A SURVEY OF LAGRANGIAN MECHANICS AND CONTROL ON LIE ALGEBROIDS AND GROUPOIDS

JORGE CORTÉS, MANUEL DE LEÓN, JUAN C. MARRERO, D. MARTÍN DE DIEGO, AND EDUARDO MARTÍNEZ

Abstract. In this survey, we present a geometric description of Lagrangian and Hamiltonian Mechanics on Lie algebroids. The flexibility of the Lie algebroid formalism allows us to analyze systems subject to nonholonomic constraints, mechanical control systems, Discrete Mechanics and extensions to Classical Field Theory within a single framework. Various examples along the discussion illustrate the soundness of the approach.

Contents

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1. Introduction

The theory of Lie algebroids and Lie groupoids has proved to be very useful in different areas of mathematics including algebraic and differential geometry, algebraic topology, and symmetry analysis. In this survey, we illustrate the wide range of applications of this formalism to Mechanics. Specifically, we show how the flexibility provided by Lie algebroids and groupoids allows us to analyze, within a single framework, different classes of situations such as systems subject to nonholonomic constraints, mechanical control systems, Discrete Mechanics and Field Theory.

The notions of Lie algebroid and Lie groupoid allow to study general Lagrangian and Hamiltonian systems beyond the ones defined on the tangent and cotangent bundles of the configuration manifold, respectively. These include systems determined by Lagrangian and Hamiltonian functions defined on Lie algebras, Lie groups, Cartesian products of manifolds, and reduced spaces.

The inclusive feature of the Lie algebroid formalism is particularly relevant for the class of Lagrangian systems invariant under the action of a Lie group of symmetries. Given a standard Lagrangian system, one associates to the Lagrangian function a Poincaré-Cartan symplectic form and an energy function using the particular geometry of the tangent bundle. The dynamics is then obtained as the Hamiltonian vector field associated to the energy function through the Poincaré-Cartan form. The reduction by the Lie group action of the dynamics of this system yields a reduced dynamics evolving on a quotient space (which is not a tangent bundle). However, the interplay between the geometry of this quotient space and the reduced dynamics is not as transparent as in the tangent bundle case. Recent efforts have lead to a unifying geometric framework to overcome this drawback. It is precisely the underlying structure of Lie algebroid on the phase space what allows a unified treatment. This idea was introduced by Weinstein [54] (see also [31]), who developed a generalized theory of Lagrangian Mechanics on Lie algebroids. He obtained the equations of motion using the linear Poisson structure on the dual of the Lie algebroid and the Legendre transformation associated with a (regular) Lagrangian. In [54], Weinstein also posed the question of whether it was possible to develop a treatment on Lie algebroids and groupoids similar to Klein's formalism for ordinary Lagrangian Mechanics [22]. This question was answered positively by E. Martínez in [38] (see also [13, 39, 41, 49]). The main notion was that of prolongation of a Lie algebroid over a mapping, introduced by P.J. Higgins and K. Mackenzie [19]. More recently, the work [26] has developed a description of Hamiltonian and Lagrangian dynamics on a Lie algebroid in terms of Lagrangian submanifolds of symplectic Lie algebroids. An alternative approach, using the linear Poisson structure on the dual of the Lie algebroid was discussed in [18].

In this paper, we present an overview of these developments. We make special emphasis on one of the main advantages of the Lie algebroid formalism: the possibility of establishing appropriate maps (called morphisms) between systems that respect the structure of the phase space, and allow to relate their respective properties. As we will show, this will allow us to present a comprehensive study of the reduction process of Lagrangian systems while remaining within the same category of mathematical objects.

We also consider nonholonomic systems (i.e., systems subject to constraints involving the velocities, see [10] for a list of references) and control systems evolving on Lie algebroids [13]. This is motivated by the renewed interest in the study of nonholonomic mechanical systems for new applications in the areas of robotics and control. In particular, we provide widely applicable tests to decide the accessibility and controllability properties of mechanical control systems defined on Lie algebroids.

We end this survey by paying attention to two recent developments in the context of Lie algebroids and groupoids: Discrete Mechanics and Classical Field Theory. Discrete Mechanics seeks to develop a complete discrete-time counterpart of the usual continuous-time treatment of Mechanics. The ultimate objective of this effort is the construction of numerical integrators for Lagrangian and Hamiltonian systems (see [37] and references therein). Up to now, this effort has been mainly focused on the case of discrete Lagrangian functions defined on the Cartesian product of the configuration manifold with itself. This Cartesian product is just an example of a Lie groupoid. Here, we review the recent developments in [35], where we proposed a complete description of Lagrangian and Hamiltonian Mechanics on Lie groupoids. In particular, this description covers the analysis of discrete systems with symmetries, and naturally produces reduced geometric integrators. Another extension that we consider is the study of Classical Field Theory on Lie algebroids. Thinking of a Lie algebroid as a substitute of the tangent bundle of a manifold, we substitute the classical notion of fibration bundle by a surjective morphism of Lie algebroids $\pi\colon E \longrightarrow F$. Then, we construct the jet space $\mathcal{J}\pi$ as the affine bundle whose elements are linear maps from a fiber of F to a fiber of E, i.e., sections of the projection π. After a suitable choice of the space of variations, we derive the Euler-Lagrange equations for this problem.

The paper is organized as follows. In Section 2 we present some basic facts on Lie algebroids, including results from differential calculus, morphisms and prolongations of Lie algebroids, and linear connections. We also introduce the notion of Lie groupoid, Lie algebroid associated to a Lie groupoid, and morphisms and prolongations of Lie groupoids. Various examples are given in order to illustrate the generality of the theory. In Section 3, we give a brief introduction to the Lagrangian formalism of Mechanics on Lie algebroids, determined by a Lagrangian function $L : E \longrightarrow \mathbb{R}$ on the Lie algebroid $\tau : E \longrightarrow M$. Likewise, we introduce the Hamiltonian formalism on Lie algebroids, determined by a Hamiltonian function $H\colon E^* \longrightarrow \mathbb{R}$, where $\tau^*\colon E^* \longrightarrow M$ is the dual of the Lie algebroid $E \longrightarrow M$. In Section 4 we introduce the class of nonholonomic Lagrangian systems. We study the existence and uniqueness of solutions, and characterize the notion of regularity of a nonholonomic system. Under this property, we derive a procedure to obtain the solution of the nonholonomic problem from the solution of the free problem by means of projection techniques. Moreover, we construct a nonholonomic bracket that measures the evolution of the observables, and we study the reduction of nonholonomic systems in terms of morphisms of Lie algebroids. In Section 5 we introduce the class of mechanical control systems defined on a Lie algebroid. We generalize the notion of affine connection control system to the setting of Lie algebroids, and introduce the notions of (base) accessibility and (base) controllability. We provide sufficient conditions to check these properties for a given mechanical control system. In Section 6, we study Discrete Mechanics on Lie groupoids. In particular, we construct the discrete Euler-Lagrange equations, discrete Poincaré-Cartan sections, discrete Legendre transformations, and Noether's theorem, and identify the preservation properties of the discrete flow. In the last section, we extend the variational formalism for Classical Field Theory to the setting of Lie algebroids. Given a Lagrangian function, we study the problem of finding critical points of the action functional when we restrict the fields to be morphisms of Lie algebroids. Throughout the paper, various examples illustrate the results. We conclude the paper by identifying future directions of research.

2. Lie algebroids and Lie groupoids

2.1. Lie algebroids. Given a real vector bundle $\tau\colon E \longrightarrow M$, let $\mathrm{Sec}(\tau)$ denote the space of the global cross sections of $\tau\colon E \longrightarrow M$. A ***Lie algebroid*** E over a manifold M is a real vector bundle $\tau\colon E \longrightarrow M$ together with a Lie bracket $[\![\cdot,\cdot]\!]$ on $\mathrm{Sec}(\tau)$ and a bundle map $\rho\colon E \longrightarrow TM$ over the identity, called ***the anchor map***, such that the homomorphism (denoted also $\rho\colon \mathrm{Sec}(\tau) \longrightarrow \mathfrak{X}(M)$) of $C^\infty(M)$-modules induced by the anchor map verifies

$$[\![X, fY]\!] = f[\![X,Y]\!] + \rho(X)(f)Y,$$

for $X, Y \in \mathrm{Sec}(\tau)$ and $f \in C^\infty(M)$. The triple $(E, [\![\cdot,\cdot]\!], \rho)$ is called ***a Lie algebroid over*** M (see [32, 33]). If $(E, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over M, then the anchor map $\rho\colon \mathrm{Sec}(\tau) \longrightarrow \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\mathrm{Sec}(\tau), [\![\cdot,\cdot]\!])$ and $(\mathfrak{X}(M), [\cdot,\cdot])$.

In what concerns to Mechanics, it is convenient to think of a Lie algebroid as a generalization of the tangent bundle of M. One regards an element a of E as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a, i.e., $v = \rho(a)$. A curve $a\colon [t_0, t_1] \longrightarrow E$ is said to be ***admissible*** if $\dot m(t) = \rho(a(t))$, where $m(t) = \tau(a(t))$ is the base curve.

Given local coordinates (x^i) in the base manifold M and a local basis of sections (e_α) of E, then local coordinates of a point $a \in E$ are (x^i, y^α) where $a = y^\alpha e_\alpha(\tau(a))$. In local form, the Lie algebroid structure is determined by the local functions ρ^i_α and $C^\gamma_{\alpha\beta}$ on M. Both are determined by the relations

$$\begin{aligned} \rho(e_\alpha) &= \rho^i_\alpha \frac{\partial}{\partial x^i}, && (2.1)\\ [\![e_\alpha, e_\beta]\!] &= C^\gamma_{\alpha\beta}\, e_\gamma && (2.2) \end{aligned}$$

and they satisfy the following equations

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \quad\text{and}\quad \sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu}\right] = 0. \tag{2.3}$$

Cartan calculus. One may define ***the exterior differential of*** E, $d\colon \mathrm{Sec}(\wedge^k\tau^*) \longrightarrow \mathrm{Sec}(\wedge^{k+1}\tau^*)$, as follows

$$\begin{aligned} d\omega(X_0,\dots,X_k) &= \sum_{i=0}^{k} (-1)^i \rho(X_i)(\omega(X_0,\dots,\widehat{X_i},\dots,X_k))\\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([\![X_i,X_j]\!], X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k), \end{aligned} \tag{2.4}$$

for $\omega \in \mathrm{Sec}(\wedge^k \tau^*)$ and $X_0, \ldots, X_k \in \mathrm{Sec}(\tau)$. d is a cohomology operator, that is, $d^2 = 0$. In particular, if $f : M \longrightarrow \mathbb{R}$ is a real smooth function then $df(X) = \rho(X)f$, for $X \in \mathrm{Sec}(\tau)$. Locally,

$$dx^i = \rho^i_\alpha e^\alpha \qquad \text{and} \qquad de^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. We may also define the Lie derivative with respect to a section X of E as the operator $\mathcal{L}_X\colon \mathrm{Sec}(\wedge^k \tau^*) \longrightarrow \mathrm{Sec}(\wedge^k \tau^*)$ given by $\mathcal{L}_X = i_X \circ d + d \circ i_X$ (for more details, see [32, 33]).

Morphisms. Let $(E, [\![\cdot,\cdot]\!], \rho)$ (resp., $(E', [\![\cdot,\cdot]\!]', \rho')$) be a Lie algebroid over a manifold M (resp., M') and suppose that $\Psi\colon E \longrightarrow E'$ is a vector bundle morphism over the map $\Psi_0\colon M \longrightarrow M'$. Then, the pair (Ψ, Ψ_0) is said to be a ***Lie algebroid morphism*** if

$$d((\Psi,\Psi_0)^*\phi') = (\Psi,\Psi_0)^*(d'\phi'), \quad \text{for all } \phi' \in \mathrm{Sec}(\wedge^k(E')^*) \text{ and for all } k, \tag{2.5}$$

where d (resp., d') is the differential of the Lie algebroid E (resp., E') (see [26]). Note that $(\Psi,\Psi_0)^*\phi'$ is the section of the vector bundle $\wedge^k E^* \longrightarrow M$ defined for $k > 0$ by

$$((\Psi,\Psi_0)^*\phi')_x(a_1,\ldots,a_k) = \phi'_{\Psi_0(x)}(\Psi(a_1),\ldots,\Psi(a_k)),$$

for $x \in M$ and $a_1,\ldots,a_k \in E_x$, and by $(\Psi,\Psi_0)^* f = f \circ \Psi_0$ for $f \in \mathrm{Sec}(\wedge^0 E'^*) = C^\infty(M')$. In the particular case when $M = M'$ and $\Psi_0 = id_M$ then (2.5) holds if and only if

$$[\![\Psi \circ X, \Psi \circ Y]\!]' = \Psi[\![X,Y]\!], \quad \rho'(\Psi X) = \rho(X), \quad \text{for } X, Y \in \mathrm{Sec}(\tau).$$

Linear connections on Lie algebroids. Let $\tau : E \longrightarrow M$ be a Lie algebroid over M. A ***connection on*** E is a $\mathbb{R}$-bilinear map $\nabla\colon \mathrm{Sec}(E) \times \mathrm{Sec}(E) \longrightarrow \mathrm{Sec}(E)$ such that

$$\nabla_{fX} Y = f \nabla_X Y \ , \qquad \nabla_X(fY) = \rho(X)(f)Y + f\nabla_X Y$$

for $f \in C^\infty(M)$ and $X, Y \in \mathrm{Sec}(E)$.

Given a local basis $\{e_\alpha\}$ of $\mathrm{Sec}(E)$ such that $X = X^\alpha e_\alpha$ and $Y = Y^\beta e_\beta$ then

$$\nabla_X Y = X^\alpha \left(\rho^i_\alpha \frac{\partial Y^\gamma}{\partial x^i} + \Gamma^\gamma_{\alpha\beta} Y^\beta \right) e_\gamma.$$

The terms $\Gamma^\gamma_{\alpha\beta}$ are called the ***connection coefficients***. The ***symmetric product*** associated with ∇ is given by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X, \qquad X, Y \in \mathrm{Sec}(E).$$

Since the connection is $C^\infty(M)$-linear in the first argument, it is possible to define the derivative of a section $Y \in \mathrm{Sec}(E)$ with respect to an element $a \in E_m$ by simply putting

$$\nabla_a Y = (\nabla_X Y)(m),$$

with $X \in \mathrm{Sec}(E)$ satisfying $X(m) = a$. Moreover, the connection allows us to take the derivative of sections along maps and, as a particular case, of sections along curves. If we have a morphism of Lie algebroids $\Phi : F \longrightarrow E$ over the map $\varphi : N \longrightarrow M$ and a section $X : N \longrightarrow E$ along φ (i.e., $X(n) \in E_{\varphi(n)}$, for $n \in N$), then X may be written as

$$X = \sum_{l=1}^p F_l (X_l \circ \varphi),$$

for some sections $\{X_1, \dots, X_p\}$ of E and for some functions $F_1, \dots, F_p \in C^\infty(N)$, and the derivative of X along φ is given by

$$\nabla_b X = \sum_{l=1}^{p} \left[(\rho_F(b)F_l)X_l(\varphi(n)) + F_l(n)\nabla_{\Phi(b)}X_l \right], \quad \text{for } b \in F_n ,$$

where ρ_F is the anchor map of the Lie algebroid $F \longrightarrow N$.

A particular case of the above general situation is the following. Let $a : I \longrightarrow E$ be an admissible curve and $b : I \longrightarrow E$ be a curve in E, both of them projecting by τ onto the same base curve in M, $\tau(a(t)) = m(t) = \tau(b(t))$. Take the Lie algebroid structure $TI \longrightarrow I$ and consider the morphism $\Phi : TI \longrightarrow E$, $\Phi(t, \dot t) = \dot t a(t)$ over $m : I \longrightarrow M$. Then one can define the derivative of $b(t)$ along $a(t)$ as $\nabla_{d/dt} b(t)$. This derivative is usually denoted by $\nabla_{a(t)} b(t)$. In local coordinates, this reads

$$\nabla_{a(t)} b(t) = \left[\frac{db^\gamma}{dt} + \Gamma^\gamma_{\alpha\beta} a^\alpha b^\beta \right] e_\gamma(m(t)), \text{ for all } t.$$

The admissible curve $a : I \longrightarrow E$ is said to be a ***geodesic*** for ∇ if $\nabla_{a(t)} a(t) = 0$ (see [13]).

Now, let $\mathcal{G} : E \times_M E \longrightarrow \mathbb{R}$ be a bundle metric on a Lie algebroid $\tau : E \longrightarrow M$. In a parallel way to the situation in the tangent bundle geometry, one can see that there is a canonical connection $\nabla^{\mathcal{G}}$ on E associated with $\mathcal{G}$. In fact, the connection $\nabla^{\mathcal{G}}$ is determined by the formula

$$\begin{aligned} 2\mathcal{G}(\nabla^{\mathcal{G}}_X Y, Z) &= \rho(X)(\mathcal{G}(Y,Z)) + \rho(Y)(\mathcal{G}(X,Z)) - \rho(Z)(\mathcal{G}(X,Y)) \\ &\quad + \mathcal{G}(X, [\![Z,Y]\!]) + \mathcal{G}(Y, [\![Z,X]\!]) - \mathcal{G}(Z, [\![Y,X]\!]), \end{aligned}$$

for $X, Y, Z \in \mathrm{Sec}(E)$. $\nabla^{\mathcal{G}}$ is a torsion-less connection and it is metric with respect to $\mathcal{G}$. In other words

$$\begin{aligned} [\![X,Y]\!] &= \nabla^{\mathcal{G}}_X Y - \nabla^{\mathcal{G}}_Y X , \\ \rho(X)(\mathcal{G}(Y,Z)) &= \mathcal{G}(\nabla^{\mathcal{G}}_X Y, Z) + \mathcal{G}(Y, \nabla^{\mathcal{G}}_X Z) . \end{aligned}$$

$\nabla^{\mathcal{G}}$ is called the ***Levi-Civita connection*** of $\mathcal{G}$ (see [13]).

Finally, suppose that $E = D \oplus D^c$, with D and D^c vector subbundles of E, and denote by $P : E \longrightarrow D$ and $Q : E \longrightarrow D^c$ the corresponding complementary projectors induced by the decomposition. Then, the ***constrained connection*** is the connection $\check\nabla$ on E defined by

$$\check\nabla_X Y = P(\nabla_X Y) + \nabla_X (QY),$$

for $X, Y \in \mathrm{Sec}(E)$ (for the properties of the constrained connection $\check\nabla$, see [13]).

Examples. We will present some examples of Lie algebroids.

1.- **Real Lie algebras of finite dimension**. Let $\mathfrak{g}$ be a real Lie algebra of finite dimension. Then, it is clear that $\mathfrak{g}$ is a Lie algebroid over a single point.

2.- **The tangent bundle**. Let TM be the tangent bundle of a manifold M. Then, the triple $(TM, [\cdot,\cdot], id_{TM})$ is a Lie algebroid over M, where $id_{TM} : TM \longrightarrow TM$ is the identity map.

3.- **Foliations**. Let $\mathcal{F}$ be a foliation of finite dimension on a manifold P and $\tau_{\mathcal{F}} : T\mathcal{F} \longrightarrow P$ be the tangent bundle to the foliation $\mathcal{F}$. Then, $\tau_{\mathcal{F}} : T\mathcal{F} \longrightarrow P$ is a Lie algebroid over P. The anchor map is the canonical inclusion $\rho_{\mathcal{F}} : T\mathcal{F} \longrightarrow TP$ and the Lie bracket on the space $\mathrm{Sec}(\tau_{\mathcal{F}})$ is the restriction to $\mathrm{Sec}(\tau_{\mathcal{F}})$ of the standard Lie bracket of vector fields on P. In particular, if $\pi : P \longrightarrow M$ is a fibration, $\tau_P : TP \longrightarrow P$ is the canonical projection and $(\tau_P)_{|V\pi} : V\pi \longrightarrow P$ is the restriction of τ_P to the vertical bundle to π, then $(\tau_P)_{|V\pi} : V\pi \longrightarrow P$ is a Lie algebroid over P.

4.- **Atiyah algebroids**. Let $p : Q \longrightarrow M$ be a principal G-bundle. Denote by $\Phi : G \times Q \longrightarrow Q$ the free action of G on Q and by $T\Phi : G \times TQ \longrightarrow TQ$ the tangent action of G on TQ. Then, one may consider the quotient vector bundle $\tau_Q|G : TQ/G \longrightarrow M = Q/G$, and the sections of this vector bundle may be identified with the vector fields on Q which are invariant under the action Φ. Using that every G-invariant vector field on Q is p-projectable and that the usual Lie bracket on vector fields is closed with respect to G-invariant vector fields, we can induce a Lie algebroid structure on TQ/G. This Lie algebroid is called ***the Atiyah algebroid associated with the principal*** G***-bundle*** $p : Q \longrightarrow M$ (see [26, 32]).

5.- **Action Lie algebroids**. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $f : M' \longrightarrow M$ be a smooth map. Then, the pull-back of E over f, $f^*E = \{ (x', a) \in M' \times E \mid f(x') = \tau(a) \}$, is a vector bundle over M' whose vector bundle projection is the restriction to f^*E of the first canonical projection $pr_1 : M' \times E \to M'$. However, f^*E is not, in general, a Lie algebroid.

Now, suppose that $\Phi\colon \mathrm{Sec}(\tau) \longrightarrow \mathfrak{X}(M')$ is an action of E on f, that is, Φ is a $\mathbb{R}$-linear map which satisfies the following conditions

$$\Phi(hX) = (h \circ f)\Phi X, \quad \Phi[\![X, Y]\!] = [\Phi X, \Phi Y], \quad \Phi X(h \circ f) = \rho(X)(h) \circ f,$$

for $X, Y \in \mathrm{Sec}(\tau)$ and $h \in C^\infty(M)$. Then, one may introduce a Lie algebroid structure $([\![\cdot,\cdot]\!]_\Phi, \rho_\Phi)$ on the vector bundle $f^*E \to M'$ which is characterized by the following conditions

$$[\![X \circ f, Y \circ f]\!]_\Phi = [\![X, Y]\!] \circ f, \quad \rho_\Phi(X \circ f) = \Phi(X), \quad \text{for} \quad X, Y \in \mathrm{Sec}(\tau). \tag{2.6}$$

The resultant Lie algebroid is denoted by $E \ltimes f$ and we call it ***an action Lie algebroid*** (for more details, see [26]).

6.- **The prolongation of a Lie algebroid over a fibration** [19, 26, 39]. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $\pi : P \longrightarrow M$ be a fibration. We consider the subset of $E \times TP$

$$\mathcal{T}^E_p P = \{ (b, v) \in E_x \times T_p P \mid \rho(b) = T_p\pi(v) \},$$

where $T\pi : TP \longrightarrow TM$ is the tangent map to π, $p \in P_x$ and $\pi(p) = x$. We will frequently use the redundant notation (p, b, v) to denote the element $(b, v) \in \mathcal{T}^E_p P$. $\mathcal{T}^E P = \cup_{p \in P} \mathcal{T}^E_p P$ is a vector bundle over P and the vector bundle projection τ^E_P is just the projection onto the first factor. The anchor of $\mathcal{T}^E P$ is the projection onto the third factor, that is, the map $\rho^\pi : \mathcal{T}^E P \longrightarrow TP$ given by $\rho^\pi(p, b, v) = v$. The projection onto the second factor will be denoted by $\mathcal{T}\pi\colon \mathcal{T}^E P \longrightarrow E$, and it is a morphism of Lie algebroids over π. Explicitly, $\mathcal{T}\pi(p, b, v) = b$.

An element $z \in \mathcal{T}^E P$ is said to be ***vertical*** if it projects to zero, that is $\mathcal{T}\pi(z) = 0$. Therefore it is of the form $(p, 0, v)$, with v a π-vertical vector tangent to P at p.

Given local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of sections of E, we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \Big(p, e_\alpha(\pi(p)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_p\Big) \qquad \text{and} \qquad \mathcal{V}_A(p) = \Big(p, 0, \frac{\partial}{\partial u^A}\Big|_p\Big).$$

If $z = (p, b, v)$ is an element of $\mathcal{T}^E P$, with $b = z^\alpha e_\alpha$, then v is of the form $v = \rho^i_\alpha z^\alpha \frac{\partial}{\partial x^i} + v^A \frac{\partial}{\partial u^A}$, and we can write

$$z = z^\alpha \mathcal{X}_\alpha(p) + v^A \mathcal{V}_A(p).$$

Vertical elements are linear combinations of $\{\mathcal{V}_A\}$.

The anchor map ρ^π applied to a section Z of $\mathcal{T}^E P$ with local expression $Z = Z^\alpha \mathcal{X}_\alpha + V^A \mathcal{V}_A$ is the vector field on P whose coordinate expression is

$$\rho^\pi(Z) = \rho^i_\alpha Z^\alpha \frac{\partial}{\partial x^i} + V^A \frac{\partial}{\partial u^A}.$$

Next, we will see that it is possible to induce a Lie bracket structure on the space of sections of $\mathcal{T}^E P$. For that, we say that a section $\tilde{X}$ of $\tau^E_P : \mathcal{T}^E P \longrightarrow P$ is ***projectable*** if there exists a section X of $\tau : E \longrightarrow M$ and a vector field $U \in \mathfrak{X}(P)$ which is π-projectable to the vector field $\rho(X)$ and such that $\tilde{X}(p) = (X(\pi(p)), U(p))$, for all $p \in P$. For such a projectable section $\tilde{X}$, we will use the following notation $\tilde{X} \equiv (X, U)$. It is easy to prove that one may choose a local basis of projectable sections of the space $\mathrm{Sec}(\tau^E_P)$.

The Lie bracket of two projectable sections $Z_1 = (X_1, U_1)$ and $Z_2 = (X_2, U_2)$ is then given by

$$[\![Z_1, Z_2]\!]^\pi(p) = (p, [\![X_1, X_2]\!](x), [U_1, U_2](p)), \qquad p \in P, \quad x = \pi(p).$$

Since any section of $\mathcal{T}^E P$ can be locally written as a linear combination of projectable sections, the definition of the Lie bracket for arbitrary sections of $\mathcal{T}^E P$ follows. In particular, the Lie brackets of the elements of the basis are

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^\pi = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma, \qquad [\![\mathcal{X}_\alpha, \mathcal{V}_B]\!]^\pi = 0 \qquad \text{and} \qquad [\![V_A, \mathcal{V}_B]\!]^\pi = 0,$$

and, therefore, the exterior differential is determined by

$$dx^i = \rho^i_\alpha \mathcal{X}^\alpha, \qquad\qquad du^A = \mathcal{V}^A,$$
$$d\mathcal{X}^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \qquad\qquad d\mathcal{V}^A = 0,$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

The Lie algebroid $\mathcal{T}^E P$ is called the ***prolongation of*** E ***over*** π or the E-***tangent bundle to*** π.

2.2. Lie groupoids.

In this section, we review the definition of a Lie groupoid and present some basic facts generalities about them (see [32, 33] for more details). A ***groupoid*** over a set M is a set G together with the following structural maps:

- A pair of maps $\alpha : G \longrightarrow M$, the ***source***, and $\beta : G \longrightarrow M$, the ***target***. These maps define the set of composable pairs
$$G_2 = \{\, (g, h) \in G \times G \mid \beta(g) = \alpha(h) \,\}.$$
- A ***multiplication*** $m : G_2 \longrightarrow G$, to be denoted simply by $m(g, h) = gh$, such that
  - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$,
  - $g(hk) = (gh)k$.
- An ***identity section*** $\epsilon : M \longrightarrow G$ such that
  - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.
- An ***inversion map*** $i : G \longrightarrow G$, to be denoted simply by $i(g) = g^{-1}$, such that
  - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

A groupoid G over a set M will be denoted simply by the symbol $G \rightrightarrows M$.

The groupoid $G \rightrightarrows M$ is said to be a ***Lie groupoid*** if G and M are manifolds and all the structural maps are differentiable with α and β differentiable submersions. If $G \rightrightarrows M$ is a Lie groupoid then m is a submersion, ϵ is an immersion and i is a diffeomorphism. Moreover, if $x \in M$, $\alpha^{-1}(x)$ (resp., $\beta^{-1}(x)$) will be said the α-***fiber*** (resp., the β-***fiber***) of x.

On the other hand, if $g \in G$ then the ***left-translation by*** $g \in G$ and the ***right-translation by*** g are the diffeomorphisms

$$\begin{array}{lll} l_g : \alpha^{-1}(\beta(g)) \longrightarrow \alpha^{-1}(\alpha(g)) & ; & h \longrightarrow l_g(h) = gh, \\ r_g : \beta^{-1}(\alpha(g)) \longrightarrow \beta^{-1}(\beta(g)) & ; & h \longrightarrow r_g(h) = hg. \end{array}$$

Note that $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$.

A vector field $\tilde{X}$ on G is said to be ***left-invariant*** (resp., ***right-invariant***) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}(g))$, for $(g,h) \in G_2$.

Now, we will recall the definition of the ***Lie algebroid associated with*** G.

We consider the vector bundle $\tau : E_G \longrightarrow M$, whose fiber at a point $x \in M$ is $(E_G)_x = V_{\epsilon(x)}\alpha = Ker(T_{\epsilon(x)}\alpha)$. It is easy to prove that there exists a bijection between the space $\mathrm{Sec}(\tau)$ and the set of left-invariant (resp., right-invariant) vector fields on G. If X is a section of $\tau : E_G \longrightarrow M$, the corresponding left-invariant (resp., right-invariant) vector field on G will be denoted $\overleftarrow{X}$ (resp., $\overrightarrow{X}$), where

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g))), \tag{2.7}$$

$$\overrightarrow{X}(g) = -(T_{\epsilon(\alpha(g))} r_g)((T_{\epsilon(\alpha(g))} i)(X(\alpha(g)))), \tag{2.8}$$

for $g \in G$. Using the above facts, we may introduce a Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket, \rho)$ on E_G, which is defined by

$$\overleftarrow{\llbracket X, Y \rrbracket} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \tag{2.9}$$

for $X, Y \in \mathrm{Sec}(\tau)$ and $x \in M$ (for more details, see [9, 32]).

Given two Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$, a ***morphism of Lie groupoids*** is a smooth map $\Psi : G \longrightarrow G'$ such that

$$(g,h) \in G_2 \Longrightarrow (\Psi(g), \Psi(h)) \in (G')_2$$

and

$$\Psi(gh) = \Psi(g)\Psi(h).$$

A morphism of Lie groupoids $\Psi : G \longrightarrow G'$ induces a smooth map $\Phi_0 : M \longrightarrow M'$ in such a way that

$$\alpha' \circ \Psi = \Phi_0 \circ \alpha, \quad \beta' \circ \Psi = \Phi_0 \circ \beta, \quad \Psi \circ \epsilon = \epsilon' \circ \Phi_0,$$

α, β and ϵ (resp., α', β' and ϵ') being the source, the target and the identity section of G (resp., G').

Suppose that (Ψ, Φ_0) is a morphism between the Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$ and that $\tau : E_G \longrightarrow M$ (resp., $\tau' : E_{G'} \longrightarrow M'$) is the Lie algebroid of G (resp., G'). Then, if $x \in M$ we may consider the linear map $\Phi_x : (E_G)_x \longrightarrow (E_{G'})_{\Phi_0(x)}$ defined by

$$\Phi_x(v_{\epsilon(x)}) = (T_{\epsilon(x)}\Psi)(v_{\epsilon(x)}), \quad \text{for } v_{\epsilon(x)} \in A_x G. \tag{2.10}$$

In fact, we have that the pair (Φ, Φ_0) is a morphism between the Lie algebroids $\tau : E_G \longrightarrow M$ and $\tau' : E_{G'} \longrightarrow M'$ (see [32, 33]).

Examples. We will present some examples of Lie groupoids.

1.- **Lie groups**. Any Lie group G is a Lie groupoid over $\{\mathfrak{e}\}$, the identity element of G. The Lie algebroid associated with G is just the Lie algebra $\mathfrak{g}$ of G.

2.- **The pair or banal groupoid**. Let M be a manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the projection onto the first factor and β is the projection onto the second factor; $\epsilon(x) = (x,x)$, for all $x \in M$, $m((x,y),(y,z)) = (x,z)$, for $(x,y),(y,z) \in M \times M$ and $i(x,y) = (y,x)$.

$M \times M \rightrightarrows M$ is called the ***pair or banal groupoid***. If x is a point of M, it follows that

$$V_{\epsilon(x)}\alpha = \{0_x\} \times T_xM \subseteq T_xM \times T_xM \cong T_{(x,x)}(M \times M).$$

Thus, the linear maps

$$\Phi_x : T_xM \longrightarrow V_{\epsilon(x)}\alpha, \quad v_x \longrightarrow (0_x, v_x),$$

induce an isomorphism (over the identity of M) between the Lie algebroids $\tau_M : TM \longrightarrow M$ and $\tau : E_{M\times M} \longrightarrow M$.

3.- **The Lie groupoid associated with a fibration**. Let $\pi : P \longrightarrow M$ be a fibration, that is, π is a surjective submersion and denote by G_π the subset of $P \times P$ given by

$$G_\pi = \{ (p, p') \in P \times P \mid \pi(p) = \pi(p') \}.$$

Then, G_π is a Lie groupoid over P and the structural maps α_π, β_π, m_π, ϵ_π and i_π are the restrictions to G_π of the structural maps of the pair groupoid $P \times P \rightrightarrows P$.

If p is a point of P it follows that

$$V_{\epsilon_\pi(p)}\alpha_\pi = \{ (0_p, Y_p) \in T_pP \times T_pP \mid (T_p\pi)(Y_p) = 0 \}.$$

Thus, if $(\tau_P)_{|V\pi} : V\pi \longrightarrow P$ is the vertical bundle to π then the linear maps

$$(\Phi_\pi)_p : V_p\pi \longrightarrow V_{\epsilon_\pi(p)}\alpha_\pi, \quad Y_p \longrightarrow (0_p, Y_p)$$

induce an isomorphism (over the identity of M) between the Lie algebroids $(\tau_P)_{|V\pi} : V\pi \longrightarrow P$ and $\tau : E_{G_\pi} \longrightarrow P$.

4.- **Atiyah or gauge groupoids**. Let $p : Q \longrightarrow M$ be a principal left G-bundle. Then, the free action $\Phi : G \times Q \longrightarrow Q$, $(g, q) \longrightarrow \Phi(g, q) = gq$, of G on Q induces, in a natural way, a free action $\Phi \times \Phi : G \times (Q \times Q) \longrightarrow Q \times Q$ of G on $Q \times Q$ given by $(\Phi \times \Phi)(g, (q, q')) = (gq, gq')$, for $g \in G$ and $(q, q') \in Q \times Q$. Moreover, one may consider the quotient manifold $(Q \times Q)/G$ which admits a Lie groupoid structure over M with structural maps given by

$$\begin{array}{lcl}
\tilde{\alpha} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q), \\
\tilde{\beta} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q'), \\
\tilde{\epsilon} : M \longrightarrow (Q \times Q)/G & ; & x \longrightarrow [(q, q)], \quad \text{if } p(q) = x, \\
\tilde{m} : ((Q \times Q)/G)_2 \longrightarrow (Q \times Q)/G & ; & ([(q, q')], [(gq', q'')]) \longrightarrow [(gq, q'')], \\
\tilde{i} : (Q \times Q)/G \longrightarrow (Q \times Q)/G & ; & [(q, q')] \longrightarrow [(q', q)].
\end{array}$$

This Lie groupoid is called ***the Atiyah (gauge) groupoid associated with the principal G-bundle*** $p : Q \longrightarrow M$ (see [31]).

If x is a point of M such that $p(q) = x$, with $q \in Q$, and $p_{Q\times Q} : Q \times Q \longrightarrow (Q \times Q)/G$ is the canonical projection then it is clear that

$$V_{\tilde{\epsilon}(x)}\tilde{\alpha} = (T_{(q,q)}p_{Q\times Q})(\{0_q\} \times T_qQ).$$

Thus, if $\tau_Q|G : TQ/G \longrightarrow M$ is the Atiyah algebroid associated with the principal G-bundle $p : G \longrightarrow M$ then the linear maps

$$(TQ/G)_x \longrightarrow V_{\tilde{\epsilon}(x)}\tilde{\alpha} \ ; \ [v_q] \longrightarrow (T_{(q,q)}p_{Q\times Q})(0_q, v_q), \text{ with } v_q \in T_qQ,$$

induce an isomorphism (over the identity of M) between the Lie algebroids $\tau : E_{(Q\times Q)/G} \longrightarrow M$ and $\tau_Q|G : TQ/G \longrightarrow M$.

5.- **Action Lie groupoids**. Let $G \rightrightarrows M$ be a Lie groupoid and $f : M' \longrightarrow M$ be a smooth map. If $M' \,{}_f\!\times_\alpha G = \{ (p, g) \in P \times G \mid f(p) = \alpha(g) \}$ then a ***right action of G on f*** is a smooth map

$$M' \,{}_f\!\times_\alpha G \longrightarrow M', \ (x', g) \longrightarrow x'g,$$

which satisfies the following relations

$$
\begin{array}{rcll}
f(x'g) &=& \beta(g), & \text{for } (x',g) \in M' \,{}_f\!\times_\alpha G, \\
(x'g)h &=& x'(gh), & \text{for } (x',g) \in M' \,{}_f\!\times_\alpha G \text{ and } (g,h) \in G_2, \text{ and} \\
x'\epsilon(f(x')) &=& x', & \text{for } x' \in M'.
\end{array}
$$

Given such an action one constructs ***the action Lie groupoid*** $M' \,{}_f\!\times_\alpha G$ over M' by defining

$$
\begin{array}{lcl}
\tilde{\alpha}_f : M' \,{}_f\!\times_\alpha G \longrightarrow M' & ; & (x',g) \longrightarrow x', \\
\tilde{\beta}_f : M' \,{}_f\!\times_\alpha G \longrightarrow M' & ; & (x',g) \longrightarrow x'g, \\
\tilde{\epsilon}_f : M' \longrightarrow M' \,{}_f\!\times_\alpha G & ; & x' \longrightarrow (x', \epsilon(f(x'))), \\
\tilde{m}_f : (M' \,{}_f\!\times_\alpha G)_2 \longrightarrow M' \,{}_f\!\times_\alpha G & ; & ((x',g),(x'g,h)) \longrightarrow (x',gh), \\
\tilde{i}_f : M' \,{}_f\!\times_\alpha G \longrightarrow M' \,{}_f\!\times_\alpha G & ; & (x',g) \longrightarrow (x'g, g^{-1}).
\end{array}
$$

Now, if $x' \in M'$, we consider the map $x' \cdot : \alpha^{-1}(f(x')) \longrightarrow M'$ given by

$$x' \cdot (g) = x'g.$$

Then, if $\tau : E_G \longrightarrow M$ is the Lie algebroid of G, the $\mathbb{R}$-linear map $\Phi\colon \mathrm{Sec}(\tau) \longrightarrow \mathfrak{X}(M')$ defined by

$$\Phi(X)(x') = (T_{\epsilon(f(x'))}x' \cdot)(X(f(x'))), \quad \text{for } X \in \mathrm{Sec}(\tau) \text{ and } x' \in M',$$

induces an action of E_G on $f : M' \longrightarrow M$. In addition, the Lie algebroid associated with the Lie groupoid $M' \,{}_f\!\times_\alpha G \rightrightarrows P$ is the action Lie algebroid $E_G \ltimes f$ (for more details, see [19]).

6.- **The prolongation of a Lie groupoid over a fibration**. Given a Lie groupoid $G \rightrightarrows M$ and a fibration $\pi : P \longrightarrow M$, we consider the set

$$P \overset{G}{\times} P \equiv P \,{}_\pi\!\times_\alpha G \,{}_\beta\!\times_\pi P = \{\, (p,g,p') \in P \times G \times P \mid \pi(p) = \alpha(g), \;\; \beta(g) = \pi(p') \,\}.$$

Then, $P \overset{G}{\times} P$ is a Lie groupoid over P with structural maps given by

$$
\begin{array}{lcl}
\alpha^\pi : P \overset{G}{\times} P \longrightarrow P & ; & (p,g,p') \longrightarrow p, \\
\beta^\pi : P \overset{G}{\times} P \longrightarrow P & ; & (p,g,p') \longrightarrow p', \\
\epsilon^\pi : P \longrightarrow P \overset{G}{\times} P & ; & p \longrightarrow (p, \epsilon(\pi(p)), p), \\
m^\pi : (P \overset{G}{\times} P)_2 \longrightarrow P \overset{G}{\times} P & ; & ((p,g,p'),(p',h,p'')) \longrightarrow (p,gh,p''), \\
i^\pi : P \overset{G}{\times} P \longrightarrow P \overset{G}{\times} P & ; & (p,g,p') \longrightarrow (p', g^{-1}, p).
\end{array}
$$

$P \overset{G}{\times} P$ is called the ***prolongation of G over*** $\pi : P \longrightarrow M$.

Now, denote by $\tau : E_G \longrightarrow M$ the Lie algebroid of G, by $E_{P \overset{G}{\times} P}$ the Lie algebroid of $P \overset{G}{\times} P$ and by $\mathcal{T}^{E_G}P$ the prolongation of $\tau : E_G \longrightarrow M$ over the fibration π. If $p \in P$ and $m = \pi(p)$, then it follows that

$$\left(E_{P \overset{G}{\times} P}\right)_p = \left\{\, (0_p, v_{\epsilon(m)}, X_p) \in T_pP \times (E_G)_m \times T_pP \;\middle|\; (T_p\pi)(X_p) = (T_{\epsilon(m)}\beta)(v_{\epsilon(m)}) \,\right\}$$

and, thus, one may consider the linear isomorphism

$$(\Phi^\pi)_p : (E_{P \overset{G}{\times} P})_p \longrightarrow \mathcal{T}^{E_G}_p P, \quad (0_p, v_{\epsilon(m)}, X_p) \longrightarrow (v_{\epsilon(m)}, X_p). \tag{2.11}$$

In addition, one may prove that the maps $(\Phi^\pi)_p$, $p \in P$, induce an isomorphism $\Phi^\pi : E_{P \overset{G}{\times} P} \longrightarrow \mathcal{T}^{E_G}P$ between the Lie algebroids $E_{P \overset{G}{\times} P}$ and $\mathcal{T}^{E_G}P$ (for more details, see [19]).

A particular case. Next, suppose that $P = E_G$ and that the map π is just the vector bundle projection $\tau : E_G \longrightarrow M$. In this case,

$$E_G \overset{G}{\times} E_G = E_G \;{}_\tau\!\times_\alpha G \;{}_\beta\!\times_\tau E_G$$

and we may define the map $\Theta : E_G \overset{G}{\times} E_G \longrightarrow V\beta \oplus_G V\alpha$ as follows

$$\Theta(u_{\epsilon(\alpha(g))}, g, v_{\epsilon(\beta(g))}) = ((T_{\epsilon(\alpha(g))}(r_g \circ i))(u_{\epsilon(\alpha(g))}), (T_{\epsilon(\beta(g))} l_g)(v_{\epsilon(\beta(g))})),$$

for $(u_{\epsilon(\alpha(g))}, g, v_{\epsilon(\beta(g))}) \in (E_G)_{\alpha(g)} \times G \times (E_G)_{\beta(g)}$. Θ is a bijective map and

$$\Theta^{-1}(X_g, Y_g) = ((T_g(i \circ r_{g^{-1}}))(X_g), g, (T_g l_{g^{-1}})(Y_g)),$$

for $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$. Thus, the spaces $E_G \overset{G}{\times} E_G$ and $V\beta \oplus_G V\alpha$ may be identified and, under this identification, the structural maps of the Lie groupoid structure on $V\beta \oplus_G V\alpha$ are given by

$$\begin{array}{lll}
\alpha^\tau : V\beta \oplus_G V\alpha \longrightarrow E_G & ; & (X_g, Y_g) \longrightarrow (T_g(i \circ r_{g^{-1}}))(X_g), \\
\beta^\tau : V\beta \oplus_G V\alpha \longrightarrow E_G & ; & (X_g, Y_g) \longrightarrow (T_g l_{g^{-1}})(Y_g), \\
\epsilon^\tau : E_G \longrightarrow V\beta \oplus_G V\alpha & ; & v_{\epsilon(x)} \longrightarrow ((T_{\epsilon(x)} i)(v_{\epsilon(x)}), v_{\epsilon(x)}), \\
i^\tau : V\beta \oplus_G V\alpha \longrightarrow V\beta \oplus_G V\alpha & ; & (X_g, Y_g) \longrightarrow ((T_g i)(Y_g), (T_g i)(X_g)),
\end{array}$$

and the multiplication $m^\tau : (V\beta \oplus_G V\alpha)_2 \longrightarrow V\beta \oplus_G V\alpha$ is

$$m^\tau((X_g, Y_g), ((T_g(r_{gh} \circ i))(Y_g), Z_h)) = ((T_g r_h)(X_g), (T_h l_g)(Z_h)).$$

This Lie groupoid structure was considered by Saunders [53]. We remark that the Lie algebroid of $E_G \overset{G}{\times} E_G \cong V\beta \oplus_G V\alpha \rightrightarrows E_G$ is isomorphic to the prolongation $\mathcal{T}^{E_G} E_G$ of E_G over $\tau : E_G \longrightarrow M$.

3. Mechanics on Lie algebroids

We recall that a symplectic section on a vector bundle $\pi \colon F \longrightarrow M$ is a section ω of $\wedge^2 \pi^*$ which is regular at every point when it is considered as a bilinear form. By a ***symplectic Lie algebroid*** we mean a pair (E, ω) where $\tau \colon E \longrightarrow M$ is a Lie algebroid and ω is a symplectic section on the vector bundle E satisfying the compatibility condition $d\omega = 0$, where d is the exterior differential of E.

On a symplectic Lie algebroid (E, ω) we can define a dynamical system for every function on the base, as in the standard case of a tangent bundle. Given a function $H \in C^\infty(M)$ there is a unique section $\sigma_H \in \mathrm{Sec}(\tau)$ such that

$$i_{\sigma_H} \omega = dH.$$

The section σ_H is said to be the ***Hamiltonian section*** defined by H and the vector field $X_H = \rho(\sigma_H)$ is said to be the ***Hamiltonian vector field*** defined by H. In this way we get the dynamical system $\dot{x} = X_H(x)$.

A symplectic structure ω on a Lie algebroid E defines a ***Poisson bracket*** $\{\ ,\ \}^\omega$ on the base manifold M as follows. Given two functions $F, G \in C^\infty(M)$ we define the bracket

$$\{F, G\}^\omega = \omega(\sigma_F, \sigma_G).$$

It is easy to see that the closure condition $d\omega = 0$ implies that $\{\ ,\ \}^\omega$ is a Poisson structure on M. In other words, if we denote by Λ the inverse of ω as bilinear form, then $\{F, G\}^\omega = \Lambda(dF, dG)$. The Hamiltonian dynamical system associated to H can be written in terms of the Poisson bracket as $\dot{x} = \{x, H\}^\omega$.

By a ***symplectomorphism*** between two symplectic Lie algebroids (E, ω) and (E', ω') we mean an isomorphism of Lie algebroids (Ψ, Ψ_0) from E to E' such that $(\Psi, \Psi_0)^* \omega' = \omega$. In this case the base map Ψ_0 is a Poisson diffeomorphism, that is, it satisfies $\Psi_0^* \{F', G'\}^{\omega'} = \{\Psi_0^* F', \Psi_0^* G'\}^\omega$, for all $F', G' \in C^\infty(M')$.

Sections 3.1 and 3.2 describe two particular and important cases of the above construction.

3.1. Lagrangian Mechanics.

In [38] (see also [26]) a geometric formalism for Lagrangian Mechanics on Lie algebroids was introduced. It is developed in the prolongation $\mathcal{T}^E E$ of a Lie algebroid E over the vector bundle projection $\tau : E \longrightarrow M$. The canonical geometrical structures defined on $\mathcal{T}^E E$ are the following:

- The ***vertical lift*** $\xi^V : \tau^* E \longrightarrow \mathcal{T}^E E$ given by $\xi^V(a,b) = (a, 0, b_a^V)$, where b_a^V is the vector tangent to the curve $a + tb$ at $t = 0$.
- The ***vertical endomorphism*** $S \colon \mathcal{T}^E E \longrightarrow \mathcal{T}^E E$ defined as follows:

$$S(a,b,v) = \xi^V(a,b) = (a, 0, b_a^V).$$

- The ***Liouville section***, which is the vertical section corresponding to the Liouville dilation vector field:

$$\Delta(a) = \xi^V(a,a) = (a, 0, a_a^V).$$

We also mention that the ***complete lift*** X^C of a section $X \in \mathrm{Sec}(E)$ is the section of $\mathcal{T}^E E$ characterized by the following properties:

(i) projects to X, i.e., $\mathcal{T}\tau \circ X^C = X \circ \tau$,
(ii) $\mathcal{L}_{X^C}\hat{\mu} = \widehat{\mathcal{L}_X \mu}$,

where by $\hat{\alpha} \in C^\infty(E)$ we denote the linear function associated to $\alpha \in \mathrm{Sec}(E^*)$.

Given a Lagrangian function $L \in C^\infty(E)$ we define the ***Cartan 1-section*** $\theta_L \in \mathrm{Sec}((\mathcal{T}^E E)^*)$ and the ***Cartan 2-section*** $\omega_L \in \mathrm{Sec}(\wedge^2(\mathcal{T}^E E)^*)$ and the ***Lagrangian energy*** $E_L \in C^\infty(E)$ as

$$\theta_L = S^*(dL), \qquad \omega_L = -d\theta_L \qquad \text{and} \qquad E_L = \mathcal{L}_\Delta L - L. \tag{3.1}$$

If (x^i, y^α) are local fibred coordinates on E, $(\rho^i_\alpha, C^\gamma_{\alpha\beta})$ are the corresponding local structure functions on E and $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ is the corresponding local basis of sections of $\mathcal{T}^E E$ then

$$S\mathcal{X}_\alpha = \mathcal{V}_\alpha, \quad S\mathcal{V}_\alpha = 0, \quad \text{for all } \alpha, \tag{3.2}$$

$$\Delta = y^\alpha \mathcal{V}_\alpha, \tag{3.3}$$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta}\right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \tag{3.4}$$

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L. \tag{3.5}$$

From (3.2), (3.3), (3.4) and (3.5), it follows that

$$i_{SX}\omega_L = -S^*(i_X \omega_L), \quad i_\Delta \omega_L = -S^*(dE_L), \tag{3.6}$$

for $X \in \mathrm{Sec}(\mathcal{T}^E E)$.

Now, a curve $t \longrightarrow c(t)$ on E is a solution of the ***Euler-Lagrange equations*** for L if

- c is admissible (that is, $\rho(c(t)) = \dot{m}(t)$, where $m = \tau \circ c$) and
- $i_{(c(t),\dot{c}(t))}\omega_L(c(t)) - dE_L(c(t)) = 0$, for all t.

If $c(t) = (x^i(t), y^\alpha(t))$ then c is a solution of the Euler-Lagrange equations for L if and only if

$$\dot{x}^i = \rho^i_\alpha y^\alpha, \qquad \frac{d}{dt}\Big(\frac{\partial L}{\partial y^\alpha}\Big) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = 0. \tag{3.7}$$

Note that if E is the standard Lie algebroid TM then the above equations are the classical Euler-Lagrange equations for $L : TM \longrightarrow \mathbb{R}$.

On the other hand, the Lagrangian function L is said to be ***regular*** if ω_L is a symplectic section, that is, if ω_L is regular at every point as a bilinear form. In such a case, there exists a unique solution Γ_L verifying

$$i_{\Gamma_L}\Omega_L - dE_L = 0 \,.$$

In addition, using (3.6), it follows that $i_{S\Gamma_L}\omega_L = i_\Delta \omega_L$ which implies that Γ_L is a SODE ***section***, that is,

$$S(\Gamma_L) = \Delta,$$

or alternatively $\mathcal{T}\tau(\Gamma_L(a)) = a$ for all $a \in E$.

Thus, the integral curves of Γ_L (that is, the integral curves of the vector field $\rho^\tau(\Gamma_L)$) are solutions of the Euler-Lagrange equations for L. Γ_L is called the ***Euler-Lagrange section*** associated with L.

From (3.4), we deduce that L is regular if and only if the matrix $W_{\alpha\beta} = \dfrac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$ is regular. Moreover, the local expression of Γ_L is

$$\Gamma_L = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha,$$

where the functions f^α satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha} f^\beta + \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho^i_\beta y^\beta + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = 0, \text{ for all } \alpha. \tag{3.8}$$

Examples.

1.- **Real Lie algebras of finite dimension**. Let $\mathfrak{g}$ be a real Lie algebra of finite dimension and $L : \mathfrak{g} \longrightarrow \mathbb{R}$ be a Lagrangian function. Then, the Euler-Lagrange equations for L are just the well-known ***Euler-Poincaré equations*** (see, for instance, [36]).

2.- **The tangent bundle**. Let $L : TM \longrightarrow \mathbb{R}$ be a standard Lagrangian function on the tangent bundle TM of M. Then, the resultant equations are the ***classical Euler-Lagrange equations*** for L.

3.- **Foliations**. If the Lie algebroid is the tangent bundle of a foliation $\mathcal{F}$ on P then one recovers the classical formalism of ***holonomic mechanics***.

4.- **Atiyah algebroids**. Let $\tau_Q|G : TQ/G \longrightarrow M$ be the Atiyah algebroid associated with a principal G-bundle $p : Q \longrightarrow M$ and $L : TQ|G \longrightarrow \mathbb{R}$ be a Lagrangian function. Then, the Euler-Lagrange equations for L are just the ***Lagrange-Poincaré equations*** (see [26]).

5.- **Action Lie algebroids**. Suppose that $\mathfrak{g}$ is a real Lie algebra of finite dimension and that $\Phi : \mathfrak{g} \times V^* \longrightarrow V^*$ is a linear representation of $\mathfrak{g}$ on V^*. If $L : \mathfrak{g} \times V^* \longrightarrow \mathbb{R}$ is a Lagrangian function on the action Lie algebroid $\mathfrak{g} \times V^* \longrightarrow V^*$ then the Euler-Lagrange equations for L are just the so-called ***Euler-Poincaré equations with advected parameters*** or the ***Euler-Poisson-Poincaré equations*** (see [20]).

3.2. Hamiltonian Mechanics. In this section, we discuss how the Hamiltonian formalism can be developed for systems evolving on Lie algebroids (for more details, see [26, 39]).

Let $\tau^* : E^* \longrightarrow M$ be the vector bundle projection of the dual bundle E^* to E. Consider the prolongation $\mathcal{T}^E E^*$ of E over τ^*,

$$\begin{aligned}\mathcal{T}^E E^* &= \{ (b, v) \in E \times TE^* \mid \rho(b) = (T\tau^*)(v) \} \\ &= \{ (a^*, b, v) \in E^* \times E \times TE^* \mid \tau^*(a^*) = \tau(b), \rho(b) = (T\tau^*)(v) \}.\end{aligned}$$

The canonical geometrical structures defined on $\mathcal{T}^E E^*$ are the following:

- The ***Liouville section*** $\Theta_E \in \mathrm{Sec}((\mathcal{T}^E E^*)^*)$ defined by

$$\Theta_E(a^*)(b, v) = a^*(b). \tag{3.9}$$

- The ***canonical symplectic section*** $\Omega_E \in \mathrm{Sec}(\wedge^2(\mathcal{T}^E E^*)^*)$ is defined by

$$\Omega_E = -d\Theta_E. \tag{3.10}$$

  where d is the differential on the Lie algebroid $\mathcal{T}^E E^*$.

Take coordinates (x^i, p_α) on E^* and denote by $\{\mathcal{Y}_\alpha, \mathcal{P}^\beta\}$ the local basis of sections $\mathcal{T}^E E^*$, with

$$\mathcal{Y}_\alpha(a^*) = \left(a^*, e_\alpha(\tau^*(a^*)), \rho^i_\alpha \frac{\partial}{\partial x^i}\right) \qquad \text{and} \qquad \mathcal{P}^\beta(a^*) = \left(a^*, 0, \frac{\partial}{\partial p_\alpha}\right).$$

In coordinates the Liouville and canonical symplectic sections are written as

$$\Theta_E = p_\alpha \mathcal{Y}^\alpha \qquad \text{and} \qquad \Omega_E = \mathcal{Y}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} p_\gamma C^\gamma_{\alpha\beta} \mathcal{Y}^\alpha \wedge \mathcal{Y}^\beta,$$

where $\{\mathcal{Y}^\alpha, \mathcal{P}_\beta\}$ is the dual basis of $\{\mathcal{Y}_\alpha, \mathcal{P}^\beta\}$.

Every function $H \in C^\infty(E^*)$ define a unique section Γ_H of $\mathcal{T}^E E^*$ by the equation

$$i_{\Gamma_H} \Omega_E = dH,$$

and, therefore, a vector field $\rho^{\tau^*}(\Gamma_H) = X_H$ on E^* which gives the dynamics. In coordinates,

$$\Gamma_H = \frac{\partial H}{\partial p_\alpha} \mathcal{Y}_\alpha - \left(\rho^i_\alpha \frac{\partial H}{\partial x^i} + p_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial p_\beta}\right) \mathcal{P}^\alpha,$$

and therefore,

$$X_H = \rho^i_\alpha \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial x^i} - \left(\rho^i_\alpha \frac{\partial H}{\partial x^i} + p_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial p_\beta}\right) \frac{\partial}{\partial p_\alpha}.$$

Thus, the ***Hamilton equations*** are

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial p_\alpha} \qquad \frac{dp_\alpha}{dt} = -\rho^i_\alpha \frac{\partial H}{\partial x^i} - p_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial p_\beta}. \tag{3.11}$$

The Poisson bracket $\{\ ,\ \}^{\Omega_E}$ defined by the canonical symplectic section Ω_E on E^* is the canonical Poisson bracket, which is known to exists on the dual of a Lie algebroid [3].

Examples.

1.- **Real Lie algebras of finite dimension**. If the Lie algebroid E is a real Lie algebra of finite dimension then the Hamilton equations are just the well-known ***Lie-Poisson equations*** (see, for instance, [36]).

2.- **The tangent bundle**. If E is the standard Lie algebroid TM and $H : T^*M \longrightarrow \mathbb{R}$ is a Hamiltonian function then the resultant equations are the ***classical Hamilton equations*** for H.

3.- **Foliations**. If the Lie algebroid is the tangent bundle of a foliation $\mathcal{F}$ then one recovers the classical formalism of ***holonomic Hamiltonian mechanics***.

4.- **Atiyah algebroids**. Let $\tau_Q|G : TQ/G \longrightarrow M = Q|G$ be the Atiyah algebroid associated with a principal G-bundle $p : Q \longrightarrow M$ and $H : T^*Q|G \longrightarrow \mathbb{R}$ be a Hamilton function. Then, the Hamilton equations for H are just the ***Hamilton-Poincaré equations*** (see [26]).

5.- **Action Lie algebroids**. Suppose that $\mathfrak{g}$ is a real Lie algebra of finite dimension, that V is a real vector space of finite dimension and that $\Phi : \mathfrak{g} \times V^* \longrightarrow V^*$ is a linear representation of $\mathfrak{g}$ on V^*. If $H : \mathfrak{g}^* \times V^* \longrightarrow \mathbb{R}$ is a Hamiltonian function on the action Lie algebroid $\mathfrak{g} \times V^* \longrightarrow V^*$ then the Hamilton equations for H are just the ***Lie-Poisson equations on the dual of the semidirect product of Lie algebras*** $\mathfrak{s} = \mathfrak{g} \circledS V$ (see [20]).

3.3. The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms.

Let $L : E \longrightarrow \mathbb{R}$ be a Lagrangian function and $\theta_L \in \mathrm{Sec}((\mathcal{T}^E E)^*)$ be the Poincaré-Cartan 1-section associated with L.

We introduce ***the Legendre transformation associated with*** L as the smooth map $Leg_L : E \longrightarrow E^*$ defined by

$$Leg_L(a)(b) = \frac{d}{dt} L(a + tb)\Big|_{t=0}, \tag{3.12}$$

for $a, b \in E_x$, where E_x is the fiber of E over the point $x \in M$. In other words $Leg_L(a)(b) = \theta_L(a)(z)$, where z is a point in the fiber of $\mathcal{T}^E E$ over the point a such that $\mathcal{T}\tau(z) = b$.

The map Leg_L is well-defined and its local expression in fibred coordinates on E and E^* is

$$Leg_L(x^i, y^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}). \tag{3.13}$$

From this local expression it is easy to prove that the Lagrangian L is regular if and only if Leg_L is a local diffeomorphism.

The Legendre transformation induces a map $\mathcal{T} Leg_L : \mathcal{T}^E E \longrightarrow \mathcal{T}^E E^*$ defined by

$$(\mathcal{T} Leg_L)(b, X_a) = (b, (T_a Leg_L)(X_a)), \tag{3.14}$$

for $a, b \in E$ and $(a, b, X_a) \in \mathcal{T}^E_a E \subseteq E_{\tau(a)} \times E_{\tau(a)} \times T_a E$, where $TLeg_L : TE \longrightarrow TE^*$ is the tangent map of Leg_L. Note that $\tau^* \circ Leg_L = \tau$ and thus $\mathcal{T} Leg_L$ is well-defined.

If we consider local coordinates on $\mathcal{T}^E E$ (resp. $\mathcal{T}^E E^*$) induced by the local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ (resp., $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$) the local expression of $\mathcal{T} Leg_L$ is

$$\mathcal{T} Leg_L(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}; z^\alpha, \rho^i_\beta z^\beta \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + v^\beta \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}). \tag{3.15}$$

The relationship between Lagrangian and Hamiltonian Mechanics is given by the following result.

Theorem 3.1. [26] *The pair $(\mathcal{T}Leg_L, Leg_L)$ is a morphism between the Lie algebroids $(\mathcal{T}^E E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and $(\mathcal{T}^E E^*, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$. Moreover, if θ_L and ω_L (respectively, Θ_E and Ω_E) are the Poincaré-Cartan 1-section and 2-section associated with L (respectively, the Liouville 1-section and the canonical symplectic section on $\mathcal{T}^E E^*$) then*

$$(\mathcal{T}Leg_L, Leg_L)^*(\Theta_E) = \theta_L, \quad (\mathcal{T}Leg_L, Leg_L)^*(\Omega_E) = \omega_L. \tag{3.16}$$

In addition, in [26], it is proved that if the Lagrangian L is ***hyperregular***, that is, Leg_L is a global diffeomorphism, then $(\mathcal{T}Leg_L, Leg_L)$ is a symplectomorphism and the Euler-Lagrange section Γ_L associated with L and the Hamiltonian section Γ_H are $(\mathcal{T}Leg_L, Leg_L)$-related, that is,

$$\Gamma_H \circ Leg_L = \mathcal{T}Leg_L \circ \Gamma_L. \tag{3.17}$$

Therefore, an admissible curve $a(t)$ on $\mathcal{T}^E E$ is a solution of the Euler-Lagrange equations if and only if the curve $\mu(t) = Leg_L(a(t))$ is a solution of the Hamilton equations.

4. Nonholonomic Lagrangian systems on Lie algebroids

4.1. Constrained Lagrangian systems.

In this section, we will discuss Lagrangian systems on a Lie algebroid $\tau : E \longrightarrow M$ subject to nonholonomic constraints. The constraints are real functions on the positions and generalized velocities which constrain the motion to some submanifold $\mathcal{M}$ of E. $\mathcal{M}$ is the ***constraint submanifold***.

We will assume that the constraints are purely nonholonomic, that is, not all the generalized velocities are allowable, although all the positions are permitted. So, we will suppose that $\pi = \tau|_{\mathcal{M}} : \mathcal{M} \longrightarrow M$ is a fibration.

The constraints are linear if they are linear functions on E or, in more geometrical terms, if $\mathcal{M}$ is a vector subbundle of E over M (Lagrangian systems subject to linear constraints were discussed in [13, 45]).

In the general case, since π is a fibration, the prolongation $\mathcal{T}^E\mathcal{M}$ is defined. We will denote by r the dimension of the fibers of $\pi : \mathcal{M} \longrightarrow M$, that is $r = \dim \mathcal{M} - \dim M$.

Now, we define the bundle $\mathcal{V} \longrightarrow \mathcal{M}$ of ***virtual displacements*** as the subbundle of τ^*E of rank r whose fiber at a point $a \in \mathcal{M}$ is

$$\mathcal{V}_a = \left\{ b \in E_{\tau(a)} \;\middle|\; b_a^V \in T_a\mathcal{M} \right\}.$$

In other words, the elements of $\mathcal{V}$ are pairs of elements $(a, b) \in E \oplus_M E$ such that

$$\frac{d}{dt}\phi(a + tb)\Big|_{t=0} = 0,$$

for every local constraint function ϕ.

We also define the bundle of ***constraint forces*** $\mathbf{\Psi}$ by $\mathbf{\Psi} = S^*((\mathcal{T}^E\mathcal{M})^\circ)$. Since π is a fibration, the transformation $S^* : (\mathcal{T}^E\mathcal{M})^0 \longrightarrow \mathbf{\Psi}$ defines an isomorphism between the vector bundles $(\mathcal{T}^E\mathcal{M})^0 \longrightarrow \mathcal{M}$ and $\mathbf{\Psi} \longrightarrow \mathcal{M}$. Therefore, the rank of $\mathbf{\Psi}$ is $s = n - r$, where n is the rank of E.

Next, suppose that $L \in C^\infty(E)$ is a regular Lagrangian function. Then, the pair $(L, \mathcal{M})$ is a ***constrained Lagrangian system***. Moreover, assuming the validity of a Chetaev's principle in the spirit of that of standard Nonholonomic Mechanics (see [25]), the solutions of the system $(L, \mathcal{M})$ are curves $t \longrightarrow c(t)$ on E such that:

- c is admissible (that is, $\rho(c(t)) = \dot{m}(t)$, where $m = \tau \circ c$),
- c is contained in $\mathcal{M}$ and,

– $i_{(c(t),\dot c(t))}\omega_L(c(t)) - dE_L(c(t)) \in \mathbf{\Psi}(c(t))$, for all t.

If (x^i, y^α) are local fibred coordinates on E, $(\rho^i_\alpha, C^\gamma_{\alpha\beta})$ are the corresponding local structure functions of E and

$$\phi^A(x^i, y^\alpha) = 0, \quad A = 1, \dots, s,$$

are the local equations defining $\mathcal{M}$ as a submanifold of E, then $\left\{\dfrac{\partial \phi^A}{\partial y^\alpha}\mathcal{X}^\alpha\right\}_{A=1,\dots,s}$ is a local basis of $\mathbf{\Psi}$. Moreover, a curve $t \longrightarrow c(t) = (x^i(t), y^\alpha(t))$ on E is a solution of the problem if and only if

$$\begin{aligned} &\dot x^i = \rho^i_\alpha y^\alpha, \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = \lambda_A \frac{\partial \phi^A}{\partial y^\alpha}, \\ &\phi^A(x^i, y^\alpha) = 0, \end{aligned} \tag{4.1}$$

where λ_A are the Lagrange multipliers to be determined.

These equations are called ***the Lagrange-d'Alembert equations for the constrained system*** $(L, \mathcal{M})$. Note that if E is the Lie algebroid TM, then the above equations are just the standard Lagrange-d'Alembert equations for the constrained system $(L, \mathcal{M})$.

Now, we will assume that the solution curves of the problem are the integral curves of a section Γ of $\mathcal{T}^E E \longrightarrow E$. Then, we may reformulate geometrically the problem as follows: we look for a section Γ of $\mathcal{T}^E E \longrightarrow E$ such that

$$\begin{aligned} &(i_\Gamma \omega_L - dE_L)|_{\mathcal{M}} \in \mathrm{Sec}(\mathbf{\Psi}), \\ &\Gamma|_{\mathcal{M}} \in \mathrm{Sec}(\mathcal{T}^E \mathcal{M}). \end{aligned} \tag{4.2}$$

If Γ is a solution of the above equations then, from (3.6), we have that

$$(i_{S\Gamma}\omega_L - i_\Delta \omega_L)|_{\mathcal{M}} = 0,$$

which implies that Γ is a SODE section along $\mathcal{M}$, that is, $(S\Gamma - \Delta)|\mathcal{M} = 0$.

4.2. Regularity, projection of the free dynamics and nonholonomic bracket.

We will discuss next the regularity of the constrained system $(L, \mathcal{M})$ (the constrained system $(L, \mathcal{M})$ is ***regular*** if equations (4.2) admit a unique solution Γ).

For this purpose, we will introduce two new vector bundles F and $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ over $\mathcal{M}$. The fibers of F and $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ at the point $a \in \mathcal{M}$ are

$$\begin{aligned} F_a &= \omega_L^{-1}(\mathbf{\Psi}_a), \\ \mathcal{T}^{\mathcal{V}}_a\mathcal{M} &= \left\{ z \in \mathcal{T}^E_a\mathcal{M} \;\middle|\; \mathcal{T}\pi(z) \in \mathcal{V}_a \right\} = \left\{ z \in \mathcal{T}^E_a\mathcal{M} \;\middle|\; S(z) \in \mathcal{T}^E_a\mathcal{M} \right\}. \end{aligned}$$

Then, one may prove the following result.

Theorem 4.1. [10] *The following properties are equivalent:*

(i) *The constrained Lagrangian system* $(L, \mathcal{M})$ *is regular.*
(ii) $\mathcal{T}^E\mathcal{M} \cap F = \{0\}$.
(iii) $\mathcal{T}^{\mathcal{V}}\mathcal{M} \cap (\mathcal{T}^{\mathcal{V}}\mathcal{M})^\perp = \{0\}$.

Here, the orthogonal complement is taken with respect to the symplectic section ω_L.

Condition (ii) (or, equivalently, (iii)) in Theorem 4.1 is locally equivalent to the regularity of the matrix

$$\left(\mathcal{C}^{AB} = \frac{\partial \phi^A}{\partial y^\alpha} W^{\alpha\beta} \frac{\partial \phi^B}{\partial y^\beta}\right)_{A,B=1,\dots,s}$$

where $(W^{\alpha\beta})$ is the inverse matrix of $\Big(W_{\alpha\beta} = \dfrac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\Big)$.

Thus, if L is a Lagrangian function of mechanical type (that is, $L(a) = \frac{1}{2}\mathcal{G}(a,a) - V(\tau(a))$, for all $a \in E$, with $\mathcal{G} : E \times_M E \longrightarrow \mathbb{R}$ a bundle metric on E and $V : M \longrightarrow \mathbb{R}$ a real function on M) then the constrained system $(L, \mathcal{M})$ is always regular.

Now, assume that the constrained Lagrangian system $(L, \mathcal{M})$ is regular. Then (ii) in Theorem 4.1 is equivalent to $(\mathcal{T}^E E)_{|\mathcal{M}} = \mathcal{T}^E \mathcal{M} \oplus F$ and we will denote by P and Q the complementary projectors defined by this decomposition

$$P_a : \mathcal{T}^E_a E \longrightarrow \mathcal{T}^E_a \mathcal{M}, \quad Q_a : \mathcal{T}^E_a E \longrightarrow F_a, \quad \text{for all } a \in \mathcal{M}.$$

Moreover, we have

Theorem 4.2. [10] *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then, the solution of the constrained dynamics is the* SODE *Γ along $\mathcal{M}$ obtained as follows*

$$\Gamma = P(\Gamma_L|\mathcal{M}).$$

On the other hand, (3) in Theorem 4.1 is equivalent to $(\mathcal{T}^E E)_{|\mathcal{M}} = \mathcal{T}^{\mathcal{V}}\mathcal{M} \oplus (\mathcal{T}^{\mathcal{V}}\mathcal{M})^\perp$ and we will denote by $\bar{P}$ and $\bar{Q}$ the corresponding projectors induced by this decomposition, that is,

$$\bar{P}_a : \mathcal{T}^E_a E \longrightarrow \mathcal{T}^{\mathcal{V}}_a \mathcal{M}, \quad \bar{Q}_a = \mathcal{T}^E_a E \longrightarrow (\mathcal{T}^{\mathcal{V}}_a \mathcal{M})^\perp, \text{ for all } a \in \mathcal{M}.$$

Theorem 4.3. [10] *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system, Γ_L (respectively, Γ) be the solution of the free (respectively, constrained) dynamics and Δ be the Liouville section of $\mathcal{T}^E E \longrightarrow E$. Then, $\Gamma = \bar{P}(\Gamma_L|\mathcal{M})$ if and only if the restriction to $\mathcal{M}$ of the vector field $\rho^\tau(\Delta)$ on E is tangent to $\mathcal{M}$.*

Note that if $\mathcal{M}$ is a vector subbundle of E then the vector field $\rho^\tau(\Delta)$ is tangent to $\mathcal{M}$. Therefore, using Theorem 4.3, it follows that

Corollary 4.4. *Under the same hypotheses as in Theorem 4.3 if $\mathcal{M}$ is a vector subbundle of E (that is, the constraints are linear) then $\Gamma = \bar{P}(\Gamma_L|\mathcal{M})$.*

Next, we will study the conservation of the Lagrangian energy for the constrained Lagrangian system $(L, \mathcal{M})$.

Since $S^* : (\mathcal{T}^E\mathcal{M})^0 \longrightarrow \Psi$ is a vector bundle isomorphism, it follows that there exists a unique section $\alpha_{(L,\mathcal{M})}$ of $(\mathcal{T}^E\mathcal{M})^0 \longrightarrow \mathcal{M}$ such that

$$i_{Q(\Gamma_L|\mathcal{M})}\omega_L = S^*(\alpha_{(L,\mathcal{M})}).$$

Moreover, we have

Theorem 4.5 (Conservation of the energy)**.** [10] *If $(L, \mathcal{M})$ is a regular constrained Lagrangian system and Γ is the solution of the dynamics then $\mathcal{L}_\Gamma(E_L|\mathcal{M}) = 0$ if and only if $\alpha_{(L,\mathcal{M})}(\Delta|\mathcal{M}) = 0$. In particular, if the vector field $\rho^\tau(\Delta)$ is tangent to $\mathcal{M}$ then $\mathcal{L}_\Gamma(E_L|\mathcal{M}) = 0$.*

Now, suppose that f and g are two smooth real functions on $\mathcal{M}$ and take arbitrary extensions to E denoted by the same letters. Then, we may define ***the nonholonomic bracket*** of f and g as follows

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_f), \bar{P}(X_g))|\mathcal{M},$$

where X_f and X_g are the Hamiltonian sections on $\mathcal{T}^E E$ associated with f and g, respectively.

The nonholonomic bracket is well-defined and, furthermore, it is not difficult to prove the following result.

Theorem 4.6 (The nonholonomic bracket)**.** [10] *The nonholonomic bracket is an almost-Poisson bracket, i.e., it is skew-symmetric and satisfies the Leibniz rule (it is a derivation in each argument with respect to the usual product of functions). Moreover, if $f \in C^\infty(\mathcal{M})$ is an observable, then the evolution $\dot{f}$ of f is given by*

$$\dot{f} = \rho^\tau(R_L)(f) + \{f, E_L|\mathcal{M}\}_{nh},$$

where R_L is the section of $\mathcal{T}^E\mathcal{M} \longrightarrow \mathcal{M}$ defined by $R_L = P(\Gamma_L|\mathcal{M}) - \bar{P}(\Gamma_L|\mathcal{M})$. In particular, if the vector field $\rho^\tau(\Delta)$ is tangent to $\mathcal{M}$ then

$$\dot{f} = \{f, E_L|\mathcal{M}\}_{nh}.$$

4.3. **Reduction.** Next, we will discuss a reduction process and its relation with Lie algebroid epimorphisms. These results will be also valid for Lie algebroids without nonholonomic constraints, simply taking $\mathcal{M} = E$ and $\mathcal{M}' = E'$ in the sequel.

Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system on a Lie algebroid $\tau : E \longrightarrow M$ and $(L', \mathcal{M}')$ be another constrained Lagrangian system on a second Lie algebroid $\tau' : E' \longrightarrow M'$. Suppose also that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \longrightarrow E'$ over a surjective submersion $\phi : M \longrightarrow M'$ such that:

i) $L = L' \circ \Phi$,
ii) $\Phi_{|\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}'$ is a surjective submersion and
iii) $\Phi(\mathcal{V}_a) = \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$.

Note that if $\mathcal{M}$ and $\mathcal{M}'$ are vector subbundles of E and E', respectively, then conditions *i)*, *ii)* and *iii)* hold if and only if

$$L = L' \circ \Phi \quad \text{and} \quad \Phi(\mathcal{M}) = \mathcal{M}'.$$

In the general case, one may introduce the map $\mathcal{T}^\Phi\Phi : \mathcal{T}^E\mathcal{M} \longrightarrow \mathcal{T}^{E'}\mathcal{M}'$ given by

$$(\mathcal{T}^\Phi\Phi)(b, v) = (\Phi(b), (T\Phi)(v)), \qquad \text{for } (b, v) \in \mathcal{T}^E\mathcal{M},$$

and we have that $\mathcal{T}^\Phi\Phi$ is a Lie algebroid epimorphism over Φ. In addition, the following results hold

Theorem 4.7 (Reduction of the constrained dynamics)**.** [10] *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system on a Lie algebroid E and $(L', \mathcal{M}')$ be a constrained Lagrangian system on a second Lie algebroid E'. Assume that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \longrightarrow E'$ over $\phi : M \longrightarrow M'$ such that conditions $i)$, $ii)$ and $iii)$ hold. Then:*

(i) *The constrained Lagrangian system $(L', \mathcal{M}')$ is regular.*
(ii) *If Γ (respectively, Γ') is the constrained dynamics for L (respectively, for L') then $\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi$.*
(iii) *If $t \longrightarrow c(t)$ is a solution of Lagrange-d'Alembert differential equations for L then $\Phi(c(t))$ is a solution of Lagrange-d'Alembert differential equations for L'.*

Theorem 4.8 (Reduction of the nonholonomic bracket)**.** [10] *Under the same hypotheses as in Theorem 4.7, we have that*

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}'_{nh} \circ \Phi$$

for $f', g' \in C^\infty(\mathcal{M}')$, where $\{\cdot,\cdot\}_{nh}$ (respectively, $\{\cdot,\cdot\}'_{nh}$) is the nonholonomic bracket for the constrained system $(L, \mathcal{M})$ (respectively, $(L', \mathcal{M}')$). In other words, $\Phi : \mathcal{M} \longrightarrow \mathcal{M}'$ is an almost-Poisson morphism when on $\mathcal{M}$ and $\mathcal{M}'$ we consider the almost-Poisson structures defined by the corresponding nonholonomic brackets.

Reduction by symmetries. Let $\phi : Q \longrightarrow M$ be a principal G-bundle and $\tau : E \longrightarrow Q$ be a Lie algebroid over Q. In addition, assume that we have an action of G on E such that the quotient vector bundle E/G is defined and the set $\mathrm{Sec}(E)^G$ of equivariant sections of E is a Lie subalgebra of $\mathrm{Sec}(E)$. Then, $E' = E/G$ has a canonical Lie algebroid structure over M such that the canonical projection $\Phi : E \longrightarrow E'$ is a fiberwise bijective Lie algebroid morphism over ϕ (see [26]).

Next, suppose that $(L, \mathcal{M})$ is a G-invariant regular constrained Lagrangian system, that is, the Lagrangian function L and the constraint submanifold $\mathcal{M}$ are G-invariant. Assume also that $\mathcal{M}$ is closed. Then, one may define a Lagrangian function $L' : E' \longrightarrow \mathbb{R}$ on E' such that

$$L = L' \circ \Phi.$$

Moreover, G acts on $\mathcal{M}$ and the set of orbits $\mathcal{M}' = \mathcal{M}/G$ of this action is a quotient manifold, that is, $\mathcal{M}'$ is a smooth manifold and the canonical projection $\Phi_{|\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}' = \mathcal{M}/G$ is a submersion. Thus, one may consider the constrained Lagrangian system $(L', \mathcal{M}')$ on E'.

Since the orbits of the action of G on E are the fibers of Φ and $\mathcal{M}$ is G-invariant, we deduce that

$$V_a(\Phi) \subseteq T_a\mathcal{M}, \text{ for all } a \in \mathcal{M},$$

$V(\Phi)$ being the vertical bundle of Φ. This implies that $\Phi_{|\mathcal{V}_a} : \mathcal{V}_a \longrightarrow \mathcal{V}'_{\Phi(a)}$ is a linear isomorphism, for all $a \in \mathcal{M}$.

Therefore, from Theorem 4.7, we conclude that the constrained Lagrangian system $(L', \mathcal{M}')$ is regular and that

$$\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi,$$

where Γ (resp., Γ') is the constrained dynamics for L (resp., L'). In addition, using Theorem 4.8, we obtain that $\Phi : \mathcal{M} \longrightarrow \mathcal{M}'$ is an almost-Poisson morphism when on $\mathcal{M}$ and $\mathcal{M}'$ we consider the almost-Poisson structures induced by the corresponding nonholonomic brackets.

4.4. **Example: a rolling ball on a rotating table.** We apply the results in this section to the case of a ball rolling without sliding on a rotating table with constant angular velocity [1, 4, 10, 30, 48]. A (homogeneous) sphere of radius $r > 0$, unit mass $m = 1$ and inertia k^2 about any axis, rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere.

Choose a Cartesian reference frame with origin at the center of rotation of the table and z-axis along the rotation axis. Let (x, y) denote the position of the point of contact of the sphere with the table. The configuration space for the sphere on the table is $Q = \mathbb{R}^2 \times SO(3)$, where $SO(3)$ may be parameterized by the Eulerian angles θ, φ and ψ. The kinetic energy of the sphere is then given by

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)).$$

With the potential energy being constant, we may put $V = 0$. Thus, the Lagrangian function L is T and the constraint equations are

$$\begin{aligned} \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi &= -\Omega y, \\ \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi &= \Omega x. \end{aligned}$$

Since the Lagrangian function is of mechanical type, the constrained system is regular. Note that the constraints are not linear and that the restriction to the constraint submanifold $\mathcal{M}$ of the Liouville vector field on TQ is not tangent to $\mathcal{M}$. Indeed, the constraints are linear if and only if $\Omega = 0$.

Next, following [4, 10], we will consider local coordinates $(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_i)_{i=1,\dots,5}$ on $TQ = T\mathbb{R}^2 \times T(SO(3))$, where

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \quad \bar{\psi} = \psi,$$
$$\pi_1 = r\dot{x} + k^2\dot{q}_2, \quad \pi_2 = r\dot{y} - k^2\dot{q}_1, \quad \pi_3 = k^2\dot{q}_3,$$
$$\pi_4 = \frac{k^2}{(k^2+r^2)}(\dot{x} - r\dot{q}_2 + \Omega y), \quad \pi_5 = \frac{k^2}{(k^2+r^2)}(\dot{y} + r\dot{q}_1 - \Omega x),$$

$(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ are the quasi-coordinates defined by

$$\dot{q}_1 = \omega_x, \quad \dot{q}_2 = \omega_y, \quad \dot{q}_3 = \omega_z,$$

and ω_x, ω_y and ω_z are the components of the angular velocity of the sphere.

Then, the constrained dynamics is the SODE Γ along $\mathcal{M}$ defined by

$$\begin{aligned} \Gamma = (P\Gamma_L|\mathcal{M}) &= (\dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{\theta}\frac{\partial}{\partial\bar{\theta}} + \dot{\varphi}\frac{\partial}{\partial\bar{\varphi}} + \dot{\psi}\frac{\partial}{\partial\bar{\psi}})|\mathcal{M} \\ &= (\dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + \dot{q}_3\frac{\partial}{\partial q_3})|\mathcal{M}. \end{aligned} \tag{4.3}$$

On the other hand, when constructing the nonholonomic bracket on $\mathcal{M}$, we find that the only non-zero fundamental brackets are

$$\begin{array}{lll} \{x, \pi_1\}_{nh} = r, & \{y, \pi_2\}_{nh} = r, & \\ \{q_1, \pi_2\}_{nh} = -1, & \{q_2, \pi_1\}_{nh} = 1, & \{q_3, \pi_3\}_{nh} = 1, \\ \{\pi_1, \pi_2\}_{nh} = \pi_3, & \{\pi_2, \pi_3\}_{nh} = \dfrac{k^2}{(k^2+r^2)}\pi_1 + \dfrac{rk^2\Omega}{(k^2+r^2)}y, & \\ \{\pi_3, \pi_1\}_{nh} = \dfrac{k^2}{(k^2+r^2)}\pi_2 - \dfrac{rk^2\Omega}{(k^2+r^2)}x, & & \end{array} \tag{4.4}$$

in which the "appropriate operational" meaning has to be attached to the quasi-coordinates q_i.

Thus, we have that

$$\dot{f} = R_L(f) + \{f, L\}_{nh}, \text{ for } f \in C^\infty(\mathcal{M}),$$

where R_L is the vector field on $\mathcal{M}$ given by

$$\begin{aligned} R_L = &\ (\frac{k^2\Omega}{(k^2+r^2)}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) + \frac{r\Omega}{(k^2+r^2)}(x\frac{\partial}{\partial q_1} + y\frac{\partial}{\partial q_2} \\ &+ x(\pi_3 - k^2\Omega)\frac{\partial}{\partial\pi_1} + y(\pi_3 - k^2\Omega)\frac{\partial}{\partial\pi_2} - k^2(\pi_1 x + \pi_2 y)\frac{\partial}{\partial\pi_3}))|\mathcal{M}. \end{aligned}$$

Note that $R_L = 0$ if and only if $\Omega = 0$.

Now, it is clear that $Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$-bundle over $\mathbb{R}^2$ and the bundle projection $\phi : Q \longrightarrow M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding Atiyah algebroid $E' = TQ/SO(3)$ over $M = \mathbb{R}^2$.

One may prove that E' is isomorphic to the real vector bundle $T\mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ in such a way that the anchor map $\rho' : E' \cong T\mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow T\mathbb{R}^2$ is just the canonical projection on the first factor. Moreover, one may choose a global basis $\{e'_i\}_{i=1,\dots,5}$ of $\mathrm{Sec}(E')$ and the only non-zero fundamental Lie brackets are

$$[\![e'_4, e'_3]\!]' = e'_5, \quad [\![e'_5, e'_4]\!]' = e'_3, \quad [\![e'_3, e'_5]\!]' = e'_4.$$

We have that the Lagrangian function $L = T$ and the constraint submanifold $\mathcal{M}$ are $SO(3)$-invariant. Consequently, L induces a Lagrangian function L' on $E' = TQ/SO(3)$ and the set of orbits $\mathcal{M}' = \mathcal{M}/SO(3)$ is a submanifold of $E' =$

$TQ/SO(3)$ in such a way that the canonical projection $\Phi|\mathcal{M} : \mathcal{M} \longrightarrow \mathcal{M}' = \mathcal{M}/SO(3)$ is a surjective submersion.

Under the identification between $E' = TQ/SO(3)$ and $T\mathbb{R}^2 \times \mathbb{R}^3$, L' is given by

$$L'(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k^2}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

where $(x, y, \dot{x}, \dot{y})$ and $(\omega_1, \omega_2, \omega_3)$ are the standard coordinates on $T\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Moreover, the equations defining $\mathcal{M}'$ as a submanifold of $T\mathbb{R}^2 \times \mathbb{R}^3$ are

$$\dot{x} - r\omega_2 + \Omega y = 0, \quad \dot{y} + r\omega_1 - \Omega x = 0.$$

So, we have the constrained Lagrangian system $(L', \mathcal{M}')$ on the Atiyah algebroid $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. Note that the constraints are not linear and that if Δ' is the Liouville section of the prolongation $\mathcal{T}^{E'}E'$ then the restriction to $\mathcal{M}'$ of the vector field $(\rho')^{\tau'}(\Delta')$ is not tangent to $\mathcal{M}'$.

Now, if we put

$$\begin{array}{lll} x' = x, & y' = y, & \\ \pi_1' = r\dot{x} + k^2\omega_2, & \pi_2' = r\dot{y} - k^2\omega_1, & \pi_3' = k^2\omega_3, \\ \pi_4' = \dfrac{k^2}{(k^2 + r^2)}(\dot{x} - r\omega_2 + \Omega y), & \pi_5' = \dfrac{k^2}{(k^2 + r^2)}(\dot{y} + r\omega_1 - \Omega x), & \end{array}$$

then $(x', y', \pi_1', \pi_2', \pi_3', \pi_4', \pi_5')$ is a system of global coordinates on $TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. In these coordinates the equations defining the submanifold $\mathcal{M}'$ are $\pi_4' = 0$ and $\pi_5' = 0$ and the canonical projection $\Phi : TQ \longrightarrow TQ/SO(3)$ is given by

$$\Phi(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (\bar{x}, \bar{y}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5). \tag{4.5}$$

Thus, if Γ' is the constrained dynamics for the system $(L', \mathcal{M}')$, it follows that (see (4.3))

$$(\rho')^{\tau'}(\Gamma') = (\dot{x}'\frac{\partial}{\partial x'} + \dot{y}'\frac{\partial}{\partial y'})|\mathcal{M}'.$$

On the other hand, from (4.4), (4.5) and Theorem 4.8, we deduce that the only non-zero fundamental nonholonomic brackets for the system $(L', \mathcal{M}')$ are

$$\begin{array}{ll} \{x', \pi_1'\}_{nh}' = r, & \{y', \pi_2'\}_{nh}' = r, \\ \{\pi_1', \pi_2'\}_{nh}' = \pi_3', & \{\pi_2', \pi_3'\}_{nh}' = \dfrac{k^2}{(k^2 + r^2)}\pi_1' + \dfrac{rk^2\Omega}{(k^2 + r^2)}y', \\ \{\pi_3', \pi_1'\}_{nh}' = \dfrac{k^2}{(k^2 + r^2)}\pi_2' - \dfrac{rk^2\Omega}{(k^2 + r^2)}x'. & \end{array}$$

Therefore, we have that

$$\dot{f}' = (\rho')^{\tau'}(R_{L'})(f') + \{f', L'\}_{nh}', \text{ for } f' \in C^\infty(\mathcal{M}'),$$

where $(\rho')^{\tau'}(R_{L'})$ is the vector field on $\mathcal{M}'$ given by

$$\begin{aligned} (\rho')^{\tau'}(R_{L'}) \quad = \quad & \{\frac{k^2\Omega}{k^2 + r^2}(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}) + \frac{r\Omega}{(k^2 + r^2)}(x'(\pi_3' - k^2\Omega)\frac{\partial}{\partial \pi_1'} \\ & + y'(\pi_3' - k^2\Omega)\frac{\partial}{\partial \pi_2'} - k^2(\pi_1'x' + \pi_2'y')\frac{\partial}{\partial \pi_3'})\}_{|\mathcal{M}'}. \end{aligned}$$

4.5. **Hamiltonian formalism.** Let $(L, \mathcal{M})$ be a constrained Lagrangian system on a Lie algebroid E and assume that the Lagrangian function L is hyperregular. Then, since the Leg_L is a diffeomorphism then, it is clear that one may develop a Hamiltonian formalism which is equivalent, via the Legendre transformation, to the Lagrangian formalism.

5. Mechanical control systems on Lie algebroids

5.1. General control systems on Lie algebroids. Consider a Lie algebroid $\tau\colon E \longrightarrow M$, with anchor map $\rho\colon E \longrightarrow TM$. Let $\{\sigma, \eta_1, \dots, \eta_k\}$ be sections of E. A ***control problem on the Lie algebroid*** $\tau\colon E \longrightarrow M$ ***with drift section*** σ ***and input sections*** $\eta_1, \dots, \eta_k$ is defined by the following equation on M,

$$\dot{m}(t) = \rho\Big(\sigma(m(t)) + \sum_{i=1}^{k} u_i(t)\eta_i(m(t))\Big), \tag{5.1}$$

where $u = (u_1, \dots, u_k) \in U$, and U is an open set of $\mathbb{R}^k$ containing 0. The function $t \mapsto u(t) = (u_1(t), \dots, u_k(t))$ belongs to a certain class of functions of time, denoted by $\mathcal{U}$, called the ***set of admissible controls***. For our purposes, we may restrict the admissible controls to be the piecewise constant functions with values in U. Notice that the trajectories of the control system are admissible curves of the Lie algebroid, and therefore they must lie on a leaf of E. It follows that if E is not transitive, then there are points that cannot be connected by solutions of any control system defined on such a Lie algebroid. In particular, the system (5.1) cannot be locally accessible at points $m \in M$ where ρ is not surjective. Since the emphasis here is put on the controllability analysis, without loss of generality we will restrict our attention to locally transitive Lie algebroids.

Denoting by $f = \rho(\sigma)$ and $g_i = \rho(\eta_i)$, $i \in 1, \dots, k$, we can rewrite the system (5.1) as

$$\dot{m}(t) = f(m(t)) + \sum_{i=1}^{k} u_i(t) g_i(m(t)), \tag{5.2}$$

which is a standard nonlinear control system on M affine in the inputs [47]. Here we make use of the additional geometric structure provided by the Lie algebroid in order to carry over the analysis of the controllability properties of the control system (5.1). We refer to [47] for a comprehensive discussion of the notions of reachable sets, accessibility algebra and computable accessibility tests.

Definition 5.1. *The* ***accessibility algebra*** $\mathcal{D}$ ***of the control system*** (5.1) ***in the Lie algebroid*** *is the smallest subalgebra of* $\mathrm{Sec}(E)$ *that contains the sections* $\{\sigma, \eta_1, \dots, \eta_k\}$.

Using the Jacobi identity, one can deduce that any element of accessibility algebra $\mathcal{D}$ is a linear combination of repeated Lie brackets of sections of the form

$$[\![\zeta_l, [\![\zeta_{l-1}, [\![\dots, [\![\zeta_2, \zeta_1]\!] \dots]\!]]\!]]\!],$$

where $\zeta_i \in \{\sigma, \eta_1, \dots, \eta_k\}$, $1 \le i \le l$ and $l \in \mathbb{N}$.

Definition 5.2. *The* ***accessibility subbundle in the Lie algebroid****, denoted by* $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \dots, \eta_k\})$*, is the vector subbundle of* E *generated by the accessibility algebra* $\mathcal{D}$*,*

$$\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \dots, \eta_k\}) = \mathrm{span}\,\{\zeta(m) \mid \zeta \text{ section of } E \text{ in } \mathcal{D}\}, \quad m \in M.$$

If the dimension of $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \dots, \eta_k\})$ is constant, then $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \dots, \eta_k\})$ is the smallest Lie subalgebroid of E that has $\{\sigma, \eta_1, \dots, \eta_k\}$ as sections.

5.2. Mechanical control systems. Let $\tau\colon E \longrightarrow M$ be a Lie algebroid, let ∇ be a connection on E, and let $\{\eta, \eta_1, \dots, \eta_k\}$ be sections of E. A ***mechanical control system on the Lie algebroid*** $\tau\colon E \longrightarrow M$ is defined by the following equation

$$\nabla_{a(t)} a(t) + \eta(m(t)) = \sum_{i=1}^{k} u_i(t)\eta_i(m(t)). \tag{5.3}$$

We will often refer to η as the potential energy term in equations (5.3). Associated with this equation, there is always a control system on the Lie algebroid $\mathcal{T}^E E \longrightarrow E$ given by

$$\dot{a}(t) = \rho^\tau\Big((\Gamma_\nabla - \eta^V)(a(t)) + \sum_{i=1}^k u_i(t)\eta_i^V(a(t))\Big), \tag{5.4}$$

where η^V (resp. η_i^V) denotes the vertical lift of η (resp. η_i) and Γ_∇ is the SODE section associated with ∇. Γ_∇ is locally given by (see [13])

$$\Gamma_\nabla = y^\alpha \mathcal{X}_\alpha - \frac{1}{2}\left(\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta}\right) y^\beta y^\gamma \mathcal{V}_\alpha.$$

There are two distinguished families within the class of mechanical control systems. We introduce them next.

Mechanical control systems. Consider a Lagrangian system with $L\colon E \longrightarrow \mathbb{R}$ of the form

$$L(a) = \frac{1}{2}\mathcal{G}(a,a) - V\circ\tau(a), \quad a\in E,$$

with $\mathcal{G}\colon E\times_M E \longrightarrow \mathbb{R}$ a bundle metric on E and V a function on M. This Lagrangian function gives rise to the Euler-Lagrange equations as explained in Section 3.1.

Consider now the situation when the Lagrangian system is subject to some external forces, represented by a collection $\{\theta_1,\dots,\theta_k\}$ of sections of E^*. Denote by $\{\eta_1,\dots,\eta_k\}$ the input sections of E determined by the control forces $\{\theta_1,\dots,\theta_k\}$ via the metric, i.e., $\theta_i(X) = \mathcal{G}(\eta_i, X)$ for all $X\in \mathrm{Sec}(E)$. If $\Gamma_{\nabla^{\mathcal{G}}}$ denotes the SODE section associated with the Levi-Civita connection $\nabla^{\mathcal{G}}$, the controlled Euler-Lagrange equations can be written as

$$\dot{a}(t) = \rho^\tau\Big(\Gamma_{\nabla^{\mathcal{G}}}(a(t)) - (\mathrm{grad}_{\mathcal{G}} V)^V(a(t)) + \sum_{i=1}^k u_i(t)\eta_i^V(a(t))\Big). \tag{5.5}$$

Here $\mathrm{grad}_{\mathcal{G}} V$ is the section of E characterized by $\mathcal{G}(\mathrm{grad}_{\mathcal{G}} V, X) = dV(X)$ for $X\in\mathrm{Sec}(E)$. Note that system (5.5) is a control problem on the Lie algebroid $\mathcal{T}^E E\longrightarrow E$ as defined in Section 5.1. Locally, the equations can be written as

$$\begin{aligned}\dot{x}^i &= \rho^i_\alpha y^\alpha,\\ \dot{y}^\alpha &= -\frac{1}{2}\left(\Gamma^\alpha_{\beta\gamma}(x) + \Gamma^\alpha_{\gamma\beta}(x)\right) y^\beta y^\gamma - \mathcal{G}^{\alpha\beta}\rho^i_\beta \frac{\partial V}{\partial x^i} + \sum_{i=1}^k u_i(t)\eta_i^\alpha(x),\end{aligned}$$

where $(\mathcal{G}_{\alpha\beta})$ are the components of the metric $\mathcal{G}$ and $(\mathcal{G}^{\alpha\beta})$ is the inverse matrix of $(\mathcal{G}_{\alpha\beta})$.

Alternatively, one can describe the dynamical behavior of the mechanical control system by means of an equation on E via the covariant derivative. An admissible curve $a: t\mapsto a(t)$ is a solution of the system (5.5) if and only if

$$\nabla^{\mathcal{G}}_{a(t)} a(t) + \mathrm{grad}_{\mathcal{G}} V(m(t)) = \sum_{i=1}^k u_i(t)\eta_i(m(t)). \tag{5.6}$$

This equation corresponds to a mechanical control system (5.3) with connection $\nabla = \nabla^{\mathcal{G}}$ and sections $\{\mathrm{grad}_{\mathcal{G}} V, \eta_1,\dots,\eta_k\}$.

Mechanical control systems with constraints. Assume a mechanical control system with data $(\mathcal{G}, V, \{\theta_1,\dots,\theta_k\})$ is subject to the constraints determined by a subbundle D of E. Consider the orthogonal decomposition $E = D\oplus D^\perp$ an the associated orthogonal projectors $P\colon E \longrightarrow D$, $Q\colon E \longrightarrow D^\perp$. Then, one can write the controlled Lagrange-d'Alembert equations as

$$P(\nabla^{\mathcal{G}}_{a(t)}a(t)) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t)P(\eta_i(m(t))), \qquad Q(a) = 0.$$

In terms of the constrained connection $\check{\nabla}_\sigma\eta = P(\nabla^{\mathcal{G}}_\sigma\eta) + \nabla^{\mathcal{G}}_\sigma(Q\eta)$, with $\sigma, \eta \in \operatorname{Sec}(E)$, the controlled equations can be rewritten as $\check{\nabla}_{a(t)}a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t)P(\eta_i(m(t)))$, $Q(a) = 0$. Since the forcing terms coming from the potential and the inputs belong to D, the solutions of the total controlled dynamics initially belonging to D also remain in D. As a consequence, an admissible curve $a : t \mapsto a(t)$ is a solution of the system (5.8) if and only if

$$\check{\nabla}_{a(t)}a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t)P(\eta_i(m(t))), \qquad a_0 \in D. \tag{5.7}$$

This equation corresponds to a mechanical control system (5.3) with connection $\nabla = \check{\nabla}$ and sections $\{P(\operatorname{grad}_{\mathcal{G}} V), P(\eta_1), \dots, P(\eta_k)\}$.

Note that one can write the controlled dynamics as a control system on the Lie algebroid $\mathcal{T}^E E \longrightarrow E$,

$$\dot{a}(t) = \rho^\tau\Big(\Gamma_{\check{\nabla}}(a(t)) - P(\operatorname{grad}_{\mathcal{G}} V)^{\scriptscriptstyle V}(a(t)) + \sum_{i=1}^{k} u_i(t)P(\eta_i)^{\scriptscriptstyle V}(a(t))\Big). \tag{5.8}$$

The coordinate expression of these equations is greatly simplified if we take a basis $\{e_\alpha\} = \{e_a, e_A\}$ of E adapted to the orthogonal decomposition $E = D \oplus D^\perp$, i.e., $D = \operatorname{span}\{e_a\}$, $\mathcal{D}^\perp = \operatorname{span}\{e_A\}$. Denoting by $(y^\alpha) = (y^a, y^A)$ the induced coordinates, the constraint equations $Q(a) = 0$ just read $y^A = 0$. The controlled equations (5.7) are then

$$\begin{aligned}
\dot{x}^i &= \rho^i_a y^a, \\
\dot{y}^a &= -\frac{1}{2}S^a_{bc}y^b y^c - \mathcal{G}^{a\beta}\rho^i_\beta\frac{\partial V}{\partial x^i} + \sum_{i=1}^{k} u_i(t)P(\eta_i)^a, \\
y^A &= 0.
\end{aligned}$$

where $S^a_{bc} = \Gamma^b_{ca} + \Gamma^c_{ba}$ are the components of the symmetric product.

5.3. Accessibility and controllability notions.

Here we introduce the notions of accessibility and controllability that are specialized to mechanical control systems on Lie algebroids. Let $m \in M$ and consider a neighborhood V of m in M. Define the set of reachable points in the base manifold M starting from m as

$$\begin{aligned}
\mathbb{R}^V_M(m,T) = \{m' \in M \mid \exists u \in \mathcal{U} &\text{ defined on } [0,T] \text{ such that the evolution of (5.4)} \\
&\text{for } a(0) = 0_m \text{ satisfies } \tau(a(t)) \in V,\ t \in [0,T] \text{ and } \tau(a(T)) = m'\}\,.
\end{aligned}$$

Alternatively, one may write $\mathbb{R}^V_M(m,T) = \tau(\mathbb{R}^{\tau^{-1}(V)}_E(0_m,T))$. Denote

$$\mathbb{R}^V_M(m, \leq T) = \bigcup_{t\leq T}\mathbb{R}^V_M(m,t).$$

Definition 5.3. *The system* (5.4) *is* ***locally base accessible from*** m *(respectively,* ***locally base controllable from*** m*) if* $\mathbb{R}^V_M(m, \leq T)$ *contains a non-empty open set of* M *(respectively,* $\mathbb{R}^V_M(m, \leq T)$ *contains a non-empty open set of* M *to which* m *belongs) for all neighborhoods* V *of* m *and all* $T > 0$*. If this holds for any* $m \in M$*, then the system is called* ***locally base accessible*** *(respectively,* ***locally base controllable****).*

In addition to the notions of base accessibility and base controllability, we shall also consider full-state accessibility and controllability starting from points of the form $0_m \in E$, $m \in M$ (note that full-state is meant here with regards to E, not to TM).

Definition 5.4. *The system* (5.4) *is* ***locally accessible from*** m ***at zero*** *(respectively,* ***locally controllable from*** m ***at zero****) if* $\mathbb{R}_E^W(0_m, \leq T)$ *contains a non-empty open set of* E *(respectively,* $\mathbb{R}_E^W(0_m, \leq T)$ *contains a non-empty open set of* E *to which* 0_m *belongs) for all neighborhoods* W *of* 0_m *in* E *and all* $T > 0$*. If this holds for any* $m \in M$*, then the system is called* ***locally accessible at zero*** *(respectively,* ***locally controllable at zero****).*

The relevance of the above definitions stems from the fact that, frequently, one needs to control a system by starting at rest. Nevertheless it is important to notice that not every equilibrium point at m corresponds to the point 0_m. Finally, we also introduce the notion of accessibility and controllability with regards to a manifold.

Definition 5.5. *Let* $\psi\colon M \longrightarrow N$ *be an open mapping. The system* (5.4) *is* ***locally base accessible from*** m ***with regards to*** N *(respectively,* ***locally base controllable from*** m *with regards to* N*) if* $\psi(\mathbb{R}_M^V(m, \leq T))$ *contains a non-empty open set of* N *(respectively,* $\psi(\mathbb{R}_M^V(m, \leq T))$ *contains a non-empty open set of* N *to which* $\psi(m)$ *belongs) for all neighborhoods* V *of* m *and all* $T > 0$*. If this holds for any* $m \in M$*, then the system is called* ***locally base accessible with regards to*** N *(respectively,* ***locally base controllable with regards to*** N*).*

Note that base accessibility and controllability with regards to M with $\mathrm{id}_M : M \longrightarrow M$ corresponds to the notions of base accessibility and controllability (cf. Definition 5.3). Moreover, if the system is base accessible, then it is base accessible with regards to N. The analogous implication for base controllability also holds true.

5.4. The structure of the control Lie algebra. The aim of this section is to show that the analysis of the structure of the control Lie algebra of affine connection control systems carried out in [29] can be further extended to control systems defined on a Lie algebroid. The enabling technical notion exploited here is that of homogeneity.

Let B be a Lie bracket formed with sections of the family $\mathcal{X} = \{\Gamma_\nabla, \eta_1^V, \ldots, \eta_k^V, \eta^V\}$. The ***degree*** of B is the number of occurrences of all its factors, and is therefore given by $\delta(B) = \delta_0(B) + \delta_1(B) + \cdots + \delta_k(B)$, where $\delta_0(B)$, $\delta_i(B)$, $i \in \{1, \ldots, k\}$, and $\delta_{k+1}(B)$ correspond, respectively, to the number of times that Γ_∇, η_i^V, $i \in \{1, \ldots, k\}$, and η^V appear in B. For each l, consider the following sets

$$\mathrm{Br}^l(\mathcal{X}) = \{B \text{ bracket in } \mathcal{X} \mid \delta(B) = l\}, \qquad \mathrm{Br}_l(\mathcal{X}) = \{B \text{ bracket in } \mathcal{X} \mid B \in \mathcal{P}_l\},$$

where $\mathcal{P}_l$ denotes the set of homogeneous sections of $\mathcal{T}E$ of degree l. The notion of ***primitive*** bracket will also be useful. Given a bracket B in $\mathcal{X}$, it is clear that we can write $B = [B_1, B_2]$, with B_i brackets in $\mathcal{X}$. In turn, we can also write $B_\alpha = [B_{\alpha 1}, B_{\alpha 2}]$ for $\alpha = 1, 2$, and continue these decompositions until we end up with elements belonging to $\mathcal{X}$. The collection of brackets $B_1, B_2, B_{11}, B_{12}, \ldots$ are called the ***components*** of B. The components of B which do not admit further decompositions are called ***irreducible***. A bracket B is called ***primitive*** if all of its components are brackets in $\mathrm{Br}_{-1}(\mathcal{X}) \cup \mathrm{Br}_0(\mathcal{X}) \cup \{\Gamma_\nabla\}$.

Consider the set $\mathcal{X}' = \{\Gamma_\nabla - \eta^V, \eta_1^V, \ldots, \eta_k^V\}$. Clearly, the elements in $\overline{\mathrm{Lie}}(\mathcal{X}')$ are linear combinations of the elements in $\overline{\mathrm{Lie}}(\mathcal{X})$. In fact, for each bracket B' of elements in $\mathcal{X}'$, let us define the subset $S(B') \subset \mathrm{Br}(\mathcal{X})$ formed by all possible brackets $B \in \mathrm{Br}(\mathfrak{X})$ obtained by replacing each occurrence of $\Gamma_\nabla - \eta^V$ in B' by either Γ_∇ or η^V. Then, one can prove by induction (cf. [28]) that

$$B' = \sum_{B \in S(B')} (-1)^{\delta_{k+1}(B)} B. \tag{5.9}$$

Reciprocally, given an element $B \in \mathrm{Br}(\mathfrak{X})$, one can determine the bracket B' of elements in $\mathfrak{X}'$ such that $B \in S(B')$ simply by substituting each occurrence of Γ_∇ or η^V in B by $\Gamma_\nabla - \eta^V$. We denote this operation by $\mathrm{pseudoinv}(B) = B'$. For each $k \in \mathbb{N}$, define the following families of sections in E,

$$\begin{aligned}
&\mathcal{C}^{(k)}_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k) = \\
&\{\sigma \in \mathrm{Sec}(E) \mid \sigma^V = B'', B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap \, \mathrm{Br}_{-1}(\mathfrak{X}) \cap \mathrm{Br}_0(\mathfrak{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B},\ B \in \mathrm{Br}^{2k-1}(\mathfrak{X}) \text{ primitive}\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{C}^{(k)}_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k) = \\
&\{\sigma \in \mathrm{Sec}(E) \mid \sigma = \sigma_{B''}, B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap \, \mathrm{Br}_{-1}(\mathfrak{X}) \cap \mathrm{Br}_0(\mathfrak{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B},\ B \in \mathrm{Br}^{2k}(\mathfrak{X}) \text{ primitive}\}.
\end{aligned}$$

Consider

$$\begin{aligned}
\mathcal{C}_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k) &= \cup_{k \in \mathbb{N}} \mathcal{C}^{(k)}_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k), \\
\mathcal{C}_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k) &= \cup_{k \in \mathbb{N}} \mathcal{C}^{(k)}_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k),
\end{aligned}$$

and denote by $C_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k)$ and $C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)$, respectively, the subbundles of the Lie algebroid E generated by the latter families.

Taking into account the previous discussion, we are now ready to compute $\overline{\mathrm{Lie}}(\{\Gamma_\nabla - \eta^V, \eta_1^V, \dots, \eta_k^V\})$ for a mechanical control system defined on a Lie algebroid.

Proposition 5.6. *([13]) Let $m \in M$. Then,*

$$\begin{aligned}
\overline{\mathrm{Lie}}(\{\Gamma_\nabla - \eta^V, \eta_1^V, \dots, \eta_k^V\}) \cap \mathrm{Ver}_{0_m}(\mathcal{T}E) &= C_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k)(m)^V, \\
\overline{\mathrm{Lie}}(\{\Gamma_\nabla - \eta^V, \eta_1^V, \dots, \eta_k^V\}) \cap \mathrm{Hor}_m(\mathcal{T}E) &= C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)(m).
\end{aligned}$$

Remark 5.7. In the absence of potential terms, i.e., $\eta = 0$, one has that

$$C_{\mathrm{ver}}(0; \eta_1, \dots, \eta_k) = \overline{\mathrm{Sym}}(\{\eta_1, \dots, \eta_k\}),\ C_{\mathrm{hor}}(0; \eta_1, \dots, \eta_k) = \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1, \dots, \eta_k\})),$$

where $\overline{\mathrm{Sym}}(\{\eta_1, \dots, \eta_k\})$ denotes the distribution obtained by closing (the distribution defined by) $\{\eta_1, \dots, \eta_k\}$ under the symmetric product associated with ∇. It is worth noticing that, in this case, $C_{\mathrm{ver}}(0; \eta_1, \dots, \eta_k) \subseteq C_{\mathrm{hor}}(0; \eta_1, \dots, \eta_k)$. This is not true in general. ◇

5.5. Accessibility and controllability tests.

In this section we merge the notions introduced in Section 5.3 with the results obtained in Section 5.4 to give tests for accessibility and controllability.

Proposition 5.8. [13] *Let $m \in M$ and assume the Lie algebroid E is locally transitive at m. Then the mechanical control system* (5.4) *is*

- *locally base accessible from m if $C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)(m) + \ker \rho = E_m$,*
- *locally accessible from m at zero if $C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)(m) + \ker \rho = E_m$ and $C_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k)(m) = E_m$.*

In order to state controllability tests, we need to introduce the notions of good and bad symmetric products. We say that a symmetric product P in the sections $\{\eta, \eta_1, \dots, \eta_k\}$ is ***bad*** if the number of occurrences of each η_i in P is even. Otherwise,

P is ***good***. Accordingly, $\langle \eta_i : \eta_i \rangle$ is bad and $\langle\langle \eta : \eta_j \rangle : \langle \eta_i : \eta_i \rangle\rangle$ is good. The following theorem gives sufficient conditions for local controllability.

Proposition 5.9. [13] *Let $m \in M$. The mechanical control system* (5.4) *is*

- *locally base controllable from m if it is locally base accessible from m and every bad symmetric product in $\{\eta, \eta_1, \dots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree and elements of* $\ker \rho$,
- *locally controllable from m at zero if it is locally accessible from m at zero and every bad symmetric product in $\{\eta, \eta_1, \dots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree.*

The corresponding tests for base accessibility and controllability with regards to a manifold can be proved in a similar way.

Proposition 5.10. [13] *Let $\psi \colon M \longrightarrow N$ be an open map. Let $m \in M$ and assume $\psi_*(\rho(E_m)) = T_{\psi(m)}N$. Then the mechanical control system* (5.4) *is*

- *locally base accessible from m with regards to N if $C_{\rm hor}(\eta; \eta_1, \dots, \eta_k)(m) + \rho^{-1}(\ker \psi_*) = E_m$,*
- *locally base controllable from m with regards to N if the system is locally base accessible from m with regards to N and every bad symmetric product in $\{\eta, \eta_1, \dots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree and elements of $\rho^{-1}(\ker \psi_*)$.*

6. Discrete Mechanics on Lie groupoids

In this section, we discuss discrete Lagrangian Mechanics on a Lie groupoid $G \rightrightarrows M$. Instead of the usual Euler-Lagrange equations (3.7) for a Lie algebroid $\tau : E \longrightarrow M$ equipped with a Lagrange function $L : E \longrightarrow \mathbb{R}$, we obtain a set of difference equations called *Discrete Euler-Lagrange equations* for a discrete Lagrangian $L_d : G \longrightarrow \mathbb{R}$ [35]. When the Lie algebroid is precisely $E = E_G$ and L_d is a suitable approximation of the continuous Lagrangian $L : E_G \longrightarrow \mathbb{R}$, then we will obtain a geometric integrator for the Euler-Lagrange equations. In the next subsections we will carefully analyze this construction and its geometric properties.

6.1. Lie algebroid structure on the vector bundle $\pi^\tau : E_G \overset{G}{\times} E_G \longrightarrow G$.

Let $G \rightrightarrows M$ be a Lie groupoid with structural maps

$$\alpha, \beta : G \longrightarrow M, \quad \epsilon : M \longrightarrow G, \quad i : G \longrightarrow G, \quad m : G_2 \longrightarrow G.$$

We know that $E_G \overset{G}{\times} E_G \rightrightarrows E_G$ is a Lie groupoid. The following theorem shows that the vector bundle $\pi^\tau : E_G \overset{G}{\times} E_G \cong V\beta \oplus_G V\alpha \longrightarrow G$ is equipped with a natural structure of Lie algebroid.

Theorem 6.1 (See Theorem 3.3 in [35]). *The vector bundle $\pi^\tau : E_G \overset{G}{\times} E_G \cong V\beta \oplus_G V\alpha \longrightarrow G$ admits a Lie algebroid structure, where the anchor map is given by*

$$\rho^{E_G \overset{G}{\times} E_G}(X_g, Y_g) = X_g + Y_g, \quad \textit{for } (X_g, Y_g) \in V_g\beta \oplus V_g\alpha, \tag{6.1}$$

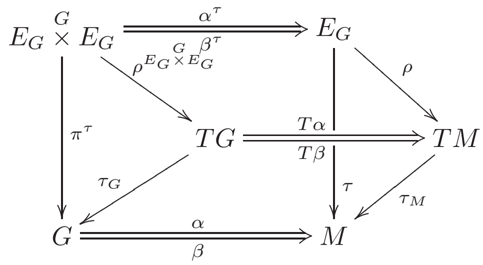
and the Lie bracket $[\![\cdot, \cdot]\!]^{E_G \overset{G}{\times} E_G}$ on the space $\mathrm{Sec}(\pi^\tau)$ *is characterized by the following relation*

$$[\![(\overrightarrow{X}, \overleftarrow{Y}), (\overrightarrow{X'}, \overleftarrow{Y'})]\!]^{E_G \overset{G}{\times} E_G} = \Big(-\overrightarrow{[\![X, X']\!]}, \overleftarrow{[\![Y, Y']\!]}\Big), \tag{6.2}$$

for $X, Y, X', Y' \in$ $\mathrm{Sec}(\tau)$.

We also remark that, if we denote by (G,α) the fibration $\alpha\colon G \longrightarrow M$ and by (G,β) the fibration $\beta\colon G \longrightarrow M$, then it is not difficult to prove that the Lie algebroid prolongations $\mathcal{T}^{E_G}(G,\alpha) \longrightarrow G$ and $\mathcal{T}^{E_G}(G,\beta) \longrightarrow G$ are isomorphic, as Lie algebroids, and both are isomorphic to the Lie algebroid $V\beta \oplus_G V\alpha \longrightarrow G$ and hence to $E_G \overset{G}{\times} E_G \longrightarrow G$ (see [35] for the details).

The following diagram shows both structures of $E_G \overset{G}{\times} E_G$,



where the vertical maps are morphisms of Lie groupoids and the horizontal maps are morphisms of Lie algebroids.

Given a section X of $E_G \longrightarrow M$, we define the sections $X^{(1,0)}$, $X^{(0,1)}$ (the β and α- lifts) and $X^{(1,1)}$ (the complete lift) of X to $\pi^\tau : E_G \overset{G}{\times} E_G \longrightarrow G$ as follows:

$$X^{(1,0)}(g) = (\overrightarrow{X}(g), 0_g), \quad X^{(0,1)}(g) = (0_g, \overleftarrow{X}(g)) \quad \text{and} \quad X^{(1,1)}(g) = (-\overrightarrow{X}(g), \overleftarrow{X}(g))$$

We can easily see that

$$\begin{aligned} [\![X^{(1,0)}, Y^{(1,0)}]\!]^{E_G \overset{G}{\times} E_G} &= -[\![X,Y]\!]^{(1,0)} \\ [\![X^{(0,1)}, Y^{(0,1)}]\!]^{E_G \overset{G}{\times} E_G} &= [\![X,Y]\!]^{(0,1)} \end{aligned} \quad \text{and} \quad [\![X^{(0,1)}, Y^{(1,0)}]\!]^{E_G \overset{G}{\times} E_G} = 0 \tag{6.3}$$

and, as a consequence,

$$\begin{aligned} [\![X^{(1,1)}, Y^{(1,0)}]\!]^{E_G \overset{G}{\times} E_G} &= [\![X,Y]\!]^{(1,0)} \\ [\![X^{(1,1)}, Y^{(0,1)}]\!]^{E_G \overset{G}{\times} E_G} &= [\![X,Y]\!]^{(0,1)} \end{aligned} \quad \text{and} \quad [\![X^{(1,1)}, Y^{(1,1)}]\!]^{E_G \overset{G}{\times} E_G} = [\![X,Y]\!]^{(1,1)}. \tag{6.4}$$

6.2. Discrete Variational Mechanics on Lie groupoids.

Discrete Lagrangian systems on Lie groupoids have a variational origin, as we explain next. A ***discrete Lagrangian system*** consists of a Lie groupoid $G \rightrightarrows M$ (the ***discrete space***) and a ***discrete Lagrangian*** $L_d : G \longrightarrow \mathbb{R}$.

Discrete Euler-Lagrange equations. For $g \in G$ fixed, we consider the set of ***admissible sequences***:

$$\mathcal{C}_g^N = \left\{ (g_1, \dots, g_N) \in G^N \;\middle|\; (g_k, g_{k+1}) \in G_2 \text{ for } k = 1, \dots, N-1 \text{ and } g_1 \dots g_n = g \right\}.$$

It is easy to show that we may identify the tangent space to $\mathcal{C}_g^N$ with

$$T_{(g_1,\dots,g_N)}\mathcal{C}_g^N \equiv \{ (v_1, \dots, v_{N-1}) \mid v_k \in (E_G)_{x_k} \text{ and } x_k = \beta(g_k), 1 \le k \le N-1 \}.$$

An element of $T_{(g_1,\dots,g_N)}\mathcal{C}_g^N$ is called an ***infinitesimal variation***. Now, we define the ***discrete action sum*** associated to the discrete Lagrangian $L_d : G \longrightarrow \mathbb{R}$ by

$$\mathcal{S}L_d((g_1, \dots, g_N) = \sum_{k=1}^{N} L_d(g_k).$$

Hamilton's principle requires that this discrete action sum be stationary with respect to all the infinitesimal variations. This requirement gives the following alternative expressions for the ***discrete Euler-Lagrange equations*** (see [35]):

$$\overleftarrow{X}(g_k)(L_d) - \overrightarrow{X}(g_{k+1})(L_d) = 0, \tag{6.5}$$

or

$$\langle dL_d, X^{(0,1)}\rangle(g_k) - \langle dL_d, X^{(1,0)}\rangle(g_{k+1}) = 0,$$

for all sections X of $\tau : E_G \longrightarrow M$. Here, d denotes the differential of the Lie algebroid $\pi^\tau : E_G \overset{G}{\times} E_G \equiv V\beta \oplus_G V\alpha \longrightarrow G$. Alternatively, we may rewrite the Discrete Euler-Lagrange equations as

$$\mathrm{d}^\circ \left[L_d \circ l_{g_k} + L_d \circ r_{g_{k+1}} \circ i \right] (\epsilon(x_k))\Big|_{(E_G)_{x_k}} = 0,$$

where $\beta(g_k) = \alpha(g_{k+1}) = x_k$, and where d° denotes the standard differential on G, that is, the differential of the Lie algebroid $\tau_G : TG \longrightarrow G$.

Thus, we may define the ***discrete Euler-Lagrange operator***:

$$D_{\mathrm{DEL}} L_d : G_2 \longrightarrow E_G^* \ ,$$

where E_G^* is the dual of E_G. This operator is given by

$$D_{\mathrm{DEL}} L_d(g,h) = \mathrm{d}^\circ \left[L_d \circ l_g + L_d \circ r_h \circ i \right] (\epsilon(x))\Big|_{(E_G)_{x_k}}$$

with $\beta(g) = \alpha(h) = x$.

Discrete Poincaré-Cartan sections. Consider the Lie algebroid $\pi^\tau : E_G \overset{G}{\times} E_G \cong V\beta \oplus_G V\alpha \longrightarrow G$, and define the ***Poincaré-Cartan 1-sections*** $\Theta^-_{L_d}, \Theta^+_{L_d} \in \mathrm{Sec}((\pi^\tau)^*)$ as follows

$$\Theta^-_{L_d}(g)(X_g, Y_g) = -X_g(L_d), \qquad \Theta^+_{L_d}(g)(X_g, Y_g) = Y_g(L_d), \tag{6.6}$$

for each $g \in G$ and $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$.

Since $dL_d = \Theta^+_{L_d} - \Theta^-_{L_d}$ and so, using $d^2 = 0$, it follows that $d\Theta^+_{L_d} = d\Theta^-_{L_d}$. This means that there exists a unique 2-section $\Omega_{L_d} = -d\Theta^+_{L_d} = -d\Theta^-_{L_d}$, that will be called the ***Poincaré-Cartan*** 2-section. This 2-section will be important to study the symplectic character of the discrete Euler-Lagrange equations.

If $\{X_i\}$ is a local basis of $\mathrm{Sec}(\tau)$ then $\{X_i^{(1,0)}, X_i^{(0,1)}\}$ is a local basis of $\mathrm{Sec}(\pi^\tau)$. Moreover, if $\{(X^i)^{(1,0)}, (X^i)^{(0,1)}\}$ is the dual basis of $\{X_i^{(1,0)}, X_i^{(0,1)}\}$, it follows that

$$\Theta^-_{L_d} = -\overrightarrow{X}_i(L_d)(X^i)^{(1,0)}, \quad \Theta^+_{L_d} = \overleftarrow{X}_i(L_d)(X^i)^{(0,1)},$$

$$\Omega_{L_d} = -\overrightarrow{X}_i(\overleftarrow{X}_j L_d)(X^i)^{(1,0)} \wedge (X^j)^{(0,1)}.$$

Discrete Lagrangian evolution operator. Let $\xi : G \longrightarrow G$ be a smooth map such that:

- $\mathrm{graph}(\xi) \subseteq G_2$, that is, $(g, \xi(g)) \in G_2$, for all $g \in G$ (ξ is a ***second order operator***).
- $(g, \xi(g))$ is a solution of the discrete Euler-Lagrange equations, for all $g \in G$, that is, $(D_{\mathrm{DEL}} L_d)(g, \xi(g)) = 0$, for all $g \in G$.

In such case

$$\overleftarrow{X}(g)(L_d) - \overrightarrow{X}(\xi(g))(L_d) = 0 \tag{6.7}$$

for every section X of E_G and every $g \in G$. The map $\xi : G \longrightarrow G$ is called a ***discrete flow*** or a ***discrete Lagrangian evolution operator for*** L_d.

Now, let $\xi : G \longrightarrow G$ be a second order operator. Then, the prolongation $\mathcal{T}\xi : V\beta \oplus_G V\alpha \longrightarrow V\beta \oplus_G V\alpha$ of ξ is the Lie algebroid morphism over $\xi : G \longrightarrow G$ defined as follows (see [35]):

$$\mathcal{T}_g\xi(X_g, Y_g) = ((T_g(r_{g\xi(g)} \circ i))(Y_g), (T_g\xi)(X_g) + (T_g\xi)(Y_g) - T_g(r_{g\xi(g)} \circ i)(Y_g)), \quad (6.8)$$

for all $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$. Moreover, from (2.7), (2.8) and (6.8), we obtain that

$$\mathcal{T}_g\xi(\overrightarrow{X}(g), \overleftarrow{Y}(g)) = (-\overrightarrow{Y}(\xi(g)), (T_g\xi)(\overrightarrow{X}(g) + \overleftarrow{Y}(g)) + \overrightarrow{Y}(\xi(g))), \quad (6.9)$$

for all X, Y sections of E_G.

Using (6.8), one may prove that (see [35]):

(i) The map ξ is a discrete Lagrangian evolution operator for L_d if and only if $(\mathcal{T}\xi, \xi)^*\Theta^-_{L_d} = \Theta^+_{L_d}$.
(ii) The map ξ is a discrete Lagrangian evolution operator for L_d if and only if $(\mathcal{T}\xi, \xi)^*\Theta^-_{L_d} - \Theta^-_{L_d} = dL_d$.
(iii) If ξ is discrete Lagrangian evolution operator then $(\mathcal{T}\xi, \xi)^*\Omega_{L_d} = \Omega_{L_d}$.

Discrete Legendre transformations. Given a Lagrangian $L_d : G \longrightarrow \mathbb{R}$ we define the ***discrete Legendre transformations*** $\mathbb{F}^-L_d : G \longrightarrow E^*_G$ and $\mathbb{F}^+L_d : G \longrightarrow E^*_G$ by

$$(\mathbb{F}^-L_d)(h)(v_{\epsilon(\alpha(h))}) = -v_{\epsilon(\alpha(h))}(L_d \circ r_h \circ i), \quad \text{for } v_{\epsilon(\alpha(h))} \in (E_G)_{\alpha(h)},$$
$$(\mathbb{F}^+L)(g)(v_{\epsilon(\beta(g))}) = v_{\epsilon(\beta(g))}(L_d \circ l_g), \text{ for } v_{\epsilon(\beta(g))} \in (E_G)_{\beta(g)}.$$

Now, we introduce the prolongations $\mathcal{T}\mathbb{F}^-L_d : E_G \overset{G}{\times} E_G \equiv V\beta \oplus_G V\alpha \longrightarrow \mathcal{T}^{E_G}E^*_G$ and $\mathcal{T}\mathbb{F}^+L_d : E_G \overset{G}{\times} E_G \equiv V\beta \oplus_G V\alpha \longrightarrow \mathcal{T}^{E_G}E^*_G$ by

$$\mathcal{T}_h\mathbb{F}^-L_d(X_h, Y_h) = (T_h(i \circ r_{h^{-1}})(X_h), (T_h\mathbb{F}^-L)(X_h) + (T_h\mathbb{F}^-L)(Y_h)),$$
$$\mathcal{T}_h\mathbb{F}^+L_d(X_h, Y_h) = (T_h l_{h^{-1}}(Y_h), (T_h\mathbb{F}^+L)(X_h) + (T_h\mathbb{F}^+L)(Y_h)),$$

for all $h \in G$ and $(X_h, Y_h) \in V_h\beta \oplus V_h\alpha$. We observe that the discrete Poincaré-Cartan 1-sections and 2-section are related to the canonical Liouville section of $(\mathcal{T}^{E_G}E^*_G)^* \longrightarrow E^*_G$ and the canonical symplectic section of $\wedge^2(\mathcal{T}^{E_G}E^*_G)^* \longrightarrow E^*_G$ by pull-back under the discrete Legendre transformations, that is,

$$(\mathcal{T}\mathbb{F}^-L_d, \mathbb{F}^-L_d)^*\Theta_{E_G} = \Theta^-_{L_d}, \qquad (\mathcal{T}\mathbb{F}^+L_d, \mathbb{F}^+L_d)^*\Theta_{E_G} = \Theta^+_{L_d},$$
$$(\mathcal{T}\mathbb{F}^-L_d, \mathbb{F}^-L_d)^*\Omega_{E_G} = \Omega_{L_d}, \qquad (\mathcal{T}\mathbb{F}^+L_d, \mathbb{F}^+L_d)^*\Omega_{E_G} = \Omega_{L_d}.$$

Discrete regular Lagrangians. A discrete Lagrangian $L_d : G \longrightarrow \mathbb{R}$ is said to be ***regular*** if the set of solutions of the discrete Euler-Lagrange equations is locally the graph of a diffeomorphism, that is, there exists locally a unique discrete Lagrangian evolution operator $\xi_{L_d} : G \longrightarrow G$ for L_d. In such a case, ξ_{L_d} is called the discrete Euler-Lagrange evolution operator. In [35] (see Theorem 4.13 in [35]), we obtained some necessary and sufficient conditions for a discrete Lagrangian on a Lie groupoid G to be regular that we summarize as follows:

L_d is regular $\iff$ The Legendre transformation $\mathbb{F}^+L_d$ is a local diffeomorphism
$\iff$ The Legendre transformation $\mathbb{F}^-L_d$ is a local diffeomorphism
$\iff$ The Poincaré-Cartan 2-section Ω_{L_d} is symplectic on the Lie algebroid $E_G \overset{G}{\times} E_G \equiv V\beta \oplus_G V\alpha \longrightarrow G$.

Locally, we deduce that L_d is regular if and only if for every local basis $\{X_i\}$ of $Sec(\tau)$ the local matrix $(\overrightarrow{X}_i(\overleftarrow{X}_j L_d))$ is regular.

Discrete Hamiltonian evolution operator. If $L_d : G \longrightarrow \mathbb{R}$ is a regular Lagrangian, then pushing forward to E_G^* with the discrete Legendre transformations, we obtain the ***discrete Hamiltonian evolution operator***, $\tilde{\xi}_{L_d} : E_G^* \longrightarrow E_G^*$ which is given by

$$\tilde{\xi}_{L_d} = \mathbb{F}^{\pm} L_d \circ \xi_{L_d} \circ (\mathbb{F}^{\pm} L_d)^{-1} \; . \tag{6.10}$$

Defining the prolongation $\mathcal{T}\tilde{\xi}_{L_d} : \mathcal{T}^{E_G} E_G^* \longrightarrow \mathcal{T}^{E_G} E_G^*$ of $\tilde{\xi}_{L_d}$ by

$$\mathcal{T}\tilde{\xi}_{L_d} = \mathcal{T}\mathbb{F}^{\pm} L_d \circ \mathcal{T}\xi_{L_d} \circ (\mathcal{T}\mathbb{F}^{\pm} L_d)^{-1},$$

we deduce that (see [35]):

$$(\mathcal{T}\tilde{\xi}_{L_d}, \tilde{\xi}_{L_d})^* \Theta_{E_G} = \Theta_{E_G} + d(L_d \circ (\mathbb{F}^- L_d)^{-1}), \quad (\mathcal{T}\tilde{\xi}_{L_d}, \tilde{\xi}_{L_d})^* \Omega_{E_G} = \Omega_{E_G}.$$

Noether's theorem. In Discrete Mechanics is also possible to relate invariance of the discrete Lagrangian under some transformation group with the existence of constants of the motion. In fact, we will say that a section X of E_G is a ***Noether's symmetry of the Lagrangian*** L_d if there exists a function $f \in C^\infty(M)$ such that

$$dL_d(X^{(1,1)}) = \beta^* f - \alpha^* f.$$

In the particular case when $dL_d(X^{(1,1)}) = -\overrightarrow{X} L_d + \overleftarrow{X} L_d = 0$, we will say that X is an ***infinitesimal symmetry*** of the discrete Lagrangian L_d.

If $L_d : G \longrightarrow \mathbb{R}$ is a regular discrete Lagrangian, by a ***constant of the motion*** we mean a function F invariant under the discrete Euler-Lagrange evolution operator ξ_{L_d}, that is, $F \circ \xi_{L_d} = F$. Then, we have the following result.

Theorem 6.2 (Discrete Noether's theorem)**.** [35] *If X is a Noether symmetry of a discrete Lagrangian L_d, then the function $F = \Theta^-_{L_d}(X^{(1,1)}) - \alpha^* f$ is a* ***constant of the motion*** *for the discrete dynamics defined by L_d.*

7. Classical Field Theory on Lie algebroids

In this section, we study Classical Field Theories on Lie algebroids. We consider a fiber bundle $\nu \colon M \longrightarrow N$, a Lie algebroid structure on a vector bundle $\tau^E_M \colon E \longrightarrow M$ and a surjective morphism of Lie algebroids $\pi \colon E \longrightarrow TN$ over ν. The physical interpretation of the above data is as follows: we will consider a field theory in which the fields are the sections of the bundle ν and the partial derivatives of the fields are parameterized by linear sections of π.

We will find the equations for the extremals of a variational problem which roughly speaking is the following: given a Lagrangian function L defined on the set of sections of π, and a volume form ω on the manifold N, we look for those morphisms of Lie algebroids which are critical points of the action functional

$$\mathcal{S}(\Phi) = \int_N L(\Phi)\, \omega.$$

This is a constrained variational problem, because we are restricting the fields Φ to be morphisms of Lie algebroids, which is a condition on the derivatives of Φ.

7.1. Jets.

We consider two vector bundles $\tau^E_M \colon E \longrightarrow M$ and $\tau^F_N \colon F \longrightarrow N$ and a surjective vector bundle map $\pi \colon E \longrightarrow F$ over the map $\nu \colon M \longrightarrow N$. Moreover, we will assume that $\nu \colon M \longrightarrow N$ is a smooth fiber bundle. We will denote by $K \longrightarrow M$ the kernel of the map π, which is a vector bundle over M. Given a point $m \in M$, if we denote $n = \nu(m)$, we have the following exact sequence $0 \longrightarrow K_m \longrightarrow E_m \longrightarrow F_n \longrightarrow 0$, and we can consider the set $\mathcal{J}_m\pi$ of splittings ϕ of such sequence. More concretely, we define the following sets $\mathcal{L}_m\pi = \{\, w \colon F_n \longrightarrow E_m \mid w \text{ is linear} \,\}$, $\mathcal{J}_m\pi = \{\, \phi \in \mathcal{L}_m\pi \mid \pi \circ \phi = \mathrm{id}_{F_n} \,\}$ and $\mathcal{V}_m\pi = \{\, \psi \in \mathcal{L}_m\pi \mid \pi \circ \psi = 0 \,\}$. Therefore

$\mathcal{L}_m\pi$ is a vector space, $\mathcal{V}_m\pi$ is a vector subspace of $\mathcal{L}_m\pi$ and $\mathcal{J}_m\pi$ is an affine subspace of $\mathcal{L}_m\pi$ modeled on the vector space $\mathcal{V}_m\pi$. By taking the union, $\mathcal{L}\pi = \cup_{m\in M}\mathcal{L}_m\pi$, $\mathcal{J}\pi = \cup_{m\in M}\mathcal{J}_m\pi$ and $\mathcal{V}\pi = \cup_{m\in M}\mathcal{V}_m\pi$, we get the vector bundle $\tilde{\pi}_{10}\colon \mathcal{L}\pi \longrightarrow M$ and the affine subbundle $\pi_{10}\colon \mathcal{J}\pi \longrightarrow M$ modeled on the vector bundle $\boldsymbol{\pi}_{10}\colon \mathcal{V}\pi \longrightarrow M$. We will also consider the projection $\pi_1\colon \mathcal{J}\pi \longrightarrow N$ defined by composition $\pi_1 = \nu\circ\pi_{10}$. An element of $\mathcal{J}_m\pi$ will be simply called a ***jet*** at the point $m \in M$ and accordingly the bundle $\mathcal{J}\pi$ is said to be the first ***jet bundle*** of π.

Notice that the standard case [52] is recovered when we have a bundle $\nu\colon M \longrightarrow N$ and one considers the standard Lie algebroids $E = TM \longrightarrow M$ and $F = TN \longrightarrow N$ together with the differential of the projection $\pi = T\nu\colon TM \longrightarrow TN$. With the standard notations, we have that $J^1\nu \equiv \mathcal{J}(T\nu)$.

Local coordinates on $\mathcal{J}\pi$ are given as follows. We consider local coordinates (x^i) on N and (x^i, u^A) on M adapted to the projection ν. We also consider local basis of sections $\{\bar{e}_a\}$ of F and $\{e_a, e_\alpha\}$ of E adapted to the projection π, that is $\pi\circ e_a = \bar{e}_a\circ\nu$ and $\pi\circ e_\alpha = 0$. In this way $\{e_\alpha\}$ is a base of sections of K. An element w in $\mathcal{L}_m\pi$ is of the form $w = (w^b_a e_b + w^\alpha_a e_\alpha)\otimes\bar{e}^a$, and it is in $\mathcal{J}_m\pi$ if and only if $w^b_a = \delta^b_a$, i.e., an element ϕ in $\mathcal{J}\pi$ is of the form $\phi = (e_a + \phi^\alpha_a e_\alpha)\otimes\bar{e}^a$. If we set $y^\alpha_a(\phi) = \phi^\alpha_a$, we have adapted local coordinates (x^i, u^A, y^α_a) on $\mathcal{J}\pi$. Similarly, an element $\psi \in \mathcal{V}_m\pi$ is of the form $\psi = \psi^\alpha_a e_\alpha\otimes\bar{e}^a$. If we set $y^\alpha_a(\psi) = \psi^\alpha_a$, we have adapted local coordinates (x^i, u^A, y^α_a) on $\mathcal{V}\pi$. As usual, we use the same name for the coordinates in an affine bundle and in the associated vector bundle.

An element $z \in \mathcal{L}^*_m\pi$ defines an affine function $\hat{z}$ on $\mathcal{J}_m\pi$ by contraction $\hat{z}(\phi) = \langle z, \phi\rangle$ where $\langle\cdot,\cdot\rangle$ is the pairing $\langle z, \phi\rangle = \mathrm{Tr}(z\circ\phi) = \mathrm{Tr}(\phi\circ z)$. Therefore, a section θ of $\mathcal{L}^*\pi$ defines a fiberwise affine function $\hat{\theta}$ on $\mathcal{J}\pi$, $\hat{\theta}(\phi) = \langle\theta_{\pi_{10}(\phi)}, \phi\rangle = \mathrm{Tr}(\theta_{\pi_{10}(\phi)}\circ\phi)$. In local coordinates, a section of $\mathcal{L}^*\pi$ is of the form $\theta = (\theta^a_b(x)e^b + \theta^a_\alpha(x)e^\alpha)\otimes\bar{e}_a$, and the affine function defined by θ is $\hat{\theta} = \theta^a_a(x) + \theta^a_\alpha(x)y^\alpha_a$.

Anchor. Consider now anchored structures on the bundles E and F, that is, we have two vector bundle maps $\rho_F\colon F \longrightarrow TN$ and $\rho_E\colon E \longrightarrow TM$ over the identity in N and M respectively. We will assume that the map π is admissible, that is $\rho_F\circ\pi = T\nu\circ\rho_E$. Therefore we have

$$\rho_F(\bar{e}_a) = \rho^i_a\frac{\partial}{\partial x^i} \qquad \text{and} \qquad \begin{cases} \rho_E(e_a) = \rho^i_a\dfrac{\partial}{\partial x^i} + \rho^A_a\dfrac{\partial}{\partial u^A}, \\ \rho_E(e_\alpha) = \rho^A_\alpha\dfrac{\partial}{\partial u^A}, \end{cases}$$

with $\rho^i_a = \rho^i_a(x)$, $\rho^A_a = \rho^A_a(x,u)$ and $\rho^A_\alpha = \rho^A_\alpha(x,u)$.

The anchor allows us to define the concept of total derivative of a function with respect to a section. Given a section $\sigma \in \mathrm{Sec}(F)$, the total derivative of a function $f \in C^\infty(M)$ with respect to σ is the function $\widehat{df\otimes\sigma}$, i.e., the affine function associated to $df\otimes\sigma \in \mathrm{Sec}(\mathcal{L}^*\pi)$. In particular, the total derivative with respect to an element $\bar{e}_a$ of the local basis of sections of F, will be denoted by $\acute{f}_{|a}$. In this way, if $\sigma = \sigma^a\bar{e}_a$ then $\widehat{df\otimes\sigma} = \acute{f}_{|a}\sigma^a$, where the coordinate expression of $\acute{f}_{|a}$ is

$$\acute{f}_{|a} = \rho^i_a\frac{\partial f}{\partial x^i} + (\rho^A_a + \rho^A_\alpha y^\alpha_a)\frac{\partial f}{\partial u^A}.$$

Notice that, for a function f in the base N, we have that $\acute{f}_{|a} = \rho^i_a\frac{\partial f}{\partial x^i}$ are just the components of df in the basis $\{\bar{e}_a\}$.

Bracket. Finally, let us assume that we have Lie algebroid structures on $\tau^F_N\colon F \longrightarrow N$ and on $\tau^E_M\colon E \longrightarrow M$, and that the projection π is a morphism of Lie algebroids. This condition implies the vanishing of some structure functions.

We have the following expressions for the brackets of elements in the basis of sections

$$[\![\bar{e}_a, \bar{e}_b]\!] = C^c_{ab}\bar{e}_c \qquad \text{and} \qquad \begin{cases} [\![e_a, e_b]\!] = C^\gamma_{ab}e_\gamma + C^c_{ab}e_c \\ [\![e_a, e_\beta[\![= C^\gamma_{a\beta}e_\gamma \\ [\![e_\alpha, e_\beta]\!] = C^\gamma_{\alpha\beta}e_\gamma, \end{cases}$$

where $C^a_{bc} = C^a_{bc}(x)$ is a basic function.

The structure functions can be conveniently combined as terms of some affine function as follows

$$Z^\alpha_{a\gamma} = C^\alpha_{a\gamma} + C^\alpha_{\beta\gamma}y^\beta_a \qquad \text{and} \qquad Z^\alpha_{ac} = C^\alpha_{ac} + C^\alpha_{\beta c}y^\beta_a.$$

In particular, such functions will appear in the Euler-Lagrange equations of the variational problem.

7.2. **Morphisms and admissible maps.** By a section of π we mean a vector bundle map Φ such that $\pi \circ \Phi = \mathrm{id}_F$, (i.e., we consider only linear sections of π (see also [16])). It follows that the base map $\phi\colon N \longrightarrow M$ is a section of ν, i.e., $\nu \circ \phi = \mathrm{id}_N$. The set of sections of π will be denoted by $\mathrm{Sec}(\pi)$. The set of those sections of π which are a morphism of Lie algebroids will be denoted by $\mathcal{M}(\pi)$. We will find local conditions for $\Phi \in \mathrm{Sec}(\pi)$ to be an admissible map between anchored vector bundles and also local conditions for Φ to be a morphism of Lie algebroids.

Taking adapted local coordinates (x^i, u^A) on M, the map ϕ has the expression $\phi(x^i) = (x^i, u^A(x))$. If we moreover take an adapted basis $\{e_a, e_\alpha\}$ of local sections of E, then the expression of Φ is given by $\Phi(\bar{e}_a) = e_a + y^\alpha_a(x)e_\alpha$, so that the map Φ is determined by the functions $\big(u^A(x), y^\alpha_a(x)\big)$ locally defined on N. The action on the dual basis is $\Phi^\star e^a = \bar{e}^a$, and $\Phi^\star e^\alpha = y^\alpha_a(x)\bar{e}^a$, and for the coordinate functions $\Phi^\star x^i = x^i$ and $\Phi^\star u^A = u^A(x)$.

The admissibility condition reads $\Phi^\star(df) = d(\Phi^\star f)$ for every function $f \in C^\infty(M)$. Taking $f = x^i$ we get an identity, while taking $f = u^A$ we get the condition

$$\rho^i_a \frac{\partial u^A}{\partial x^i} = \rho^A_a + \rho^A_\alpha y^\alpha_a.$$

In addition to the admissibility condition, the morphism condition reads $\Phi^\star d\theta = d(\Phi^\star\theta)$ for every section θ of E^*. For $\theta = e^a$ we get an identity, while for $\theta = e^\alpha$ we find the

$$\rho^i_b \frac{\partial y^\alpha_c}{\partial x^i} - \rho^i_c \frac{\partial y^\alpha_b}{\partial x^i} - y^\alpha_a C^a_{bc} + C^\alpha_{\beta\gamma}y^\beta_b y^\gamma_c + C^\alpha_{b\gamma}y^\gamma_c - C^\alpha_{c\gamma}y^\gamma_b + C^\alpha_{bc} = 0.$$

7.3. **Variational Calculus.** In what follows in this paper we consider the case where the Lie algebroid F is the tangent bundle $F = TN$ with $\rho_F = \mathrm{id}_{TN}$ and $[\cdot,\cdot]$ the usual Lie bracket of vector fields on N. The Lie algebroid E remains a general Lie algebroid. Moreover, for local expressions on F, the local basis of sections of F which we will consider is a basis of coordinate vector fields $\bar{e}_i = \frac{\partial}{\partial x^i}$, so that $\rho^i_a = \delta^i_a$ and $C^a_{bc} = 0$.

Variational problem. Given a Lagrangian function $L \in C^\infty(\mathcal{J}\pi)$ and a volume form $\omega \in \bigwedge^r(TN)$, where $r = \dim(N)$, we consider the following variational problem: find the critical points of the action functional $\mathcal{S}(\Phi) = \int_N L(\Phi)\,\omega$ defined on the set of sections of π which are moreover morphisms of Lie algebroids, that is, defined on the set $\mathcal{M}(\pi)$. Here by $L(\Phi)$ we mean the function $n \mapsto L(\Phi_n)$, where $\Phi_n \in \mathcal{J}\pi$ is the restriction of Φ the fiber $F_n = T_nN$.

It is important to notice that the above variational problem is a constrained problem, not only because the condition $\pi \circ \Phi = \mathrm{id}_F$, which can be easily solved, but because of the condition of Φ being a morphism of Lie algebroids, which is a condition on the derivatives of Φ. Taking coordinates on N such that the volume form is $\omega = dx^1 \wedge \cdots \wedge dx^r$, the problem is to find the critical points of

$$\int_N L(x^i, u^A(x), y^\alpha_a(x))\, dx^1 \wedge \cdots \wedge dx^r,$$

subject to the constraints

$$\frac{\partial u^A}{\partial x^a} = \rho^A_a + \rho^A_\alpha y^\alpha_a \quad \text{and} \quad \frac{\partial y^\alpha_c}{\partial x^b} - \frac{\partial y^\alpha_b}{\partial x^c} + C^\alpha_{b\gamma} y^\gamma_c - C^\alpha_{c\gamma} y^\gamma_b + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_c + C^\alpha_{bc} = 0.$$

The first method one can try to solve the problem is to use Lagrange multipliers. Nevertheless, one has no warranties that all solutions to this problem are normal (i.e., not strictly abnormal). In fact, in simple cases such as the problem of a heavy top [38], one can easily see that there will be strictly abnormal solutions. Therefore we take another approach, which consists of finding explicitly finite variations of a solution, that is, defining a curve in $\mathcal{M}(\pi)$ starting at the given solution.

Variations and infinitesimal variations. In order to find admissible variations, we consider sections of E and the associated flow. With the help of this flow we can transform morphisms of Lie algebroids into morphisms of Lie algebroids.

Flow defined by a section. We recall that every section of a Lie algebroid has an associated local flow composed of morphisms of Lie algebroids [42, 34]. More explicitly, given a section σ of a Lie algebroid E, there exists a local flow $\Phi_s\colon E \longrightarrow E$ such that

$$\mathcal{L}_\sigma \theta = \frac{d}{ds} \Phi^\star_s \theta \Big|_{s=0},$$

for any section θ of $\bigwedge E$. Moreover, for every fixed s, the map Φ_s is a morphism of Lie algebroids, and the base map $\phi_s\colon M \longrightarrow M$, the (ordinary) flow of the vector field $\rho(\sigma) \in \mathfrak{X}(M)$.

Complete lift of a section. In this section we will define the lift of a projectable section of E to a vector field on $\mathcal{J}\pi$, in a similar way to the definition of the first jet prolongations of a projectable vector field in the standard theory of jet bundles [52].

We consider a section σ of a Lie algebroid E projectable over a section η of F. We denote by Ψ_s the flow on E associated to σ and by Φ_s the flow on F associated to η. We recall that, for every fixed s, the maps Ψ_s and Φ_s are morphisms of Lie algebroids. Moreover, the base maps ψ_s and ϕ_s, are but the flows of the vector fields $\rho_E(\sigma)$ and $\rho_F(\eta)$, respectively.

The projectability of the section implies the projectability of the flow. It follows that (locally, in the domain of the flows) we have defined a map $\mathcal{L}\Psi_s\colon \mathcal{L}\pi \longrightarrow \mathcal{L}\pi$ by means of $\mathcal{L}\Psi_s(w) = \Psi_s \circ w \circ \Phi_{-s}$. By restriction of $\mathcal{L}\Psi_s$ to $\mathcal{J}\pi$ we get a map $\mathcal{J}\Psi_s$, which is a local flow in $\mathcal{J}\pi$. We will denote by $X^{(1)}_\sigma$ the vector field on $\mathcal{J}\pi$ generating the flow $\mathcal{J}\Psi_s$. The vector field $X^{(1)}_\sigma$ will be called the ***complete lift*** to $\mathcal{J}\pi$ of the section σ. Since $\mathcal{J}\Psi_s$ projects to the flow ψ_s it follows that the vector field $X^{(1)}_\sigma$ projects to the vector field $\rho_E(\sigma)$ in M.

Locally, a section $\sigma = \sigma^a e_a + \sigma^\alpha e_\alpha$ is projectable if $\sigma^a = \sigma^a(x^i)$ depends only on x^i. Its complete lift $X^{(1)}_\sigma$ has the local expression

$$X^{(1)}_\sigma = \sigma^a \frac{\partial}{\partial x^a} + (\rho^A_a \sigma^a + \rho^A_\alpha \sigma^\alpha) \frac{\partial}{\partial u^A} + \sigma^\alpha_a \frac{\partial}{\partial y^\alpha_a},$$

where $\sigma^\alpha_a = \acute{\sigma}^\alpha_{|a} + Z^\alpha_{ab}\sigma^b + Z^\alpha_{a\beta}\sigma^\beta - y^\alpha_b \left(\acute{\sigma}^b_{|a} + \sigma^c C^b_{ac} \right)$. In particular, if σ projects to the zero section, i.e., $\sigma^a = 0$, we have

$$X^{(1)}_\sigma = \rho^A_\alpha \sigma^\alpha \frac{\partial}{\partial u^A} + \left(\acute{\sigma}^\alpha_{|a} + Z^\alpha_{a\beta}\sigma^\beta \right) \frac{\partial}{\partial y^\alpha_a}.$$

Euler-Lagrange equations. Let $\Phi \in \mathcal{M}(\pi)$ be a critical point of $\mathcal{S}$. In order to find admissible variations we consider a π-vertical section σ of E. Its flow $\Psi_s\colon E \longrightarrow E$ projects to the identity in $F = TN$. Moreover we will require σ to have compact support. Since for every fixed s, the map Ψ_s is a morphism of Lie algebroids, it follows that the map $\Phi_s = \Psi_s \circ \Phi$ is a section of π and a morphism of Lie algebroids, that is, $s \mapsto \Phi_s$ is a curve in $\mathcal{M}(\pi)$. Using this kind of variations we have the following result.

Theorem 7.1. [42] *Select local coordinates such that the volume form is expressed as $\omega = dx^1 \wedge \cdots \wedge dx^r$. A map Φ is a critical section of $\mathcal{S}$ if and only if the components y^α_a of Φ satisfy the system of partial differential equations*

$$\begin{aligned}
&\frac{\partial u^A}{\partial x^a} = \rho^A_a + \rho^A_\alpha y^\alpha_a,\\
&\frac{\partial y^\alpha_a}{\partial x^b} - \frac{\partial y^\alpha_b}{\partial x^a} + C^\alpha_{b\gamma} y^\gamma_a - C^\alpha_{a\gamma} y^\gamma_b + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_a + C^\alpha_{ba} = 0,\\
&\frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = \frac{\partial L}{\partial y^\gamma_a} Z^\gamma_{a\alpha} + \frac{\partial L}{\partial u^A}\rho^A_\alpha.
\end{aligned}$$

Proof. Recall that by $L(\Phi)$ we mean the function in N given by $L(\Phi)(n) = L(\Phi_n)$, where Φ_n is the restriction of $\Phi\colon F \longrightarrow E$ to the fiber F_n. The function $L(\Phi_s)$ is

$$L(\Phi_s)(n) = L(\Psi_s \circ \Phi_n) = L(\mathcal{J}\Psi_s(\Phi_n)) = (\mathcal{J}\Psi_s^* L)(\Phi)(n),$$

and therefore the variation of the action along the curve $s \mapsto \Phi_s$ is

$$0 = \frac{d}{ds}\mathcal{S}(\Phi_s)\Big|_{s=0} = \int_N \frac{d}{ds} L(\Phi_s)\Big|_{s=0}\omega = \int_N (\mathcal{L}_{X^{(1)}_\sigma} L)(\Phi)\,\omega.$$

Taking into account the local expression of $X^{(1)}_\sigma$ for a π-vertical σ, we have that

$$\begin{aligned}
\mathcal{L}_{X^{(1)}_\sigma} L &= \rho^A_\alpha \sigma^\alpha \frac{\partial L}{\partial u^A} + \left(\frac{d\sigma^\alpha}{dx^a} + Z^\alpha_{a\beta}\sigma^\beta\right)\frac{\partial L}{\partial y^\alpha_a}\\
&= \sigma^\alpha\left[\rho^A_\alpha \frac{\partial L}{\partial u^A} + Z^\gamma_{a\alpha}\frac{\partial L}{\partial y^\gamma_a} - \frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right)\right] + \frac{d}{dx^a}\left(\sigma^\alpha \frac{\partial L}{\partial y^\alpha_a}\right).
\end{aligned}$$

Let us denote by δL the expression with components

$$\delta L_\alpha = \frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) - Z^\gamma_{a\alpha}\frac{\partial L}{\partial y^\gamma_a} - \rho^A_\alpha \frac{\partial L}{\partial u^A},$$

and by J_σ the $(r-1)$-form (along π_1) $J_\sigma = \sigma^\alpha \frac{\partial L}{\partial y^\alpha_a}\omega_a$ with $\omega_a = i_{\frac{\partial}{\partial x^a}}\omega$. Then we have that

$$0 = \frac{d}{ds}\mathcal{S}(\Phi_s)\Big|_{s=0} = -\int_N (\delta L_\alpha\,\sigma^\alpha)\omega + \int_N d(J_\sigma \circ \Phi).$$

Since σ has compact support the second term vanishes by Stokes theorem. Moreover, since the section σ is arbitrary, by the fundamental theorem of the Calculus of Variations, we get $\delta L = 0$, which are the Euler-Lagrange equations. Notice that the first two equations in the above statement are nothing but the morphism conditions. □

7.4. Examples.

Standard case. In the standard case, we consider a bundle $\nu\colon M \longrightarrow N$, the standard Lie algebroids $F = TN$ and $E = TM$ and the tangent map $\pi = T\nu\colon TM \longrightarrow TN$. Then we have that $\mathcal{J}\pi = J^1\nu$. When we choose coordinate basis of vector fields (i.e., of sections of TN and TM) we recover the equations for the standard first-order field theory. Moreover, if we consider a different basis, what we get are the equations for a first-order field theory written in pseudo-coordinates [5, 14, 42].

Time-dependent Mechanics. In [50, 51] we developed a theory of Lagrangian Mechanics for time dependent systems defined on Lie algebroids, where the base manifold is fibered over the real line $\mathbb{R}$. Since time-dependent Mechanics is nothing but a 1-dimensional field theory, our results must reduce to that.

The morphism condition is just the admissibility condition so that, if we write $x^0 \equiv t$ and $y^\alpha_0 \equiv y^\alpha$, the Euler-Lagrange equations are

$$\frac{du^A}{dt} = \rho^A_0 + \rho^A_\alpha y^\alpha, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \frac{\partial L}{\partial y^\gamma}(C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} y^\beta) + \frac{\partial L}{\partial u^A}\rho^A_\alpha,$$

in full agreement with [44]. In particular, for an autonomous system on a Lie algebroid $V \longrightarrow Q$, one considers $E = T\mathbb{R}\times V \longrightarrow \mathbb{R}\times M$ with π the projection onto the first factor $T\mathbb{R}$. Our results provide yet another indication of the variational character of autonomous mechanical systems on Lie algebroids.

Topological field theories. Given a closed r-form Ω on a Lie algebroid $V \longrightarrow Q$, we can define a topological field theory as follows. For an r-dimensional manifold N we consider $F = TN \longrightarrow N$, $E = TN \times V \longrightarrow N \times Q$ and π the projection onto the first factor TN. The Lagrangian of the theory is $L(\Phi) = \Phi^\star\Omega$. Then it is easy to see that the Euler-Lagrange equations reduce to the morphism condition. In this way, one can cope with systems such as Poisson σ-models or Chern-Simons theories [2, 42, 43].

Systems with symmetry. The case of a system with symmetry is very important in Physics. We consider a principal bundle $\nu\colon P \longrightarrow M$ with structure group G and we set $N = M$, $F = TN$ and $E = TP/G$ (the Atiyah algebroid of P), with $\pi([v]) = T\nu(v)$. Sections of π are just principal connections on P and a section is a morphism if and only if it is a flat connection. The kernel K is just the adjoint bundle $(P\times\mathfrak{g})/G \longrightarrow M$. By an adequate choice of a local basis of sections of F, K and E one easily find the covariant Euler-Poincaré equations [7, 8]. The covariant Lagrange-Poincaré equations [6] can also be recovered within this formalism.

8. Future work

We have illustrated the generality of the theory of Lie algebroids and groupoids in a wide range of situations, from nonholonomic Lagrangian and Hamiltonian systems and mechanical control systems to Discrete Mechanics and extensions to Field Theory. Current and future directions of research include the following:

- *Hamilton-Jacobi equation for a Hamiltonian system on a Lie algebroid:* It would be interesting to continue with the study started in [26] of Hamilton-Jacobi theory for Hamiltonian systems on Lie algebroids. In particular, it would be interesting to introduce a suitable definition of a local (global) complete integral of the Hamilton-Jacobi equation. The idea would be that the knowledge of an integral of the equation would allow the "direct determination" of some integral curves of the corresponding Hamiltonian vector field.
- *Geometric formalism for Vakonomic Mechanics on Lie algebroids:* An interesting topic to study is the case of constrained variational problems on Lie algebroids. In this case, to derive the equations of motion for a Lagrangian system subject to nonholonomic constraints, one invokes a variational principle, rather than the Lagrange-D'Alembert's principle (cf. Section 4). The differential equations obtained, called vakonomic equations, are in general different. From an optimal control perspective, it seems interesting to generalize the formalism developed in [11].

Mechanical control systems on Lie algebroids: Topics of interest related to mechanical control systems on Lie algebroids include the investigation of controllability tests along relative equilibria and the study of systems that include gyroscopic and dissipative forces.

Discrete Mechanics on Lie groupoids: We are currently studying the construction of geometric integrators for mechanical systems on Lie algebroids. We have introduced the exact discrete Lagrangian in the Lie groupoid formalism, and are discussing different types of discretizations of continuous Lagrangians and their numerical implementation. We plan to explore natural extensions to forced systems and to systems with holonomic and nonholonomic constraints as in [12, 27].

Classical Field Theory and Lie algebroids: In [16, 44] the authors have introduced the notion of a Lie affgebroid structure [17, 41, 44] (see also [21]). They have developed a Lagrangian (and Hamiltonian) formalism on Lie affgebroids, which generalizes some classical formalisms for time-dependent Mechanics and, in addition, may be applied to other situations. Since time-dependent Mechanics is a 1-dimensional field theory, it would be interesting to define the notion of a "Lie multialgebroid", as a generalization of the notion of a Lie affgebroid. This mathematical object should encode the geometric structure necessary to develop field theories. The first example of a Lie multialgebroid¡ should be $\mathcal{J}\pi$. The notion of a Lie multialgebroid will potentially allow to study other aspects of the theory, such as Tulczyjew's triples associated with a Lie multialgebroid and Hamilton-Jacobi equation for classical field theories on Lie multialgebroids.

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Jorge Cortés: Applied Mathematics and Statistics, Baskin School of Engineering, University of California, Santa Cruz, California 95064, USA

E-mail address: jcortes@ucsc.edu

M. de León: Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

E-mail address: mdeleon@imaff.cfmac.csic.es

Juan C. Marrero: Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain

E-mail address: jcmarrer@ull.es

D. Martín de Diego: Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

E-mail address: d.martin@imaff.cfmac.csic.es

Eduardo Martínez: Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address: emf@unizar.es