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A Supersymmetric Extension of the Kadomtsev-Petviashvili Hierarchy

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Abstract. An extension of the Kadomtsev-Petviashvili hierarchy by odd variables is given. Conservation laws and formal integrability are proved.

0. Introduction

It is well known that integrable systems usually admit a natural extension by odd variables, see e.g. [1–4].

In the articles [5, 6] an infinite hierarchy of nonlinear differential equations was studied with the property that many known completely integrable systems can be obtained from this hierarchy by various reductions. It is called the Kadomtsev-Petviashvili hierarchy and it can be succinctly written in the following way. Let x be a space variable and $t_1, t_2, \ldots$ an infinite system of time variables. Denote by $u_{-1}, u_{-2}, \ldots$ an infinite set of functions depending on $x, t_1, t_2, \ldots$. Set $\partial = \partial/\partial x$ and introduce a formal pseudo-differential operator $L = \partial + \sum_{i=1}^{\infty} u_{-i}\partial^{-i}$. The Kadomtsev-Petviashvili (KP) hierarchy is

$$\partial_i L = [L^i{}_+, L], \quad \partial_i = \partial/\partial t_i, \tag{1}$$

where A_+ is the differential part of an operator A. If in addition $L^2 = (L^2)_+$, Eq. (1) becomes the i^{th} equation of the Korteweg-de Vries hierarchy (KdV itself corresponding to $i = 3$).

The objective of this note is to introduce a new system of equations for an infinite set of even and odd functions, depending on an even-odd pair of space variables (x, ξ) and even-odd times $(\tau_1, t_2, \tau_3, t_4, \ldots)$. We shall call this system the supersymmetric Kadomtsev-Petviashvili (SKP) hierarchy, since KP is its natural reduction. We shall show that SKP shares the standard properties of "completely integrable" systems, e.g. has infinitely many conservation laws, is formally solvable by the Zakharov-Shabat method, and can be reduced to Lax- and Gelfand-Dikii (Dickey) equations. In a subsequent publication we hope to discuss the solitons and algebraic type solutions of SKP, as well as the transformation groups for this hierarchy.

Section 1 contains definitions and statements of principal results. The proofs are given in Sect. 2.

1. Definitions and Results

1.1. Pseudodifferential Operators on the $1|1$ *Line*

The reader may consult [7 or 8] for the background in superalgebra and supergeometry.

We fix an even variable x and odd one ξ; in general we set $\tilde{X}=0$ (respectively 1), if X is even (respectively odd). Set $\theta=\partial/\partial\xi+\xi\,\partial/\partial x$. Then $\theta^2=\frac{1}{2}[\theta,\theta]=\partial$. (We recall that the supercommutator is defined by $[X,Y]=XY-(-1)^{\tilde{X}\tilde{Y}}YX$.)

Let B be a $\mathbb{Z}_2$-graded ring, on which θ acts as an odd superderivation, $\widetilde{\theta b}=\tilde{b}+1$, $\theta(bc)=(\theta b)c+(-1)^{\tilde{b}}b\theta c$. We shall write $\theta^i b=b^{[i]}$. The ring of formal pseudodifferential operators $B((\theta^{-1}))$ consists of the formal series $L=\sum\limits_{i\leqq m} b_i\theta^i$, $b_i\in B$. It is $\mathbb{Z}_2$-graded by $\widetilde{b\theta^i}=\tilde{b}+\tilde{i}$. We set $L_+=\sum\limits_{0\leqq i\leqq m} b_i\theta^i$, $L_-=\sum\limits_{i\leqq -1} b_i\theta^i$.

The multiplication in $B((\theta^{-1}))$ is best described by means of "superbinomial coefficients" $\begin{bmatrix} m \\ k \end{bmatrix}$, $k,m\in\mathbb{Z}$. First recall the ordinary coefficients: $\binom{m}{k}=0$ for $k>m$, 1 for $k=m$, $(k+1)\dots m/(m-k)!$ for $k<m$. Now we set

$$\begin{bmatrix} j \\ k \end{bmatrix}=\begin{cases} 0 & \text{for} \quad k>j \quad \text{and for} \quad (j,k)\equiv(0,1)\bmod 2 \\ \\ \dbinom{[j/2]}{[k/2]} & \text{for} \quad k\leqq j, \quad (j,k)\not\equiv(0,1)\bmod 2, \end{cases} \tag{2}$$

Then we define

$$\theta^j\circ a=\sum_{k\leqq j}\begin{bmatrix} j \\ k \end{bmatrix}(-1)^{\tilde{a}k}a^{[j-k]}\theta^k, \tag{3}$$

$$\sum b_j\theta^j\circ\sum a_\ell\theta^\ell=\sum_{j,k,\ell}\begin{bmatrix} j \\ k \end{bmatrix}(-1)^{\tilde{a}_\ell k}b_j a_\ell^{[j-k]}\theta^{k+\ell} \tag{4}$$

[we sometimes use $\circ$ for multiplication in $B((\theta^{-1}))$ to distinguish e.g. $\theta^j\circ a$ from $\theta^j a=a^{[j]}$].

1.2. Flows with Even and Odd Times

The evolution in KP (1) is given by specifying the derivations with respect to the commuting family of the flows ∂_i. The evolution in SKP is defined with respect to a non-abelian Lie superalgebra of flows θ_i, $i\geqq 1$, $\tilde{\theta}_i=\tilde{i}$ with the commutation relations

$$[\theta_{2i},\theta_{2j}]=0,\quad [\theta_{2i},\theta_{2j-1}]=0,\quad [\theta_{2i-1}\,\theta_{2j-1}]=2\theta_{2i+2j-2}\,. \tag{5}$$

The choice of (5) is motivated by the fact that $\theta_i \to \theta^i$ is a representation of (5), in the same manner as $\partial_i \to \partial^i$ is a representation of the abelian Lie algebra of KP.

Let us now introduce the even and odd time variables $(\tau_1, t_2, \tau_3, \ldots)$, and the following representation of (5) which we shall use in what follows:

$$\theta_{2i} = \partial/\partial t_{2i}, \quad \theta_{2i-1} = \partial/\partial\tau_{2i-1} + \sum_{j=1}^{\infty} \tau_{2j-1}\partial/\partial t_{2i+2j-2}. \tag{6}$$

With minor changes we could also use a more general representation

$$\theta_{2i-1} = \partial/\partial\tau_{2i-1} + \sum_{j=1}^{\infty} T_{2j-1}^{2i}(\tau)\,\partial/\partial t_{2j},$$

where T are some odd polynomials with constant coefficients such that

$$\partial/\partial\tau_{2i-1} T_{2k-1}^{2\ell} + \partial/\partial\tau_{2k-1} T_{2i-1}^{2\ell} = 2\delta_{\ell, i+k-1}.$$

1.3. The Definition of the SKP Hierarchy

Let us assume now that the Lie superalgebra (5) is represented by superderivations of the ring B, supercommuting with θ. This means that the elements of B can be informally considered as functions of x, ξ, τ_i, t_j. We shall extend the action of θ_i to $B((\theta^{-1}))$ in the obvious way.

Now we shall consider an odd pseudodifferential operator

$$\Lambda = \theta + \sum_{i=0}^{\infty} u_{-i}\theta^{-i}, \quad \tilde{u}_{-i} = \tilde{i} + 1, \quad u_i \in B,$$

and we shall call the SKP hierarchy the following system:

$$\left.\begin{aligned} \theta_{2i}\Lambda &= -[\Lambda^{2i}{}_+, \Lambda], \\ \theta_{2i-1}\Lambda &= -[\Lambda^{2i-1}{}_+, \Lambda] + 2\Lambda^{2i}. \end{aligned}\right\} \tag{7}$$

To explain the appearance of $-2\Lambda^{2i}$ in (7), recall that the ordinary KP (1) can be written also in the form $\partial_j L = -[L^j{}_-, L]$ since $[L^j{}_+ + L^j{}_-, L] = [L^j, L] = 0$. In our case $[\Lambda^j{}_+ + \Lambda^j{}_-, \Lambda] = [\Lambda^j, \Lambda] = 0$ for even j, but it is $2\Lambda^{j+1}$ for odd j. Hence the equivalent form of (7) is

$$\theta_j\Lambda = [\Lambda^j{}_-, \Lambda]. \tag{8}$$

Notice that (8) implies $\theta_j(u_{-1} + \frac{1}{2}u_0^{[1]}) = 0$. This is analogous to the conservation of the $(n-1)^{\text{th}}$ coefficient in the ordinary Lax equations.

To rewrite SKP once more, we introduce the formal differential

$$d = \sum_{i \geqq 1} (dt_{2i}\partial/\partial t_{2i} + d\tau_{2j-1}\partial/\partial\tau_{2j-1}).$$

Using (6), we have

$$\partial/\partial\tau_{2j-1} = \theta_{2j-1} - \sum_{k=1}^{\infty} \tau_{2k-1}\theta_{2j+2k-2},$$

and therefore (7) and (8) are equivalent to the equations

$$d\Lambda = -[U_\Lambda{}^+, \Lambda] + 2\sum_{j=1}^{\infty} d\tau_{2j-1}\Lambda^{2j} = [U_\Lambda{}^-, \Lambda], \tag{9}$$

where

$$U_\Lambda{}^{\pm} = \sum_{i=1}^{\infty} dt_{2i}\Lambda_{\pm}^{2i} + \sum_{j=1}^{\infty} d\tau_{2j-1}\left(\Lambda^{2j-1}_{\pm} - \sum_{k=1}^{\infty} \tau_{2k-1}\Lambda^{2j+2k-2}_{\pm}\right). \tag{10}$$

1.4. The Superresidue and the Berezin Integral

From now on we assume that B is supercommutative. We set

$$\mathrm{res}_\theta\left(\sum_{i \leqq m} b_i\theta^i\right) = b_{-1}\,.$$

We remind the reader that the Berezin integral $\int f(x,\xi)\,d(x,\xi)$ is well defined if $\frac{\partial}{\partial\xi} f$ as a function of x has a compact support (or quickly vanishes at infinity). In this case $\int f(x,\xi)\,d(x,\xi) = \int\left(\frac{\partial}{\partial\xi} f\right)dx$. In particular $\int f(x,\xi)\,d(x,\xi)=0$ if $\partial f/\partial\xi = \partial g/\partial x$, where g quickly vanishes at infinity. The following lemma is an analogue of a well known fact in the theory of the KdV hierarchy (cf. [10, Chap. 2, Lemma 3.3]).

1.5. Lemma. a) *Let $P, Q \in B((\theta^{-1}))$. Then there exists a universal polynomial $F = F(P, Q)$ depending on the coefficients of P, Q and their θ-derivatives, such that*

$$\mathrm{res}_\theta[P,Q] = \theta F(P,Q)\,,$$

b) $\int \mathrm{res}_\theta[P,Q]d(x,\xi) = 0$ *in the sense that*

$$\frac{\partial}{\partial\xi}\mathrm{res}_\theta[P,Q] = \frac{\partial}{\partial x}\left(1 - \xi\frac{\partial}{\partial\xi}\right)F(P,Q)\,.$$

1.6. Theorem on the Conservation Laws. *The SKP equations for an operator Λ imply the equations*

$$\theta_i R_n + \theta S_{i,n} = 0, \qquad i, n \geqq 1\,,$$

where

$$R_n = \mathrm{res}_\theta \Lambda^n, \qquad S_{i,n} = -F(\Lambda^i{}_-, \Lambda^n)\,,$$

F being defined in Lemma 1.5.

1.7. The Zakharov-Shabat Formalism

In this section we assume that the ring B has an appropriate topology. Then it may happen that the multiplication in $B((\theta^{-1}))$ can be defined by continuity on a certain subset of the doubly infinite formal series $B\{\{\theta^{-1}\}\} = \left\{\sum_i b_i\theta^i,\ i \in \mathbb{Z} \mid b_i \in B\right\}$.

We shall show that certain identities in such a ring make it possible to construct a solution of SKP. Consider a differential form-valued operator

$$U_\theta = \sum_{i=1}^{\infty} dt_{2i}\theta^{2i} + \sum_{j=1}^{\infty} d\tau_{2j-1}\left(\theta^{2j-1} - \sum_{k=1}^{\infty} \tau_{2k-1}\theta^{2j+2k-2}\right)$$

[cf. Eq. (10)]. Let $\Psi \in B\{\{\theta^{-1}\}\}$, $\tilde{\Psi}=0$. Assume that the following relations are valid:

$$d\Psi = -U_\theta \psi\,, \tag{11}$$

$$\Psi = V^{-1}Y,\ V = 1 + \sum_{k=1}^{\infty} v_{-k}\theta^{-k},\ Y = \sum_{\ell=0}^{\infty} y_\ell \theta^\ell\,, \tag{12}$$

where $\tilde{V}=\tilde{Y}=0$. The relation (11) is a system of linear differential equations with constant coefficients, and (12) is the decomposition of a Lie superalgebra element into a product of the upper and lower triangular parts, or a solution of the formal Riemann-Hilbert problem.

1.8. Theorem. *If Y is invertible in $B\{\{\theta^{-1}\}\}$ and verifies $(Y^{-1})_- = 0$, then $\Lambda = V\theta V^{-1}$ is a solution of SKP.*

We point out that the "dressed" operator Λ verifies the relation $u_{-1} + \frac{1}{2}u_0{}^{[1]} = 0$.

Our next results concern the reductions of SKP to the Lax-Gelfand-Dikii hierarchies [9, 10].

1.9. Theorem on the Fractional Powers. *Let $L = \theta^N + \sum_{n \leqq N-1} u_n\theta^n$. If $N > 0$, $N \equiv 1(2)$, then there exists a unique odd operator $\Lambda = \theta + \sum_{i \leqq 0} v_i\theta^i$ such that $\Lambda^N = L$.*

For $N \equiv 0 \bmod 2$ both existence and uniqueness need not be true.

1.10. Variational Formalism

In the following we shall consider the coefficients u_i of the differential operator L as differentially independent variables. To this end we set $B = A[u_i{}^{[j]}]$, $\widetilde{u_i^{[j]}} = \tilde{u}_i + \tilde{j}$, where $u_i{}^{[j]}$ are algebraically independent variables and A is a supercommutative ring with an action of θ. We shall extend the action of θ to B, setting $\bar{\theta}|A = \theta$, $\bar{\theta}u_j{}^{[i]} = u_j{}^{[i+1]}$, and in what follows we shall denote $\bar{\theta}$ again by θ. The Euler-Lagrange operators are defined by the formula

$$\frac{\delta P}{\delta u_i} = \sum_k (-1)^{\tilde{u}_i k + \frac{k(k+1)}{2}} \left(\frac{\partial P}{\partial u_i{}^{[k]}}\right)^{[k]}. \tag{13}$$

Furthermore, we introduce the supersymmetric Gelfand-Dikii operator (cf. [9, 10]),

$$\gamma = \Gamma - \Gamma^*\,,$$

$$\Gamma_{bc} = \sum_{a \geqq 0} (-1)^{(c+1)(b+1)} \begin{bmatrix} a+b \\ b \end{bmatrix} u_{a+b+c+1}\theta^a\,. \tag{14}$$

Here Γ^* is defined from the adjointness property

$$\sum_{b,c} \int (\Gamma_{bc}x_c)y_b\, d(x,\xi) = \sum_{b,c} \int (\Gamma_{bc}^* y_c)x_b\, d(x,\xi)\,,$$

where $\tilde{x}_i = \tilde{y}_i = \tilde{u}_i = \tilde{i} + 1$. See (21) for the explicit formula.

The operator γ is a highly formal object, having an infinite order and infinite matrix coefficients, but it becomes finite if we set $u_N = 1$, $u_i = 0$ for $i \geqq N+1$. Denote the result by $\gamma^{(N)}$.

1.11. Theorem on the Lax Equations. *Let* $N \equiv p \equiv 1 \bmod 2$, $p \geqq N$, $L = \theta^N + \sum_{n \leqq N-1} u_n \theta^n$, $\tilde{\Lambda} = 1$, $\Lambda^N = L$, $\Lambda = \theta + \sum_{i \leqq 0} v_i \theta^i$. *Moreover, let*

$$H\left(\frac{p}{N}\right) = \frac{N}{p} \operatorname{res}_\theta \Lambda^p .$$

Then the SKP equations for Λ *imply the Gelfand-Dikii equations for the coefficients of* L:

$$\theta_{p-N} u_b = -\sum_c \gamma_{bc}^{(N)} \frac{\delta}{\delta u_c} H\left(\frac{p}{N}\right). \tag{15}$$

2. The Proofs

2.1. Some Identities with Superbinomial Coefficients

a) The coefficients $\begin{bmatrix} j \\ k \end{bmatrix}$ for $0 \leqq j$, $k \leqq 7$ constitute the matrix:

$j \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	0	1					
3	1	1	1	1				
4	1	0	2	0	1			
5	1	1	2	2	1	1		
6	1	0	3	0	3	0	1	
7	1	1	3	3	3	3	1	1

It is clear how to continue it further.

The identity $(-1)^k \cdot \binom{m}{k} \cdot k = (-1)^m \binom{-k}{-m} \cdot m$ (which follows from the definition) implies

$$\begin{bmatrix} m \\ -l-1 \end{bmatrix} = (-1)^{\left[\frac{m}{2}\right] + \left[\frac{\ell}{2}\right] + 1} \begin{bmatrix} \ell \\ -m-1 \end{bmatrix} \quad \text{for} \quad \ell, m \in \mathbb{Z}. \tag{16}$$

b) The most direct way to describe multiplication in $B((\theta^{-1}))$ is to consider (4) as the definition. Then it is necessary to prove the associativity law: $(P \circ Q) \circ R = P \circ (Q \circ R)$. Collecting together similar terms we can reduce it to a bilinear identity in $\begin{bmatrix} j \\ k \end{bmatrix}$, and then prove it directly. Another way is to use the well-known construction of multiplication in $B((\partial^{-1}))$ by means of the ordinary binomial coefficients and the fact that $B((\theta^{-1}))=B((\partial^{-1}))[\theta]$ with evident commutation relations,

$$\theta(a\,\partial^m)=a^{[1]}\theta^{2m}+(-1)^{\tilde{a}}a\theta^{2m+1}.$$

2.2. Proof of Lemma 1.5

The statement b) follows from a) because $\frac{\partial}{\partial\xi}\theta F(P,Q)=\frac{\partial}{\partial x}\left(1-\xi\frac{\partial}{\partial\xi}\right)F(P,Q)$.

To prove a) it is sufficient to consider the case $P=a\theta^m$, $Q=b\theta^\ell$. From (4) we find

$$a\theta^m\circ b\theta^\ell=a\sum_{k\leqq m}\begin{bmatrix} m\\ k\end{bmatrix}(-1)^{\tilde{b}k}b^{[m-k]}\theta^{k+\ell}.$$

Hence

$$\mathrm{res}_\theta(a\theta^m\circ b\theta^\ell)=\begin{bmatrix} m\\ -\ell-1\end{bmatrix}(-1)^{\tilde{b}(\ell+1)}ab^{[m+\ell+1]}.$$

By the symmetry

$$\mathrm{res}_\theta(b\theta^\ell\circ a\theta^m)=\begin{bmatrix} \ell\\ -m-1\end{bmatrix}(-1)^{\tilde{a}(m+1)}ba^{[m+\ell+1]}.$$

Using these two formulas and (16) we find

$$\mathrm{res}_\theta[P,Q]=\begin{bmatrix} m\\ -\ell-1\end{bmatrix}(-1)^{\tilde{b}(\ell+1)}(ab^{[m+\ell+1]}+(-1)^Ma^{[m+\ell+1]}b), \tag{17}$$

$$M=\tilde{a}(m+\ell+1)+m\ell+\left[\frac{m}{2}\right]+\left[\frac{\ell}{2}\right].$$

The right-hand side of (17) is represented as the complete θ-derivative in two different ways depending on the parity of $m+\ell+1$ $\left(\text{the case } (m,\ell)\equiv(0,0)\bmod 2\right.$ needs no attention, since then $\left.\begin{bmatrix} m\\ -\ell-1\end{bmatrix}=0\right)$:

$$ab^{[m+\ell+1]}+(-1)^Ma^{[m+\ell+1]}b$$

$$=\begin{cases}\theta\left(\sum\limits_{i=0}^{m+\ell}(-1)^{\tilde{a}(i+1)+\left[\frac{i}{2}\right]}a^{[i]}b^{[m+\ell-i]}\right) & \text{if}\quad m+\ell+1\equiv 1\bmod 2\\ \theta^2\left(\sum\limits_{i=0}^{\frac{m+\ell-1}{2}}(-1)^ia^{[2i]}b^{[m+\ell-2i-1]}\right) & \text{if}\quad m+\ell+1\equiv 0\bmod 2.\end{cases}$$

2.3. Proof of Lemma 1.6

From (8) we find $\theta_i\Lambda^n=[\Lambda^i{}_-,\Lambda^n]$. Therefore

$$\theta_i\,\mathrm{res}_\theta\Lambda^n=\mathrm{res}_\theta\theta_i\Lambda^n=\mathrm{res}_\theta[\Lambda^i{}_-,\Lambda^n]=\theta F(\Lambda^i{}_-,\Lambda^n).$$

2.4. Proof of Theorem 1.8

In the notation of Sect. 1.7 we construct the following operators from the data (11) and (12):

$$\Lambda = V \circ \theta \circ V^{-1}, U^+ = (dY) \circ Y^{-1}, U^- = -(dV) \circ V^{-1}.$$

On the other hand, we construct from Λ "connection forms" $U_\Lambda^\pm$ by the formula (10). We shall check that if the condition of Theorem 1.8 is fulfilled, then $U^\pm = -U_\Lambda^\pm$. Really, using that Ψ is invertible in view of (12), we find from (11),

$$\begin{aligned} -U_\theta &= d\Psi \circ \Psi^{-1} = d(V^{-1} \circ Y) \circ Y^{-1} \circ V \\ &= (-V^{-1} dV \circ V^{-1} \circ Y + V^{-1} \circ dY) Y^{-1} \circ V \\ &= V^{-1}(-dV \circ V^{-1} + dY \circ Y^{-1}) V = V^{-1}(U^- + U^+) V. \end{aligned}$$

Hence

$$U^+ + U^- = -VU_\theta V^{-1} = -\sum_{i=1}^{\infty} dt_{2i}\Lambda^{2i} + \sum_{j=1}^{\infty} d\tau_{2j-1}\left(\Lambda^{2j-1} - \sum_{k=1}^{\infty} \tau_{2k-1}\Lambda^{2j+2k-2}\right)$$

and, consequently, $U^\pm = -U_\Lambda^\pm$, because U^+ is a differential operator, and U^- is an integral one. On the other hand, by the definition of Λ,

$$\begin{aligned} d\Lambda &= d(V \circ \theta \circ V^{-1}) = dV \circ \theta \circ V^{-1} + V \circ \theta \circ V^{-1} \circ dV \circ V^{-1} \\ &= [dV \circ V^{-1}, V \circ \theta \circ V^{-1}] = -[U^-, \Lambda] = [U_\Lambda^-, \Lambda], \end{aligned}$$

i.e. the SKP hierarchy in the form (9).

2.5. Proof of Theorem 1.9

Let us construct Λ using successive approximations. Assume that at the r^{th} step, $r \geqq -1$, we have proved the following statement:

there exists an operator $X_r = \theta + O(1)$ *such, that* $X_r^N = L + O(\theta^{N-r-2})$*; it is defined uniquely* $\bmod O(\theta^{-(r+1)})$.

For $r = -1$, evidently, $X_r = \theta$. To make the inductive step we set $X_{r+1} = X_r + x_{r+1}\theta^{-(r+1)}$. Then

$$X_{r+1}^N = X_r^N + \sum_{i=0}^{N-1} X_r^i \circ x_{r+1}\theta^{-(r+1)} \circ X_r^{N-i-1} + R_r.$$

Now we compute the right-hand side sum modulo $O(\theta^{N-r-3})$:

$$\begin{aligned} \Sigma &= \sum_{i=0}^{N-1} (-1)^i x_{r+1}\theta^{-(r+1)} \circ X_r^{N-1} \bmod O(\theta^{N-r-3}) \\ &= x_{r+1}\theta^{N-r-2} \bmod O(\theta^{N-r-3}). \end{aligned}$$

(We use the fact that N is odd only in this place: if N is even the linear term in x_{r+1} disappears.) The remainder is the sum of the products of $j \leqq N-2$ terms X_r and of $N-j$ terms $x_{r+1}\theta^{-(r+1)}$. Therefore its order with respect to θ is not more than

$\max_{j \leqq N-2} [j-(N-j)(r+1)] = N-2r-4 \leqq N-r-3$. Hence x_{r+1} is defined uniquely from the condition $X_{r+1}^N = L + O(\theta^{N-2-3})$. To finish the proof we set $\Lambda = \lim_{r\to\infty} X_r$.

We give two examples concerning even N.

a) Let $L = \theta^{2n} + \sum_{i \leqq 2n-1} u_i \theta^i$, $\tilde{L} = 0$. If $\operatorname{res}_\theta L \notin \theta B$, then $L \neq \Lambda^{2n}$ for any odd Λ. In fact, $\Lambda^{2n} = \frac{1}{2}[\Lambda^{2n-1}, \Lambda]$, so $\operatorname{res}_\theta \Lambda^{2n}$ should be a θ-derivative in view of Lemma 1.5.

b) In the case $L = \theta^2$ the following nonuniqueness of the square root arises: we can take Λ in the form $\theta + u_0 - \frac{1}{2} u_0^{[1]} \theta^{-1}{}_+ + \sum_{i \leqq -2} u_i \theta^i$, where the u_i satisfy the following conditions:

$$\begin{cases} u_{i,x} = 0\,, \\ u_{2p-1,\xi} = 0\,, \\ u_{2p,\xi} + 2u_{2p-1} + \sum_{\ell+m=p+1} (u_{2\ell-1} u_{2m,\xi} + u_{2\ell-1} u_{2m-1}) = 0\,. \end{cases}$$

It is possible, for instance, to choose arbitrary coefficients u_{2p} independent of x and then to compute u_{2p-1} by induction. In fact, the first series of equations follows from the equality

$$0 = [\Lambda^2, \Lambda] = [\partial_x, \Lambda] = \sum_{i \leqq 0} u_{i,x} \theta^i\,.$$

Coefficients of $\frac{1}{2}[\Lambda, \Lambda]$ at odd powers of θ give us the following equations:

$$u_{2p-1,\xi} + \sum_{m+n=p+1} u_{2m-1} u_{2n-1,\xi} = 0\,.$$

Now, inductively we get $u_{2p-1,\xi} = 0$. Finally, the coefficients at $\theta^{2\ell}$, $\ell \leqq 0$, lead to the last series of equations.

2.6. More on the Variational Formalism

Using the notations from Sect. 1.10 we consider the B-module of relative differentials $\Omega = \Omega^1 B/A$. It is freely generated by the elements $\delta u_i^{[j]}$. We assume that $\tilde{\delta} = 0$ (that is $\delta\tilde{P} = \tilde{P}$). The derivation θ acts on Ω by the formula $\theta(\delta P) = \delta(P^{[1]})$. Therefore we can define a bimodule $\Omega((\theta^{-1}))$ over the ring $B((\theta^{-1}))$ using the formula (4) for the exterior multiplication. The associativity axiom is verified by repeating the reasoning from Sect. 2.1b). The map $\delta : B((\theta^{-1})) \to \Omega((\theta^{-1}))$ is defined termwise. It shares the property $\delta(P \circ Q) = \delta P \circ Q + P \circ \delta Q$.

Let $\omega \in \Omega$ be a variational differential. It has a unique representation in the form $\omega = \omega_0 + \theta\omega_1$, where $\omega_0 \in \oplus B\,\delta u_i$. The existence is proved by the classical procedure of integrating by parts

$$a\,\delta u_i^{[j+1]} = (-1)^{\tilde{a}+1}(a^{[1]}\,\delta u_i^{[j]} - \theta(a\,\delta u_i^{[j]}))\,. \tag{18}$$

The uniqueness follows from the fact: if $\theta\omega_1 \neq 0$, then $\theta\omega_1 \notin \oplus\, B\,\delta u_i$ because the term $a\,\delta u_i^{[j]}$ in ω_1 with maximal j gives the term $a\,\delta u_i^{[j+1]}$ in $\theta\omega_1$ which cannot cancel.

It is easy to derive from (18) the following lemma, giving the invariant description of the Euler-Lagrange operators (13):

2.7. Lemma. *Let $P \in B$ and $\delta P = \sum \delta u_i Q_i + \theta \omega_1$. Then $Q_i = \frac{\delta P}{\delta u_i}$.*

To prove Theorem 1.11 we need the following facts:

2.8. Lemma. a) *Let $\Lambda \in B((\theta^{-1}))$, $\tilde{\Lambda} = 1$, $\Lambda = \theta + \sum_{i \leqq 0} v_i \theta^i$, $p \geqq 1$. Then*

$$\delta\, \mathrm{res}_\theta \Lambda^p \equiv \begin{Bmatrix} p\, \mathrm{res}_\theta(\delta\Lambda \circ \Lambda^{p-1}) & \text{if} & p \equiv 1(2) \\ 0 & \text{if} & p \equiv 0(2) \end{Bmatrix} \bmod \theta\Omega .$$

b) *Let $L = \Lambda^N$. Then*

$$\delta\, \mathrm{res}_\theta \Lambda^p \equiv \frac{p}{N} \mathrm{res}_\theta(\delta L \circ \Lambda^{p-N}) \bmod \theta\Omega \quad \text{if} \quad N \equiv p \equiv 1 \bmod 2 .$$

Proof. a) We have

$$\delta\Lambda^p = \sum_{i=0}^{p-1} \Lambda^i \circ \delta\Lambda \circ \Lambda^{p-i-1},$$

$$\Lambda^i \circ \delta\Lambda \circ \Lambda^{p-i-1} = (-1)^{i(p-i)}(\delta\Lambda \circ \Lambda^{p-1} - [\delta\Lambda \circ \Lambda^{p-i-1}, \Lambda^i]) .$$

Furthermore $\sum_{i=0}^{p-1} (-1)^{i(p-i)} = p$ if $p \equiv 1(2)$ and 0 if $p \equiv 0(2)$. So the statement a) follows from Lemma 1.5, remaining undoubtedly valid when it is applied to commutators in

$$[\Omega((\theta^{-1})), B((\theta^{-1}))] .$$

b) Similarly

$$\delta\Lambda^N \circ \Lambda^{p-N} = \sum_{i=0}^{N-1} \Lambda^i \circ \delta\Lambda \circ \Lambda^{p-i-1} \equiv N\, \delta\Lambda \circ \Lambda^{p-1} \bmod \theta\Omega \quad \text{if} \quad N \equiv 1(2) .$$

This finishes the proof.

2.9. Lemma. *Let $N \equiv 1(2)$, $L = \theta^N + \sum_{i=0}^{N-1} u_i \theta^i$, $\tilde{L} = 1$, u_i being independent differential variables. Let Λ be the odd N^{th} power root from L, constructed in Sect. 2.5. We define the differential polynomials $v_k\left(\frac{p}{N}\right)$ in u_i by the formula*

$$\Lambda^p_- = (L^{p/N})_- = \sum_{k \leqq 0} \theta^{k-1} \circ v_k\left(\frac{p}{N}\right).$$

Then if $p \equiv 1 \bmod 2$, we have for $k \leqq N-1$

$$\frac{\delta}{\delta u_k} v_0\left(\frac{p}{N}\right) = \frac{p}{N}(-1)^{k+1} v_{-k}\left(\frac{p-N}{N}\right). \tag{19}$$

Proof. Since $v_0\left(\frac{p}{N}\right)$ is even, $\operatorname{res}_\theta \Lambda^p = v_0\left(\frac{p}{N}\right)$. From the previous lemma we have

$$\begin{aligned}
\delta v_0\left(\frac{p}{N}\right) &= \delta \operatorname{res}_\theta \Lambda^p \equiv \frac{p}{N} \operatorname{res}_\theta(\delta L \circ \Lambda^{p-N}) \\
&= \frac{p}{N} \operatorname{res}_\theta\left(\sum_{j=0}^{N-1} \delta u_j \theta^j \circ \sum_{k \leqq 0} \theta^{k-1} \circ v_k\left(\frac{p-N}{N}\right)\right) \\
&= \frac{p}{N} \operatorname{res}_\theta\left(\sum_{j=0}^{N-1} \delta u_j \theta^{-1} \circ v_{-j}\left(\frac{p-N}{N}\right)\right) \\
&= \frac{p}{N} \sum_{j=0}^{N-1} \delta u_j (-1)^{j+1} v_{-j}\left(\frac{p-N}{N}\right).
\end{aligned}$$

To finish the proof we use Lemma 2.6.

2.10. The End of the Proof of Theorem 1.11

From the SKP equations for Λ we derive the following equations for L:

$$-\theta_{p-N} L = [L, \Lambda^{p-N}{}_{-}],$$

or

$$-\sum_{b=0}^{N-1} (\theta_{p-N} u_b)\theta^b = \left[\sum_{\ell=0}^{N} u_\ell \theta^\ell, \sum_{j \geqq 0} \theta^{-j-1} \circ v_{-j}\left(\frac{p-N}{N}\right)\right]. \tag{20}$$

We shall show that the coefficient at θ^b in the first member of the commutator is equal to $\sum_c \Gamma_{bc} \frac{\delta}{\delta u_c} v_0\left(\frac{p}{N}\right)$, where Γ_{bc} are defined in (14). Really

$$[L, \Lambda^{p-N}{}_{-}] = -[L, \Lambda^{p-N}{}_{+}] = [L, \Lambda^{p-N}{}_{-}]_{+}\,.$$

Hence it is sufficient to compute $(L\Lambda^{p-N}{}_{-})_+$ and $(\Lambda^{p-N}{}_{-}L)_+$. Using Lemma 2.9 we have

$$\begin{aligned}
(L\Lambda^{p-N}{}_{-})_+ &= \left(\sum_{\ell=0}^{N} u_\ell \theta^\ell \circ \sum_{j \geqq 0} (-1)^{j+1} \theta^{-(j+1)} \frac{\delta}{\delta u_j} H\left(\frac{p}{N}\right)\right)_+ \\
&= \left(\sum_{\ell=0}^{N} \sum_{j \geqq 0} (-1)^{j+1} u_\ell \sum_b \begin{bmatrix} \ell - j - 1 \\ b \end{bmatrix} (-1)^{b(j+1)}\right. \\
&\quad \left.\cdot \left(\frac{\delta}{\delta u_j} H\left(\frac{p}{N}\right)\right)^{[\ell - j - 1 - b]} \theta^b\right)_+ \\
&= \sum_{\substack{a,b,c \geqq 0 \\ a+b+c+1 \leqq N}} (-1)^{(c+1)(b+1)} u_{a+b+c+1} \begin{bmatrix} a+b \\ b \end{bmatrix} \left(\frac{\delta}{\delta u_c} H\left(\frac{p}{N}\right)\right)^{[a]} \theta^b\,.
\end{aligned}$$

Here we have denoted $\ell - j - b - 1 = a, j = c$. Computing $(\Lambda^{p-N}{}_{-}L)_{+}$, we obtain at first

$$\sum_{\substack{a,b,c\geq 0\\ a+b+c+1\leq N}} \begin{bmatrix} -c-1 \\ -a-c-1 \end{bmatrix} (-1)^{ab} \left(u_{a+b+c+1} \frac{\delta}{\delta u_c} H\left(\frac{p}{N}\right)\right)^{[a]} \theta^b .$$

Then we transform the superbinomial coefficient using (16):

$$\begin{bmatrix} -c-1 \\ -a-c-1 \end{bmatrix} = (-1)^{\left[\frac{-c-1}{2}\right]+\left[\frac{a+c}{2}\right]+1} \begin{bmatrix} a+c \\ c \end{bmatrix}.$$

If a or c are even, then the sign is equal to $(-1)^{\left[\frac{a}{2}\right]}$. If a and c are both odd then the coefficients vanish. Finally

$$(\Lambda^{p-N}{}_{-}L)_{+} = \sum_{\substack{a,b,c\geq 0\\ a+b+c+1\leq N}} (-1)^{\left[\frac{a}{2}\right]+ab} \begin{bmatrix} a+c \\ c \end{bmatrix} \left(u_{a+b+c+1} \frac{\delta}{\delta u_c} H\left(\frac{p}{N}\right)\right)^{[a]} \circ \theta^b . \tag{21}$$

The reader can check that the operators we have computed are adjoint.

2.11. Examples. Set $L = \Lambda^4 = \Lambda_{+}^4 = \theta^4 + v_1\theta + v_0$, $\tilde{v}_1 = 1$, $\tilde{v}_0 = 0$. Then Lax's equation $L_t = [L_{+}^{3/2}, L]$ is equivalent to

$$\left.\begin{aligned} v_{0t} &= \partial_x(v_{0xx}/4 + 3v_0{}^2/4 + 3v_1v_0{}^{[1]}/4)\,, \\ v_{1t} &= \partial_x(v_{1xx}/4 + 3v_1v_1{}^{[1]}/4 + 3v_1v_0/2)\,. \end{aligned}\right\} \tag{22}$$

Setting here $v_1 = 0$ we get KdV. On the other hand, setting $v_0 = 0$ and $v_1 = w_1 + \xi w_0$, $\tilde{w}_1 = 1$, $\tilde{w}_0 = 0$, we get from (22)

$$\left.\begin{aligned} w_{0t} &= \partial_x(w_{0xx}/4 + 3w_0{}^2/4 - 3w_1w_{1x}/4) \\ w_{1t} &= \partial_x(w_{1xx}/4 + 3w_0w_1/4)\,. \end{aligned}\right\} \tag{23}$$

Setting $w_1 = 0$ here we again get KdV. The first Eq. (23) after rescaling coincides with the first equation of Kupershmidt's SKdV, but the second equations are different (cf. [3]).

References

1. Saveliev, M.: Integrable supermanifolds and nonlinear systems. Preprint IHEP 84–20, Serpukhov (1984)
2. Chaichian, M., Kulish, P.: On the method of inverse scattering problem and Bäcklund transformations for supersymmetric equations. Phys. Lett. **78** B, 413 (1978)
3. Kupershmidt, B.: A super-Korteweg-de Vries equation: an integrable system. Preprint UTSI, Tullahoma (1984)
4. d'Auria, R., Sciuto, S.: Group theoretical construction of two-dimensional supersymmetric models. Nucl. Phys. B **171**, 189 (1980)
5. Date, E., Kashiwara, M., Miwa, T.: Vertex operators and τ-functions. Transformation groups for soliton equations. II. Proc. Jpn. Acad. Sci. **57** A, 387 (1981)

6. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Operator approach to the Kadomtsev-Petviashvili equation. Transformation groups for soliton equations. III. J. Phys. Soc. Jpn. **50**, 3866 (1981)
7. Wess, J.: Supersymmetry-supergravity. In: Lecture Notes in Physics, Vol. 77. Berlin, Heidelberg, New York: Springer 1978
8. Leites, D.: Introduction to the theory of supermanifolds. Russ. Math. Surv. **35**, 3 (1980)
9. Gelfand, I., Dikii, L.: Fractional powers of operators and hamiltonian systems. Funct. Analiz **10**, 13 (1976)
10. Manin, Yu.: Algebraic aspects of differential equations. J. Sov. Math. **11**, 1 (1979)

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