A Superposition Principle in Quantum Logics

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Abstract. A new definition of the superposition principle in quantum logics is given, which enables us to define the sectors. It is shown that the superposition principle holds only in the irreducible quantum logics.

There are several formulations of the superposition principle in physics [1, 2, 3]. In the quantum logic approach to quantum mechanics Gudder [4] postulated that the superposition principle in quantum logic (L, M) holds, if there is an isomorphism between the logic L and the set $\mathcal{M}$ of all "closed" subsets of the set of states M.

We shall try to find another formulation of the superposition principle, which will enable us to define the "sectors" analogical to those defined in the C^*-algebras [3].

First we shall give definitions of the basic concepts. We shall use the expression "quantum logic" in the same sense as in [4].

Let L be a *logic*, that is an orthocomplemented partially ordered set with the first and last elements 0 and 1, respectively, which has the following orthomodularity property:

$a \leq b$, a, $b \in L$ implies that there is a $d \in L$, $d \leq a'$, such that $b = a \vee d$; and in which $\vee a_i \in L$ for all sequences $\{a_i\}$ of mutually disjoint elements in L.

The orthocomplementation $a \to a'$ in L has the following properties:
(i) $(a')' = a$; (ii) $a \leq b$ *implies* $b' \leq a'$; (iii) $a \vee a' = 1$, where $a, b \in L$.
The elements $a, b \in L$ are *disjoint*, written $a \perp b$, if $a \leq b'$.
The elements $a, b \in L$ are *compatible*, written $a \leftrightarrow b$, if there are a_1, b_1, c in L, mutually disjoint and such that $a = a_1 \vee c$ and $b = b_1 \vee c$. We shall assume that if a, b, c are mutually compatible, then $a \leftrightarrow b \vee c$.

A map $m : L \to [0, 1]$ which satisfies

(1) $m(1) = 1$,

(2) $m(\vee a_i) = \sum_i m(a_i)$

if $\{a_i\}$ is a sequence of mutually disjoint elements of L, is a *state* on L. If m is a state which cannot be written in the form $m = cm_1 + (1-c)m_2$, where $0 < c < 1$ and m_1, m_2 are distinct states, then m is called a *pure state*. Let M be a set of states

and P the set of all pure states in M. If $a \in L$, $m \in P$, we define $P_a = \{m \in P : m(a) = 1\}$, $L_m = \{a \in L : m(a) = 1\}$. As in [4], we call (L, M) a *quantum logic* if $P_a \subset P_b$ implies $a \leq b$, and $L_{m_1} \subset L_{m_2}$ implies $m_1 = m_2$. If $p, q \in M$, then $m_0 \in M$ is a *superposition* of the states p, q if $p(a) = 0$, $q(a) = 0$ imply $m_0(a) = 0$.

Definition 1. *A set $S \subset P$ will be said to be closed under superpositions if it contains every pure superposition of any pair of its elements.*

If S is not closed under superpositions, let $\Lambda(S)$ denote the smallest subset of P closed under superpositions and containing S.

Definition 2. *We say that a superposition principle holds in (L, M) if there is an $r \in \Lambda(\{p, q\})$, $r \neq p$, $r \neq q$ for any pair p, q in P, $p \neq q$.*

That is, a superposition of every two distinct pure states always produces a new pure state.

We shall say that the pair $p, q \in P$ allows only the classical superpositions if $\Lambda(\{p, q\}) = \{p, q\}$. That is, a superposition of p, q does not produce any new pure state.

From the foregoing we see that our definition of the superposition principle is different from Guder's definition.

For $S \subset P$ Gudder [4] has defined $\bar{S}$ as the set $\bar{S} = \{r \in P : S(a) = 0 \Rightarrow r(a) = 0\}$, where he writes $S(a) = 0$ if $m(a) = 0$ for all $m \in S$. It is shown that if $S = \emptyset$ or $S = \{p\}$, then $\bar{S} = S$.

Theorem 1. *The map $S \to \Lambda(S)$ has the following properties:*
1. *If $S = \{p, q\}$, then $\Lambda(S) = \bar{S}$.*
2. *If $S_1 \subset S_2 \subset P$, then $\Lambda(S_1) \subset \Lambda(S_2)$.*
3. *If $S_\alpha \subset P$, $\alpha \in A$, then $\bigcap_\alpha \Lambda(S_\alpha)$ is closed under superpositions and $\Lambda\left(\bigcap_\alpha S_\alpha\right) \subset \bigcap_\alpha \Lambda(S_\alpha)$.*

4. *If $S_\alpha \subset P$, $\alpha \in A$, then $\bigcup_\alpha \Lambda(S_\alpha) \subset \Lambda\left(\bigcup_\alpha S_\alpha\right)$.*

Proof. 1. If $r, s \in \overline{\{p, q\}}$ and $m \in \overline{\{r, s\}}$, then if $p(a) = 0$, $q(a) = 0$ there is also $r(a) = 0$, $s(a) = 0$, which implies $m(a) = 0$. That is $m \in \overline{\{p, q\}}$ and $\overline{\{r, s\}} \subset \overline{\{p, q\}}$, from which it follows that $\overline{\{p, q\}}$ is closed under superpositions; therefore $\Lambda\{p, q\} \subset \overline{\{p, q\}}$. On the other hand, $\overline{\{p, q\}} \subset \Lambda(p, q)$ is evident.

2. If $S_1 \subset S_2$, then $S_1 \subset \Lambda(S_2)$, and $\Lambda(S_1) \subset \Lambda(S_2)$ follows from the minimality of $\Lambda(S_1)$.

3. $S_\alpha \subset \Lambda(S_\alpha)$ for every α, this implies $\bigcap_\alpha S_\alpha \subset \bigcap_\alpha \Lambda(S_\alpha)$. As $\bigcap_\alpha \Lambda(S_\alpha)$ is clearly closed under superpositions, we have $\Lambda\left(\bigcap_\alpha S_\alpha\right) \subset \bigcap_\alpha \Lambda(S_\alpha)$.

4. $S_\alpha \subset \bigcup_\alpha S_\alpha$ imply $\Lambda(S_\alpha) \subset \Lambda\left(\bigcup_\alpha S_\alpha\right)$ by the property 3, and from this we have $\bigcup_\alpha \Lambda(S_\alpha) \subset \Lambda\left(\bigcup_\alpha S_\alpha\right)$.

Definition 3. *The set $S \subset P$ is a sector if*
1. $S = \Lambda(S)$;
2. *If $p, q \in S$, then there is an $r \in \Lambda(\{p, q\})$, $r \neq p$, $r \neq q$;*
3. *If $q \in P$, $q \notin S$, then $\Lambda(\{p, q\}) = \{p, q\}$ for any $p \in S$.*

Roughly speaking, a sector is the maximal subset of P closed under superpositions, for which the superposition principle holds.

Theorem 2. *If S_1, $S_2 \subset P$, $S_1 \neq S_2$ are sectors, then $S_1 \cap S_2 = \emptyset$.*

Proof. Let there exist $p, q \in P$ such that $p \in S_1$, $p \in S_2$ and $q \in S_1$, $q \notin S_2$. Then $p \in S_1$, $q \in S_1$ imply the existence of an $r \in \Lambda(\{p, q\})$, $r \neq p$, $r \neq q$; while $p \in S_2$, $q \notin S_2$ imply $\Lambda(\{p, q\}) = \{p, q\}$.

It can be seen from [4] and from *Theorem 1*, 1. that in the separable classical logic there are no sectors.

Let C be the set of all elements of L which are compatible with all the other elements, that is $C = \{a : a \leftrightarrow b \text{ for all } b \in L\}$. C is called the *center* of L. Varadarajan [5] has shown that C is a Boolean sub-σ-algebra of L.

A logic L is called *irreducible* if its center consists only of 0 and 1.

If p is a pure state and $c \in C$, then $p(c) = 1$ or $p(c) = 0$ [6].

Theorem 3. *If the superposition principle in the quantum logic (L, M) holds, then L is irreducible.*

Proof. Let $c \in C$, $c \neq 0$, $c \neq 1$. Let $p, q \in P$ and $p(c) = 1$, $q(c) = 0$. Let $r \in \Lambda(\{p, q\})$. Since $a \leftrightarrow c$ if $a \in L$, we can write $a = (a \wedge c) \vee (a \wedge c')$. From $p(c') = 0$ we have $p(a \wedge c') = 0$, that is $p(a) = p(a \wedge c)$. Analogically $q(c) = 0$ implies $q(a) = q(a \wedge c')$ for all $a \in L$. As $r \in P$, we have $r(c) = 1$ or $r(c) = 0$. Let $r(c) = 1$; then $r(a) = r(a \wedge c)$ for all $a \in L$. If $b \in L$, $p(b) = 0 = p(b \wedge c)$, then also $r(b) = r(b \wedge c) = 0$, because $q(b \wedge c) = 0$ for all $b \in L$ and r is the superposition of p and q. Thus we have $L_p \subset L_r$, which implies $p = r$. If $r(c) = 0$, then $r(c') = 1$ and we can show in the same way that $q = r$. Thus we have found the states $p, q \in P$ such that $\Lambda(\{p, q\}) = \{p, q\}$ and the superposition principle does not hold. The converse statement to *Theorem 3* does not seem to be so easy to prove. Let us prove it for a special case.

Theorem 4. *Let (L, M) be a quantum logic. If*
1. *L is irreducible;*
2. *L satisfies Piron's axioms [7];*
3. *$p(a) = 1$, $p(b) = 1$ imply $p(a \wedge b) = 1$ for all $p \in P$ and $a, b \in L$,*
then the superposition principle holds in (L, M). We note that assumption 3 holds both in the Hilbert space logic and the separable classical logic.

Proof. If L is irreducible and satisfies Piron's axioms, then for any pair u, v of atoms in L there is an atom $z \in L$ such that $z \leq u \vee v$.

If $a \in L$, $a \neq 0$, then there must be at least one state $m \in P$ with $m(a) = 1$. If it is not so, then from $P_a = \emptyset$ we have $P_a = P_0$, from which it follows that $a = 0$. Let $p \in P_u$ and $q \in P_v$. Then $p \neq q$, because if $p = q$, then from assumption 3 it follows that $p(u \wedge v) = 1$, which is impossible for $u \wedge v = 0$. Let $r \in P_z$, then $r \neq p$, $r \neq q$ for the same reasons. If $a \in L$, $p(a) = 1$, $q(a) = 1$, then also $p(u \wedge a) = 1$, $q(v \wedge a) = 1$.

However, $u \wedge a = 0$ or u, because u is an atom. Hence $u \wedge a = u$, that is $u \leq a$. Analogously $v \leq a$ and $z \leq u \vee v \leq a$ imply that $r(a) = 1$. Thus we have shown that $r \in \Lambda(\{p, q\})$.

The center C of a logic L is discrete if there exists an at most countable set $\{c_n\}_{n \in D}$ of mutually disjoint elements of C such that (i) $\bigvee_n c_n = 1$, (ii) C consists precisely of all the lattice sums $\bigvee_{n \in Z} c_n$, where Z is an arbitrary subset of D. The $c_n - s$ are called the *atoms* of C.

Let L be a logic with the discrete center C. Then given any $a \in L$ we can write $a = a_1 \vee a_2 \vee \ldots$, where $a_j \leq c_j$ and the a_j are uniquely determined (in fact $a_j = a \wedge c_j$). If we define $L_j = L_{[0,c_j]} = \{b : b \in L, b \leq c_j\}$, then L_j are irreducible logics (c_j is the first element of L_j). The logic L may thus be thought as a direct sum of the irreducible logics L_j. Varadarajan [6] proved the following statement:

Theorem 5 (Varadarajan). *Let L be any logic and C its center. Let P be the set of pure states of L. Suppose that C is discrete and that $\{c_n ; n \in D\}$ is the family of atoms of C. Then for each $n \in D$, $L_{[0,c_n]} = L_n$ is an irreducible logic. If p is a pure state of L_n and we define $\tilde{p}$ on L by*

$$\tilde{p}(a) = p(a \wedge c_n) \quad (a \in L),$$

then $\tilde{p}$ is a pure state of L. If

$$P_n = \{\tilde{p} : p \text{ a pure state of } L_n\},$$

then the $P_n : n \in D$ are disjoint subsets of P and $P = \bigcup_{n \in D} P_n$.

Let the assumptions of *Theorem 5* be satisfied. Let $\tilde{p}, \tilde{q} \in P_n$ for some $n \in D$ and $r \in \Lambda(\{p, q\})$. Then $\tilde{p}(c_n) = 1$, $\tilde{q}(c_n) = 1$ imply $r(c_n) = 1$, while $\tilde{p}(c_m) = 0$, $\tilde{q}(c_m) = 0$ imply $r(c_m) = 0$ for every $m \neq n$. Then for any $a \in L$ we have $r(a) = r\left(\bigvee_j a \wedge c_j\right) = \sum_j r(a \wedge c_j) = r(a \wedge c_n)$. From this we see that $r \in P_n$ and therefore $\Lambda(P_n) = P_n$ for all $n \in D$.

On the other hand if $\tilde{p} \in P_n$ and $\tilde{q} \in P_m$, $m \neq n$, then $\tilde{p}(c_n) = 1$, $\tilde{q}(c_n) = 0$ and from the proof of *Theorem 3* it follows that $\Lambda(\{\tilde{p}, \tilde{q}\}) = \{\tilde{p}, \tilde{q}\}$. Thus we have proved the following statement.

Theorem 6. *Let (L, M) be a quantum logic, where L is a logic with the discrete center and let $P = \cup P_n$. Then*
1. $\Lambda(P_n) = P_n$;
2. $p \in P_n$, $q \in P_m$, $m \neq n$ imply $\Lambda(\{p, q\}) = \{p, q\}$.

Let L be a logic with the discrete center C; the L can be written as the direct sum of the logics $L_i = L[0, c_i]$. Let M be a set of states on L. Then, analogously as in [8], we can find to each $m \in M$ the states m_i on L_i such that

$$m(b) = \sum_{\{i : m(c_i) \neq 0\}} m_i(b \wedge c_i) m(c_i) \quad (b \in L).$$

It is sufficient to set

$$m_i(b \wedge c_i) = m(b \wedge c_i)/m(c_i) \quad \text{if} \quad m(c_i) \neq 0.$$

If $m(c_i) = 0$, then we can set $m_i(\cdot) = 0$. Let M_i be the sets of all such m_i. It is evident that if $\tilde{p} \in P_i$, then p is a pure state in M_i.

Theorem 7. *If (L, M) is a quantum logic, it follows that (L_i, M_i), $i \in D$ are quantum logics as well.*

Proof. If $a \in L_i$, m is a pure state on L_i, then let $P_a^i = \{m \text{ a pure state on } L_i : m(a) = 1\}$, $L_m^i = \{a \in L_i, m(a) = 1\}$. We have to show that $P_a^i \subset P_b^i$ implies $a \leq b$ and $L_{m_1}^i \subset L_{m_2}^i$ implies $m_1 = m_2$. Let $\tilde{m} \in P$, then $\tilde{m} \in \bigcup_i P_i$. If $\tilde{m} \in P_j$, $j \neq i$, then $\tilde{m}(a) = 0$ for $a \in L_i$. Therefore $\tilde{m} \in P_a$ if and only if $m \in P_a^i$, that is $P_a^i \subset P_b^i$ implies $P_a \subset P_b$, $a, b \in L_i$.

If $a \in L_m^i$, m is a pure state on L_i, then $\tilde{m}(a) = m(a \wedge c_i) = 1$, that is $a \in L_{\tilde{m}}$. On the other hand, if $a \in L$, $a \in L_{\tilde{m}}$, than $\tilde{m}(a) = m(a \wedge c_i) = 1$, that is $a \wedge c_i \in L_m^i$. Thus $L_{m_1}^i \subset L_{m_2}^i$ implies $L_{\tilde{m}_1} \subset L_{\tilde{m}_2}$, that is $m_1 = m_2$.

From *Theorems 6* and *7* it follows that if the irreducible quantum logics (L_i, M_i) are such that the superposition principle holds in them, then P_i are exactly the sectors in P.

References

1. Dirac, P. A. M.: Principy kvantovoj mechaniki. Gosudarstvennoje izdatel'stvo fiziko-matematičeskij literatury. Moskva 1960
2. Jauch, J. M.: Foundations of Quantum Mechanics. Reading, Mass.: Addison-Wesley 1968
3. Roberts, J. E., Roepstorff, G.: Commun.-Math. Phys. **11**, 321 (1969)
4. Gudder, S. P.: J. Math. Phys. **11**, 1037 (1970)
5. Varadarajan, V. S.: Commun. Pure Appl. Math. XV, 189 (1962)
6. Varadarajan, V. S.: Geometry of quantum theory. Princeton, N.Y.: Van Nostrand 1968
7. Piron, C.: Helv. Phys. Acta **37**, 439 (1964)
8. Gudder, S. P.: Pac. J. Math. **19**, 81 (1966)