A SUPERREFLEXIVE BANACH SPACE WHICH
DOES NOT ADMIT COMPLEX STRUCTURE

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ABSTRACT. We construct an infinite-dimensional superreflexive real Banach space which does not admit complex structure and consequently is not isomorphic to the Cartesian square of any Banach space. We also construct a variant of Bourgain's example of a complex Banach space with nonunique complex structure and state a number of open problems about structure of Banach spaces and their linear groups.

1. Introduction and main results. The main result of this paper is the following

THEOREM 1.1. *There exists an infinite-dimensional superreflexive real Banach space which does not admit complex structure. Moreover, it can be chosen to be a subspace of a Banach lattice, which is 2-convex and q-concave for any $q > 2$ (resp. 2-concave and p-convex for any $p < 2$).*

We have the immediate

COROLLARY 1.2. *There exists an infinite-dimensional superreflexive real Banach space which is not isomorphic to the Cartesian square of any Banach space.*

It should be observed that the James' space J [6] constitutes a nonreflexive example for the main statements of Theorem 1.1 and Corollary 1.2. Also, examples of superreflexive spaces which are not isomorphic to *their* Cartesian squares have been known for some time (see [3, 1]). However, the space from [3] is known to be isomorphic to *some* Cartesian square (see also Problem 7.4).

The proof of Theorem 1.1 is based on the following finite-dimensional result:

PROPOSITION 1.3. *Given $q \in [2, \infty]$ and $n \in \mathbf{N}$ there exists $Y = Y_q^n \subset L_q$ with $\dim Y = n$ such that if $Z = Y \oplus_2 H$ (H any Hilbert space) and if $A: Z \to Z$ satisfies $A^2 = -I$, then $\|A\| \geq c_0 n^{(1/2 - 1/q)/2}$, where c_0 is a numerical constant.*

Proposition 1.3 is proved using the results from [19], in turn influenced by [5 and 18]. The procedure of "glueing" finite-dimensional spaces into an infinite-dimensional one is taken from [2]; similar schemes were previously employed e.g. in [3, 7 or 8].

The paper is organized as follows: §2 explains notation and terminology. §3 presents known results and preliminary lemmas. §4 contains the proof of Proposition 1.3. In §5 we derive Theorem 1.1 from Proposition 1.3. In §6 we give a sketch

of an alternate proof of the result of J. Bourgain [**2**] on nonuniqueness of complex structure in normed spaces. §7 lists open problems.

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2. Notation and terminology. We use standard Banach space notation, as can be found e.g. in [**11**]. If B is a measurable subset of $\mathbf{R}^n$, we will denote by $\mathrm{vol}(B)$ the usual n-dimensional Lebesgue measure of B and by $\mathrm{ac}(B)$ the absolute convex hull of B. If X is a normed space, we will denote by $\|\cdot\|_X$ its norm; by $L(X)$ the space of bounded linear operators on X, endowed with the usual operator norm $\|\cdot\|_{L(X)}$ (which we may also denote by $\|\cdot: X \to X\|$); similarly $L(X, Y)$. $L_{\mathbf{R}}(X)$ is the space of $\mathbf{R}$-linear operators on the (possibly complex) space X. $B(X)$ is the unit ball of X; $B(l_p^n) = B_p^n$. If (X_j) is a sequence of normed spaces, $(\bigoplus_j X_j)_{l_p}$ is the space of sequences $(x_j), x_j \in X_j$, for which the norm $\|(x_j)\| = \|(\|x_j\|_{X_j})\|_{l_p}$ is finite; we will also denote by $X_1 \oplus_p X_2$ the l_p-sum of just two spaces.

For $T \in L(X, Y)$, $\pi_p(T)$ denotes the usual p-absolutely summing norm of T (see [**11**, vol. II, pp. 63–64]). We need to know that if $T \in L(X, Y)$ and $S \in L(Y, Z)$, then $\pi_p(T) \leq \min\{\|S\|\pi_p(T), \pi_p(S)\|T\|\}$. We say that X satisfies the Grothendieck theorem with constant C iff $\pi_1(T) \leq C\|T\|_{L(X, l_2)}$ for $T \in L(X, l_2)$ (note that we always have $\pi_p(T) \leq \pi_1(T)$ for $p \geq 1$). If H is a Hilbert space and $T \in L(H)$, T compact, we denote by $(s_j(T))_{j=1}^{\dim H}$ the sequence of s-numbers of T (i.e. the eigenvalues of $|T| \stackrel{\mathrm{df}}{=} (T^*T)^{1/2}$, counted with multiplicity and arranged in the nonincreasing order). $\mathrm{hs}(T) = \|(s_j(T))\|_2$ is the Hilbert-Schmidt norm of T; we have $\mathrm{hs}(T) = \pi_2(T)$. Furthermore, if $T \in L(H)$ is compact and $(s_j(T)) \in l_1$, we define the quasi-norm

$$\|T\|_{C_0} = \sum_j \frac{s_j(T)}{1 + s_j(T)}.$$

$\|\cdot\|_{C_0}$ satisfies the triangle inequality, but is not positively homogeneous. For our purposes it is essentially sufficient to know that if $S \in L(l_2^n)$ and $\|S\|_{C_0} \leq \delta n$, then there exists $F \in l_2^n$ with $\mathrm{codim} F < \sqrt{\delta} n$ such that $\|Sx\|_2 \leq \sqrt{\delta}\|x\|_2$ for $x \in F$. Also let us note that if $s_k(S) = a$, then $\|Sx\|_2 \leq a\|x\|_2$ on a subspace of codimension $< k$ and $\|Sx\|_2 \geq a\|x\|_2$ on a subspace of dimension $\geq k$.

Finally, to avoid unnecessary repetitions, let us agree that the letters c, c_1, c', etc. will denote universal constants while e.g. $c(\delta)$ will denote a constant depending only on a parameter δ.

3. Known results and preliminary lemmas. The following known result is the starting point of this paper:

THEOREM 3.1 [**19**, THEOREM 1.5]. *Given $\delta \in (0, 1]$ and $n \in \mathbf{N}$, there exists a norm $\|\cdot\|_{X_1^n}$ on $\mathbf{R}^n$ satisfying*

(a) *X_1^n is isometric to a quotient of l_1^N for some $N \leq 2n$,*

(b) *$\|\cdot\|_2 \leq \|\cdot\|_{X_1^n} \leq \|\cdot\|_1$,*

(c) *X_1^n satisfies the Grothendieck theorem with constant c_1,*

(d) *if $\|T\|_{L(X_1^n)} \leq c(\delta)\sqrt{n}$, then $\|T - \lambda I\|_{C_0} \leq \delta n$ for some $\lambda \in \mathbf{R}$.*

REMARK 3.2. Conditions (a) and (b) are clearly equivalent to

(e)
$$\begin{cases} B(X_1^n) = \mathrm{ac}\{e_1, e_2, \ldots, e_n, x_{n+1}, \ldots, x_N\}, \text{ where } (e_j) \text{ is the} \\ \text{standard unit vector basis and } \|x_j\|_2 \leq 1 \text{ for } n < j \leq N. \end{cases}$$

In particular, it then follows that $[\mathrm{vol}(B(X_1^n))/\mathrm{vol}(n^{-1/2}B_2^n)]^{1/n} \leq (8e^3/\pi)^{1/2} < 7.2$ (see e.g. [17, Lemma 6.5]).

REMARK 3.3. The fact that X_1^n can be chosen to satisfy the Grothendieck theorem with universal constant c_1 (and also to have cotype 2 constant $\leq c_1$ and other properties not used here) follows from the "random" choice of x_j's; the kernel of the quotient map from (a) is then, with large probability, nearly Euclidean (cf. [19, §5(a) or 18, Remark 4.5; 9, 14]). In fact for our purposes it is sufficient to use the classical Grothendieck theorem on operators from l_1 to l_2 since, as one can show, the quotient map is a "good" isomorphism on some very large (e.g. $.9n$-dimensional) "nearly" Euclidean subspace of l_1^N.

COROLLARY 3.4. *Given* $p \in [1, 2]$, $\delta \in (0, 1]$ *and* $n \in \mathbf{N}$, *there exists a norm* $\| \cdot \|_{X_p^n}$ *on* $\mathbf{R}^n$ *such that*

(i) X_p^n *is isometric to a quotient of* l_p^N *with* $N \leq 2n$,
(ii) *if* $\|T\|_{L(X_p^n)} \leq c(\delta)n^{1/p-1/2}$, *then* $\|T - \lambda I\|_{C_0} \leq \delta n$ *for some* $\lambda \in \mathbf{R}$,
(iii) $n^{1/p-1/2}\| \cdot \|_2 \geq \| \cdot \|_{X_p^n}$ *(equivalently* $n^{1/2-1/p}B_2^n \subset B(X_p^n)$*)*,
(iv) $[\mathrm{vol}(B(X_p^n))/\mathrm{vol}(n^{1/2-1/p}B_2^n)]^{1/n} \leq c_2$,
(v) X_p^n *satisfies the Grothendieck theorem with constant* $c_1 n^{1-1/p}$.

PROOF. Consider the same quotient space as in Theorem 3.1, but with respect to the l_p^N-norm rather than the l_1^N-norm. Since $\| \cdot \|_{l_1^N} \geq \| \cdot \|_{l_p^N} \geq N^{1/p-1}\| \cdot \|_{l_1^N}$, we have also

(1)
$$\| \cdot \|_{X_1^n} \geq \| \cdot \|_{X_p^n} \geq N^{1/p-1}\| \cdot \|_{X_1^n},$$

and so (ii) and (v) follow. By Remark 3.2, the quotient map sends the standard unit vector basis of $\mathbf{R}^N$ onto $e_1, e_2, \ldots, e_n, x_{n+1}, \ldots, x_N \in \mathbf{R}^n$ and so $B_p^n \subset B(X_p^n)$; thus $\| \cdot \|_{X_p^n} \leq \| \cdot \|_{l_p^n}$ $(\leq n^{1/p-1/2}\| \cdot \|_{l_2^n})$; this shows (iii). Finally, (1) implies that $B(X_p^n) \subset N^{1-1/p}B(X_1^n)$. Since $N \leq 2n$, this combined with the second part of Remark 3.1 proves (iv) with $c_2 = 2^{1-1/p} \cdot 7.2 < 11$.

REMARK 3.5. In the terminology of [20], (iii) and (iv) mean that the "volume ratio" of X_p^n with respect to the ellipsoid $n^{1/2-1/p}B_2^n$ does not exceed c_2. Hence, by [20, (*), p. 368 or 15, Remark 5], there exists a subspace E of X_p^n with $\dim E \geq \frac{1}{2}n$ such that for $x \in E$

(2)
$$n^{1/p-1/2}\|x\|_2 \geq \|x\|_{X_p^n} \geq c_3 n^{1/p-1/2}\|x\|_2,$$

where $c_3 = (4\pi c_2^2)^{-1}$.

We also need the following

LEMMA 3.6. *Let* $(H, | \cdot |)$ *be an* n-*dimensional Hilbert space and* $D \in L(H)$ *be such that, for some* $\lambda \in \mathbf{R}$, $\|D - \lambda I\|_{C_0} \leq \delta n$. *Then there exists a subspace* F *of* H *with* $\mathrm{codim}\, F < 2\sqrt{\delta}n$ *such that* $|(D^2 + I)x| \geq (1 - 2\delta)|x|$ *for* $x \in F$.

PROOF. Let $F_0 \subset H$ be a subspace with $\mathrm{codim}\, F_0 < \sqrt{\delta}n$ such that $|Dx - \lambda x| \leq \sqrt{\delta}|x|$ for $x \in F_0$. Then $F = F_0 \cap D^{-1}(F_0)$ is the required subspace.

4. Proof of Proposition 1.3. By duality, it is enough to prove the assertion of Proposition 1.3 with Y replaced by a quotient of L_p, where $1/p = 1 - 1/q$. Specifically, let us consider $Z = X_p^n \oplus_2 H$, where X_p^n is obtained via Corollary 3.4 applied with $\delta = 2^{-6}$. Set $c_0 = \min\{c(2^{-6}), (4c_3 c_1)^{-1/2}\}$, where $c(2^{-6})$, c_1 and c_3 come from Corollary 3.4(ii), (v) and (2), respectively. To prove the Proposition let us assume, to the contrary, that there exist $A \in L(Z)$ with $A^2 = -I$ and $\|A\| < c_0 n^{(1/2-1/q)/2}$. Write A as a 2×2 matrix with operator entries

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} X_p^n \\ H \end{matrix}.$$

Clearly,

(3) $\|A_{ij}\| < c_0 n^{(1/2-1/q)/2} = c_0 n^{(1/p-1/2)/2}$

for $i, j = 1, 2$ and the corresponding operator norms. In particular, Corollary 3.4(ii) implies that there exists $\lambda \in \mathbf{R}$ such that $\|A_{11} - \lambda I_{X_p^n}\|_{C_0} < 2^{-6}n$ and so, by Lemma 3.6 applied with $D = A_{11}$ and $\delta = 2^{-6}$, we get that there exists a subspace F of X_p^n (identified with $\mathbf{R}^n$) with codim $F < n/4$ such that

(4) $\|(A_{11}^2 + I)x\|_2 \geq (1 - 2^{-5})\|x\|_2 \quad \text{for } x \in F.$

On the other hand, $A^2|_{X_p^n} = A_{11}^2 + A_{12}A_{21} = -I_{X_p^n}$ and so, denoting $R = -A_{12}A_{21}$, we have

(5) $A_{11}^2 + I_{X_p^n} = R.$

By Corollary 3.4(v) and (3),

$$\pi_2(R) \leq \|A_{12}\| \cdot \pi_2(A_{21}) \leq c_1 n^{1-1/p}\|A_{12}\| \cdot \|A_{21}\| \leq c_1 c_0^2 n^{1/2}.$$

Now choose $E_1 \subset X_p^n$ with dim $E_1 \geq n/2$ so that (2) holds for $x \in RE_1$ ($E_1 = R^{-1}E$ with E given by Remark 3.5 works). Of course, the left inequality in (2) holds for all $x \in X_p^n$ (Corollary 3.4(iii)). It thus follows that $\pi_2(R|_{E_1}: (E_1, \|\cdot\|_2) \to l_2^n) = \mathrm{hs}(R|_{E_1}) \leq c_3 c_1 c_0^2 n^{1/2} \leq \frac{1}{4}n^{1/2}$ (note our choice of c_0). Consequently, there exists a further subspace $E_2 \subset E_1$ with dim $E_2 \geq \frac{1}{2}$ dim $E_1 \geq \frac{1}{4}n$ such that

(6) $\|Rx\|_2 \leq \sqrt{4/n}\,\mathrm{hs}(R|_{E_1})\|x\|_2 \leq \frac{1}{2}\|x\|_2 \quad \text{for } x \in E_2.$

This (remember (5)) contradicts (4) and proves Proposition 1.3.

5. Proof of Theorem 1.1. In this section we deduce Theorem 1.1 from Proposition 1.3. The former one will follow immediately from the following

PROPOSITION 5.1. *There exist a sequence of integers (n_k) and a sequence of reals $q_k \downarrow 2$ such that if $Y = (\bigoplus Y_{q_k}^{n_k})_{l_2}$, where the Y_q^n's are the spaces given by Proposition 1.3, then there is no operator $A \in L(Y)$ such that $A^2 = -I$.*

PROOF. We follow the argument from [2]. However, since we work with subspaces of L_q-spaces, we do not have to go through an interpolation argument.
Define sequences (n_k) and (q_k) inductively by
(1°) $q_1 = n_1 = 4$,
(2°) if $(q_j), (n_j)$ are defined for $j < k$, choose $q_k \in (2, q_{k-1})$ so that

(i) $n_{k-1}^{1/2-1/q_k} \le 2$

and then choose $n_k > n_{k-1}$ so that

(ii) $c_0 n_k^{(1/2-1/q_k)/2} n_{k-1}^{-1/2} > k$,

where c_0 is the constant from Proposition 1.3.

Observe that since $n_k \uparrow \infty$, (i) implies that $q_k \downarrow 2$ (we could also require this additionally).

Now suppose that there exists $A \in L(Y)$ with $A^2 = -I$ and fix $k > \|A\|$. Denote by P_j the natural projection of Y onto $Y_{q_j}^{n_j}$. Set $Z' = Y_{q_k}^{n_k} + A Y_{q_k}^{n_k}$. Then $A Z' = Z'$ and $Z' = Y_{q_k}^{n_k} \oplus_2 F'$, where $F' = (I - P_k)Z' = (I - P_k)A Y_{q_k}^{n_k}$. Clearly, $\dim F' \le n_k$ and $F' \subset (\bigoplus_{j \ne k} P_j F')_{l_2}$. We have, for $j < k$,

$$d(P_j F', l_2^{\dim P_j F'}) \le (\dim P_j F')^{1/2} \le n_j^{1/2} \le n_{k-1}^{1/2};$$

similarly, for $j > k$,

$$d(P_j F', l_2^{\dim P_j F'}) \le (\dim P_j F')^{1/2-1/q_j} \le n_k^{1/2-1/q_k} \le 2$$

(we used $2°$(i) and [**11**]); this shows that

$$d(F', l_2^{\dim F'}) \le \max\{2, n_{k-1}^{1/2}\} = n_{k-1}^{1/2}.$$

Hence if $Z = Y_{q_k}^{n_k} \oplus_2 l_2^{\dim F'}$, then $d(Z, Z') \le n_{k-1}^{1/2}$. Combining this with Proposition 1.3 we see that

$$\|A|_{Z'}\|_{L(Z')} \ge c_0 n_k^{(1/2-1/q_k)/2} n_{k-1}^{-1/2} > k$$

by $2°$(ii); a contradiction. This proves Proposition 5.1 and concludes the proof of Theorem 1.1.

REMARK 5.2. It is clear from the proof that Proposition 5.1 (and hence also Theorem 1.1 and Corollary 1.2) remains true if we replace Y by $Y \oplus H$, where H is any Hilbert space.

6. A variant of Bourgain's example of a space with nonunique complex structure.
In this section we indicate how, arguing analogously as in §§3–5, one can reprove the following results of J. Bourgain [**2**].

THEOREM 6.1. *There exists a (superreflexive) complex Banach space X such that if one defines new multiplication by complex scalars by $\lambda \odot x = \bar{\lambda} x$ and denotes the resulting new complex Banach space by $\overline{X}$, then X and $\overline{X}$ are not isomorphic. As a consequence, there exists a real Banach space, which admits two nonisomorphic complex structures.*

Similarly as in [**2**], the argument is based on some finite-dimensional fact (slightly stronger than the corresponding result from [**2**]).

PROPOSITION 6.2. *Given $q \in [2, \infty]$ and $n \in \mathbf{N}$ there exists (a complex Banach space) $Y = \tilde{Y}_q^n \subset L_q$ with $\dim Y = n$ such that if $Z = Y \oplus_2 H$ (H any complex Hilbert space) and $T: Z \to Z$ is invertible and complex antilinear (i.e. T is $\mathbf{R}$-linear and $T(\lambda x) = \bar{\lambda} T x$ for $\lambda \in \mathbf{C}$ and $x \in Z$), then $\|T\| \cdot \|T^{-1}\| \ge c_0' n^{1/2-1/q}$.*

Once this is shown, we argue exactly as in [**2**] (and very similarly as in §3 of this paper, Proposition 6.2 playing the role of Proposition 1.3) to prove that $X = (\bigoplus \tilde{Y}_{q_k}^{n_k})_{l_2}$ verifies Theorem 6.1 for the appropriate choice of (n_k) and (q_k).

Also the (nearly "soft") proof of Proposition 6.2 goes along similar lines as the proof of Proposition 1.3. First [**19**, Lemma 3.5A, Theorem 1.4, cf. §5(a)] readily imply (cf. the proof of Corollary 3.4) the existence, for given $p \in [1, 2]$, $\delta \in (0, 1]$ and $n \in \mathbf{N}$, of a (complex) norm $\| \cdot \|_{\tilde{X}_p^n}$ on $\mathbf{C}^n$ satisfying conditions (i), (iii), (iv) and (v) of Corollary 3.4 ((v) for $\mathbf{R}$-linear operators) and additionally

(ii)′ $\begin{cases} \text{if } S: \mathbf{C}^n \to \mathbf{C}^n \text{ is } \mathbf{C}\text{-antilinear and at least } r \geq \delta n \text{ } s\text{-numbers} \\ \text{of } S \text{ are } \geq \alpha, \text{ then } \|S\|_{L_{\mathbf{R}}(\tilde{X}_p^n)} \geq c'(\delta) \cdot \alpha \cdot n^{(1/p-1/2)/2}, \end{cases}$

where, for the purpose of determining s-numbers, S is thought of as $\mathbf{R}$-linear (note here that in the language of [**19**] S is $\mathbf{C}$-antilinear iff it is $\mathbf{R}$-linear and $SA = -AS$, where $A: \mathbf{C}^n \to \mathbf{C}^n$ is given by $Ax = ix$).

By duality, it is again enough to show that the assertion of Proposition 6.2 holds with Z replaced by $Z' = \tilde{X}_p^n \oplus_2 H$, where $\tilde{X}_p^n$ is obtained as described above (with $\delta = 1/2$) and $1/p = 1 - 1/q$. To this end, let $T: Z' \to Z'$ be $\mathbf{C}$-antilinear and invertible and, to argue by contradiction, suppose that

(7) $$\|T\| \cdot \|T^{-1}\| < \min \left\{ \frac{1}{2} \left(c' \left(\frac{1}{2} \right) \right)^2, c'' \right\} n^{1/p-1/2},$$

where the choice of c'' will be indicated later (note that $1/p - 1/2 = 1/2 - 1/q$). Consider the matrix representations

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{matrix} \tilde{X}_p^n \\ H \end{matrix}, \qquad T^{-1} = \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} \begin{matrix} \tilde{X}_p^n \\ H \end{matrix}.$$

Denote by k the smallest integer $\geq n/2$ and set $a = s_k(T_{11})$, $a' = s_k(T'_{11})$. Then, by (ii)′ and (7),

(8) $$aa' < 1/2.$$

On the other hand (cf. the remark at the end of §2), there exists (an $\mathbf{R}$-linear subspace) $F \subset \tilde{X}_p^n$, codim $F = k - 1 < n/2$ (resp. $F' \subset \tilde{X}_p^n$) such that

(9) $$\|T_{11}x\|_2 \leq a\|x\|_2 \quad \text{for } x \in F$$

and, respectively,

(9′) $$\|T'_{11}x\|_2 \leq a'\|x\|_2 \quad \text{for } x \in F'.$$

Now observe that

(10) $$I_{\tilde{X}_p^n} = T_{11}T'_{11} + T_{12}T'_{21}$$

and, similarly as in the argument leading to (6), choose a (real or complex) subspace E_1 of $\tilde{X}_p^n$ with (real) dim $E_1 \geq \frac{3}{2}n$ such that $T_{12}T'_{21}E_1$ is "nearly" Euclidean in the sense of (2) (we use our analogues of (iii) and (iv) from Corollary 3.4 and [**15**, Théorème 8] (or [**16**, Remark 2 or **17**, Proposition 3]). Then hs$(T_{12}T'_{21}|_{E_1}) \leq \frac{1}{4}n^{1/2}$ if c'' in (7) was chosen small enough. Accordingly, there exists a further (real) subspace $E_2 \subset E_1$, dim $E_2 \geq n$, such that, for $x \in E_2$,

$$\|T_{12}T'_{21}x\|_2 \leq \sqrt{\frac{2}{n}} \cdot \frac{1}{4}n^{1/2}\|x\|_2 = \frac{\sqrt{2}}{4}\|x\|_2.$$

This, combined with (8), (9), (9)$'$ and (10), shows that if $x \in E_2 \cap F' \cap T_{11}^{-1}(F)$ (a nontrivial subspace), then

$$\|x\|_2 = \|T_{11}T'_{11}x + T_{12}T'_{21}x\|_2 \leq \left(aa' + \frac{\sqrt{2}}{4}\right)\|x\|_2 \leq \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)\|x\|_2,$$

a contradiction if $x \neq 0$. This shows Proposition 6.2 and concludes the proof of Theorem 6.1.

REMARK 6.3. As was communicated to the author by N. J. Kalton, the example [8] of a (closed) subspace of L_p ($p \in [0,1)$), every endomorphism in which is a multiple of the identity, can be modified to yield a like complex space such that every **R**-linear endomorphism in it is a (complex) multiple of the identity. It can also be modified to yield the isometric version of Theorem 6.1.

7. Open problems.

Problem 7.1. Let X be a (real) Banach lattice (resp. a space with unconditional basis). Does X admit complex structure? Is X isomorphic to Y^2 for some Banach space Y? Lattice isomorphic to Y^2 for some Banach lattice Y? It should be noticed that the spaces constructed in this paper do not even have the so-called local unconditional structure (see [4]).

Problem 7.2. Does there exist a real Banach space nonisomorphic to a Hilbert space which admits unique complex structure?

Problem 7.3. In [19] the "worst" possible finite-dimensional examples for nonexistence (resp. nonuniqueness) of complex structure were given. Under what nontrivial assumptions do we have positive results? In particular, if a space has a 1-unconditional basis?

Problem 7.4. Can we replace in Corollary 1.2 "$X \not\sim Y^2$" by "$X \not\sim Y^m$" for given $m > 2$? Any $m > 2$?

Problem 7.5. Under what general assumptions about a group of operators on l_2^n, does some element of the group not admit "good" approximation by a multiple of identity in the quasi-norm $\|\cdot\|_{C_0}$ (equivalently, satisfies the condition $(M_{k,\alpha})$ from [19] for large k, α)? For groups acting irreducibly on l_2^n this was essentially done in [12]; [19] provides other examples. Similarly, under what general assumptions about a group Γ does there exist a Banach space X such that there are no representations of Γ acting on X? Note that the assertion of Theorem 1.1 from this paper can be restated as follows. "There is no representation (of the circle group) $\sigma: S^1 \to L(X)$ with $\sigma(-1) = -I$" and one can ask under what general assumptions on a group (resp. representation) a statement like this is possible (this may be related to [13]). Also recall the well-known question about existence of a Banach space, all bounded linear operators on which one of the form $\lambda I + \Delta$ with Δ compact (resp. nuclear) (cf. Remark 6.3 and [8]). This seems to be quite hard; one can ask instead for an example of a space X for which the quotient algebra of $L(X)$ by the ideal of compact (resp. nuclear) operators is "small".

Problem 7.6. Do we have a complex version of Corollary 1.2? It would presumably follow if we had a complex version of Theorem 1.5 from [19].

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