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A survey of results on integral trees and integral graphs

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Abstract

A graph G is called *integral* if all zeros of the characteristic polynomial $P(G, x)$ are integers. Our purpose is to determine or to characterize which graphs are integral. This problem was posed by Harary and Schwenk in 1974. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some interesting families of graphs. So far, there are many results on some particular classes of integral graphs. For example, some results for integral graphs with maximum degree 3 and 4 have been obtained by Cvetković (1974, 1975, 1998), Schwenk (1978), Simić (1986, 1995, 1998, 2001, 2002), Balińska (1999, 2001, 2002, 2004) and others. An infinite family of integral complete tripartite graphs was constructed by Roitman (1984). Results on integral graphs which belong to the class $\overline{\alpha K_{a,b}}$, $\overline{\alpha K_a \cup \beta K_b}$ or $\overline{\alpha K_a \cup \beta K_{b,b}}$ were presented by Lepović (2003, 2004). Trees present another important family of graphs for which the problem has been considered by Harary (1974), Schwenk (1974, 1979), Watanabe (1979), Li(1987, 1999, 2000, 2002, 2004), Liu(1988), Cao (1988, 1991), Yuan(1998), Híc(1997, 1998, 2003), Wang (1999, 2000, 2002, 2004) and others. Moreover, several graph operations (e.g., Cartesian product, strong sum and product) on integral graphs can be used for constructing infinite families of integral graphs with increasing order. In this paper, I shall give a survey of results on integral trees and integral graphs. Our main contributions on integral graphs during the last years concern the following topics:

- Integral trees with diameters 4, 5, 6 and 8.
- Integral complete r-partite graphs.
- Integral nonregular bipartite graphs.
- Integral regular graphs.
- Other integral graphs.

Finally, I propose several open problems for further study.

Key Words: Integral graph, integral tree, graph spectrum.

AMS Subject Classification (2000): 05C05, 11D09, 11D41.

1 Contents

First of all, we will give an outline of this contribution. The following four aspects will be considered.

- Basic definitions.
- History of integral graphs.
- Our main results on integral trees and integral graphs.
- Some open questions on integral trees and integral graphs.

2 Basic definitions

Throughout this paper we shall consider only simple graphs (*i.e.* finite undirected graphs without loops or multiple edges). We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. All other terminology and notations can be found in Cvetković, Doob & Sachs [18].

(1) The *adjacency matrix* $A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge.

(2) The *characteristic polynomial* of G is the polynomial $P(G, x) = det(xI_n - A(G))$, where I_n denotes the $n \times n$ identity matrix.

(3) A graph G is called *integral* if all zeros of the characteristic polynomial $P(G, x)$ are integers.

(4) The spectrum of $A(G)$ is also called the *spectrum* of G. If the eigenvalues are ordered by $\lambda_1 > \lambda_2 > \ldots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ m_1 & m_2 & \ldots & m_r \end{pmatrix} \text{ or } Spec(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}\}.$$

Example 1. The complete graph K_4 is shown in Figure 1.

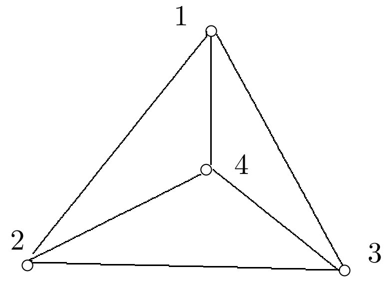


Figure 1: The complete graph K_4

The adjacency matrix $A(K_4)$ of the complete graph K_4 can be written as the 4×4 symmetric matrix:

$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

The characteristic polynomial of this matrix is easily computed:

$$P(K_4, x) = det(xI_4 - A(K_4)) = (x+1)^3(x-3),$$

its spectrum is

$$Spec(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

Our purpose is to give a survey of results on the problem to determine or characterize which graphs are integral. This problem was posed by Harary and Schwenk in 1974 (see [27]).

Problem: Which graphs have integral spectra?

There are many simple examples of integral graphs (some of them are given in the following table).

Example 2.

Graphs	Eigenvalues	Respective multiplicities	Integral
K_n	$n-1, -1$	$1, n-1$	all
$CP(n)$	$2n-2, 0, -2$	$1, n, n-1$	all
$K_{\frac{n}{k},\frac{n}{k},\ldots,\frac{n}{k}}$	$n-\frac{n}{k}, 0, -\frac{n}{k}$	$1, n-k, k-1$	all
P_n	$2\cos(\frac{k\pi}{n+1})$, $k = 1, 2, \ldots, n$	$1, 1, \ldots, 1$	P_2
C_n	$2\cos(\frac{2k\pi}{n})$, $k = 1, 2, \ldots, n$	2,1,1,...,1,2 for n even 1,1,...,1,2 for n odd	C_3, C_4, C_6
$K_{m,n}$	$\sqrt{mn}, 0, -\sqrt{mn}$	$1, m+n-2, 1$	$mn = c^2$

where K_n denotes the complete graph; $CP(n) = \overline{nK_2}$ denotes the cocktail-party graph; $K_{n/k,n/k,\ldots,n/k}$ denotes the complete multipartite graph; the path P_2 (P_2 is the only integral path in the set of paths P_n with n vertices); the circuits C_3, C_4 and C_6 (the three circuits are the only integral circuits in the set of circuits C_n with n vertices); the complete bipartite graph $K_{m,n}$ ($K_{m,n}$ is integral if and only if mn is a perfect square); the stars $K_{1,n}$ with $n = p^2$ ($p = 1, 2, 3, \ldots$), and so on.

3 History of integral graphs

3.1 The study on theory of graph spectra

The study on integral graphs is an active research field in the theory of graph spectra. The study on theory of graph spectra is presented in the following monographs.

- ([46]) Schwenk and Wilson, On the eigenvalues of a graph, (1978).
- ([18]) Cvetković, Doob and Sachs, *Spectra of Graphs–Theory and Application*, (1980).
- ([17]) Cvetković, Doob, Gutman and Torgašev, *Recent Results in the Theory of Graph Spectra* , (1988).
- ([9]) Biggs, *Algebraic Graph Theory*, (1993).
- ([8]) Beineke, Cameron and Wilson, *Topics in Algebraic Graph Theory* (2004).

3.2 Results on integral graphs

Next we will discuss results on integral graphs.

Since the spectrum of a disconnected graph is the union of the spectra of its components, in any investigation of integral graphs it is sufficient to consider connected graphs only.

Some of the well known graph operations, when applied to integral graphs, result in new integral graphs and thus can be used to generate an arbitrary number of them. Let us look at the following three graph operations:

The *product* (or *conjunction*) $G \times H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G \times H$ if and only if x is adjacent to y in G and a is adjacent to b in H.

The *sum* (or *Cartesian product*) $G + H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G + H$ if and only if either $x = y$ and a is adjacent to b in H or $a = b$ and x is adjacent to y in G.

The *strong sum* (or *strong product*) $G \bigoplus H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G \bigoplus H$ if and only if a is adjacent to b in H and either x is adjacent to y in G or $x = y$.

The results are the following:

If λ_i, $(i = 1, 2, \ldots, n)$ and μ_j $(j = 1, 2, \ldots, m)$ are the eigenvalues of G and H, respectively, then

(1) the *product* $G \times H$ has eigenvalues $\lambda_i \mu_j$,

(2) the *sum* $G + H$ has eigenvalues $\lambda_i + \mu_j$,

(3) the *strong sum* $G \bigoplus H$ has eigenvalues $\lambda_i \mu_j + \lambda_i + \mu_j$,

(in all these cases $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$). Thus, these three operations preserve the integrality.

In general, the problem of characterizing integral graphs seems to be very difficult. Since there is no general characterization (besides the definition) of these graphs, the problem of finding (or characterizing) integral graphs has to be treated for some special interesting classes of graphs.

So far, there are many results on some particular classes of integral graphs.

1. Integral 3-regular graphs

All integral connected cubic graphs were obtained by Cvetković and Bussemaker [16, 12], and independently in 1976 by Schwenk [45]. There are exactly thirteen connected cubic integral graphs. In fact, at the same time and independently, this result was reported (and published a bit later) by Schwenk. It is interesting that the research techniques used by different authors were also somewhat different; among others, Bussemaker and Cvetković combined the aid of a computer with theoretical reasoning, while Schwenk achieved the result completely by hand and pencil.

- ([16]) Cvetković, Cubic integral graphs, (1975).
- ([12]) Bussemaker and Cvetković, There are exactly 13 connected cubic integral graphs, (1976).
- ([45]) Schwenk, Exactly thirteen connected cubic graphs have integral spectra, (1978).

Cvetković [16] proved that the set of connected regular integral graphs of any fixed degree is finite. Similarly, the set of connected integral graphs with bounded vertex degrees is finite.

2. Integral complete graphs

An infinite family of integral complete tripartite graphs was constructed by Roitman in 1984 (see also [44]). In 2004, we give a sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs (see also [55]).

- ([44]) Roitman, An infinite family of integral graphs, (1984).
- ([55]) Wang, Li and Hoede, Integral complete r-partite Graphs, (2004).

3. Nonregular nonbipartite integral graphs with maximum degree four

Radosavljević and Simić determined all 13 connected nonregular nonbipartite integral graphs with maximum degree four (it was the title of the report published in 1986 [42], the full version appeared in 1995 [47]). The corresponding problem for bipartite graphs is not yet solved.

- ([42]) Radosavljević and Simić, There are just thirteen connected nonregular nonbipartite integral graphs having maximum vertex degree four (a shortened report), (1986).
- ([47]) Simić and Radosavljević, The nonregular, nonbipartite, integral graphs with maximum degree four, (1995).

4. Integral 4-regular graphs

From 1998 to now, 4-regular integral graphs began to attract some attention. Cvetković, Simić and Stevanović [20] found 1888 possible spectra of 4-regular bipartite integral graphs. A list of the 65 known 4-regular connected integral graphs was given in [20].

In 1999, Stevanović [49] obtained nonexistence results for some of these potential spectra.

In 2003, Stevanović [48] determined all 24 connected 4-regular integral graphs avoiding +3 and -3 in the spectrum.

- ([20]) Cvetković,Simić and Stevanović, 4-regular integral graphs, (1998).
- ([49]) Stevanović, Nonexistence of some 4-regular integral graphs, (1999).
- ([48]) Stevanović, 4-regular integral graphs avoiding ± 3 in the spectrum,(2003).

5. Constructions of integral graphs

In 2000, Wang, Li and Zhang [61] studied some constructions on integral graphs and obtained integral graphs K_n^t, $K_{a,b}^t$, $K_{a,a,\ldots,a}^t$, etc.

- ([61]) Wang, Li and Zhang, Construction of integral graphs, (2000).

6. Some graphs with few distinct eigenvalues

Bridges and Mena [10, 11] investigated some graphs with exactly three distinct eigenvalues in 1979 and 1981. Van Dam, Haemers *et al.* further studied nonregular or regular graphs with few different eigenvalues in [21, 22, 23, 24, 41].

7. Connected integral graphs with the number of vertices up to 12

Balińska *et al.* [1] proved in 1999 that there are exactly 150 connected integral graphs up to 10 vertices. The results of all connected integral graphs on 11 and 12 vertices can be found in ([3, 4]). Note that a few errata in the article [2] appeared in [19].

$n:$	1	2	3	4	5	6	7	8	9	10	11
i_n	1	1	1	2	3	6	7	22	24	83	113

Table 1: The numbers i_n of connected integral graphs with n vertices

8. Nonregular bipartite integral graphs

Balińska and Simić also studied some results of the nonregular, bipartite, integral graphs with maximum degree 4 in [5, 6, 7]. Wang, Li and Hoede investigated nonregular bipartite integral graphs in [56].

9. Integral bipartite semiregular graphs

Zhang and Zhou [71] obtained in 2003 some new integral graphs based on the study of bipartite semiregular graphs.

- Zhang and Zhou, On some classes of integral graphs (2003).

10. Some results on integral graphs which belong to special classes

Lepović [31, 32, 33, 34] obtained in 2003 and 2004 some results on integral graphs which belong to the class $\overline{\alpha K_{a,b}}$, $\overline{\alpha K_a \cup \beta K_b}$ or $\overline{\alpha K_a \cup \beta K_{b,b}}$.

A good survey on integral graphs:

Balińska, Cvetković, Radosavljević, Simić and Stevanović [2] presented in 2002 a survey of results on integral graphs and on the corresponding proof techniques.

- Balińska, Cvetković, Radosavljević, Simić, Stevanović, A survey on integral graphs, (2002).

3.3 Results on integral trees

Trees are another important and interesting family of graphs. In the initial paper [27] of Harary and Schwenk integral trees were mentioned as well, while considerable results on this topic were firstly published by Watanabe and Schwenk in [63, 64].

- ([27]) Harary and Schwenk, Which graphs have integral spectra? (1974).
- ([63]) Watanabe, Note on integral trees, (1979).
- ([64]) Watanabe and Schwenk, Integral starlike trees, (1979).

Then, again in 1987, starting with the article [36] of Li and Lin, a group of Chinese mathematicians began to present their results. In 1987, Li and Lin [36] gave answers to the three questions proposed by A.J. Schwenk (see also [15]), studied integral trees with diameters 4, 5 and 6, and discovered some new infinite sets of integral trees with diameters 4 and 6. Finally, they raised several open problems.

- ([36]) Li and Lin, On integral trees problems, (1987).

In that paper [36], integral trees with diameter five were mentioned for the first time, and they presented a theorem in the form of a necessary and sufficient condition for such a tree to be integral, but they were not able to find any example. The first integral tree of diameter 5 was found by Liu in 1988 [39].

The first integral tree of diameter 8 was found by Híc and Nedela [29] in 1998. Híc and Nedela firstly constructed infinitely many integral trees with diameter 8, and obtained some positive and negative results about the questions on balanced integral trees. He also proved that there are no balanced integral trees of diameter $4k+1$ for $k \geq 1$. Wang, Li and Liu [59] also constructed independently some families of integral trees with diameter 8 by using a different method in 1999.

- ([29]) Híc and Nedela, Balanced integral trees, (1998).
- ([59]) Wang, Li and Liu, Integral trees with diameter 6 or 8, (1999).

In 1998, Yuan [68] gave a necessary condition for trees $S(r; m_i)$ of diameter 4 to be integral, and constructed many new classes of such integral trees. In addition, some basic questions about integral trees with diameter 4 were posed in [68].

- ([68]) Yuan, Integral trees of diameter 4, (1998).

Then, Zhang & Tan [69] and Li, Yang & Wang [35] in 2000 gave a further useful sufficient and necessary condition for graphs to be such integral trees. Some questions proposed by Yuan in [68] were answered in [35, 43, 65, 66, 69, 70]. Wang, Li *et al.* also investigated integral trees with diameters 4, 5, 6 and 8 in [51, 52, 53, 54, 60, 62].

In 2003, Híc and Pokorný [30] investigated integral balanced rooted trees of diameters 4, 6, 8 and 10. An infinite class of integral balanced rooted trees with diameter 10 was given. But the problem of the existence of integral balanced rooted trees of arbitrarily large diameter remains open.

We shall give some details on results of integral trees in Table 2.

D	E	authors	year	trees
2	yes	Harary, Schwenk [27]	74	K_{1,n^2}
3	yes	Harary, Schwenk [27]	74	$T[2,2], T[6,6]$
		Watanabe [63, 64], Schwenk [64]	79	$T[m,r]$
		Liu [39]	88	$T[m,r]$
		Cao [13]	88	$T[m,r]$(all)
4	yes	Harary,Schwenk [27]	74	$T(k^2-1,1)$
		Watanabe,Schwenk [64]	79	$T(m,t)$(all)
		Li, Lin [36]	87	$K^t_{1,m}$,$S(r;m_i)$
		Cao [13], Liu [40]	88	$S(r;m_i)$
		Yuan [68], Ren [43]	98,00	$S(r;m_i)$
		Xu [65, 66]	96,97	$S(r;m_i)$
		Li,Yang,Wang [35]	00	$S(r;m_i)$
		Zhang,Wei [70], Zhang,Tan [69]	98,00	$S(r;m_i)$
		Wang,Li,Zhang [62]	04	$S(r;m_i)$
		Watanabe [63], Wang,Li [53]	79,00	$K_{1,s} \bullet T(m,t)$
		Yao [67], Wang,Li,Yao [60]	01,02	$K_{1,s} \bullet T(m,t)$
		Xu [66], Yuan [68]	97,98	$K_{1,s} \bullet S(r;m_i)$
		Zhang,Tan [69]	00	$K_{1,s} \bullet S(r;m_i)$
		Wang,Li,Zhang [62]	04	$K_{1,s} \bullet S(r;m_i)$
5	yes	Li, Lin [36]	87	research(no)
		Liu [39, 40]	88	first $T^t[m,r]$
		Cao [13]	88	a special case (no)
		Cao [14]	91	$T^t[m,r]$
		Li [38]	98	$T^t[m,r]$
		Híc, Nedela [29]	98	balanced (no)
		Wang,Li [54]	03	$[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$
6	yes	Watanabe,Schwenk [64]	79	$T^t(m,r)$
		Liu [40],Cao [14]	88,91	$T^t(m,r)$
		Li,Wang [54]	03	$T^t(m,r)$
		Li,Lin [36]	87	$T(r,m,t)$
		Cao [13], Liu [39]	88	$T(r,m,t)$
		Híc [28], Híc, Nedela [29]	97,98	$T(r,m,t)$
		Wang,Li,Liu [59]	99	$T(r,m,t)$
		Yao [67]	01	$T(r,m,t)$
		Wang,Li,Yao [60]	02	$T(r,m,t)$

Table 2: (a) D and E denote the diameter and existence,respectively

D	E	authors	year	trees
6	yes	Híc,Pokorný [30]	03	$T(r,m,t)$
		Wang,Li,Zhang [62]	04	$T(r,m,t)$
		Wang,Li [53]	00	$K_{1,s} \bullet T(r,m,t)$
		Wang,Li,Yao [60]	02	$K_{1,s} \bullet T(r,m,t)$
		Wang,Li,Zhang [62]	04	$K_{1,s} \bullet T(r,m,t)$
		Wang,Broersma,Hoede,Li [51]	04	$T(p,q) \bullet T(r,m,t)$
		Wang,Broersma,Hoede,Li [51]	04	$K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$
7	?	?	?	
8	yes	Híc, Nedela [29]	98	$T(q,r,m,t)$
		Wang,Li, Liu [59]	99	$T(q,r,m,t)$
		Wang,Li,Yao [60]	02	$T(q,r,m,t)$
		Híc,Pokorný [30]	03	$T(q,r,m,t)$
		Wang,Li,Zhang [62]	04	$T(q,r,m,t)$
		Wang,Broersma,Hoede,Li, and Still [52]	04	$K_{1,s} \bullet T(q,r,m,t)$
9	?	Híc, Nedela [29]	98	balanced (no)
10	yes	Híc,Pokorný [30]	03	$T(s,q,r,m,t)$
$4k+1$	no	Híc, Nedela [29]	98	only balanced
$d \geq 11$	?	?	?	?

Table 2: (b) D and E denote the diameter and existence,respectively

4 Our main results on integral trees and integral graphs

4.1 Our main results on integral trees

Recently, we obtained several results on integral trees.

(1) ([59]) Wang, Li and Liu, Integral trees with diameter 6 or 8, *Electron. Notes Discrete Math.* (1999).

(2) ([53]) Wang and Li, Some new classes of integral trees with diameters 4 and 6, *Australas. J. Combin.* (2000).

(3) ([60]) Wang, Li and Yao, Integral trees with diameters 4, 6 and 8, *Australas. J. Combin.* (2002).

(4) ([62]) Wang, Li and Zhang, Families of integral trees with diameters 4, 6 and 8, *Discrete Appl. Math.* (2004).

(5) ([54]) Wang and Li, Integral trees with diameters 5 and 6, accepted for publication in *Discrete Math.*

(6) ([51]) Wang, Broersma, Hoede and Li, Integral trees of diameter 6, (2004), preprint.

1. "Wang, Li and Zhang, Families of integral trees with diameters 4, 6 and 8, *Discrete Appl. Math.* **(2004)."**

In [62], some new families of integral trees with diameters 4, 6 and 8 are given. Most of these classes are infinite. They are different from those in the literature. We believe that this new contribution to the research of integral trees is useful for constructing other integral trees. At the same time, some results on the interrelations between integral trees of various diameters are obtained for the first time. These results generalize some of the well-known results or theorems on integral trees. Finally, we propose several basic open problems on integral trees for further study.

Example 1. For any positive integer n, the following results have been derived:

(1) It is shown in [64, 53] that $G = T(280, 9)$ of diameter 4 and $K_{1,r} \bullet G = K_{1,36} \bullet T(280, 9)$ of diameter 4 are integral. So the trees $r * G = T(36, 280, 9)$ of diameter 6 and $T(36n^2, 280n^2, 9n^2)$ of diameter 6 are integral too.

(2) It is shown in [28, 29, 60] that $G = T(144, 105, 16)$ of diameter 6 and $K_{1,s} \bullet G = K_{1,676} \bullet T(144, 105, 16)$ of diameter 6 are integral. So the trees $r * G = T(676, 144, 105, 16)$ of diameter 8 and $T(676n^2, 144n^2, 105n^2, 16n^2)$ of diameter 8 are integral too.

2. "Integral trees with diameters 5, 6 and 8"

In [51, 52, 54], we determine the characteristic polynomials of the trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$, $T(p,q) \bullet T(r,m,t)$, $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(q,r,m,t)$ with diameters 5, 6, 6 and 8, respectively. We also obtain first sufficient and necessary conditions for these trees to be integral by using number theory and computer search. All these classes are infinite and different from those in the literature. We also prove that the problem of finding such integral trees is equivalent to the problem of solving some Diophantine equations. These results generalize results of the existing literature. In particular integral trees of the type $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$, $T(p,q) \bullet T(r,m,t)$, $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(q,r,m,t)$ are obtained for the first time. We further present some new results on interrelations between integral trees of various diameters.

For example, integral trees of diameter 5 were mentioned for the first time in [36], but the authors were not able to find any example. Infinitely many integral trees $T^t[m,r]$ of diameter 5 were first constructed by Liu in [39]. Later Cao obtained general results on these classes by using the solutions of some Pell equations in [14], and then Li obtained more general results on these classes by using the solutions of more general quadratic Diophantine equations in [38]. The authors of [29] proved that there are no balanced integral trees $T(1; n_{2k}, n_{2k-1}, \ldots, n_1)$ of diameter $4k+1$ for $k \geq 1$. Here the structure of integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is found for the first time [54].

Example 2.

(1) For the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5, we obtain the smallest integral tree $[K_{1,2} \bullet T(3,4)] \ominus T(3,1)$ of diameter 5 in this class. Its characteristic polynomial is $P([K_{1,2} \bullet T(3,4)] \ominus T(3,1), x) = x^{11}(x^2-1)^3(x^2-4)^3(x^2-9)$ with order 25 (See [54]).

(2) For the tree $T^t[m, r]$ of diameter 5, Liu [39] obtained the smallest integral tree $T^{144}[50, 98]$ of diameter 5 in this class with order 21750.

4.2 Our main results on integral graphs

Recently, we obtained several results on integral graphs.

(1) ([61]) Wang, Li and Zhang, Construction of integral graphs, *Appl. Math. J. Chinese Univ. Ser. B* (2000).

(2) ([55]) Wang, Li and Hoede, Integral complete r-partite Graphs, *Discrete Math.* (2004).

(3) ([57]) Wang, Li and Hoede, Two classes of integral regular graphs, accepted for publication in *Ars Combinatoria*.

(4) ([58]) Wang, Li and Hoede, Eigenvalues of a special kind of symmetric block circulant matrices, *Appl. Math. J. Chinese Univ. Ser. B* (2004).

(5) ([56]) Wang, Li and Hoede, Nonregular bipartite integral graphs, (2003), preprint.

(6) ([52]) Wang, Broersma, Hoede, Li and Still, Families of integral graphs,(2004), preprint.

For example.

1. "Wang, Li and Hoede, Integral complete r-partite Graphs, ***Discrete Math.*** **(2004)."**

An infinite family of integral complete tripartite graphs was constructed by M. Roitman in 1984 (see [44]), where he mentioned the general problem of finding integral complete multipartite graphs. He conjectured that for $r > 3$ there exist an infinite number of integral complete r-partite graphs. However, he did not find such integral graphs. Balińska and Simić [5] remarked in 2001 that the general problem seems to be intractable. In this paper [55], we give a sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs. We will show that the problem of finding such integral graphs is equivalent to solving certain Diophantine equations. Roitman's result on the integral complete tripartite graphs is generalized in this chapter. We finally propose several basic open problems for further study.

2. "Wang, Li and Hoede, Nonregular bipartite integral graphs, (2003), preprint."

In [56], we construct fifteen classes of larger integral graphs from known smaller ones. These classes consist of nonregular and bipartite graphs. Their spectra and characteristic polynomials are obtained from matrix theory. Their integral property is established by using number theory and computer search. All these classes are infinite and different from those in the literature. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving Diophantine equations. We believe that these results are useful for constructing other integral graphs. They generalize some results of Balińska and Simić. Finally, we propose several open problems for further study.

For example, the graphs in Figures 2 and 3 are nonregular bipartite integral graphs with maximum degree four (The graphs in Figure 2 are integral graphs with number of vertices up to 16 (see also [5])). Fifteen classes of graphs in Figures 4 and 5 are constructed from the integral graphs in Figures 2 and 3.

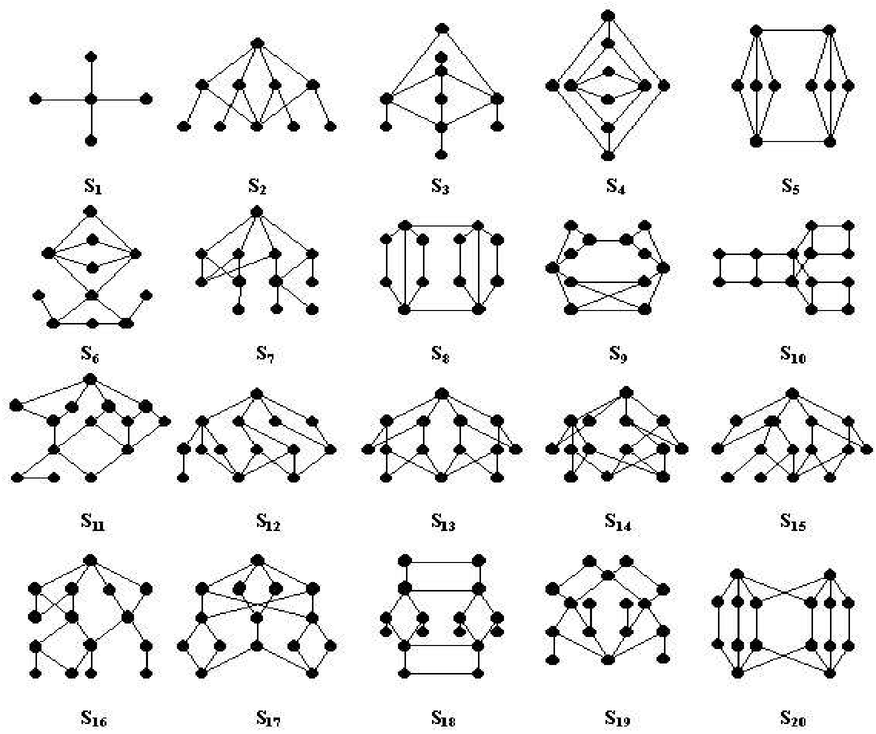


Figure 2: Nonregular bipartite integral graphs with maximum degree four and at most 16 vertices.

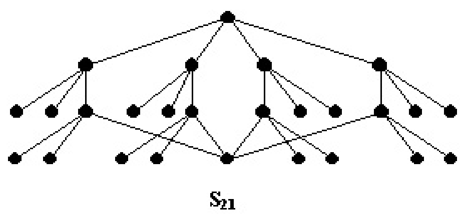


Figure 3: A nonregular bipartite integral graph with maximum degree four and 26 vertices.

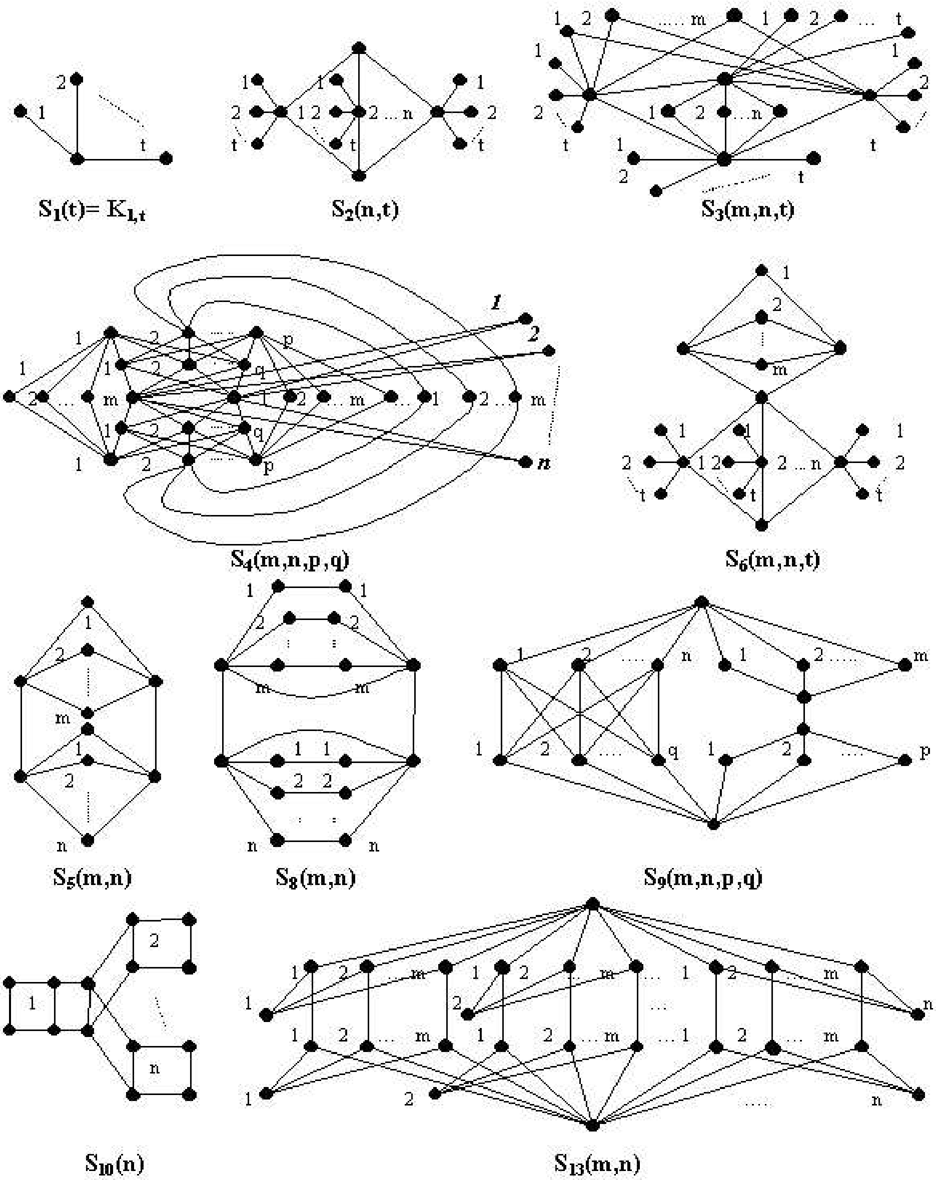


Figure 4: Nonregular bipartite graphs.

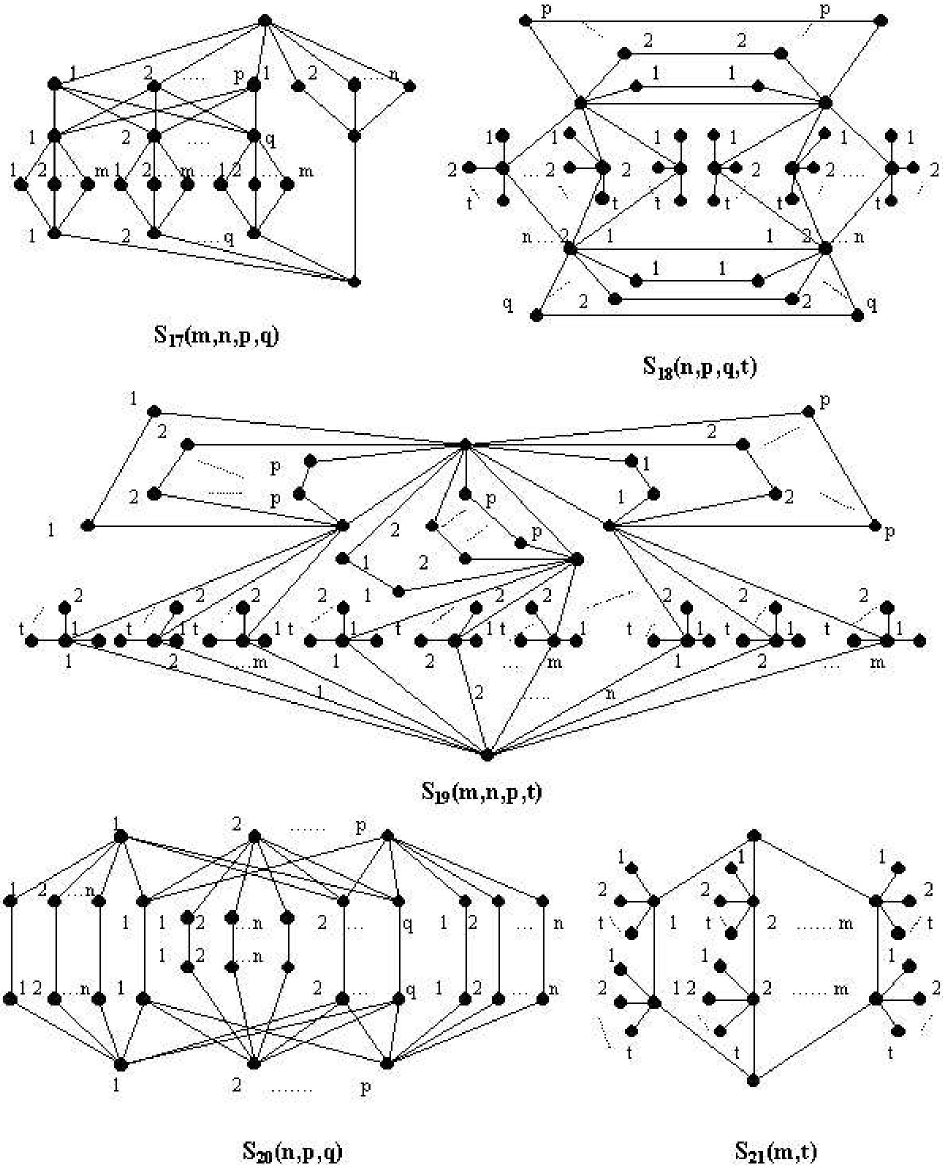


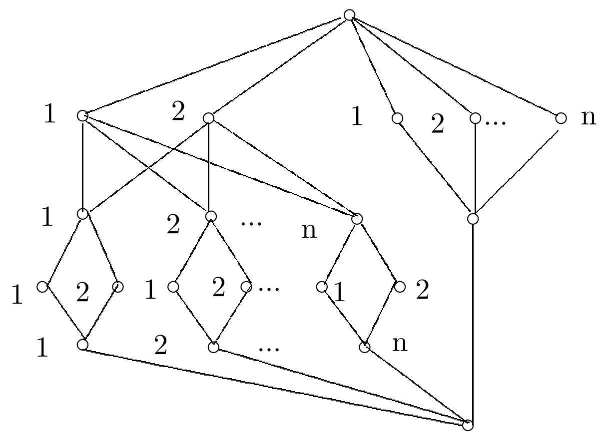
Figure 5: Nonregular bipartite graphs.

Techniques used in the proofs.

- Matrix theory.
- Number theory.
- Computer search.

Example 3.

(1) Construct a graph



$S_{17}(m, n, p, q) = S_{17}(2, n, 2, n)$

(2) The characteristic polynomial of this graph $S_{17}(2, n, 2, n)$ can be written:
$P(S_{17}(2, n, 2, n), x) = |xI_{5n+5} - A(S_{17}(2, n, 2, n))|$.

(3) Using matrix theory to obtain
$P(S_{17}(2, n, 2, n), x) = x^{3n+1}(x+2)^{n-1}(x-2)^{n-1}(x^2-n-2)(x^2+x-2n-2)(x^2-x-2n-2)$, $(n \geq 1)$.

(4) From (3) we see:
The graph $S_{17}(2, n, 2, n)$ is integral if and only if $n = l^2 - 2$ and $2n + 2 = k(k+1)$.

(5) By number theory we have to find solutions of Diophantine equation
$n = l^2 - 2$, where $2l^2 - 2 = k(k+1)$.

(6) A computer search finds the following solutions:

n	l	k	n	l	k	n	l	k
2	2	2	14	4	5	119	11	15
527	23	32	4094	64	90	/	/	/

Integral graphs $S_{17}(2, n, 2, n)$, where $1 \leq k \leq 100$.

3. " Wang, Broersma, Hoede, Li and Still, Families of integral graphs, (2004), preprint."

In [52], the graphs $K_{1,r} \bullet K_n$, $r * K_n$, $K_{1,r} \bullet K_{m,n}$ and $r * K_{m,n}$ are studied. We determine the characteristic polynomials of the four classes of graphs and obtain sufficient and necessary conditions for these graphs to be integral by using number theory and computer search. All these classes are infinite. We also give some new cospectral graphs and cospectral integral graphs.

4. "Wang, Li and Hoede, Two classes of integral regular graphs, accepted for publication in *Ars Combinatoria*."

In [57], we study the spectra and characteristic polynomials of two classes of regular graphs. We derive the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. These graphs are not only integral but also Laplacian integral. The discovery of these integral graphs is a new contribution to the research of integral graphs. These results generalize some results of Harary and Schwenk [27].

5 Some questions on integral trees and integral graphs

5.1 Some questions on integral trees

Integral trees of diameters 1, 2, 3, 4, 5, 6, 8 and 10 have been constructed in the literature. We suggest the following questions.

Question 5.1. *Are there integral trees of arbitrarily large diameter?*

Question 5.2. *Are there integral trees of diameter 7, 9, 11, 12,...?*

Question 5.3. *Can we give all integral trees of diameters 4, 5, 6, 8 and 10?*

5.2 Some questions on integral complete r-partite graphs

Assume that the number of distinct integers of $p_1, p_2, \ldots, p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < \ldots < p_s$. Suppose that a_i is the multiplicity of p_i for each $i = 1, 2, \ldots, s$. The complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$=$K_{p_1,\ldots,p_1,p_2,\ldots,p_2,\ldots,p_s,\ldots,p_s}$ is also denoted by $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_i p_i$. We posed some questions on integral complete r-partite graphs in [55].

Question 5.4. *Are there integral complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1 \cdot p_1, \ldots, a_s \cdot p_s}$ with arbitrarily large s?*

Question 5.5. *Are there integral complete r-partite graphs $K_{p_1,\ldots,p_r} = K_{a_1\cdot p_1,\ldots,a_s\cdot p_s}$, with $a_1 = a_2 = \ldots = a_s = 1$, and $s \geq 4$?*

Question 5.6. *When $s = 3, 4, \ldots$, can we give a better (see [55] for details) sufficient and necessary condition for $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ to be integral ?*

5.3 Some other questions on integral graphs

Question 5.7. *Can we determine integral 5-regular (6,7, ... regular) graphs?*

Question 5.8. *Can we determine all integral graphs on 13(14,15,...) vertices?*

Question 5.9. *What are properties of integral graphs with maximum degree 5* (6, 7, 8, ...*) ?*

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