A survey of hypertoric geometry and topology

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Abstract. Hypertoric varieties are quaternionic analogues of toric varieties, important for their interaction with the combinatorics of matroids as well as for their prominent place in the rapidly expanding field of algebraic symplectic and hyperkähler geometry. The aim of this survey is to give clear definitions and statements of known results, serving both as a reference and as a point of entry to this beautiful subject.

Given a linear representation of a reductive complex algebraic group G, there are two natural quotient constructions. First, one can take a geometric invariant theory (GIT) quotient, which may also be interpreted as a Kähler quotient by a maximal compact subgroup of G. Examples of this sort include toric varieties (when G is abelian), moduli spaces of spacial polygons, and, more generally, moduli spaces of semistable representations of quivers. A second construction involves taking an algebraic symplectic quotient of the cotangent bundle of V, which may also be interpreted as a hyperkähler quotient. The analogous examples of the second type are hypertoric varieties, hyperpolygon spaces, and Nakajima quiver varieties.

The subject of this survey will be hypertoric varieties, which are by definition the varieties obtained from the second construction when G is abelian. Just as the geometry and topology of toric varieties is deeply connected to the combinatorics of polytopes, hypertoric varieties interact richly with the combinatorics of hyperplane arrangements and matroids. Furthermore, just as in the toric case, the flow of information goes in both directions.

On one hand, Betti numbers of hypertoric varieties have a combinatorial interpretation, and the geometry of the varieties can be used to prove combinatorial results. Many purely algebraic constructions involving matroids acquire geometric meaning via hypertoric varieties, and this has led to geometric proofs of special cases of the g-theorem for matroids [HSt, 7.4] and the Kook-Reiner-Stanton convolution formula [PW, 5.4]. Future plans include a geometric interpretation of the Tutte polynomial and of the phenomenon of Gale duality of matroids [BLP].

On the other hand, hypertoric varieties are important to geometers with no interest in combinatorics simply because they are among the most explicitly understood examples of algebraic symplectic or hyperkähler varieties, which are becoming increasingly prevalent in many areas of mathematics. For example, Nakajima's quiver varieties include resolutions of Slodowy slices and Hilbert schemes of points on ALE spaces, both of which play major roles in modern representation theory. Moduli spaces of Higgs bundles are currently receiving a lot of attention in string theory, and character varieties of fundamental groups of surfaces and 3-manifolds have become an important tool in low-dimensional topology. Hypertoric

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varieties are useful for understanding such spaces partly because their geometries share various characteristics, and partly through explicit abelianization theorems, such as those stated and conjectured in Section 3.4.

Our main goal is to give clear statements of the definitions and selected theorems that already appear in the literature, along with explicit references. With the exception of Theorem 3.2.2, for which we give a new proof, this article does not contain any proofs at all. Section 1 covers the definition of hypertoric varieties, and explains their relationship to hyperplane arrangements. Section 2 gives three different constructions of unions of toric varieties that are equivariantly homotopy equivalent to a given hypertoric variety. These constructions have been extremely useful from the standpoint of computing algebraic invariants, and can also make hypertoric varieties more accessible to someone with a background in toric geometry but less experience with algebraic symplectic or hyperkähler quotients. Finally, Section 3 in concerned with the cohomology of hypertoric varieties, giving concrete form to the general principle that hypertoric geometry is intricately related to the combinatorics of matroids.

Section 2 assumes a familiarity with toric varieties, but Sections 1 and 3 can both be read independently of Section 2. The main quotient construction of Section 1.1 is logically self-contained, but may be fairly opaque to a reader who is not familiar with geometric invariant theory. Two alternative interpretations of this construction are given in Remarks 1.1.1 and 2.1.6, or one can take it as a black box and still get a sense of the combinatorial flavor of the subject.

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1 Definitions and basic properties

Hypertoric varieties can be considered either as algebraic varieties or, in the smooth case, as hyperkähler manifolds. In this section we give a constructive definition, with a strong bias toward the algebraic interpretation. Section 1.1 proceeds in greater generality than is necessary for hypertoric varieties so as to unify the theory with that of other algebraic symplectic quotients, most notably Nakajima quiver varieties.

1.1 Algebraic symplectic quotients

Let G be a reductive algebraic group over the complex numbers acting linearly and effectively on a finite-dimensional complex vector space V, and let $\mathfrak{g}$ be its Lie algebra. The cotangent bundle

$$T^*V \cong V \times V^*$$

carries a natural algebraic symplectic form Ω. The induced action of G on T^*V is hamiltonian, with moment map

$$\mu : T^*V \to \mathfrak{g}^*$$

given by the equation

$$\mu(z,w)(x) = \Omega(x \cdot z, w) \quad \text{for all } z \in V,\ w \in V^*,\ x \in \mathfrak{g}.$$

Suppose given an element $\lambda \in Z(\mathfrak{g}^*)$ (the part of $\mathfrak{g}^*$ fixed by the coadjoint action of G), and a multiplicative character $\alpha : G \to \mathbb{C}^\times$, which may be identified with an element of $Z(\mathfrak{g}^*_{\mathbb{Z}})$ by taking its derivative at the identity element of G.[2] The fact that λ lies in $Z(\mathfrak{g}^*)$ implies that G acts on $\mu^{-1}(\lambda)$. Our main object of study in this survey will be the algebraic symplectic quotient[3]

$$\mathfrak{M}_{\alpha,\lambda} := T^*V /\!\!/\!\!/\!\!/_{\alpha,\lambda} G := \mu^{-1}(\lambda) /\!\!/_{\alpha} G.$$

Here the second quotient is a projective GIT quotient

$$\mu^{-1}(\lambda) /\!\!/_{\alpha} G := \operatorname{Proj} \bigoplus_{m=0}^{\infty} \left\{ f \in \operatorname{Fun}\left(\mu^{-1}(\lambda)\right) \,\middle|\, \nu(g)^* f = \alpha(g)^m f \ \text{ for all } g \in G \right\}, \tag{1}$$

where Fun denotes the ring of global algebraic functions, and $\nu(g)$ is the automorphism of $\mu^{-1}(\lambda)$ defined by g.

This quotient may be defined in a more geometric way as follows. A point $(z,w) \in \mu^{-1}(\lambda)$ is called **α-semistable** if there exists a function f on $\mu^{-1}(\lambda)$ and a positive integer m such that $\nu(g)^* f = \alpha(g)^m f$ for all $g \in G$ and $f(z,w) \neq 0$. It is called **α-stable** if it is α-semistable and its G-orbit in the α-semistable set is closed with finite stabilizers. Then the stable and semistable sets

$$\mu^{-1}(\lambda)^{\alpha-st} \subseteq \mu^{-1}(\lambda)^{\alpha-ss} \subseteq \mu^{-1}(\lambda)$$

are nonempty and Zariski open, and there is a surjection

$$\mu^{-1}(\lambda)^{\alpha-ss} \twoheadrightarrow \mathfrak{M}_{\alpha,\lambda}$$

with (z,w) and (z',w') mapping to the same point if and only if the closures of their G-orbits intersect in $\mu^{-1}(\lambda)^{\alpha-ss}$. In particular, the restriction of this map to the stable locus is nothing but the geometric quotient by G. For an introduction to geometric invariant theory that explains the equivalence of these two perspectives, see [P2].

[2]Strictly speaking, an element of $Z(\mathfrak{g}^*_{\mathbb{Z}})$ only determines a character of the connected component of the identity of G. It can be checked, however, that the notion of α-stability defined below depends only on the restriction of α to the identity component, therefore we will abusively think of α as sitting inside of $Z(\mathfrak{g}^*_{\mathbb{Z}})$.

[3]The algebraic symplectic quotient is sometimes denoted by three slashes rather than four. The reason for four is that it reduces dimension by four times the dimension of the compact form of G, as is apparent from Remark 1.1.1.

Remark 1.1.1 The algebraic symplectic quotient defined above may also be interpreted as a hyperkähler quotient. The even dimensional complex vector space T^*V admits a complete hyperkähler metric, and the action of the maximal compact subgroup $G_{\mathbb{R}} \subseteq G$ is **hyperhamiltonian** (also called **trihamiltonian**), meaning that it is hamiltonian with respect to all three of the real symplectic forms on T^*V. Then $\mathfrak{M}_{\alpha,\lambda}$ is naturally diffeomorphic to the hyperkähler quotient of T^*V by $G_{\mathbb{R}}$, in the sense of [HKLR], at the value $(\alpha, \operatorname{Re}\lambda, \operatorname{Im}\lambda) \in \mathfrak{g}^*_{\mathbb{R}} \otimes \mathbb{R}^3$. This was the original perspective on both hypertoric varieties [BD] and Nakajima quiver varieties [N1]. For more on this perspective in the hypertoric case, see Konno's survey in this volume [K4, §3].

We note that if $\alpha = 0$ is the trivial character of G, then Equation (1) simplifies to

$$\mathfrak{M}_{0,\lambda} = \operatorname{Spec} \operatorname{Fun}\left(\mu^{-1}(\lambda)\right)^G.$$

Furthermore, since $\mathfrak{M}_{\alpha,\lambda}$ is defined as the projective spectrum of a graded ring whose degree zero part is the ring of invariant functions on $\mu^{-1}(\lambda)$, we always have a projective morphism

$$\mathfrak{M}_{\alpha,\lambda} \twoheadrightarrow \mathfrak{M}_{0,\lambda}. \tag{2}$$

This morphism may also be induced from the inclusion of the inclusion

$$\mu^{-1}(\lambda)^{\alpha-ss} \subseteq \mu^{-1}(\lambda) = \mu^{-1}(\lambda)^{0-ss}.$$

From this we may conclude that it is generically one-to-one, and therefore a partial resolution. When $\lambda = 0$, we have a distinguished point in $\mathfrak{M}_{0,0}$, namely the image of $0 \in \mu^{-1}(0)$ under the map induced by the inclusion of the invariant functions into the coordinate ring of $\mu^{-1}(0)$. The preimage of this point under the morphism (2) is called the **core** of $\mathfrak{M}_{\alpha,0}$, and will be further studied (in the case where G is abelian) in Section 2.1.

On the other extreme, if λ is a regular value of μ, then G will act locally freely on $\mu^{-1}(\lambda)$. In this case *all* points will be α-stable for any choice of α, and the GIT quotient

$$\mathfrak{M}_{\lambda} = \mu^{-1}(\lambda)/\!\!/G$$

will simply be a geometric quotient. In particular, the morphism (2) becomes an isomorphism. Both the case of regular λ and the case $\lambda = 0$ will be of interest to us.

We call a pair (α, λ) **generic** if $\mu^{-1}(\lambda)^{\alpha-st} = \mu^{-1}(\lambda)^{\alpha-ss}$. In this case the moment map condition tells us that the stable set is smooth, and therefore that the quotient $\mathfrak{M}_{\alpha,\lambda}$ by the locally free G-action has at worst orbifold singularities. Using the hyperkähler quotient perspective of Remark 1.1.1, one can prove the following Proposition. (See [K3, 2.6] or [HP1, 2.1] in the hypertoric case, and [N1, 4.2] in the case of quiver varieties; the general case is no harder than these.)

Proposition 1.1.2 *If (α, λ) and (α', λ') are both generic, then the two symplectic quotients $\mathfrak{M}_{\alpha,\lambda}$ and $\mathfrak{M}_{\alpha',\lambda'}$ are diffeomorphic.*

Remark 1.1.3 If G is semisimple, then $Z(\mathfrak{g}^*) = \{0\}$, and (unless G is finite) it will not be possible to choose a regular value $\lambda \in Z(\mathfrak{g}^*)$, nor a nontrivial character α. We will very soon specialize, however, to the case where G is abelian. In this case $Z(\mathfrak{g}^*) = \mathfrak{g}^*$, the regular values form a dense open set, and the characters of G form a full integral lattice $\mathfrak{g}^*_{\mathbb{Z}} \subseteq \mathfrak{g}^*$.

1.2 Hypertoric varieties defined

Let $\mathfrak{t}^n$ be the coordinate complex vector space of dimension n with basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$, and let $\mathfrak{t}^d$ be a complex vector space of dimension d containing a d-dimensional lattice $\mathfrak{t}^d_{\mathbb{Z}}$. Though $\mathfrak{t}^d_{\mathbb{Z}}$ is necessarily isomorphic to the standard integer lattice $\mathbb{Z}^d$, we will not choose such an isomorphism. Let $\{a_1, \ldots, a_n\} \subset \mathfrak{t}^d_{\mathbb{Z}}$ be a collection of nonzero vectors such that the map $\mathfrak{t}^n \to \mathfrak{t}^d$ taking ε_i to a_i is surjective. Let $k = n - d$, and let $\mathfrak{t}^k$ be the kernel of this map. Then we have an exact sequence

$$0 \longrightarrow \mathfrak{t}^k \overset{\iota}{\longrightarrow} \mathfrak{t}^n \longrightarrow \mathfrak{t}^d \longrightarrow 0, \tag{3}$$

which exponentiates to an exact sequence of tori

$$0 \longrightarrow T^k \longrightarrow T^n \longrightarrow T^d \longrightarrow 0. \tag{4}$$

Here $T^n = (\mathbb{C}^\times)^n$, T^d is a quotient of T^n, and $T^k = \ker\big(T^n \to T^d\big)$ is a subgroup with Lie algebra $\mathfrak{t}^k$, which is connected if and only if the vectors $\{a_i\}$ span the lattice $\mathfrak{t}^d_{\mathbb{Z}}$ over the integers. Note that every algebraic subgroup of T^n arises in this way.

The torus T^n acts naturally via coordinatewise multiplication on the vector space $\mathbb{C}^n$, thus so does the subtorus T^k. For $\alpha \in (\mathfrak{t}^k)^*_{\mathbb{Z}}$ a multiplicative character of T^k and $\lambda \in (\mathfrak{t}^k)^*$ arbitrary, the algebraic symplectic quotient

$$\mathfrak{M}_{\alpha,\lambda} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{\alpha,\lambda} T^k$$

is called a **hypertoric variety**.

The hypertoric variety $\mathfrak{M}_{\alpha,\lambda}$ is a symplectic variety of dimension $2d$ which admits a complete hyperkähler metric. The action of the quotient torus $T^d = T^n/T^k$ on $\mathfrak{M}_{\alpha,\lambda}$ is hamiltonian with respect to the algebraic symplectic form, and the action of the maximal compact subtorus $T^d_{\mathbb{R}}$ is hyperhamiltonian. In the original paper of Bielawski and Dancer [BD] the hyperkähler perspective was stressed, and the spaces were referred to as "toric hyperkähler manifolds". However, since we have worked frequently with singular reductions as well as with fields of definition other than the complex numbers (see for example [HP1, P3, PW]), we prefer the term hypertoric varieties.

Remark 1.2.1 In the hypertoric case, the diffeomorphism of Proposition 1.1.2 can be made $T^d_{\mathbb{R}}$-equivariant [HP1, 2.1].

1.3 Hyperplane arrangements

The case in which $\lambda = 0$ will be of particular importance, and it is convenient to encode the data that were used to construct the hypertoric variety $\mathfrak{M}_{\alpha,0}$ in terms of an arrangement of affine hyperplanes with some additional structure in the real vector space $(\mathfrak{t}^d)^*_{\mathbb{R}} = (\mathfrak{t}^d)^*_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. A **weighted, cooriented, affine hyperplane** $H \subseteq (\mathfrak{t}^d)^*_{\mathbb{R}}$ is an affine hyperplane along with a choice of nonzero integer normal vector $a \in \mathfrak{t}^d_{\mathbb{Z}}$. Here "affine" means that H need not pass through the origin, and "weighted" means that a is not required to be primitive. Let $r = (r_1, \ldots, r_n) \in (\mathfrak{t}^n)^*$ be a lift of α along ι^*, and let

$$H_i = \{x \in (\mathfrak{t}^d)^*_{\mathbb{R}} \mid x \cdot a_i + r_i = 0\}$$

be the weighted, cooriented, affine hyperplane with normal vector $a_i \in (\mathfrak{t}^d)^*_{\mathbb{Z}}$. (Choosing a different r corresponds to simultaneously translating all of the hyperplanes by a vector in $(\mathfrak{t}^d)^*_{\mathbb{Z}}$.) We will denote the collection $\{H_1, \ldots, H_n\}$ by $\mathcal{A}$, and write

$$\mathfrak{M}(\mathcal{A}) = \mathfrak{M}_{\alpha,0}$$

for the corresponding hypertoric variety. We will refer to $\mathcal{A}$ simply as an **arrangement**, always assuming that the weighted coorientations are part of the data.

Remark 1.3.1 We note that we allow repetitions of hyperplanes in our arrangement ($\mathcal{A}$ may be a multi-set), and that a repeated occurrence of a particular hyperplane is *not* the same as a single occurrence of that hyperplane with weight 2. On the other hand, little is lost by restricting one's attention to arrangements of distinct hyperplanes of weight one.

Since each hyperplane H_i comes with a normal vector, it seems at first that it would make the most sense to talk about an arrangement of half-spaces, where the i^{th} half-space consists of the set of points that lie on the positive side of H_i with respect to a_i. The reason that we talk about hyperplanes rather than half-spaces is the following proposition, proven in [HP1, 2.2].

Proposition 1.3.2 *The T^d-variety $\mathfrak{M}(\mathcal{A})$ does not depend on the signs of the vectors a_i.*

In other words, if we make a new hypertoric variety with the same arrangement of weighted hyperplanes but with some of the coorientations flipped, it will be T^d-equivariantly isomorphic to the hypertoric variety with which we started.[4]

We call the arrangement $\mathcal{A}$ **simple** if every subset of m hyperplanes with nonempty intersection intersects in codimension m. We call $\mathcal{A}$ **unimodular** if every collection of d

[4]In [HP1] we consider an extra $\mathbb{C}^\times$ action on $\mathfrak{M}(\mathcal{A})$ that *does* depend on the coorientations.

linearly independent vectors $\{a_{i_1}, \ldots, a_{i_d}\}$ spans $\mathfrak{t}^d$ over the integers. An arrangement which is both simple and unimodular is called **smooth**. The following proposition is proven in [BD, 3.2 & 3.3].

Proposition 1.3.3 *The hypertoric variety $\mathfrak{M}(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth.*

For the remainder of the paper, let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a **central** arrangement, meaning that $r_i = 0$ for all i, so that all of the hyperplanes pass through the origin. Then $\mathfrak{M}(\mathcal{A})$ is the singular affine variety $\mathfrak{M}_{0,0}$. Let $\tilde{\mathcal{A}} = \{\tilde{H}_1, \ldots, \tilde{H}_n\}$ be a **simplification** of $\mathcal{A}$, by which we mean an arrangement defined by the same vectors $\{a_i\} \subset \mathfrak{t}^d$, but with a different choice of $r \in (\mathfrak{t}^n)^*$, such that $\tilde{\mathcal{A}}$ is simple. This corresponds to translating each of the hyperplanes in $\mathcal{A}$ away from the origin by some generic amount. Then $\mathfrak{M}(\tilde{\mathcal{A}})$ maps T-equivariantly to $\mathfrak{M}(\mathcal{A})$ by Equation (2), and Proposition 1.3.3 tell us that it is in fact an "orbifold resolution", meaning a projective morphism, generically one-to-one, in which the source has at worst orbifold singularities. The structure of this map is studied extensively in [PW].

1.4 Toward an abstract definition

The definition of a hypertoric variety in Section 1.2 is constructive, modeled on the definition of toric varieties as GIT quotients of the form $\mathbb{C}^n /\!\!/_\alpha T^k$, or equivalently as symplectic quotients by compact tori. In the case of toric varieties, there are also abstract definitions. In the symplectic world, one defines a toric orbifold to be a symplectic orbifold of dimension $2d$ along with an effective Hamiltonian action of a compact d-torus, and proves that any connected, compact toric orbifold arises from the symplectic quotient construction [De, LT]. In the algebraic world, one defines a toric variety to be a normal variety admitting a torus action with a dense orbit, and then proves that any semiprojective[5] toric variety with at worst orbifold singularities arises from the GIT construction. This idea goes back to [Co], and can be found in this language in [HSt, 2.6].

It is natural to ask for such an abstract definition and classification theorem for hypertoric varieties, either from the standpoint of symplectic algebraic geometry or that of hyperkähler geometry. In the hyperkähler setting, this task was achieved by Bielawski with Theorems 1.4.1 and 1.4.2 below. A **Taub-NUT deformation** of a hypertoric variety $\mathfrak{M}$ is a hyperkähler quotient of $\mathfrak{M} \times T^*\mathbb{C}^m$ by the noncompact group $\mathbb{R}^m$, where $\mathbb{R}^m$ acts on $T^*\mathbb{C}^m$ by first projecting onto the torus $T^m_{\mathbb{R}}$, and on $\mathfrak{M}$ via some homomorphism $\mathbb{R}^m \to T^d_{\mathbb{R}}$ which is injective on Lie algebras. Bielawski shows that such a deformation is canonically $T^d_{\mathbb{R}}$-equivariantly diffeomorphic to $\mathfrak{M}$, but with a different hyperkähler structure. The following theorem appears in [Bi, 3].

[5]Hausel and Sturmfels call a toric variety semiprojective if it is projective over its affinization and has at least one torus fixed point.

Theorem 1.4.1 *Any complete, connected, hyperkähler manifold of real dimension $4d$ which admits an effective, hyperhamiltonian action of the compact torus $T^d_{\mathbb{R}}$ is equivariantly isometric to a Taub-NUT deformation of a hypertoric variety.*

A riemannian manifold M of dimension ℓ is said to have **euclidean volume growth** if there exists a point $p \in M$ such that the volume of the unit ball of radius r centered at p grows like r^ℓ. The fact that a hypertoric variety $\mathfrak{M}$ has Euclidean volume growth can be deduced by examining the measure on $(\mathfrak{t}^d)^*_{\mathbb{R}} \otimes \mathbb{R}^3$ obtained by pushing forward the riemannian measure on $\mathfrak{M}$ along the hyperkähler moment map. Bielawski proves the converse of this fact in [Bi, 4].

Theorem 1.4.2 *A Taub-NUT deformation of a hypertoric variety is equivariantly isometric to a hypertoric variety if and only if it has Euclidean volume growth.*

It would be nice to find an analogous theorem that intrinsically categorizes hypertoric varieties (smooth or singular) in the algebraic category. Such a theorem should look something like the following.

Conjecture 1.4.3 *Any connected, symplectic, algebraic variety which is projective over its affinization and admits an effective, hamiltonian action of the algebraic torus T^d is equivariantly isomorphic to a Zariski open subset of a hypertoric variety.*

2 Homotopy models

In this section we fix the vector configuration $\{a_1, \dots a_n\} \subseteq \mathfrak{t}^d_{\mathbb{Z}}$, consider three spaces that are T^d-equivariantly homotopy equivalent to the hypertoric variety $\mathfrak{M}_{\alpha,\lambda}$ for generic choice of (α, λ). Each space is essentially toric rather than hypertoric in nature, and therefore may provide a way to think about hypertoric varieties in terms of more familiar objects. Recall that if $\lambda = 0$ then $\mathfrak{M}_{\alpha,\lambda} = \mathfrak{M}(\tilde{\mathcal{A}})$ for a simple hyperplane arrangement $\tilde{\mathcal{A}}$, in which the positions of the hyperplanes (up to simultaneous translation) are determined by α. If, on the other hand, λ is a regular value, then $\mathfrak{M}_{\alpha,\lambda} = \mathfrak{M}_\lambda$ is independent of α.

2.1 The core

Recall from Section 1.3 that we have an equivariant orbifold resolution

$$\mathfrak{M}(\tilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A}),$$

and from Section 1.1 that the fiber $\mathfrak{L}(\tilde{\mathcal{A}}) \subseteq \mathfrak{M}(\tilde{\mathcal{A}})$ over the most singular point of $\mathfrak{M}(\mathcal{A})$ is called the **core** of $\mathfrak{M}(\tilde{\mathcal{A}})$. The primary interest in the core comes from the following proposition, originally proven in [BD, 6.5] from the perspective of Proposition 2.1.4.

Proposition 2.1.1 *The core $\mathfrak{L}(\tilde{\mathcal{A}})$ is a $T^d_{\mathbb{R}}$-equivariant deformation retract of $\mathfrak{M}(\tilde{\mathcal{A}})$.*

Remark 2.1.2 In fact, Proposition 2.1.1 holds in the greater generality of Section 1.1, for algebraic symplectic quotients $\mathfrak{M}_{\alpha,0}$ by arbitrary reductive groups [P1, 2.8]. The cores of Nakajima's quiver varieties play an important role in representation theory, because the fundamental classes of the irreducible components form a natural basis for the top nonvanishing homology group of $\mathfrak{M}_{\alpha,0}$, which may be interpreted as a weight space of an irreducible representation of a Kac-Moody algebra [N2, 10.2]. Cores of hypertoric and quiver varieties share many geometric properties with the nilpotent cone of moduli spaces of Higgs bundles over a curve, first studied by Laumon [La].

We now give a toric interpretation of $\mathfrak{L}(\tilde{\mathcal{A}})$. For any subset $U \subseteq \{1, \ldots, n\}$, let

$$P_U = \{x \in (\mathfrak{t}^d)^*_{\mathbb{R}} \mid x \cdot a_i + r_i \geq 0 \text{ if } i \in U \text{ and } x \cdot a_i + r_i \leq 0 \text{ if } i \notin U\}. \tag{5}$$

Thus P_U is the polyhedron "cut out" by the cooriented hyperplanes of $\tilde{\mathcal{A}}$ after reversing the coorientations of the hyperplanes with indices in U. Since $\tilde{\mathcal{A}}$ is a weighted arrangement, P_U is a labeled polytope in the sense of [LT]. Let

$$\mathcal{E}_U = \{(z, w) \in T^*\mathbb{C}^n \mid w_i = 0 \text{ if } i \in U \text{ and } z_i = 0 \text{ if } i \notin U\}$$

and

$$\mathfrak{X}_U = \mathcal{E}_U /\!\!/_{\alpha} T^k.$$

Then $\mathcal{E}_U \subseteq \mu^{-1}(0)$, and therefore

$$\mathfrak{X}_U = \mathcal{E}_U /\!\!/_{\alpha} T^k \subseteq \mu^{-1}(0) /\!\!/_{\alpha} T^k = \mathfrak{M}(\tilde{\mathcal{A}}).$$

The following proposition is proven in [BD, 6.5], but is stated more explicitly in this language in [P1, 3.8].

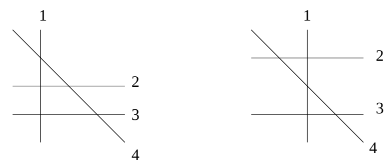
Proposition 2.1.3 *The variety $\mathfrak{X}_U$ is isomorphic to the toric orbifold classified by the weighted polytope P_U.*

It is not hard to see that the subvariety $\mathcal{E}_U /\!\!/_{\alpha} T^k \subseteq \mathfrak{M}(\tilde{\mathcal{A}})$ lies inside the core $\mathfrak{L}(\tilde{\mathcal{A}})$ of $\mathfrak{M}(\tilde{\mathcal{A}})$. In fact, these subvarieties make up the entire core, as can be deduced from [BD, §6].

Proposition 2.1.4 $\mathfrak{L}(\tilde{\mathcal{A}}) \;=\; \bigcup_{P_U \text{ bounded}} \mathcal{E}_U /\!\!/_{\alpha} T^k \;\subseteq\; \mathfrak{M}(\tilde{\mathcal{A}}).$

Thus $\mathfrak{L}(\tilde{\mathcal{A}})$ is a union of compact toric varieties sitting inside the hypertoric $\mathfrak{M}(\tilde{\mathcal{A}})$, glued together along toric subvarieties as prescribed by the combinatorics of the polytopes P_U and their intersections in $(\mathfrak{t}^d)^*_{\mathbb{R}}$.

Example 2.1.5 Consider the two hyperplane arrangement pictured below, with all hyperplanes having primitive normal vectors. Note that there are two primitive vectors to choose from for each hyperplane (one must choose a direction), but the corresponding hypertoric varieties and their cores will be independent of these choices by Proposition 1.3.2. In the



first picture, the core consists of a $\mathbb{C}P^2$ (the toric variety associated to a triangle) and a $\mathbb{C}P^2$ blown up at a point (the toric variety associated to a trapezoid) glued together along a $\mathbb{C}P^1$ (the toric variety associated to an interval). In the second picture, it consists of two copies of $\mathbb{C}P^2$ glued together at a point.

Remark 2.1.6 Each of the core components $\mathcal{E}_U$ is a lagrangian subvariety of $\mathfrak{M}(\tilde{\mathcal{A}})$, therefore its normal bundle in $\mathfrak{M}(\tilde{\mathcal{A}})$ is isomorphic to its cotangent bundle. Furthermore, each $\mathcal{E}_U$ has a T^d-invariant algebraic tubular neighborhood in $\mathfrak{M}(\tilde{\mathcal{A}})$ (necessarily isomorphic to the total space of $T^*\mathfrak{X}_U$), and these neighborhoods cover $\mathfrak{M}(\tilde{\mathcal{A}})$. Thus $\mathfrak{M}(\tilde{\mathcal{A}})$ is a union of cotangent bundles of toric varieties, glued together equivariantly and symplectically in a manner prescribed by the combinatorics of the bounded chambers of $\tilde{\mathcal{A}}$. It is possible to take Proposition 2.1.3 and Equation (5) as a definition of $\mathfrak{X}_U$, and this remark as a definition of $\mathfrak{M}(\tilde{\mathcal{A}})$. The affine variety $\mathfrak{M}(\mathcal{A})$ may then be defined as the spectrum of the ring of global functions on $\mathfrak{M}(\tilde{\mathcal{A}})$.

Remark 2.1.7 Though Propositions 2.1.1, 2.1.3, and 2.1.4 appear in the literature only for $\tilde{\mathcal{A}}$ simple, this hypothesis should not be necessary.

2.2 The Lawrence toric variety

Let

$$\mathfrak{B}(\tilde{\mathcal{A}}) = T^*\mathbb{C}^n /\!\!/_{\alpha} T^k.$$

This variety is a GIT quotient of a vector space by the linear action of a torus, and is therefore a toric variety. Toric varieties that arise in this way are called **Lawrence toric varieties**. The following proposition is proven in [HSt, §6].

Proposition 2.2.1 *The inclusion*

$$\mathfrak{M}(\tilde{\mathcal{A}}) = \mu^{-1}(0) /\!\!/_{\alpha} T^k \hookrightarrow T^*\mathbb{C}^n /\!\!/_{\alpha} T^k = \mathfrak{B}(\tilde{\mathcal{A}})$$

is a $T^d_{\mathbb{R}}$-equivariant homotopy equivalence.

This Proposition is proven by showing that any toric variety retracts equivariantly onto the union of those T^d-orbits whose closures are compact. In the case of the Lawrence toric variety, this is nothing but the core $\mathfrak{L}(\tilde{\mathcal{A}})$.

2.3 All the GIT quotients at once

Given $\alpha \in (\mathfrak{t}^k)^*_{\mathbb{Z}}$, we may define **stable** and **semistable** sets

$$(\mathbb{C}^n)^{\alpha-st} \subseteq (\mathbb{C}^n)^{\alpha-ss} \subseteq \mathbb{C}^n$$

as in Section 1.1, and the toric variety $\mathfrak{X}_\alpha = \mathbb{C}^n /\!\!/_\alpha T^k$ may be defined as the categorical quotient of $(\mathbb{C}^n)^{\alpha-ss}$ by T^k. In analogy with Section 1.1, we will call α **generic** if the α-stable and α-semistable sets of $\mathbb{C}^n$ coincide. In this case the categorical quotient will be simply a geometric quotient, and $\mathfrak{X}_\alpha$ will be the toric orbifold corresponding to the polytope $P_\emptyset$ of Section 2.1. We consider two characters to be equivalent if their stable sets are the same, and note that there are only finitely many equivalence classes of characters, given by the various possible combinatorial types of $P_\emptyset$ for different simplifications $\tilde{\mathcal{A}}$ of $\mathcal{A}$. Let $\alpha_1, \ldots, \alpha_m$ be a complete list of representatives of equivalence classes for which[6] $\emptyset \neq (\mathbb{C}^n)^{\alpha-st} = (\mathbb{C}^n)^{\alpha-ss}$.

Let $(\mathbb{C}^n)^{\ell f}$ be the set of vectors in $\mathbb{C}^n$ on which T^k acts locally freely, meaning with finite stabilizers. For any character α of T^k, the stable set $(\mathbb{C}^n)^{\alpha-st}$ is, by definition, contained in $(\mathbb{C}^n)^{\ell f}$. Conversely, every element of $(\mathbb{C}^n)^{\ell f}$ is stable for some generic α [P4, 1.1], therefore

$$(\mathbb{C}^n)^{\ell f} = \bigcup_{i=1}^{m} (\mathbb{C}^n)^{\alpha_i-st}.$$

We define the *nonhausdorff* space

$$\mathfrak{X}^{\ell f} = (\mathbb{C}^n)^{\ell f}/\,T^k = \bigcup_{i=1}^{m} (\mathbb{C}^n)^{\alpha_i-st}/T^k = \bigcup_{i=1}^{m} \mathfrak{X}_{\alpha_i}$$

to be the union of the toric varieties $\mathfrak{X}_{\alpha_i}$ along the open loci of commonly stable points.

For an arbitrary $\lambda \in (\mathfrak{t}^k)^*$, consider the projection

$$\pi_\lambda : \mu^{-1}(\lambda) \hookrightarrow T^*\mathbb{C}^n \to \mathbb{C}^n.$$

The following proposition is proven in [P2, 1.3].

Proposition 2.3.1 *If λ is a regular value of μ, then π_λ has image $(\mathbb{C}^n)^{\ell f}$, and is naturally a T^n-equivariant affine bundle of dimension d.*

Corollary 2.3.2 *The variety $\mathfrak{M}_\lambda = \mu^{-1}(\lambda)/T^k$ is a T^d-equivariant affine bundle over $\mathfrak{X}^{\ell f} = (\mathbb{C}^n)^{\ell f}/T^k$.*

It follows from Corollary 2.3.2 that the natural projection $\mathfrak{M}_\lambda \to \mathfrak{X}^{\ell f}$ is a T^d-equivariant weak homotopy equivalence, meaning that it induces isomorphisms on all homotopy and

[6]Though $\mu^{-1}(\lambda)^{\alpha-st}$ is never empty, $(\mathbb{C}^n)^{\alpha-st}$ sometimes is.

homology groups. It is not a homotopy equivalence in the ordinary sense because it does not have a homotopy inverse–in particular, it does not admit a section.

Example 2.3.3 Consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ by the formula $t \cdot (z_1, z_2) = (tz_1, t^{-1}z_2)$. A multiplicative character of $\mathbb{C}^\times$ is given by an integer α, and that character will be generic if and only if that integer is nonzero. The equivalence class of generic characters will be given by the sign of that integer, so we let $\alpha_1 = -1$ and $\alpha_2 = 1$. The corresponding stable sets will be

$$(\mathbb{C}^2)^{\alpha_1-st} = \mathbb{C}^2 \smallsetminus \{z_1 = 0\} \text{ and } (\mathbb{C}^2)^{\alpha_2-st} = \mathbb{C}^2 \smallsetminus \{z_2 = 0\}.$$

The corresponding toric varieties $\mathfrak{X}_{\alpha_1}$ and $\mathfrak{X}_{\alpha_2}$ will both be isomorphic to $\mathbb{C}$, and $\mathfrak{X}^{\ell f} = \mathfrak{X}_{\alpha_1} \cup \mathfrak{X}_{\alpha_2}$ will be the (nonhausdorff) union of two copies of $\mathbb{C}$ glued together away from the origin, as so:

The moment map

$$\mu : \mathbb{C}^2 \times (\mathbb{C}^2)^\vee \to (\mathfrak{t}^k)^* \cong \mathbb{C}$$

is given in coordinates by $\mu(z, w) = z_1 w_1 - z_2 w_2$. The hypertoric variety $\mathfrak{M}_\alpha = \mu^{-1}(0) /\!\!/ \mathbb{C}^\times$ at a generic character is isomorphic to $T^*\mathbb{C}P^1$, and its core is the zero section $\mathbb{C}P^1$. It is diffeomorphic to $\mathfrak{M}_\lambda = \mu^{-1}(\lambda)/\mathbb{C}^\times$, which is, by Corollary 2.3.2, an affine bundle over $\mathfrak{X}^{\ell f}$. If we trivialize this affine bundle over the two copies of $\mathbb{C}$, we may write down a family of affine linear maps $\rho_z : \mathbb{C} \to \mathbb{C}$ such that, over a point $0 \neq z \in \mathbb{C}$, the fibers of the two trivial bundles are glued together using ρ_z. Doing this calculation, we find that $\rho_z(w) = w + z^{-2}$.

Remark 2.3.4 Both Proposition 2.1.1 and Corollary 2.3.2 show that a hypertoric variety is equivariantly (weakly) homotopy equivalent to a union of toric orbifolds. In the case of Proposition 2.1.1 those toric orbifolds are always compact, and glued together along closed toric subvarieties. In the case of Corollary 2.3.2 those toric orbifolds may or may not be compact, and are glued together along Zariski open subsets to create something that has at worst orbifold singularities, but is not Hausdorff. In general, there is no relationship between the collection of toric varieties that appear in Proposition 2.1.1 and those that appear in Corollary 2.3.2.

Remark 2.3.5 Corollary 2.3.2 generalizes to abelian quotients of cotangent bundles of arbitrary varieties, rather than just vector spaces [P4, 1.4]. A more complicated statement for nonabelian groups was used by Crawley-Boevey and Van den Bergh [CBVdB] to prove a conjecture of Kac about counting quiver representations over finite fields.

3 Cohomology

In this Section we discuss the cohomology of the orbifold $\mathfrak{M}(\tilde{\mathcal{A}})$ and the intersection cohomology of the singular variety $\mathfrak{M}(\mathcal{A})$, focusing on the connection to the combinatorics of matroids. In Section 3.4 we explain how hypertoric varieties can be used to compute cohomology rings of nonabelian algebraic symplectic quotients, as defined in Section 1.1. There are a number of results on the cohomology of hypertoric varieties that we won't discuss, including computations of the intersection form on the L^2-cohomology of $\mathfrak{M}(\tilde{\mathcal{A}})$ [HSw] and the Chen-Ruan orbifold cohomology ring of $\mathfrak{M}(\tilde{\mathcal{A}})$ [GH, JT].

3.1 Combinatorial background

A **simplicial complex** Δ on the set $\{1,\ldots,n\}$ is a collection of subsets of $\{1,\ldots,n\}$, called faces, such that a subset of a face is always a face. Let $f_i(\Delta)$ denote the number of faces of Δ of cardinality i, and define the **h-polynomial**

$$h_\Delta(q) := \sum_{i=0}^{d} f_i q^i (1-q)^{d-i},$$

where d is the cardinality of the largest face of Δ. Although the numbers $f_i(\Delta)$ are themselves very natural to consider, it is unclear from the definition above why we want to encode them in this convoluted way. The following equivalent construction of the h-polynomial is less elementary but better motivated.

To any simplicial complex one associates a natural graded algebra, called the **Stanley-Reisner ring**, defined as follows:

$$\mathcal{SR}(\Delta) := \mathbb{C}[e_1,\ldots,e_n] \Big/ \left\langle \prod_{i\in S} e_i \;\Big|\; S \notin \Delta \right\rangle.$$

In order to agree with the cohomological interpretation that we will give to this ring in Theorem 3.2.2, we let the generators e_i have degree 2. Consider the Hilbert series

$$\operatorname{Hilb}(\mathcal{SR}(\Delta), q) := \sum_{i=0}^{\infty} \dim \mathcal{SR}^{2i}(\Delta) q^i,$$

which may be expressed as a rational function in q. The following proposition (see [St, §II.2]) says that the h-polynomial is the numerator of that rational function.

Proposition 3.1.1 $\operatorname{Hilb}(\mathcal{SR}(\Delta), q) = h_\Delta(q)/(1-q)^d$.

3.2 Cohomology of $\mathfrak{M}(\tilde{\mathcal{A}})$

Let $\Delta_{\mathcal{A}}$ be the simplicial complex consisting of all sets $S \subseteq \{1,\ldots,n\}$ such that the normal vectors $\{a_i \mid i \in S\}$ are linearly independent. This simplicial complex is known as the **matroid complex** associated to $\mathcal{A}$. The Betti numbers of $\mathfrak{M}(\tilde{\mathcal{A}})$ were computed in [BD, 6.7], but the following combinatorial interpretation was first observed by [HSt, 1.2]. Let

$$\mathrm{Poin}_{\mathfrak{M}(\tilde{\mathcal{A}})}(q) = \sum_{i=0}^{d} \dim H^{2i}(\mathfrak{M}(\tilde{\mathcal{A}}))\, q^i$$

be the even degree Poincaré polynomial of $\mathfrak{M}(\tilde{\mathcal{A}})$.

Theorem 3.2.1 *The cohomology of $\mathfrak{M}(\tilde{\mathcal{A}})$ vanishes in odd degrees, and*

$$\mathrm{Poin}_{\mathfrak{M}(\tilde{\mathcal{A}})}(q) = h_{\Delta_{\mathcal{A}}}(q).$$

Theorem 3.2.1 is a consequence of the following stronger result.

Theorem 3.2.2 *There is a natural isomorphism of graded rings $H^*_{T^d}(\mathfrak{M}(\tilde{\mathcal{A}})) \cong \mathcal{SR}(\Delta_{\mathcal{A}})$.*

The action of T^d on $\mathfrak{M}(\tilde{\mathcal{A}})$ is equivariantly formal [K1, 2.5], therefore the Hilbert series of $H^*_{T^d}(\mathfrak{M}(\tilde{\mathcal{A}}))$ is equal to $\mathrm{Poin}_{\mathfrak{M}(\tilde{\mathcal{A}})}(q)/(1-q)^d$, and Theorem 3.2.1 follows immediately from Proposition 3.1.1. Theorem 3.2.2 was proven for $\tilde{\mathcal{A}}$ smooth in [K1, 2.4] from the perspective of Section 2.1, and in the general case in [HSt, 1.1] from the perspective of Section 2.2. Here we give a new, very short proof, from the perspective of Section 2.3.

Proof of 3.2.2: By Proposition 1.1.2, Remark 1.2.1, and Corollary 2.3.2,

$$H^*_{T^d}(\mathfrak{M}(\tilde{\mathcal{A}})) \cong H^*_{T^d}(\mathfrak{M}_\lambda) \cong H^*_{T^d}(\mathfrak{X}^{\ell f}) \cong H^*_{T^n}((\mathbb{C}^n)^{\ell f}). \tag{6}$$

Given a simplicial complex Δ on $\{1,\ldots,n\}$, Buchstaber and Panov build a T^n-space $\mathcal{Z}_\Delta$ called the moment angle complex with the property that $H^*_{T^n}(\mathcal{Z}_\Delta) \cong \mathcal{SR}(\Delta)$ [BP, 7.12]. In the case of the matroid complex $\Delta_{\mathcal{A}}$, there is a T^n-equivariant homotopy equivalence $\mathcal{Z}_{\Delta_{\mathcal{A}}} \simeq (\mathbb{C}^n)^{\ell f}$ [BP, 8.9], which completes the proof. □

Remark 3.2.3 There is no problem with working with the singular (equivariant) cohomology of the non-hausdorff space $\mathfrak{X}^{\ell f}$, but the faint of heart may replace $H^*_{T^d}(\mathfrak{X}^{\ell f})$ with $H^*_{T^n}(\mu^{-1}(\lambda))$ in Equation (6) and be left with a string of isomorphisms that makes explicit use of Proposition 2.3.1 rather than Corollary 2.3.2.

3.3 Intersection cohomology of $\mathfrak{M}(\mathcal{A})$

The singular hypertoric variety $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}_{0,0}$ is contractible, hence its ordinary cohomology is trivial. Instead, we consider intersection cohomology, a variant of cohomology introduced by Goresky and MacPherson which is better at probing the topology of singular varieties [GM1, GM2]. Let

$$\mathrm{Poin}_{\mathfrak{M}(\mathcal{A})}(q) = \sum_{i=0}^{d-1} \dim IH^{2i}(\mathfrak{M}(\mathcal{A}))\, q^i$$

be the even degree intersection cohomology Poincaré polynomial of $\mathfrak{M}(\mathcal{A})$. We will interpret this polynomial combinatorially with a theorem analogous to Theorem 3.2.1.

A minimal nonface of $\Delta_\mathcal{A}$ is called a **circuit**. Given an ordering σ of $\{1,\ldots,n\}$, define a **σ-broken circuit** to be a circuit minus its smallest element with respect to the ordering σ. The **σ-broken circuit complex** $\mathrm{bc}_\sigma\Delta_\mathcal{A}$ is defined to be the collection of subsets of $\{1,\ldots,n\}$ that do not contain a σ-broken circuit. Though the simplicial complex $\mathrm{bc}_\sigma\Delta_\mathcal{A}$ depends on the choice of σ, its h-polynomial does not. The following theorem was proved by arithmetic methods in [PW, §4].

Theorem 3.3.1 *The intersection cohomology of $\mathfrak{M}(\mathcal{A})$ vanishes in odd degrees, and*

$$\mathrm{Poin}_{\mathfrak{M}(\mathcal{A})}(q) = h_{\mathrm{bc}_\sigma\Delta_\mathcal{A}}(q).$$

Given the formal similarity of Theorems 3.2.1 and 3.3.1, it is natural to ask if there is an analogue of Theorem 3.2.2 in the central case. The most naive guess is that the equivariant cohomology $IH^*_{T^d}(\mathfrak{M}(\mathcal{A}))$ is naturally isomorphic to the Stanley-Reisner ring $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$, but this guess is problematic for two reasons. The first is that intersection cohomology generally does not admit a ring structure, and therefore such an isomorphism would be surprising. The second and more important problem is that the ring $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$ depends on σ, while the vector space $IH^*_{T^d}(\mathfrak{M}(\mathcal{A}))$ does not. Since the various rings $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$ for different choices of σ are not naturally isomorphic to each other, they cannot all be naturally isomorphic to $IH^*_{T^d}(\mathfrak{M}(\mathcal{A}))$, even as vector spaces. These problems can be addressed and resolved by the following construction.

Let $R(\mathcal{A}) = \mathbb{C}[a_1^{-1},\ldots,a_n^{-1}]$ be the subring of the ring of all rational functions on $\mathbb{C}^n$ generated by the inverses of the linear forms that define the hyperplanes of $\mathcal{A}$. There is a surjective map φ from $\mathbb{C}[e_1,\ldots,e_n]$ to $R(\mathcal{A})$ taking e_i to a_i^{-1}. Given a set $S \subseteq \{1,\ldots,n\}$ and a linear relation of the form $\sum_{i\in S} c_i a_i = 0$, the element $k_S = \sum_{i\in S} c_i \prod_{j\in S\smallsetminus\{i\}} e_j$ lies in the kernel $I(\mathcal{A})$ of φ, and in fact $I(\mathcal{A})$ is generated by such elements. Since k_S is clearly homogeneous, $R(\mathcal{A})$ is a graded ring, with the usual convention of $\deg e_i = 2$ for all i. The following proposition, proven in [PS, 4], states that the ring $R(\mathcal{A})$ is a simultaneous deformation of the various Stanley-Reisner rings $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$.

Proposition 3.3.2 *The set $\{k_S \mid S \text{ a circuit }\}$ is a universal Gröbner basis for $I(\mathcal{A})$, and the choice of an ordering σ of $\{1,\ldots,n\}$ defines a flat degeneration of $R(\mathcal{A})$ to the Stanley-Reisner ring $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$.*

Example 3.3.3 Let $d=2$, identify $\mathfrak{t}^d_\mathbb{R}$ with $\mathbb{R}^2$, and let

$$a_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad a_2 = a_3 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \text{and } a_4 = \begin{pmatrix}-1\\-1\end{pmatrix}.$$

The two arrangements pictured in Example 2.1.5 are two different simplifications of the resulting central arrangement $\mathcal{A}$. We then have

$$R(\mathcal{A}) \cong \mathbb{C}[e_1, \ldots, e_4] \,/\, \langle\, e_2 - e_3, \;\; e_1e_2 + e_1e_4 + e_2e_4, \;\; e_1e_3 + e_1e_4 + e_3e_4 \,\rangle\,.$$

By taking the initial ideal with respect to some term order, we get the Stanley-Reisner ring of the corresponding broken circuit complex.

In Theorem 3.3.4, proven in [BrP], we show that $R(\mathcal{A})$ replaces the Stanley-Reisner ring in the "correct" analogue of Theorem 3.2.2.

Theorem 3.3.4 *Suppose that $\mathcal{A}$ is unimodular. The equivariant intersection cohomology sheaf $\mathbf{IC}_{T^d}(\mathfrak{M}(\mathcal{A}))$ admits canonically the structure of a ring object in the bounded equivariant derived category of $\mathfrak{M}(\mathcal{A})$. This induces a ring structure on $IH^*_{T^d}(\mathfrak{M}(\mathcal{A}))$, which is naturally isomorphic to $R(\mathcal{A})$.*

Remark 3.3.5 The problems of classifying h-polynomials of matroid complexes and their broken circuit complexes remain completely open. Hausel and Sturmfels explore the restrictions on $h_{\Delta_{\mathcal{A}}}(q)$ imposed by Theorem 3.2.2 in [HSt, §7], and Webster and the author consider the combinatorial implications of applying the decomposition theorem for perverse sheaves to the map $\mathfrak{M}(\tilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ [PW, §5]. In both cases one obtains results which admit independent, purely combinatorial proofs, but which are illuminated by their geometric interpretations.

3.4 Abelianization

As in Section 1.1, let G be a reductive complex algebraic group acting linearly on a complex vector space V, and let $T \subseteq G$ be a maximal torus. We need the further technical assumption that V has no nonconstant T-invariant functions, which is equivalent to asking that any GIT quotient of V by T is projective. The inclusion of T into G induces a surjection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*$, which restricts to an inclusion of $Z(\mathfrak{g}^*)$ into $\mathfrak{t}^*$. Thus a pair of parameters $(\alpha, \lambda) \in Z(\mathfrak{g}^*_{\mathbb{Z}}) \times Z(\mathfrak{g}^*)$ may be interpreted as parameters for T as well as for G. Suppose given $\alpha \in Z(\mathfrak{g}^*_{\mathbb{Z}})$ such that $(\alpha, 0)$ is generic for both G and T, so that the symplectic quotients

$$\mathfrak{M}_{\alpha,0}(G) \;\text{ and }\; \mathfrak{M}_{\alpha,0}(T)$$

are both orbifolds. Our first goal for this section is to describe the cohomology of $\mathfrak{M}_{\alpha,0}(G)$ in terms of that of $\mathfrak{M}_{\alpha,0}(T)$.

Both $\mathfrak{M}_{\alpha,0}(G)$ and $\mathfrak{M}_{\alpha,0}(T)$ inherit actions of the group $\mathbb{C}^\times$ induced by scalar multiplication on the fibers of the cotangent bundle of V. Let

$$\Phi(G) : H^*_{G\times\mathbb{C}^\times}(T^*V) \to H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(G))$$

and

$$\Phi(T) : H^*_{T\times\mathbb{C}^\times}(T^*V) \to H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(T))$$

be the **equivariant Kirwan maps**, induced by the $\mathbb{C}^\times$-equivariant inclusions of $\mu_G^{-1}(0)^{\alpha-st}$ and $\mu_T^{-1}(0)^{\alpha-st}$ into T^*V. The map $\Phi(T)$ is known to be surjective [HP1, 4.5], and $\Phi(G)$ is conjectured to be so, as well. The abelian Kirwan map $\Phi(T)$ makes the equivariant cohomology ring $H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(T))$ into a module over $H^*_{T\times\mathbb{C}^\times}(T^*V)$. The Weyl group $W = N(T)/T$ acts both on the source and the target of $\Phi(T)$, and the map is W-equivariant.

Let $\Delta \subseteq \mathfrak{t}^*$ be the set of roots of G (not to be confused with the simplicial complexes Δ that we discussed earlier), and consider the W-invariant class

$$e \;=\; \prod_{\beta\in\Delta} \beta\,(x-\beta) \;\in\; \operatorname{Sym}\mathfrak{t}^* \otimes \mathbb{C}[x] \;\cong\; H^*_{T\times\mathbb{C}^\times}(T^*V).$$

The following theorem was proven in [HP, 2.4].

Theorem 3.4.1 *If $\Phi(G)$ is surjective, then there is a natural isomorphism*

$$H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(G)) \;\cong\; H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(T))^W \big/ \operatorname{Ann}(e),$$

where $\operatorname{Ann}(e)$ *is the ideal of classes annihilated by* e.

Remark 3.4.2 We note that the abelian quotient $\mathfrak{M}_{\alpha,0}(T)$ is a hypertoric variety, but the $\mathbb{C}^\times$ that acts here is *not* a subtorus of the torus whose action we have considered previously in this paper. (In particular, the action of $\mathbb{C}^\times$ is not hamiltonian.) This extra $\mathbb{C}^\times$-action was studied in [HP1], and the equivariant cohomology ring $H^*_{\mathbb{C}^\times}(\mathfrak{M}_{\alpha,0}(T))$ was given an explicit combinatorial description in [HP1, 4.5] and [HH, 3.5]. Thus, modulo surjectivity of the Kirwan map, Theorem 3.4.1 tells us how to compute the cohomology ring of arbitrary symplectic quotients constructed in the manner of Section 1.1. In [HP, §4], this method was applied to compute the $\mathbb{C}^\times$-equivariant cohomology rings of hyperpolygon spaces, a result which originally appeared in [HP2, 3.2] as an extension of the nonequivariant computation in [K2, 7.1].

Although the proof of Theorem 3.4.1 uses the $\mathbb{C}^\times$-action in a crucial way, Hausel has conjectured a simpler, nonequivariant version. Let $\Phi_0(G)$ be the map obtained from $\Phi(G)$ by setting the equivariant parameter x to zero, and let

$$e_0 \;=\; \prod_{\beta\in\Delta} \beta \;\in\; \operatorname{Sym}\mathfrak{t}^* \;\cong\; H^*_T(T^*V).$$

Note that e_0 is *not* the class obtained from e by setting x to zero, rather it is a square root of that class.

Conjecture 3.4.3 *If Φ_0 is surjective, then there is a natural isomorphism*

$$H^*(\mathfrak{M}_{\alpha,0}(G)) \;\cong\; H^*(\mathfrak{M}_{\alpha,0}(T))^W \big/ \operatorname{Ann}(e_0).$$

We end by combining Conjecture 3.4.3 with Theorem 3.3.4 to produce a conjecture that would put a ring structure on the intersection cohomology groups of $\mathfrak{M}_{0,0}(G)$. The hypothesis that $\mathcal{A}$ be unimodular in Theorem 3.3.4 is equivalent to requiring that the orbifold resolution $\mathfrak{M}(\tilde{\mathcal{A}})$ of $\mathfrak{M}(\mathcal{A})$ is actually smooth. The analogous assumption in this context is that $\mathfrak{M}_{\alpha,0}(G)$ and $\mathfrak{M}_{\alpha,0}(T)$ are smooth for generic choice of α.

Conjecture 3.4.4 *Suppose that $\mathfrak{M}_{\alpha,0}(G)$ and $\mathfrak{M}_{\alpha,0}(T)$ are smooth for generic $(\alpha, 0)$. Then The intersection cohomology sheaf $\mathbf{IC}(\mathfrak{M}_{0,0}(G))$ admits canonically the structure of a ring object in the bounded derived category of $\mathfrak{M}_{0,0}(G)$, and there is a natural ring isomorphism*

$$IH^*(\mathfrak{M}_{0,0}(G)) \;\cong\; IH^*(\mathfrak{M}_{0,0}(T))^W \big/ \operatorname{Ann}(e_0).$$

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