hat{K}$ as acting upon certain fields on $\Gamma\backslash\mathscr{H}$, sections of an appropriate vector bundle. It can be conveniently represented in the form

$$\hat{K}_\Gamma\varphi(Z)=\int_U dV(W)K(Z,W)\varphi(W), \tag{10}$$

where the integral is taken over a fundamental domain U of Γ, and the kernel is

$$K(Z,W)=\sum_{\gamma\in\Gamma} k(Z,\gamma W)j_\gamma^m(W),$$

k being defined by (9). In fact,

$$\hat{K}\varphi(Z)=\int_{\mathscr{H}} dV(W)k(Z,W)\varphi(W)$$
$$=\sum_\gamma\int_{\gamma U} dV(W)k(Z,W)\varphi(W)=\sum_\gamma\int_U dV(W)K(Z,W)\varphi(W).$$

Using an easily checked relation

$$K(\gamma Z,\gamma' W)=j_\gamma^m(Z)K(Z,W)j_{\gamma'}^{-m}(W),$$

we can convince ourselves that (10) is a well defined operator on A^m. We shall define its (super)trace by the formula

$$\mathrm{str}\,\hat{K}_\Gamma=\int_U dV(Z)K(Z,Z)=\sum_\gamma\int_U dV(Z)k(Z,\gamma Z)j_\gamma^m(Z). \tag{11}$$

The right-hand side sum can be transformed as follows. First, calculate the partial sum over a conjugacy class in Γ. This can be done explicitly since the integral reduces to that over a simple fundamental domain of a cyclic subgroup in Γ, which centralizes γ and is generated by a primitive element γ_0, whose power is γ. Second, sum up over conjugacy classes. To apply the result to the Laplacians, we then check that a function of $\hat{\Box}_m(\Gamma)$, satisfying certain conditions, is an integral operator of the form (10).

We give now some details of calculations.

Set

$$a_U(\gamma)=\int_U dV(Z)k(Z,\gamma Z)j_\gamma^m(Z).$$

From the functional equations for k, j_γ^m it follows that $a_U(s\gamma s^{-1})=a_{s^{-1}U}(\gamma)$.

To calculate the sum over a conjugacy class of $\gamma\in\Gamma$, $\gamma\neq\mathrm{id}$, denote by $Z(\gamma)$ the centralizer of γ. Let $\Gamma\backslash Z(\gamma)$ be a fixed system of coset representatives and let $A_U(\gamma)=\sum_{\gamma'\in\{\gamma\}} a_U(\gamma')$. We have:

$$A_U(\gamma)=\sum_{\gamma'\in\{\gamma\}} a_U(\gamma')=\sum_{s\in\Gamma\backslash Z(\gamma)} a_U(s\gamma s^{-1})=\int_{\substack{\cup s^{-1}U\\ s\in\Gamma\backslash Z(\gamma)}} dV(Z)k(Z,Z)j_\gamma^m(Z).$$

Now $Z(\gamma)=\{\gamma_0^n\,|\,n\in\mathbb{Z}\}$ and $\bigcup s^{-1}U$ is a fundamental domain for $Z(\gamma)$. We can assume that $\gamma_0=\mathrm{diag}(1,N_0^{1/2},N_0^{-1/2})$, and to integrate over a convenient fundamental domain. In this way we get:

$$\begin{aligned}A_U(\gamma)=&\int_1^{N_0} dy\int_{-\infty}^{\infty} dx\int d\theta_1 d\theta_2\frac{1}{y+\theta_1\theta_2}\\ &\times\left(\frac{(N-1)x-i(N+1)y-2i\theta_1\theta_2N^{1/2}}{(N-1)x+i(N+1)y+2i\theta_1\theta_2N^{1/2}}\right)^{m/2}\\ &\times\left(\Phi\left(\frac{(N-1)^2(x^2+y^2)}{N(y+\theta_1\theta_2)}\right)+(2-N^{1/2}-N^{-1/2})\theta_1\theta_2\Psi\left(\frac{(N-1)^2(x^2+y^2)}{N(y+\theta_1\theta_2)}\right)\right).\end{aligned}$$

In order to integrate out odd variables, it is convenient to change variables

$$\theta_1\theta_2=y\tilde\theta_1\tilde\theta_2,\qquad x=\tilde x(N^{1/2}-N^{-1/2})(y+\theta_1\theta_2).$$

After a calculation we get finally

$$\begin{aligned}A_U(\gamma)=&\frac{\ln N_0}{N^{1/2}-N^{-1/2}}\\ &\times((2-N^{1/2}-N^{-1/2})(Q_1(N)-\chi(\gamma)Q_3(N))+(N+N^{-1}-2)Q_2(N)),\end{aligned}$$

where

$$Q_1(y)=\int dx\Psi(x^2+y+y^{-1}-2)\frac{(x-iy^{1/2}-iy^{-1/2})^{m/2}}{(x+iy^{1/2}+iy^{-1/2})^{m/2}},$$

$$Q_2(y)=-2\int dx\Phi'(x^2+y+y^{-1}-2)\frac{(x-iy^{1/2}-iy^{-1/2})^{m/2}}{(x+iy^{1/2}+iy^{-1/2})^{m/2}},$$

$$Q_3(y)=im\int dx\Phi(x^2+y+y^{-1}-2)\frac{x}{x^2+y+y^{-1}+2}\,\frac{(x-iy^{1/2}-iy^{-1/2})^{m/2}}{(x+iy^{1/2}+iy^{-1/2})^{m/2}}.$$

A direct calculation shows that the summand, corresponding to $\{\gamma\}=\mathrm{id}$, equals $4\pi(g-1)\Phi(0)$.

The following key result establishes a relation between Laplacians and the Selberg integral operators.

Proposition. *Assume that the function Φ, Ψ, defining the kernel K, have a compact support or quickly decrease. Then there exists a function $h(s)$, such that $\hat{K}_\Gamma\varphi=h(s)\varphi$, if $\hat{\Box}_m\varphi=s\varphi$. It can be defined by the formula*

$$h(s)=\int_0^\infty dy\,y^{s-3/2}(Q_1(y)+(y-1)Q_2(y)+\sqrt{y}Q_3(y)). \tag{12}$$

Proof. We shall treat in some detail the case $m=0$. Let $\varphi=A+iy^{-1}B\theta\bar{\theta}$ be an even superfunction with $\hat{\Box}_0\varphi=s$, without linear in θ terms. This means that A and B satisfy the relations (2). Apply to φ the operator $\hat{K}$ with the kernel $\Phi(R)+\Psi(R)r$. Using the relation

$$2iYD\bar{D}(r\Phi(R))=\Phi(R)(1+r)+\Phi'(R)Rr,$$

we get

$$\begin{aligned}\tilde{\varphi}:=\hat{K}\varphi&=\int dVr(\Psi(R)-\Phi(R)-\Phi'(R)R+\Phi(R)\Box_0)\\&=\int dVr((s-1)\Phi(R)-\Phi'(R)R+\Psi(R)).\end{aligned}$$

Now integrate out odd variables:

$$\begin{aligned}\tilde{\varphi}&=\int\frac{d^2w}{(\operatorname{Im}w)^2}\tilde{\Psi}(R_0)(A(w)+(2y)^{-1}isA(w)\theta_1\theta_2)\\&=(1+is(2y)^{-1}\theta_1\theta_2)(\hat{\hat{K}}A)(z),\end{aligned} \tag{13}$$

where

$$\tilde{\Psi}(R_0)=(s-1)\Phi(R_0)-R_0\Phi'(R_0)+\Psi(R_0) \tag{14}$$

and $R_0=|z-w|^2(\operatorname{Im}z\cdot\operatorname{Im}w)^{-1}$ is the classical two-point invariant. Now, an operator $\hat{\hat{K}}$ on the Lobachevsky plane, whose kernel depends only on R_0, is in fact a function of the Laplace operator Δ (cf. e.g. [4]). It follows that $\hat{\hat{K}}$ multiplies φ by

$$\Lambda(s)=\int_0^\infty dy\,y^{s-3/2}\tilde{Q}(y+y^{-1}-2), \tag{15}$$

where

$$\tilde{Q}(y)=\int_0^\infty dx\,\tilde{\Psi}(x^2+y)$$

(see [4] for more details). Using (15), we get (12).

The case $m\neq 0$ can be treated similarly. We omit rather tedious calculations.

The case of "odd" functions φ (i.e. those depending linearly on $\theta, \bar{\theta}$) looks even more cumbersome, if treated directly. However, the following simple remark reduces the investigation of odd eigenfunctions of $\hat{\Box}_m$ to that of even ones. Namely, if φ is an even eigenfunction of $\hat{\Box}_{m+1}$, then $\psi=Y^{1+m/2}DY^{-(1+m)/2}\varphi$ is an odd eigenfunction of $\hat{\square}_m$, and we have an identity

$$\int dV_w J^m(Z,W)(\Phi(R)+r\Psi(R))(Y_w^{1+m/2}D(Y_w^{-(1+m)/2}\varphi(W)))$$
$$=Y^{1+m/2}DY^{-(1+m)/2}\int J^{m+1}(Z,W)(\hat{\Phi}+r\hat{\Psi})\varphi(W)dV_w\,,$$

where $\hat{\Phi}(R)$, $\hat{\Psi}(R)$ are certain functions that can be explicitly expressed through $\Phi(R)$ and $\Psi(R)$, $Y_w=\mathrm{Im}(w-1/2v\bar{v})$.

Summing up, we see that for appropriate h the operator $h(\hat{\square}(\Gamma))$ equals to an integral operator $\hat{K}_\Gamma$ of the form (10), whose kernel $k(Z,W)=\Phi(R)+r\Psi(R)$ is related to h by the relation

$$h(s)=\int_{-\infty}^{\infty} du e^{u(s-1/2)} \int_{e^u+e^{-u}-2}^{\infty} dx(\Psi(x)+(e^u-1)\Phi'(x))(x-e^u-e^{-u}+2)^{-1/2}\,. \tag{16}$$

Using (16), we can obtain a superanalog of the Selberg trace formula. Namely, put

$$g(u)=(2\pi)^{-1}\int_{-\infty}^{\infty} dr e^{-iru}h(1/2+ir)\,.$$

A calculation shows that

$$A_U(\gamma)=\frac{\ln N_0}{N^{1/2}-N^{-1/2}}G(\ln N,\chi(\gamma))\,, \tag{17}$$

where

$$G(u,\chi)=g(u)+g(-u)-\chi e^{-u/2}g(u)-\chi e^{u/2}g(-u)\,. \tag{18}$$

Our final expression, the Selberg supertrace formula, takes form

$$\mathrm{str}\, h(\hat{\square})=A+\sum_{\{\gamma\}}\frac{\ln N_{\gamma_0}}{N_\gamma^{1/2}-N_\gamma^{-1/2}}G(\ln N_\gamma,\chi(\gamma))$$
$$=A+\sum_{\{\gamma_0\}}\sum_{k=1}^{\infty}\frac{\ln N_{\gamma_0}}{N_{\gamma_0}^{k/2}-N_{\gamma_0}^{-k/2}}G(k\ln N_{\gamma_0},\chi^k(\gamma_0))\,, \tag{19}$$

where the sum is taken over conjugacy classes of Γ, γ_0 is a primitive element, whose power lies in $\{\gamma\}$, G is defined by (18), A is a constant, depending only on h and the genus. For $m=0$ we have $A=\int_{-\infty}^{\infty}\frac{1}{2}i(g-1)h(r)\,\mathrm{th}\,\pi r dr$.

In order to justify our formal calculations, it is necessary to impose certain restrictions on the function h.

As in Hejhal [4], it suffices to postulate

$$h(r)=h(1/2+ir)\in C^\infty(\mathbb{R})\,,$$
$$h(r)=O(r^{-2})\,,\qquad |r|\to\infty\,.$$

In particular, for $h(s)=\exp t(s+m/2)^2$, we have

$$G(u,\chi)=(4\pi t)^{-1/2}e^{-(4t)^{-1}u^2}(e^{(m+1)u/2}+e^{-(m+1)u/2}-\chi e^{mu/2}-\chi e^{-mu/2})\,,$$

from which (2.8) follows.

We turn now to the application of the trace formula to determinants, appearing in the fermionic string measure. We start with the operator $P_\Gamma = -\square_m^2(\Gamma) + a^2$ for a large. We get $\det' \square_m$ from $\det P_\Gamma$ by analytic continuation in a, and a limit $a \to 0$, followed by taking a square root. We take again two copies of generic fuchsian groups Γ, Γ' and consider the following expression:

$$\begin{aligned}
&\ln \det P_\Gamma - \ln \det P_{\Gamma'} \\
&= \lim_{\varepsilon \to 0} - \int_\varepsilon^\infty t^{-1} dt \operatorname{str}(e^{-tP_\Gamma - etP_{\Gamma'}}) \\
&= \lim_{\varepsilon \to 0} - \int_\varepsilon^\infty t^{-1} dt (\operatorname{str}(e^{-tP_\Gamma}) - \alpha e^{-tb} - \operatorname{str}(e^{-tP_{\Gamma'}}) + \alpha e^{-tb}) \\
&= - \int_0^\infty t^{-1} dt (\operatorname{str}(e^{-tP_\Gamma}) - \alpha e^{-tb}) + (\Gamma \to \Gamma'),
\end{aligned} \tag{20}$$

where $\alpha = (4\pi)^{-1}(m+1)(g-1)$, b is a positive constant. Now we apply the Selberg supertrace formula to the function $h(s)$, given by

$$h_\varepsilon(s) = \int_\varepsilon^\infty t^{-1} dt (e^{ts^2} - \alpha e^{-tb}). \tag{21}$$

We get

$$\begin{aligned}
&\ln \det P_\Gamma - \ln \det P_{\Gamma'} \\
&= - \int_0^\infty dt (4\pi t^3)^{-1/2} e^{-ta^2} \sum_{\{\gamma\}} \frac{\ln N_{\gamma_0}}{N_\gamma^{1/2} - N_\gamma^{-1/2}} e^{-(4t)^{-1} \ln^2 N_\gamma} \\
&\quad \times (N_\gamma^{-(m+1)/2} + N_\gamma^{(m+1)/2} - \chi(\gamma) N_\gamma^{-m/2} - \chi(\gamma) N_\gamma^{m/2}) + (\Gamma \to \Gamma').
\end{aligned}$$

Taking into account the relations

$$\int_0^\infty dt (4\pi t^3)^{-1/2} e^{-a^2 t - b^2 t^{-1}} = (2b)^{-1} e^{-2ab}$$

and

$$\begin{aligned}
\ln Z_{\Gamma_\infty}(s) &= \sum_{\{\gamma_0\}} \sum_{k=0}^\infty \ln(1 - N_{\gamma_0}^{-s-k}) \\
&= - \sum_{\{\gamma_0\}} \sum_{n=1}^\infty n^{-1} (1 - N_{\gamma_0}^{-n})^{-1} N_{\gamma_0}^{-ns},
\end{aligned}$$

we obtain our final relation

$$\begin{aligned}
&\ln \det P_\Gamma - \ln \det P_{\Gamma'} \\
&= \ln Z_\Gamma(a + 1 + m/2) + \ln Z(a - m/2) \\
&\quad - \ln \tilde{Z}_\Gamma(a + (m+1)/2) - \ln \tilde{Z}_\Gamma(a + (1-m)/2) - (\Gamma \to \Gamma').
\end{aligned}$$

Appendix. On Supergeometry

Superspaces

A basic notion of the supergeometry is that of a superspace. A superspace is a pair (X, O_X), where X is a topological space and O_X is a sheaf of supercommutative rings, whose stalks at points of X are local rings. Superspaces form a category: morphisms are defined in a standard way as pairs (f, φ), where $f: X \to Y$ is a continuous map and $\varphi: O_Y \to f_*(O_X)$ is a morphism of sheaves. In notation, O_X and φ are often omitted. Sections of O_X over an open subset are called superfunctions. A morphism $X \to Y$ is sometimes called a (X-)point of Y.

The category of all superspaces is too large, and in practice we deal with three most important subcategories: differentiable supermanifolds, complex analytic superspaces (or supermanifolds), superschemes (or algebraic supermanifolds). To define them, it is convenient first to define the reduction of (X, O_X) as $(X, O_X/J)$, where J is the sheaf of all nilpotents in O_X, including all odd superfunctions. It is also denoted X_{red}. Now, (X, O_X) is a differentiable supermanifold, if X_{red} is a differentiable manifold, and O_X is locally isomorphic to a grassmannian algebra over O_X/J. Similarly, (X, O_X) is an analytic superspace, if X_{red} is an analytic space and, more precisely, $(X, O_{X,0})$ is an analytic space such that $O_{X,1}$ is a coherent sheaf of $O_{X,1}$-modules. Both definitions are equivalent to the commoner ones, using atlases.

Using this language, one can draw from the rich mathematical literature the relevant methods and results, without inventing half-baked ad hoc definitions.

E.g. a general deformation theory of complex superspaces was recently developed by A. Vaintrob (thesis; Moscow University, 1986). Its application to the moduli superspaces will be given in a separate publication.

The notion of superspace serves well in all situations where only finite-dimensional supergeometry is involved. However, supersymmetric quantum field theory, in particular, that of quantum fermionic strings, leads to the consideration of functional superspaces. The basics of infinite-dimensional supergeometry are less well understood. In particular, the most important new phenomenon, that of bosonization of fermionic dimensions, is only observed in examples, although its investigation may deeply change our understanding of the superstring path integrals.

Anyway, if we adopt a traditional point of view, the moduli superspace arises as a quotient space of an infinite dimensional superspace with respect to an infinite-dimensional gauge supergroup. To define it properly, one can adopt a different definition of a superspace, as a functor of its points.

Functor of Points

We can consider a superspace (X, O_X) as a union of sets of its Y-points, $X(Y)$, where Y runs over all superspaces. If we add to this data the information about maps $X(Y) \to X(Y')$, induced by all morphisms $Y \to Y'$, we get what is technically called a representable functor from the category of all superspaces to the category of sets. It encodes in fact all information necessary to reconstruct (X, O_X). If we describe X by local coordinates and take as Y only spectra of supercommutative rings, or even of grassmannian algebras, we get the notion essentially coinciding with the physicists' usage.

This approach to supergeometry was developed by Schwarz in [17, 18]. It has immediate application to our problem, the construction of the moduli space. In fact, it is easy to define precisely the functor of points of the gauge supergroup, the functor of points of the superriemannian metrics and then the corresponding quotient functor. The latter is quite adequate for the solution of certain local questions, such as calculation of the tangent space at smooth points. However, it has one major drawback: it is not representable, i.e. it is not of the form $M(Y)$ for a superspace M, while to reduce the path integral to a finite-dimensional one we seemingly need just that. Doubtless, a remedy for this can be found, but we leave it for the future.

We now turn to an explanation of the fermionic string supergeometry in the language of superspaces. We shall work mostly with complex analytic supermanifolds.

Superconformal Structure

Let M be a complex supermanifold of dimension $1|1$. We denote by $\mathscr{T}M$ the sheaf of holomorphic vector superfields on it. It is a locally free sheaf of rank $1|1$, whose sections in a local coordinate system are $a\partial/\partial z + b\partial/\partial\zeta$. A superconformal structure on M is defined by giving a locally free subsheaf $\mathscr{T}_1 M \subset \mathscr{T}M$ of rank $0|1$ with the following property: if D is an odd vector field, local base of $\mathscr{T}_1 M$, then $\{D, D^2\}$ is a local base of $\mathscr{T}M$. A pair, consisting of M and a superconformal structure on M, is called a superconformal manifold.

(In [3, 6] such a pair is called a superconformal manifold without odd parameters or simplest superconformal manifold.)

Examples

a. Let $M = \mathbb{C}^{1|1}$, (z, θ) a global coordinate system on M. A standard superconformal structure on M is given by $D = \partial/\partial\theta + \theta\partial/\partial z$. In fact, $D^2 = \partial/\partial z$.

b. Let $M = \mathbb{P}^{1|1}$ with homogeneous coordinates (z_1, z_2, ζ). This projective superspace is covered by two coordinate neighbourhoods $U: (z, \theta) = (z_1 z_2^{-1}, \zeta z_2^{-1})$, $U': (z', \theta') = (-z^{-1}, \theta z^{-1})$. Superconformal structures, defined by $D = \partial/\partial\theta + \theta\partial/\partial z$ on U and $D' = \partial/\partial\theta' + \theta'\partial/\partial z'$ on U', coincide on $U \cap U'$, since $D' = zD$. This superconformal structure will also be called standard.

Superconformal Structures and Theta Characteristics

Let N be a complex manifold of dimension 1. A theta characteristics on N is a pair (I, α), where I is an invertible sheaf on N and $\alpha: I^2 \xrightarrow{\sim} \Omega^1 N$ is an isomorphism (we denote by $\Omega^1 N$ the sheaf of holomorphic differentials). Two pairs (I, α) and (I', α') define the same theta characteristics, if there is an isomorphism $\beta: I \xrightarrow{\sim} I'$ such that $\alpha = \alpha' \circ \beta^2$.

Each theta characteristics on N defines a superconformal manifold $M = (N, O_N \oplus \prod I)$ (i.e. odd superfunctions on M are sections of I and $M_{\text{red}} = N$). A superconformal structure on M is given by vector fields $D = \partial/\partial\theta + \theta\partial/\partial z$, where z is a local coordinate on N and θ is a section of I such that $\alpha(\theta^{\otimes 2}) = dz$. This structure is well defined, and depends only on the isomorphism class of (I, α).

Conversely, let M be a $1|1$-dimensional complex supermanifold with a superconformal structure $\mathscr{T}_1 M$. Set $N = M_{\text{red}}$, $J = \mathscr{T}_1 MN$, $I = J^{-1}$. We shall show that I defines a theta characteristics. In fact, the supercommutator, here denoted $[\,,\,]$, defines the following map of sheaves:

$$\varphi:\mathscr{T}_1M\otimes\mathscr{T}_1M\to\mathscr{T}_0M=\mathscr{T}M/\mathscr{T}_1M\,,$$
$$\varphi(D\otimes D')=1/2[D,D']\bmod\mathscr{T}_1M\,.$$

From the definition of the superconformal structure it follows that φ is an isomorphism. As is easily seen, $\mathscr{T}_0M|N=\mathscr{T}N$. Hence φ induces an isomorphism $\varphi|N:J^{\otimes 2}\xrightarrow{\sim}\mathscr{T}N$. The dual isomorphism is what we need.

The described constructions are mutually inverse and define a bijection between the isomorphism classes of the following objects:

a) Theta characteristics on the Riemann surface N.

b) Superconformal manifolds M with $M_{\mathrm{red}}=N$.

We recall the following classical facts. Let N be a compact Riemann surface of genus $g\geqq 0$. It has 2^{2g} theta-characteristics I. The parity of $\dim H^0(N,I)$ is called the parity of I. The number of even (respectively odd) characteristics equals $2^{2g-1}+2^{g-1}$ (respectively $2^{2g-1}-2^{g-1}$). For a general N we have $\dim H^0(N,I)=0$, if I is even, 1, if I is odd. A holomorphic deformation of (N,I) does not change parity, although can change $\dim H^0(N,I)$.

In particular, for $g=0$ there is only one theta characteristics. Hence any superconformal supermanifold of genus 0 is isomorphic to $\mathbb{P}^{1|1}$ with the standard structure described above. However, there is a whole family of different, although isomorphic, superconformal structures on $\mathbb{P}^{1|1}$, corresponding to different choices of projective coordinates modulo the action of the superprojective group.

Similarly, there are many superconformal structures on $\mathbb{C}^{1|1}$, all isomorphic to the standard one.

Families of Superconformal Manifolds

A (complex) family of superconformal manifolds is a pair $(\pi,\mathscr{T}_1)$, where π is a morphism $M\to S$ of complex superspaces which locally on M is a superfibration with $1|1$-dimensional fibre, and $\mathscr{T}_1$ is a locally free rank $0|1$ subsheaf of the relative tangent sheaf: $\mathscr{T}_1\subset\mathscr{T}M/S$. Moreover, for a local base D of $\mathscr{T}_1$, $\{D,D^2\}$ should be a local base of $\mathscr{T}M/S$.

This is a proper formalization of the notion of a superconformal manifold, depending onmparameters, the parameters being coordinate (or general) superfunctions on the base superspace S.

In the main text we work also with real analytic families and consider differentiable superfunctions on complex supermanifolds. The necessary changes in definitions are self-evident.

It is convenient sometimes to omit S in notation and to consider a family of superconformal manifolds as a superconformal manifold. The following result, applicable also to families, shows that we know already the general local description of superconformal structures. We shall call a (relative) local coordinate system $Z=(z,\theta)$ on M associated with a superconformal structure $\mathscr{T}_1$, if $D=D_Z:=\partial/\partial\theta+\theta\partial/\partial z$ is a base of $\mathscr{T}_1$.

Lemma. *Every superconformal structure in a neighbourhood of each point has associated local coordinates. Let Z,Z' be two local coordinates, $D=D_Z$, $\omega=\omega_Z$*

$=dz-\theta d\theta$, D',ω' *similarly defined by* Z'. *Then the following properties are equivalent:*

a) Z *and* Z' *are associated with one and the same superconformal structure, or, in other words, are related by a superconformal transformation.*

b) $Dz'=\theta' D\theta'$.

c) $D=FD'$, *where* $F=F_Z^{Z'}$ *is an invertible even superfunction.*

d) $\omega'=G\omega$, *where* G *is an invertible even superfunction.*

If these conditions are satisfied, we have $F=D\theta'$, $G=F^2=\partial z'/\partial z+\theta'\partial\theta'/\partial z$.

Sketch of Proof. The first statement is proved as in [20] for differentiable supermanifolds. The rest is verified as follows:

$$\begin{array}{ccc} \text{a)} \overset{\text{def}}{\Leftrightarrow} \text{c)} & \Rightarrow & \text{value of } F \\ \Uparrow & & \Downarrow \\ \text{d)} & \Leftarrow & \text{b)} \end{array}$$

with the help of the identity

$$\omega'=\omega(\partial z'/\partial z+\theta'\partial\theta'/\partial z)+d\theta(Dz'-\theta' D\theta').$$

The signs are defined by the following conventions: d is an odd operator and the ring of differential forms is supercommutative. (In the physical literature the Wess convention is often used: d is even, ring of forms is superanticommutative.)

Automorphisms of $\mathbb{P}^{1|1}$

Let $A=A_0\oplus A_1$ be a supercommutative ring. The group $GL(1|2;A)$ consists of matrices of the form

$$T=\begin{pmatrix} D & \beta & \alpha \\ \delta & a & b \\ \gamma & c & d \end{pmatrix}; \quad a,b,c,d\in A_0; \quad \alpha,\beta,\gamma,\delta\in A_1$$

with D and $ad-bc$ invertible in A_0.

We have

$$\operatorname{Ber} T=\left(D-(\beta\alpha)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} \delta \\ \gamma \end{pmatrix}\right)(ad-bc)^{-1}.$$

Set $SL(1|2)=\{T\,|\,\operatorname{Ber} T=1\}$. Embed $GL(1,A)=A_0^*$ into $GL(1|2)$ as diagonal scalar matrices. Since $\operatorname{Ber}(T\cdot\operatorname{Ber} T)=1$, the map

$$GL(1|2,A)\to SL(1|2,A)\times A_0^*: T\mapsto(T\cdot\operatorname{Ber} T, \operatorname{Ber} T^{-1})$$

is a group isomorphism. In particular, we may and will identify the projective linear group $PGL(1|2)$ with $SL(1|2)$.

Defining the action of $GL(1|2)$ on the projective coordinates of $\mathbb{P}^{1|1}$

$$\begin{pmatrix} \zeta' \\ z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} D & \beta & \alpha \\ \delta & a & b \\ \gamma & c & d \end{pmatrix}\begin{pmatrix} \zeta \\ z_1 \\ z_2 \end{pmatrix},$$

we see that $SL(1|2)$ identifies in this way with the automorphism group of $\mathbb{P}^{1|1}$ in the sense of holomorphic (or algebraic) supergeometry.

However, not every element of $SL(1|2)$ stabilizes the standard superconformal structure on $\mathbb{P}^{1|1}$, described earlier. Set

$$C(1|2)=\{T\in GL(1|2)\,|\,D^2=ad-bc+3\alpha\beta;\ D\gamma=d\beta-c\alpha;\ D\delta=b\beta-a\alpha\},$$
$$SC(1|2)=C(1|2)\cap SL(1|2).$$

Proposition. *$T\in GL(1|2)$ transforms into itself the standard superconformal structure on $\mathbb{P}^{1|1}$, iff $T\in C(1|2)$. In particular, $SC(1|2)$ is the automorphism supergroup of the superconformal manifold $\mathbb{P}^{1|1}$.*

Proof. Consider the action of T on the coordinate patch (z,θ):

$$z'=\frac{az+b+\delta\theta}{cz+d+\gamma\theta}=\frac{A}{B},\qquad \theta'=\frac{D\theta+\alpha+\beta z}{cz+d+\gamma\theta}=\frac{\Gamma}{B}.$$

Now, (z',θ') is associated with the initial superconformal structure, iff $D_Z z'=D_Z\theta'\cdot\theta'$, i.e.

$$D_Z\Gamma\cdot\Gamma=BD_ZA-D_ZB\cdot A.$$

After some calculation we arrive at the equations, describing $C(1|2)$.

Laplacians

Let $(M,\mathcal{T}_1M)$ be a superconformal manifold. Denote by T_1 the sheaf of differentiable complex vector fields generated by $\mathcal{T}_1M$. Set $B^{(p,q)}=\Gamma(T_1^{-p}\otimes\bar{T}_1^{-q})$. A superriemannian structure on M is a section of $T_1/U(1)$, say, e. Since $e\bar{e}\in B^{(-1,-1)}$, we can define a scalar product on $B^{(p,q)}$ by

$$\langle a,b\rangle=\int dVa\bar{b}(e\bar{e})^{p+q},$$

where dV is the canonical volume form, associated to e.

Since the sheaf T_1 (respectively $\bar{T}_1$) can be described by means of holomorphic transition functions, we can define, by analogy with $\partial,\bar{\partial}$, differential operators $D_p,\bar{D}_p$, and also their conjugates with respect to e:

$$\bar{D}_p:B^{(p,0)}\leftrightarrows B^{(p,-1)}:\bar{D}_p^+,$$
$$D_p:B^{(0,p)}\leftrightarrows B^{(-1,p)}:D_p^+.$$

Finally, their composition gives the Laplacians:

$$\Box_p:=\bar{D}_p^+\bar{D}_p:B^{(p,0)}\to B^{(p,0)},$$
$$\bar{\Box}_p:=D_p^+D_p:B^{(0,p)}\to B^{(0,p)}.$$

References

1. Polyakov, A.M.: Phys. Lett. **103** B, 207, 211 (1981)
2. Alvarez, O.: Nucl. Phys. B **216**, 125 (1983)

3. Baranov, M.A., Schwarz, A.S.: Pisma ŽETP **42**, 340 (1985)
4. Hajhal, D.: The Selberg trace formula for $PSL(2,\mathbb{R})$. Lecture Notes in Mathematics, Vol. 548. Berlin, Heidelberg, New York: Springer 1976
5. D'Hoker, E., Phong, D.H.: Columbia Univ. preprint CU-TP-323
6. Baranov, M.A., Frolov, I.V., Schwarz, A.S.: Theor. Math. Phys. **16**, 202 (1985)
7. Alessandrini, A.: Nuovo Cimento **2** A, 321 (1971)
8. Restuccia, A., Taylor, J.G.: Kings College Preprint, 1985
9. Mandelstam, S.: Seminar presented at Workshop on Strings at Santa Barbara, August 1985
10. Gilbert, G.: Univ. of Texas Preprint UTTG-23-85
11. Belavin, A., Knizhnik, V., Morozov, A., Perelomov, A.: ITEP preprint ITEP-59, 1986
12. Manin, Yu.I.: Pisma ŽETP **43**, 161 (1986)
13. Wolpert, S.A.: Bull. AMS **11**, 189 (1984); Ann. Math. **117**, 207 (1983)
14. Abikoff, W.: The real analytic theory of Teichmüller space. Berlin, Heidelberg, New York: Springer 1980
15. Baranov, M.A., Schwarz, A.S.: Int. J. Mod. Phys. (submitted)
16. Volkov, D.V., Zeltukhin, A.A., Pashnev, A.I.: In: Nonlocal, nonlinear, nonrenormalizable field theories. Moscow: Dubna 1976, p. 272
17. Schwarz, A.S.: Commun. Math. Phys. **87**, 37 (1982)
18. Schwarz, A.S.: Theor. Math. Phys. **60**, 37 (1984)
19. Berezin, F.A.: Introduction to algebra and calculus with anticommuting variables. Moscow University 1983
20. Shander, V.I.: Funkt. Anal. Prilozh. **14**, 89 (1980)
21. Howe, P.: J. Phys. A **12**, 393 (1979)
    Martinec, E.: Phys. Rev. D **28**, 2604 (1983)
22. Moore, G., Nelson, P., Polchinski, J.: Harvard Univ. preprint, HUTP-85/A90; Univ. of Texas Preprint UTTG-38-85
23. Baranov, M.A., Manin, Yu.I., Frolov, I.V., Schwarz, A.S.: Yad. Fiz. **43**, 1053 (1986)

Communicated by A. Jaffe

Received March 27, 1987